3

The principle of superposition and its application to acoustic wave fields in configurations with geometrical symmetry

When geometrical symmetry is present in the configuration in which an acoustic wave field is to be analysed, it is always advantageous, and sometimes (especially in large-scale numerical calculations) is an absolute neccessity, to take this symmetry into account explicitly. Assuming that the configuration is linear in its acoustic behaviour, the principle of superposition yields the tool to do this. Through this principle, the total acoustic wave field is decomposed into a number of partial constituents which have the property that they only need to be calculated in a subdomain of the configuration (by the type of symmetry determined), while their superposition yields the total wave field in the entire configuration under consideration. The procedure is here elucidated for the cases of a plane of symmetry, a line of symmetry, and a point of symmetry. The indicated types of symmetry can be handled by elementary methods. For more complicated classes of symmetry, the systematic use of the methods of the theory of groups in abstract algebra is to be preferred.

3.1 The principle of superposition

Let us consider the acoustic wave field in a configuration that is linear in its acoustic behaviour. Let $\{p^{\mathrm{A}}, v_r^{\mathrm{A}}, \theta^{\mathrm{A}}, \Phi_k^{\mathrm{A}}\}$ be the field that is generated by the given source distributions $\{f_k^{\mathrm{A}}, q^{\mathrm{A}}\}$. Similarly, let $\{p^{\mathrm{B}}, v_r^{\mathrm{B}}, \theta^{\mathrm{B}}, \Phi_k^{\mathrm{B}}\}$ be the field that is generated by the given source distributions $\{f_k^{\mathrm{B}}, q^{\mathrm{B}}\}$ in the same configuration, i.e. the constitutive coefficients or relaxation functions remain the same. Then, $\{\alpha p^{\mathrm{A}} + \beta p^{\mathrm{B}}, \alpha v_r^{\mathrm{A}} + \beta v_r^{\mathrm{B}}, \alpha\theta^{\mathrm{A}} + \beta\theta^{\mathrm{B}}, \alpha\Phi_k^{\mathrm{A}} + \beta\Phi_k^{\mathrm{B}}\}$ results as the field that would be generated by the source distributions $\{\alpha f_k^{\mathrm{A}} + \beta f_k^{\mathrm{B}}, \alpha q^{\mathrm{A}} + \beta q^{\mathrm{B}}\}$, where α and β are arbitrary constants. This result is known as the *principle of superposition*; the proof follows directly from Equations (2.7-22) and (2.7-23), together with the assumed linearity of the constitutive relations and the assumed uniqueness of the causal solution of the wave problem once the sources and the medium properties have been specified. In Sections 3.2–3.4 this principle is used to write the total acoustic wave field in a configuration with geometrical symmetry as a superposition of constituents each of which needs to be calculated only in a subdomain of the configuration (by the type of symmetry determined).

3.2 Symmetry with respect to a plane

In this section we consider the acoustic wave field in a configuration that is linear in its acoustic behaviour and has a *plane of symmetry*. An orthogonal, Cartesian reference frame is chosen such that the plane of symmetry coincides with $\{\boldsymbol{x}\in\mathcal{R}^3;\ -\infty<x_1<\infty,\ -\infty<x_2<\infty,\ x_3=0\}$ (Figure 3.2-1).

Now, the acoustic wave field in the configuration will be written as the superposition of a suitably defined "odd" part, and a suitably defined "even" part, "odd" and "even" being defined in our case by the type of symmetry that the acoustic wave field shows with respect to the plane $\{x_3=0\}$. By definition, a scalar function of position is denoted as "odd" with respect to the plane $\{x_3=0\}$, if its values in two points in space that are each other's image in the plane $\{x_3=0\}$ are each other's opposite, while a scalar function of position is denoted as "even" with respect to the plane $\{x_3=0\}$ if its values in two points in space that are each other's image in

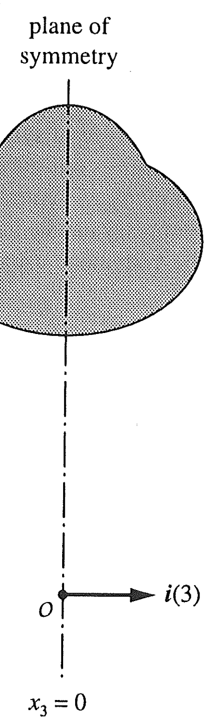

Figure 3.2-1 Configuration with a plane of symmetry $\{x_3=0\}$.

the plane $\{x_3 = 0\}$ are equal. To define "odd" and "even" for vector functions of position, we first consider the position vector $\boldsymbol{x}$, a purely geometrical quantity.

By definition, the position vectors from the origin (which is located in the plane of symmetry) to a point in space and to the image of this point with respect to the plane $\{x_3 = 0\}$ belong to the class of "even" vector functions. Apparently, the operation of imaging in the plane $\{x_3 = 0\}$, when applied to an "even" vector, leaves the part of this vector parallel to the plane of symmetry unchanged, and changes the part normal to the plane of symmetry into its opposite (Figure 3.2-2).

For an "odd" vector function, the operation of imaging in the plane of symmetry leaves the part of this vector normal to the plane of symmetry unchanged, while the part of the vector parallel to the plane of symmetry changes into its opposite (Figure 3.2-3).

These two cases serve as a guide for the definition of "odd" and "even" insofar as acoustic wave-field quantities are concerned.

