5

Acoustic radiation from sources in an unbounded, homogeneous, isotropic fluid

In this chapter we calculate the acoustic wave field that is causally related to the action of sources of bounded extent in an unbounded homogeneous, isotropic fluid that is linear, time invariant and locally reacting in its acoustic behaviour. The wave-field quantities (acoustic pressure and particle velocity) are determined with the aid of a spatial Fourier transformation method that is applied to the complex frequency-domain coupled acoustic wave equations that are discussed in Chapter 4. The acoustic scalar and vector source potentials are employed. Several applications are given. In particular, the radiation from monopole transducers and dipole transducers is discussed, both in the complex frequency domain and in the space–time domain, and the solution to the initial-value problem (Cauchy problem) for a lossless fluid is given.

5.1 The coupled acoustic wave equations and their solution in the angular wave-vector domain

The complex frequency-domain acoustic pressure $\hat{p}$ and particle velocity $\hat{v}_r$ in a homogeneous and isotropic fluid satisfy the complex frequency-domain coupled acoustic wave-field equations (see Equations (4.5-1) and (4.5-2))

$$\partial_k \hat{p} + \hat{\zeta}\hat{v}_k = \hat{f}_k , \tag{5.1-1}$$

$$\partial_r \hat{v}_r + \hat{\eta}\hat{p} = \hat{q} , \tag{5.1-2}$$

in which the longitudinal acoustic impedance $\hat{\zeta}$ and the transverse acoustic admittance $\hat{\eta}$ per length of the fluid are given by

$$\hat{\zeta} = s\hat{\mu} , \tag{5.1-3}$$

$$\hat{\eta} = s\hat{\chi} , \tag{5.1-4}$$

for a fluid with relaxation,

$$\hat{\zeta} = s\rho , \tag{5.1-5}$$

$$\hat{\eta} = s\kappa \,, \tag{5.1-6}$$

for an instantaneously reacting fluid, and

$$\hat{\zeta} = K + s\rho \,, \tag{5.1-7}$$

$$\hat{\eta} = \Gamma + s\kappa \,, \tag{5.1-8}$$

for a fluid with frictional-force/bulk-viscosity acoustic losses. Since the fluid is homogeneous, $\hat{\zeta} = \hat{\zeta}(s)$ and $\hat{\eta} = \hat{\eta}(s)$ are independent of position. We assume that the volume source density of force $\hat{f}_k$ and the volume source density of injection rate $\hat{q}$ only differ from zero in some bounded subdomain $\mathcal{D}^T$ of $\mathcal{R}^3$; $\mathcal{D}^T$ is the spatial support of the source distributions and is referred to as the source domain of the transmitted (or radiated) wave field (Figure 5.1-1). The influence of non-zero initial field values is assumed to have been incorporated into the volume source densities.

To solve Equations (5.1-1) and (5.1-2), we subject these equations to a three-dimensional Fourier transformation over the entire configuration space $\mathcal{R}^3$ (see Section B.2). The usefulness of this operation is associated with the property of shift invariance of the medium in all Cartesian directions. In accordance with Appendix B the spatial Fourier transformation is written as

$$\{\tilde{p},\tilde{v}_r\}(\mathrm{j}\boldsymbol{k},s) = \int_{\boldsymbol{x}\in\mathcal{R}^3} \exp(\mathrm{j}k_s x_s)\{\hat{p},\hat{v}_r\}(\boldsymbol{x},s)\,\mathrm{d}V \,, \tag{5.1-9}$$

where j denotes the imaginary unit and

$$\boldsymbol{k} = k_1\boldsymbol{i}(1) + k_2\boldsymbol{i}(2) + k_3\boldsymbol{i}(3) \qquad \text{with } \boldsymbol{k}\in\mathcal{R}^3 \tag{5.1-10}$$

is the *angular wave vector* in three-dimensional Fourier-transform or $\boldsymbol{k}$ space. According to Fourier's theorem, inversely