In an acoustic wave field, the acoustic pressure p, the cubic dilatation θ and the volume source density of injection rate q are present as scalar functions of position (and time), and the particle velocity v_r, the mass flow density Φ_k and the volume source density of force f_k are present as vector functions of position (and time). In accordance with the convention for scalar functions and for vector functions of the position-vector type, we define an "odd" acoustic wave field (to be denoted by the superscript "o") as one for which p, θ and q are odd scalar functions of x_3, and v_r, Φ_k and f_k are odd vector functions of x_3. Similarly, we define an "even" acoustic wave field (to be denoted by the superscript "e") as one for which p, θ and q are even scalar

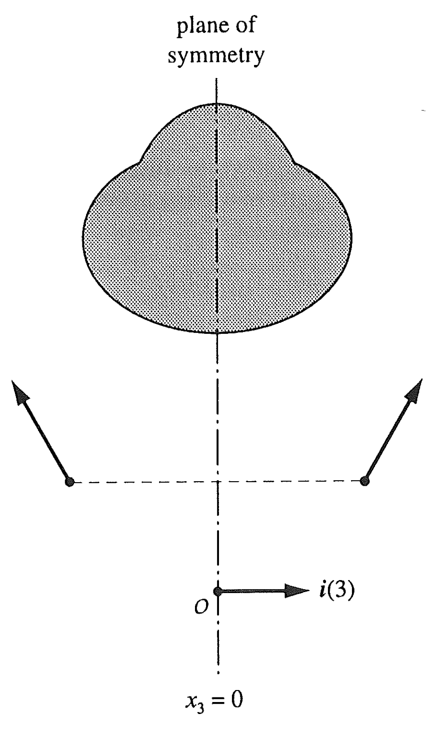

Figure 3.2-2 Even vector function in a configuration with a plane of symmetry $\{x_3=0\}$.

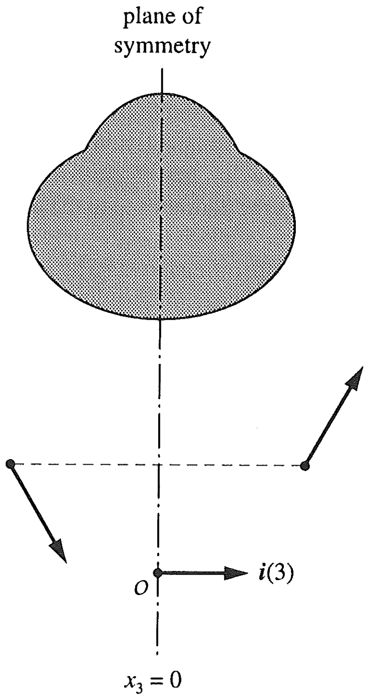

Figure 3.2-3 Odd vector function in a configuration with a plane of symmetry $\{x_3=0\}$.

functions of x_3, and v_r, Φ_k and f_k are even vector functions of x_3. In accordance with this, we construct the "odd" and the "even" parts of a given acoustic wave field as shown below.

For the "odd" part we have the definitions:

$$p^{o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[p(x_1,x_2,x_3,t) - p(x_1,x_2,-x_3,t)]\,, \tag{3.2-1}$$

with a similar definition for $\{\theta^{o},q^{o}\}$; and

$$v_1^{o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_1(x_1,x_2,x_3,t) - v_1(x_1,x_2,-x_3,t)]\,, \tag{3.2-2}$$

$$v_2^{o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_2(x_1,x_2,x_3,t) - v_2(x_1,x_2,-x_3,t)]\,, \tag{3.2-3}$$

$$v_3^{o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_3(x_1,x_2,x_3,t) + v_3(x_1,x_2,-x_3,t)]\,, \tag{3.2-4}$$

with a similar definition for $\{\Phi_k^{o},f_k^{o}\}$.

For the "even" part we have the definitions:

$$p^{e}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[p(x_1,x_2,x_3,t) + p(x_1,x_2,-x_3,t)]\,, \tag{3.2-5}$$

with a similar definition for $\{\theta^{e},q^{e}\}$; and

$$v_1^{e}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_1(x_1,x_2,x_3,t) + v_1(x_1,x_2,-x_3,t)]\,, \tag{3.2-6}$$

$$v_2^{e}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_2(x_1,x_2,x_3,t) + v_2(x_1,x_2,-x_3,t)]\,, \tag{3.2-7}$$

$$v_3^{\mathrm{e}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_3(x_1,x_2,x_3,t) - v_3(x_1,x_2,-x_3,t)]\,, \tag{3.2-8}$$

with a similar definition for $\{\Phi_k^{\mathrm{e}}, f_k^{\mathrm{e}}\}$.

With the aid of Equations (3.2-1)–(3.2-8) it can easily be verified that

$$\{p,v_r,\theta,\Phi_k,q,f_k\} = \{p,v_r,\theta,\Phi_k,q,f_k\}^{\mathrm{o}} + \{p,v_r,\theta,\Phi_k,q,f_k\}^{\mathrm{e}}. \tag{3.2-9}$$

The property that the thus defined "odd" and "even" acoustic wave fields indeed separately satisfy the acoustic wave equations, the constitutive relations, and the pertaining boundary conditions is, for homogeneous, isotropic media, easily verified.

For inhomogeneous and/or anisotropic media, the constitutive coefficients or relaxation functions must show certain symmetry properties that must be compatible with the physics of the problem. In the procedure of checking, one has to use the property that differentiations with respect to x_1, x_2 and t map odd/even functions of x_3 into odd/even functions of x_3, while differentiation with respect to x_3 maps odd/even functions of x_3 into even/odd functions of x_3.

For a medium whose constitutive relations are given by Equations (2.7-24) and (2.7-25), for a plane of symmetry $\{x_3 = 0\}$ to be present, the constitutive coefficients must exhibit the following symmetries:

$$\rho_{1,1}(x_1,x_2,x_3) = +\,\rho_{1,1}(x_1,x_2,-x_3)\,, \tag{3.2-10}$$

$$\rho_{1,2}(x_1,x_2,x_3) = +\,\rho_{1,2}(x_1,x_2,-x_3)\,, \tag{3.2-11}$$

$$\rho_{1,3}(x_1,x_2,x_3) = -\,\rho_{1,3}(x_1,x_2,-x_3)\,, \tag{3.2-12}$$

$$\rho_{2,1}(x_1,x_2,x_3) = +\,\rho_{2,1}(x_1,x_2,-x_3)\,, \tag{3.2-13}$$

$$\rho_{2,2}(x_1,x_2,x_3) = +\,\rho_{2,2}(x_1,x_2,-x_3)\,, \tag{3.2-14}$$

$$\rho_{2,3}(x_1,x_2,x_3) = -\,\rho_{2,3}(x_1,x_2,-x_3)\,, \tag{3.2-15}$$

$$\rho_{3,1}(x_1,x_2,x_3) = -\,\rho_{3,1}(x_1,x_2,-x_3)\,, \tag{3.2-16}$$

$$\rho_{3,2}(x_1,x_2,x_3) = -\,\rho_{3,2}(x_1,x_2,-x_3)\,, \tag{3.2-17}$$

$$\rho_{3,3}(x_1,x_2,x_3) = +\,\rho_{3,3}(x_1,x_2,-x_3)\,, \tag{3.2-18}$$

and

$$\kappa(x_1,x_2,x_3) = +\,\kappa(x_1,x_2,-x_3)\,. \tag{3.2-19}$$

Note that these relations are compatible with the physics of the problem, in the sense that they do not prevent the volume density of mass from being a positive definite tensor of rank two or the compressibility of being a positive scalar.

Relations similar to Equations (3.2-10)–(3.2-19) hold for the constitutive relaxation functions $\mu_{k,r}$ and χ; the relationships then have to hold for all elapsed times.

From the definitions, it is clear that once the "odd" and the "even" acoustic wave fields are known in the half-space $\{x \in \mathcal{R}^3;\ -\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ 0 < x_3 < \infty\}$, they are known in all space. Hence, Equation (3.2-9) yields the value of the total acoustic wave field in all space, once the "odd" and "even" parts of the wave field have been calculated in the half-space $\{x_3 > 0\}$ only. In analytic calculations, the latter property reveals a number of structural properties of the wave-field expressions, while in the numerical calculations less storage capacity and less computation time are needed if the field is to be computed in a less extended domain (see Exercise 3.2-3).

Acoustic wave-field values in the plane of symmetry

Some of the quantities associated with the "odd" and "even" parts of the acoustic wave field have the value zero in the plane of symmetry $\{x_3 = 0\}$. Inspection of Equations (3.2-1)–(3.2-4) reveals that if x_1 and x_2 are such that in the neighbourhood of the point $\{x_1,x_2,,0\}$ the wave field is continuous, we have

$$\lim_{x_3 \to 0} \{p^{\mathrm{o}}, v_1^{\mathrm{o}}, v_2^{\mathrm{o}}\} = 0 , \tag{3.2-20}$$

with similar relations for $\{\theta^{\mathrm{o}}, q^{\mathrm{o}}, \Phi_1^{\mathrm{o}}, \Phi_2^{\mathrm{o}}, f_1^{\mathrm{o}}, f_2^{\mathrm{o}}\}$. Inspection of Equations (3.2-5)–(3.2-8) reveals that if x_1 and x_2 are such that in the neighbourhood of the point $\{x_1,x_2,,0\}$ the wave field is continuous, we have

$$\lim_{x_3 \to 0} v_3^{\mathrm{e}} = 0 , \tag{3.2-21}$$

with a similar relation for $\{\Phi_3^{\mathrm{e}}, f_3^{\mathrm{e}}\}$.

In view of Equation (3.2-20), one often says that for the "odd" part of the acoustic wave field the plane of symmetry $\{x_3 = 0\}$ acts as a perfectly compliant wall (i.e. a wall that sustains no pressure) (Figure 3.2-4), while in view of Equation (3.2-21) one says that for the "even" part of the acoustic wave field the plane of symmetry $\{x_3 = 0\}$ acts as an immovable, perfectly rigid wall (Figure 3.2-5).

With regard to the constitutive coefficients, Equations (3.2-10)–(3.2-19) lead, in case the coefficients are continuous in the neighbourhood of the plane $\{x_3 = 0\}$, to

$$\lim_{x_3 \to 0} \{\rho_{1,3}, \rho_{2,3}, \rho_{3,1}, \rho_{3,2}\} = 0 . \tag{3.2-22}$$

The procedure outlined above can be executed in the reverse order if the actual configuration consists of a half-space $\{x \in \mathcal{R}^3;\ -\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ 0 < x_3 < \infty\}$ at the boundary $\{x_3 = 0\}$ of which either of the boundary conditions

$$\lim_{x_3 \to 0} p = 0 \tag{3.2-23}$$

or

$$\lim_{x_3 \to 0} v_3 = 0 \tag{3.2-24}$$

applies. In this case we extend the actual configuration into the half-space $\{x \in \mathcal{R}^3;$ $-\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ -\infty < x_3 < 0\}$ by assigning to the medium in this half-space constitutive coefficients that are compatible with Equations (3.2-10)–(3.2-19). Suppose, now, that in the thus extended configuration, which fills the entire three-dimensional space, the causal acoustic wave motion generated by given source distributions can be determined. Let $\{p^{\infty}, v_r^{\infty}, \theta^{\infty}, \Phi_k^{\infty}\}$ be the acoustic wave field generated in the extended configuration, as it is excited by the source distributions $\{q, f_k\}$ in the half-space $x_3 > 0$. Then, the total wave motion in the half-space $x_3 > 0$ is twice the "odd" part of the wave motion in the extended configuration if the boundary condition of Equation (3.2-23) applies and twice the "even" part of the wave motion in the extended configuration if the boundary condition of Equation (3.2-24) applies.

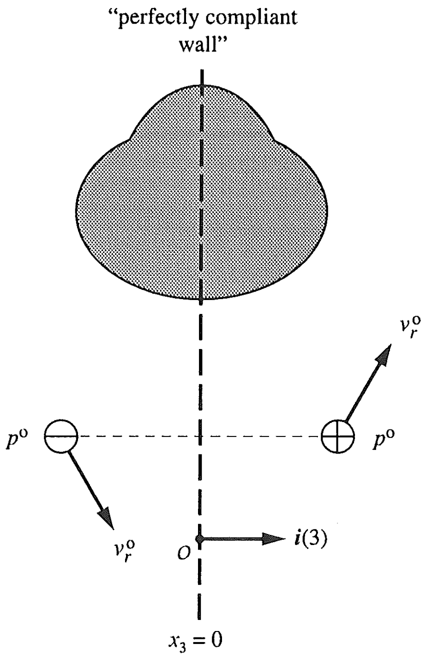

Figure 3.2-4 "Odd" acoustic wave field in a configuration with a plane of symmetry $\{x_3=0\}$; the plane of symmetry is an equivalent "perfectly compliant wall".