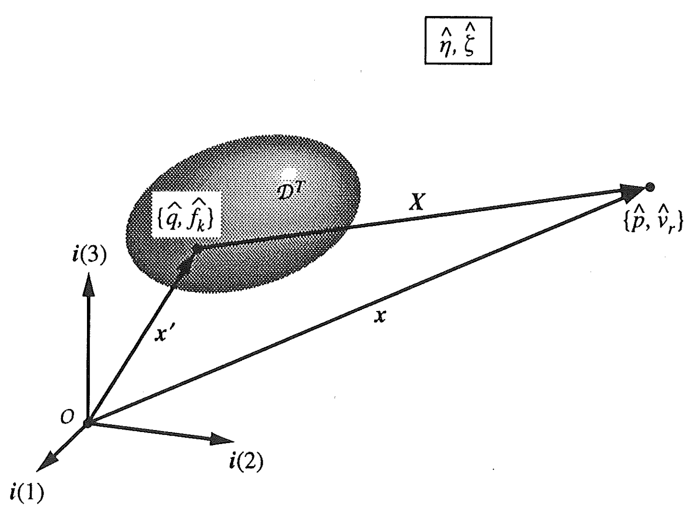

Figure 5.1-1 Sources $\{\hat{q},\hat{f}_k\}$ at position $\boldsymbol{x}'\in\mathcal{D}^T$ (source domain) generate acoustic radiation in a homogeneous, isotropic fluid with complex frequency-domain acoustic parameters $\{\hat{\eta},\hat{\zeta}\}$. The wave field $\{\hat{p},\hat{v}_r\}$ is observed at $\boldsymbol{x}\in\mathcal{R}^3$.

$$\{\hat{p},\hat{v}_r\}(x,s) = (2\pi)^{-3} \int_{k\in\mathcal{R}^3} \exp(-jk_s x_s)\{\tilde{p},\tilde{v}_r\}(jk,s)\, dV . \tag{5.1-11}$$

For the spatial derivatives we employ the relation

$$\int_{x\in\mathcal{R}^3} \exp(jk_s x_s)\partial_m\{\hat{p},\hat{v}_r\}(x,s)\, dV = -jk_m\{\tilde{p},\tilde{v}_r\}(jk,s) , \tag{5.1-12}$$

where it has been taken into account that $\hat{p}$ and $\hat{v}_r$ will, due to causality, show, for $\mathrm{Re}(s) > 0$, an exponential decay as $|x| \to \infty$. (In Section 5.3 this will be shown indeed to be true.) With this, Equations (5.1-1) and (5.1-2) transform into

$$-jk_k\tilde{p} + \hat{\zeta}\tilde{v}_k = \tilde{f}_k , \tag{5.1-13}$$

$$-jk_r\tilde{v}_r + \hat{\eta}\tilde{p} = \tilde{q} , \tag{5.1-14}$$

where

$$\{\tilde{q},\tilde{f}_k\}(jk,s) = \int_{x\in\mathcal{D}^T} \exp(jk_s x_s)\{\hat{q},\hat{f}_k\}(x,s)\, dV \tag{5.1-15}$$

is the spatial Fourier transform of the source distributions. (Note that the integration on the right-hand side is only extended over the source domain $\mathcal{D}^T$.)

To solve $\{\tilde{p},\tilde{v}_k\}$ from the linear, algebraic equations (5.1-13) and (5.1-14), we eliminate $\tilde{v}_k$ from them. To this end, $\tilde{v}_k$ is solved from Equation (5.1-13) to yield

$$\tilde{v}_k = \hat{\zeta}^{-1}(\tilde{f}_k + jk_k\tilde{p}) . \tag{5.1-16}$$

Replacing in the latter equation the subscript k by r, substituting the result in Equation (5.1-14) and multiplying through by $\hat{\zeta}$, the following equation for the acoustic pressure $\tilde{p}$ in angular wave-vector space is obtained:

$$(k_r k_r + \hat{\eta}\hat{\zeta})\tilde{p} = \hat{\zeta}\tilde{q} + jk_r\tilde{f}_r . \tag{5.1-17}$$

From Equation (5.1-17) the expression for the acoustic pressure in angular wave-vector space is obtained as

$$\tilde{p} = \tilde{G}(\hat{\zeta}\tilde{q} + jk_r\tilde{f}_r) , \tag{5.1-18}$$

in which $\tilde{G}$ is given by

$$\tilde{G} = \frac{1}{k_m k_m + \hat{\eta}\hat{\zeta}} . \tag{5.1-19}$$

Substituting the expression for $\tilde{p}$ of Equation (5.1-18) in Equation (5.1-16), the expression for the particle velocity $\tilde{v}_k$ in angular wave-vector space is obtained:

$$\tilde{v}_k = \hat{\zeta}^{-1}\hat{f}_k + jk_k\tilde{G}\tilde{q} + \hat{\zeta}^{-1}jk_k\, jk_r\tilde{G}\tilde{f}_r . \tag{5.1-20}$$