Exercises

Exercise 3.2-1

Prove that the "odd" and "even" parts of the acoustic wave fields do not separately carry acoustic power across the symmetry plane $\{x_3 = 0\}$. (*Hint*: Use the expression for the acoustic Poynting vector.) (Note that the total acoustic wave field in general does carry acoustic power across the symmetry plane $\{x_3 = 0\}$.)

Exercise 3.2-2

Verify that the acoustic wave fields and sources whose spatial distributions are themselves already "odd" or "even" reproduce themselves, assuming that the constitutive relations have the appropriate symmetry.

Exercise 3.2-3

Let a discretised version of the acoustic wave problem in a certain domain in space (and at each time) contain N unknowns. Assume that, through the discretisation procedure, these N unknowns satisfy a system of N linear, algebraic equations. Counting the number of arithmetic

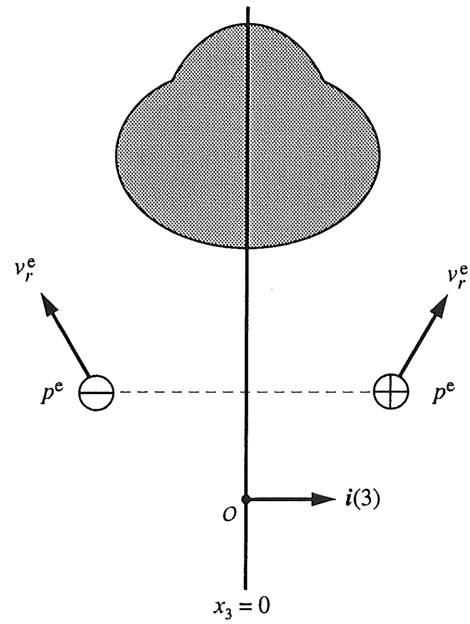

Figure 3.2-5 "Even" acoustic wave field in a configuration with a plane of symmetry $\{x_3=0\}$; the plane of symmetry is an equivalent "immovable perfectly rigid wall".

floating-point operations (multiplications and divisions) to be carried out to solve the problem, $O(N^3)$ such operations are involved. By what factor is the computation time reduced if a plane of symmetry is present in the configuration and the corresponding symmetry properties are used in the computation?

Answer: By a factor of 4.

3.3 Symmetry with respect to a line

In this section we consider the acoustic wave field in a configuration that is linear in its acoustic behaviour and has a *line of symmetry*. An orthogonal, Cartesian reference frame is chosen such that the line of symmetry coincides with $\{x \in \mathcal{R}^3;\ x_1 = 0, x_2 = 0,\ -\infty < x_3 < \infty\}$ (Figure 3.3-1). Again, each wave-field quantity in the configuration will be written as the superposition of a suitably defined "odd" part and a suitably defined "even" part, "odd" and "even" being defined in our case by the type of symmetry with respect to the line $\{x_1 = 0, x_2 = 0\}$. By definition, a scalar function of position is denoted as "odd" with respect to the line $\{x_1 = 0, x_2 = 0\}$, if its values at two points in space that are each other's image in the line $\{x_1 = 0, x_2 = 0\}$ are each other's opposite, while a scalar function of position is denoted as "even" with respect to the line $\{x_1 = 0, x_2 = 0\}$ if its values at two points in space that are each other's image in the line

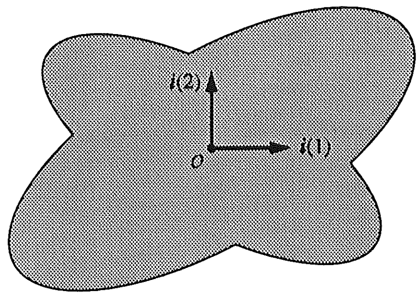

Figure 3.3-1 Configuration with a line of symmetry $\{x_1=0, x_2=0\}$.

$\{x_1=0,x_2=0\}$ are equal. To define "odd" and "even" for vector functions of position, we again borrow the pertaining result from the one applying to the position vector $\boldsymbol{x}$. By definition, the position vectors from the origin (that is located on the line of symmetry) to a point in space and to its image in the line $\{x_1=0,x_2=0\}$ belong to the class of "even" vectors. Obviously, the operation of imaging in the line $\{x_1=0,x_2=0\}$, when applied to an "even" vector, leaves the parts of this vector parallel to the line of symmetry unchanged, and changes the part in the plane normal to the line of symmetry into its opposite (Figure 3.3-2). For an "odd" vector function, the operation of imaging in the line of symmetry leaves the part of this vector in the plane normal to the line of symmetry unchanged, while the part of the vector parallel to the line of symmetry is changed into its opposite (Figure 3.3-3). So far, this applies to vectors that have the nature of a position vector in space.

For an acoustic wave field we want to introduce "odd" and "even" fields that separately can satisfy the acoustic wave equations, the constitutive relations, and the pertaining boundary conditions. Conventionally, this is done as follows: for an "odd" acoustic wave field (to be denoted by the superscript "o") the quantities $\{p^{\mathrm{o}},\theta^{\mathrm{o}},q^{\mathrm{o}}\}$ are taken to be odd scalar functions of position, while $\{v_r^{\mathrm{o}},\Phi_k^{\mathrm{o}},f_k^{\mathrm{o}}\}$ are taken to be odd vector functions of position; for an "even"

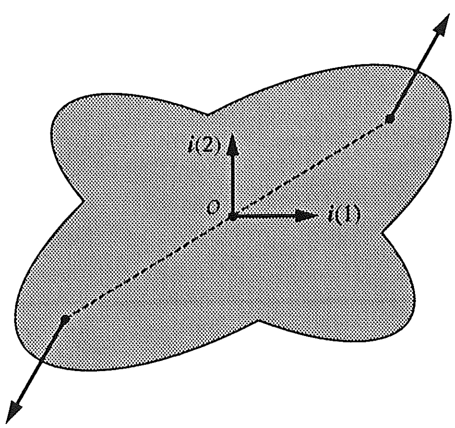

Figure 3.3-2 "Even" vector function in a configuration with a line of symmetry $\{x_1=0, x_2=0\}$.

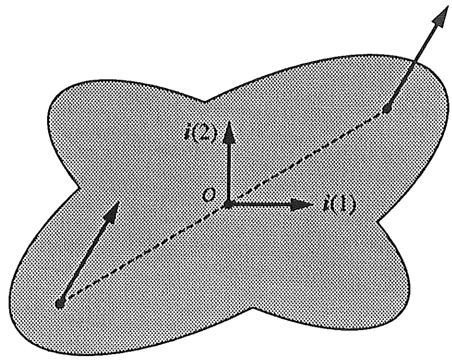

Figure 3.3-3 "Odd" vector function in a configuration with a line of symmetry $\{x_1=0, x_2=0\}$.

acoustic wave field (to be denoted by the superscript "e") the quantities $\{p^{\mathrm{e}},\theta^{\mathrm{e}},q^{\mathrm{e}}\}$ are taken to be even scalar functions of position, while $\{v_r^{\mathrm{e}},\Phi_k^{\mathrm{e}},f_k^{\mathrm{e}}\}$ are taken to be even vector functions of position (Figures 3.3-4 and 3.3-5).

In accordance with this, we have for an "odd" acoustic wave field the definitions:

$$p^{\mathrm{o}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[p(x_1,x_2,x_3,t) - p(-x_1,-x_2,x_3,t)]\,, \tag{3.3-1}$$

with a similar definition for $\{\theta^{\mathrm{o}},q^{\mathrm{o}}\}$; and

$$v_1^{\mathrm{o}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_1(x_1,x_2,x_3,t) + v_1(-x_1,-x_2,x_3,t)]\,, \tag{3.3-2}$$

$$v_2^{\mathrm{o}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_2(x_1,x_2,x_3,t) + v_2(-x_1,-x_2,x_3,t)]\,, \tag{3.3-3}$$

$$v_3^{\mathrm{o}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_3(x_1,x_2,x_3,t) - v_3(-x_1,-x_2,x_3,t)]\,, \tag{3.3-4}$$

with a similar definition for $\{\Phi_k^{\mathrm{o}},f_k^{\mathrm{o}}\}$.

For an "even" acoustic wave field we have the definitions:

$$p^{\mathrm{e}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[p(x_1,x_2,x_3,t) + p(-x_1,-x_2,x_3,t)]\,, \tag{3.3-5}$$

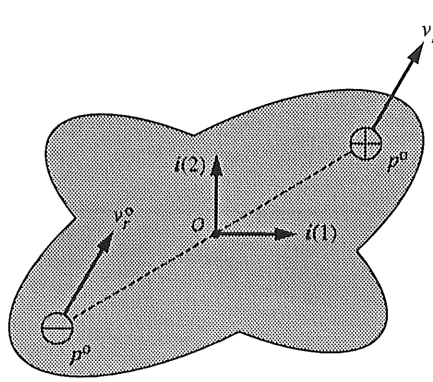

Figure 3.3-4 "Odd" acoustic wave field in a configuration with a line of symmetry $\{x_1=0, x_2=0\}$.

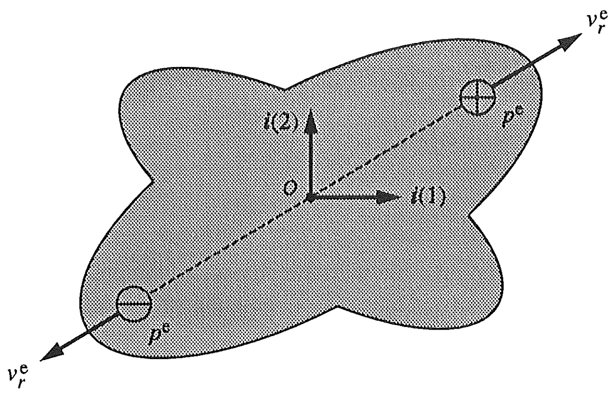

Figure 3.3-5 "Even" acoustic wave field in a configuration with a line of symmetry $\{x_1=0, x_2=0\}$.

with a similar definition for $\{\theta^e, q^e\}$; and

$$v_1^e(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_1(x_1,x_2,x_3,t) - v_1(-x_1,-x_2,x_3,t)]\,, \tag{3.3-6}$$

$$v_2^e(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_2(x_1,x_2,x_3,t) - v_2(-x_1,-x_2,x_3,t)]\,, \tag{3.3-7}$$

$$v_3^e(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_3(x_1,x_2,x_3,t) + v_3(-x_1,-x_2,x_3,t)]\,, \tag{3.3-8}$$

with similar definitions for $\{\Phi_k^e, f_k^e\}$. With the aid of Equations (3.3-1)–(3.3-8) it can be verified that

$$\{p,v_r,\theta,\Phi_k,q,f_k\} = \{p,v_r,\theta,\Phi_k,q,f_k\}^o + \{p,v_r,\theta,\Phi_k,q,f_k\}^e. \tag{3.3-9}$$

The property that the thus defined "odd" and "even" acoustic wave fields indeed satisfy separately the acoustic wave equations, the constitutive relations and the pertaining boundary conditions is, for homogeneous, isotropic media, easily verified. For inhomogeneous and/or anisotropic media, the constitutive coefficients or relaxation functions must show certain symmetry properties that must be compatible with the physics of the problem. In the procedure of checking, one has to use the property that differentiations with respect to x_3 and t map odd/even functions of x_1 and x_2 into odd/even functions of x_1 and x_2, while differentiation with respect to x_1 or x_2 maps odd/even functions of x_1 and x_2 into even/odd functions of x_1 and x_2.

For a medium whose constitutive relations are given by Equations (2.7-24) and (2.7-25), the constitutive coefficients must, for a line of symmetry $\{x_1 = 0, x_2 = 0\}$ to be present, exhibit the following symmetries:

$$\rho_{1,1}(x_1,x_2,x_3) = +\rho_{1,1}(-x_1,-x_2,x_3)\,, \tag{3.3-10}$$

$$\rho_{1,2}(x_1,x_2,x_3) = +\rho_{1,2}(-x_1,-x_2,x_3)\,, \tag{3.3-11}$$

$$\rho_{1,3}(x_1,x_2,x_3) = -\rho_{1,3}(-x_1,-x_2,x_3)\,, \tag{3.3-12}$$

$$\rho_{2,1}(x_1,x_2,x_3) = +\rho_{2,1}(-x_1,-x_2,x_3)\,, \tag{3.3-13}$$

$$\rho_{2,2}(x_1,x_2,x_3) = +\rho_{2,2}(-x_1,-x_2,x_3)\,, \tag{3.3-14}$$

$$\rho_{2,3}(x_1,x_2,x_3) = -\rho_{2,3}(-x_1,-x_2,x_3)\,, \tag{3.3-15}$$

$$\rho_{3,1}(x_1,x_2,x_3) = -\,\rho_{3,1}(-x_1,-x_2,x_3)\,, \tag{3.3-16}$$

$$\rho_{3,2}(x_1,x_2,x_3) = -\,\rho_{3,2}(-x_1,-x_2,x_3)\,, \tag{3.3-17}$$

$$\rho_{3,3}(x_1,x_2,x_3) = +\,\rho_{3,3}(-x_1,-x_2,x_3)\,, \tag{3.3-18}$$

and

$$\kappa(x_1,x_2,x_3) = +\,\kappa(-x_1,-x_2,x_3)\,. \tag{3.3-19}$$

Note that these relations are compatible with the physics of the problem in the sense that they do not prevent the volume density of mass from being a positive definite tensor of rank two or the compressibility from being a positive scalar.

Relations similar to Equations (3.3-10)–(3.3-19) hold for the constitutive relaxation functions $\mu_{k,r}$ and χ, these relationships then have to hold for all elapsed times.

From the definitions it is clear that the "odd" and the "even" acoustic wave fields are known in all space once they are known in any half-space bounded by a plane through $\{x_1 = 0, x_2 = 0\}$. Hence, Equation (3.3-9) yields the value of the total acoustic wave field in all space once the "odd" and "even" parts of the wave field have been calculated in the relevant half-space. In analytic calculations, the latter property in general reveals a number of structural properties of the wave-field expressions, while in numerical calculations less storage capacity and less computation time are needed if the field is to be computed in a less extended domain.

Acoustic wave-field values on the line of symmetry

Some of the acoustic wave-field components of the "odd" and "even" parts have the value zero on the line of symmetry $\{x_1 = 0, x_2 = 0\}$. Inspection of Equations (3.3-1)–(3.3-4) reveals that if x_3 is such that in the neighbourhood of the point $\{0, 0, x_3\}$ the wave field is continuous, we have

$$\lim_{x_1\to 0, x_2\to 0}\,\{\overset{\mathrm{o}}{p},\overset{\mathrm{o}}{v}_3\} = 0\,, \tag{3.3-20}$$

with a similar relation for $\{\overset{\mathrm{o}}{\theta},\overset{\mathrm{o}}{q},\overset{\mathrm{o}}{\Phi}_3,\overset{\mathrm{o}}{f}_3\}$. Inspection of Equations (3.3-5)–(3.3-8) reveals that if x_3 is such that in the neighbourhood of the point $\{0, 0, x_3\}$ the wave field is continuous, we have

$$\lim_{x_1\to 0, x_2\to 0}\{\overset{\mathrm{e}}{v}_1,\overset{\mathrm{e}}{v}_2\} = 0\,, \tag{3.3-21}$$

with similar relations for $\{\overset{\mathrm{e}}{\Phi}_1,\overset{\mathrm{e}}{\Phi}_2,\overset{\mathrm{e}}{f}_1,\overset{\mathrm{e}}{f}_2\}$.

Exercises

Exercise 3.3-1

Prove that the "odd" and "even" parts of the acoustic wave field do not separately carry a net acoustic power across any plane through the line of symmetry $\{x_1 = 0, x_2 = 0\}$. (*Hint*: Use the expression for the acoustic Poynting vector in two points of the plane under consideration that are each other's image in the line $\{x_1 = 0, x_2 = 0\}$.) (Note that the total acoustic wave field does in general carry a net acoustic power across such a plane.)

Exercise 3.3-2

Verify that the acoustic wave fields and sources whose spatial distributions are themselves already "odd" or "even", reproduce themselves, assuming that the constitutive relations have the appropriate symmetry.

Exercise 3.3-3

Let a discretised version of the acoustic wave problem in a certain domain in space (and at each time) contain N unknowns. Assume that, through the discretisation procedure, these N unknowns satisfy a system of N linear algebraic equations. Counting the number of arithmetic floating-point operations (multiplications and divisions) to be carried out to solve the problem, $O(N^3)$ such operations are involved. By what factor is the computation time reduced if a line of symmetry is present in the configuration and the corresponding symmetry properties are used in the computation?

Answer: By a factor of 4.

3.4 Symmetry with respect to a point

In this section we consider the acoustic wave field in a configuration that is linear in its acoustic behaviour and has a *point of symmetry*. An orthogonal, Cartesian reference frame is chosen such that the point of symmetry coincides with $\{\boldsymbol{x} \in \mathcal{R}^3; x_1 = 0, x_2 = 0, x_3 = 0\}$ (Figure 3.4-1).

As before, each wave-field quantity in the configuration will be written as the superposition of a suitably defined "odd" part, and a suitably defined "even" part, "odd" and "even" being defined in our case by the type of symmetry with respect to the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$. By definition, a scalar function of position is denoted as "odd" with respect to the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$, if its values at two points in space that are each other's image in the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$ are each other's opposite, while a scalar function of position is denoted as "even" with respect to the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$, if its values at two points in space that are each other's image in the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$ are equal. To define "odd" and "even" for vector functions of position, we again borrow the pertaining result from the one applying to the position vector $\boldsymbol{x}$. By definition, the position vectors from the origin (that is located at the point of symmetry) to a point in space and to its image in the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$, belong to the class of "even" vectors. Obviously, the operation of imaging in the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$, when applied to an "even" vector, changes all components into their opposites (Figure 3.4-2). For an "odd" vector function, the operation of imaging in the point of symmetry leaves all components of this vector unchanged (Figure 3.4-3). So far, this applies to vectors that have the nature of a position vector in space.

Now, for an acoustic wave field we want to introduce "odd" and "even" fields that separately can satisfy the acoustic wave equations, the constitutive relations, and the pertaining boundary conditions. Conventionally, this is done as follows: for an "odd" acoustic wave field (to be

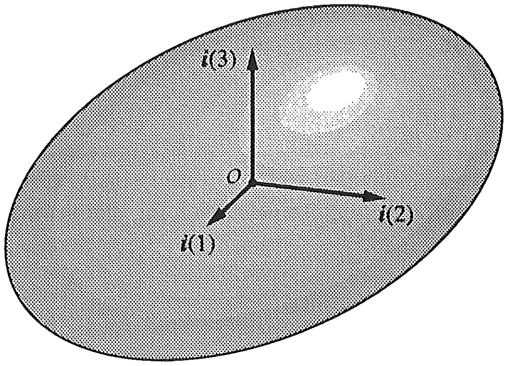

Figure 3.4-1 Configuration with a point of symmetry $\{x_1=0, x_2=0, x_3=0\}$.

denoted by the superscript "o") the quantities $\{p^{\rm o},\theta^{\rm o},q^{\rm o}\}$ are taken to be odd scalar functions of position, while $\{v_r^{\rm o},\Phi_k^{\rm o},f_k^{\rm o}\}$ are taken to be odd vector functions of position; for an "even" acoustic wave field (to be denoted by the superscript "e") the quantities $\{p^{\rm e},\theta^{\rm e},q^{\rm e}\}$ are taken to be even scalar functions of position, while $\{v_r^{\rm e},\Phi_k^{\rm e},f_k^{\rm e}\}$ are taken to be even vector functions of position (Figures 3.4-4 and 3.4-5).

In accordance with this, we have for an "odd" acoustic wave field the definitions:

$$p_1^{\rm o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[p(x_1,x_2,x_3,t) - p(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-1}$$

with a similar definition for $\{\theta^{\rm o},q^{\rm o}\}$; and

$$v_1^{\rm o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_1(x_1,x_2,x_3,t) + v_1(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-2}$$

$$v_2^{\rm o}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_2(x_1,x_2,x_3,t) + v_2(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-3}$$

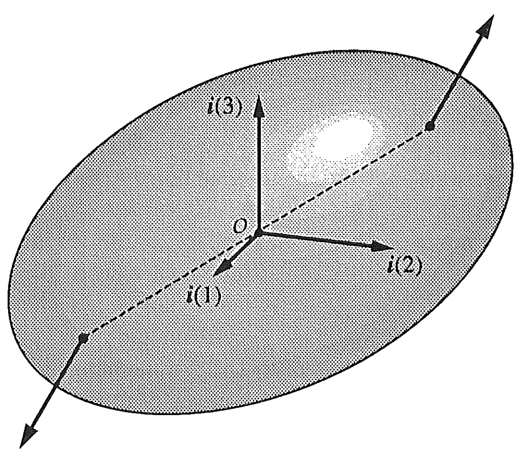

Figure 3.4-2 "Even" vector function in a configuration with a point of symmetry $\{x_1=0, x_2=0, x_3=0\}$.

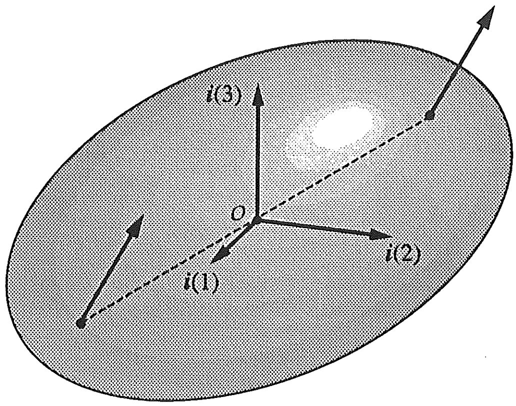

Figure 3.4-3 "Odd" vector function in a configuration with a point of symmetry $\{x_1{=}0, x_2{=}0, x_3{=}0\}$.

$$v_3^{\mathrm{o}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_3(x_1,x_2,x_3,t) + v_3(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-4}$$

with a similar definition for $\{\Phi_k^{\mathrm{o}}, f_k^{\mathrm{o}}\}$.

For an "even" acoustic wave field we have the definitions:

$$p^{\mathrm{e}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[p(x_1,x_2,x_3,t) + p(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-5}$$

with a similar definition for $\{\theta^{\mathrm{e}}, q^{\mathrm{e}}\}$; and

$$v_1^{\mathrm{e}}(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_1(x_1,x_2,x_3,t) - v_1(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-6}$$

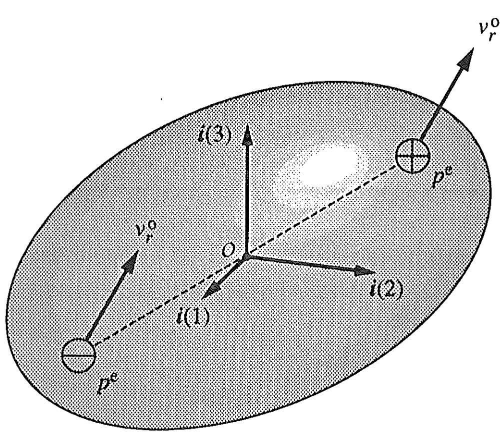

Figure 3.4-4 "Odd" acoustic wave field in a configuration with a point of symmetry $\{x_1{=}0, x_2{=}0, x_3{=}0\}$.

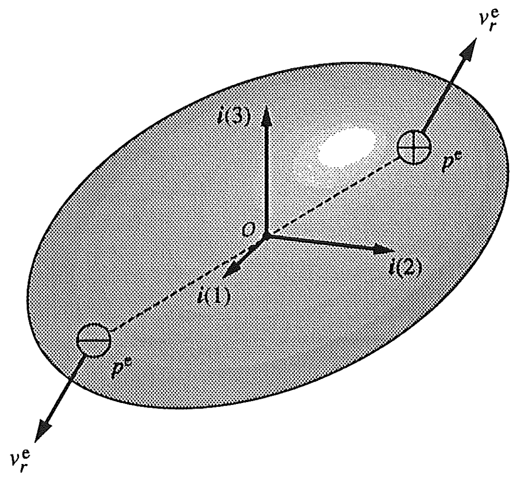

Figure 3.4-5 "Even" acoustic wave field in a configuration with a point of symmetry $\{x_1=0, x_2=0, x_3=0\}$.

$$v_2^e(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_2(x_1,x_2,x_3,t) - v_2(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-7}$$

$$v_3^e(x_1,x_2,x_3,t) = \tfrac{1}{2}\,[v_3(x_1,x_2,x_3,t) - v_3(-x_1,-x_2,-x_3,t)]\,, \tag{3.4-8}$$

with similar definitions for $\{\Phi_k^e, f_k^e\}$. With the aid of Equations (3.4-1)–(3.4-8) it is easily verified that

$$\{p,v_r,\theta,\Phi_k,q,f_k\} = \{p,v_r,\theta,\Phi_k,q,f_k\}^o + \{p,v_r,\theta,\Phi_k,q,f_k\}^e. \tag{3.4-9}$$

The property that the thus defined "odd" and "even" acoustic wave fields indeed satisfy separately the acoustic wave equations, the constitutive relations and the pertaining boundary conditions is, for homogeneous, isotropic media, easily verified. For inhomogeneous and/or anisotropic media, the constitutive coefficients or relaxation functions must show certain symmetry relations that must be compatible with the physics of the problem. In the procedure of checking, one has to use the property that differentiation with respect to t maps odd/even functions of x_1, x_2, x_3 into odd/even functions of x_1, x_2, x_3, while differentiations with respect to x_1, x_2, or x_3 map odd/even functions of x_1, x_2, and x_3 into even/odd functions of x_1, x_2, and x_3.

For a medium whose constitutive relations are given by Equations (2.7-24) and (2.7-25), the constitutive coefficients must, for a point of symmetry $\{x_1 = 0, x_2 = 0, x_3 = 0\}$ to be present, exhibit the following symmetries:

$$\rho_{1,1}(x_1,x_2,x_3) = +\rho_{1,1}(-x_1,-x_2,-x_3)\,, \tag{3.4-10}$$

$$\rho_{1,2}(x_1,x_2,x_3) = +\rho_{1,2}(-x_1,-x_2,-x_3)\,, \tag{3.4-11}$$

$$\rho_{1,3}(x_1,x_2,x_3) = +\rho_{1,3}(-x_1,-x_2,-x_3)\,, \tag{3.4-12}$$

$$\rho_{2,1}(x_1,x_2,x_3) = +\rho_{2,1}(-x_1,-x_2,-x_3)\,, \tag{3.4-13}$$

$$\rho_{2,2}(x_1,x_2,x_3) = +\rho_{2,2}(-x_1,-x_2,-x_3)\,, \tag{3.4-14}$$

$$\rho_{2,3}(x_1,x_2,x_3) = +\rho_{2,3}(-x_1,-x_2,-x_3)\,, \tag{3.4-15}$$

$$\rho_{3,1}(x_1,x_2,x_3) = +\rho_{3,1}(-x_1,-x_2,-x_3)\,, \tag{3.4-16}$$

$$\rho_{3,2}(x_1,x_2,x_3) = +\rho_{3,2}(-x_1,-x_2,-x_3)\,, \tag{3.4-17}$$

$$\rho_{3,3}(x_1,x_2,x_3) = +\rho_{3,3}(-x_1,-x_2,-x_3)\,, \tag{3.4-18}$$

and

$$\kappa(x_1,x_2,x_3) = +\kappa(-x_1,-x_2,-x_3)\,. \tag{3.4-19}$$

Note that these relations are compatible with the physics of the problem, in the sense that they do not prevent the volume density of mass from being a positive definite tensor of rank two or the compressibility from being a positive scalar.

Relations similar to Equations (3.4-10)–(3.4-19) hold for the constitutive relaxation functions $\mu_{k,r}$ and χ; these relations have to hold for all elapsed times.

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Acoustic wave-field values at the point of symmetry

Some of the field components of the "odd" and "even" parts have the value zero at the point of symmetry $\{x_1 = 0, x_2 = 0, x_3 = 0\}$. Inspection of Equations (3.4-1)–(3.4-4) reveals that, if in the neighbourhood of the point $\{0,0,0\}$ the field is continuous, then

$$\lim_{x_1\to 0, x_2\to 0, x_3\to 0} p^{\rm o} = 0\,, \tag{3.4-20}$$

with a similar relation for $\{\theta^{\rm o}, q^{\rm o}\}$. Inspection of Equations (3.4-5)–(3.4-8) reveals that, if in the neighbourhood of the point $\{0,0,0\}$ the field is continuous, then

$$\lim_{x_1\to 0, x_2\to 0, x_3\to 0} \{v_1^{\rm e}, v_2^{\rm e}, v_3^{\rm e}\} = 0\,, \tag{3.4-21}$$

with a similar relation for $\{\Phi_1^{\rm e}, \Phi_2^{\rm e}, \Phi_3^{\rm e}, f_1^{\rm e}, f_2^{\rm e}, f_3^{\rm e}\}$.

Exercises

Exercise 3.4-1

Prove that the "odd" and "even" parts of the acoustic wave field do not separately carry a net acoustic power across any plane through the point of symmetry $\{x_1 = 0, x_2 = 0, x_3 = 0\}$. (*Hint*:

Use the expression for the acoustic Poynting vector in two points of the plane under consideration that are each other's image in the point $\{x_1 = 0, x_2 = 0, x_3 = 0\}$.) (Note that the total acoustic wave field does in general carry a net acoustic power across such a plane.)

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Let a discretised version of the acoustic wave problem in a certain domain in space (and at each time) contain N unknowns. Assume that, through the discretisation procedure, these N unknowns satisfy a system of N linear, algebraic equations. Counting the number of arithmetic floating-point operations (multiplications and divisions) to be carried out to solve the problem, $O(N^3)$ such operations are involved. By what factor is the computation time reduced if a point of symmetry is present in the configuration and the corresponding symmetry properties are used in the computation?

Answer: By a factor of 4.