To elucidate the structural dependence of the acoustic wave-field quantities on the source distributions that generate them, we introduce the angular wave-vector domain volume injection source scalar potential

$$\tilde{\Phi}^q = \tilde{G}\tilde{q} \tag{5.1-21}$$

and the angular wave-vector domain force source vector potential

$$\tilde{\Phi}_r^f = \tilde{G}\tilde{f}_r . \tag{5.1-22}$$

In terms of these, Equation (5.1-18) can be rewritten as

$$\tilde{p} = \hat{\zeta}\tilde{\Phi}^q + \mathrm{j}k_r\tilde{\Phi}_r^f \tag{5.1-23}$$

and Equation (5.1-20) as

$$\tilde{v}_k = \hat{\zeta}^{-1}\tilde{f}_k + \mathrm{j}k_k\tilde{\Phi}^q + \hat{\zeta}^{-1}\mathrm{j}k_k\,\mathrm{j}k_r\tilde{\Phi}_r^f . \tag{5.1-24}$$

With this, the angular wave-vector domain expressions for the acoustic pressure and the particle velocity have been fully determined. In Sections 5.2 and 5.3 the inverse spatial Fourier transformation of these expressions will be carried out; this will result in the complex frequency-domain expressions for the acoustic pressure and the particle velocity of the acoustic wave field radiated by the sources. Once the function $\hat{G} = \hat{G}(\boldsymbol{x},s)$ that corresponds to $\tilde{G} = \tilde{G}(\mathrm{j}\boldsymbol{k},s)$ has been determined, elementary rules of the spatial Fourier transformation suffice to construct the total expressions. In Section 5.2, $\hat{G}$ is determined by evaluating directly the Fourier inversion integral applied to $\tilde{G}$.

Compatibility relations

Upon carrying out the operation $\varepsilon_{i,n,k}(-\mathrm{j}k_n)$ on Equation (5.1-13), we arrive at the compatibility relation

$$\varepsilon_{i,n,k}k_n\tilde{v}_k = \hat{\zeta}^{-1}\varepsilon_{i,n,k}k_n\tilde{f}_k \tag{5.1-25}$$

to be satisfied by $\tilde{v}_k$. Substitution of the expression for $\tilde{v}_k$ of Equation (5.1-20) shows that Equation (5.1-25) is automatically satisfied.

5.2 The Green's function of the scalar Helmholtz equation

In this section, the function $\hat{G} = \hat{G}(\boldsymbol{x},s)$ whose angular wave-vector space counterpart $\tilde{G} = \tilde{G}(\mathrm{j}\boldsymbol{k},s)$ was introduced in Equation (5.1-19) will be determined. This function can be considered as the basic wave function (Green's function) of the scalar Helmholtz equation. The starting point for the evaluation of $\hat{G}$ is the inverse Fourier transformation

$$\hat{G}(\boldsymbol{x},s) = (2\pi)^{-3}\int_{\boldsymbol{k}\in\mathcal{R}^3} \exp(-\mathrm{j}k_s x_s)\tilde{G}(\mathrm{j}\boldsymbol{k},s)\,\mathrm{d}V , \tag{5.2-1}$$

in which

$$\tilde{G}(\mathrm{j}\boldsymbol{k},s) = \frac{1}{k_m k_m + \hat{\gamma}^2} , \tag{5.2-2}$$

with

$$\hat{\gamma} = (\hat{\eta}\hat{\xi})^{1/2} . \tag{5.2-3}$$

In Equation (5.2-3), $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$, since $\hat{\xi} = \hat{\xi}(s)$ and $\hat{\eta} = \hat{\eta}(s)$ share this property. By applying the standard rules of the spatial Fourier transformation (see Appendix B), it is easily verified that $\tilde{G} = \tilde{G}(\mathrm{j}\boldsymbol{k},s)$ is the three-dimensional Fourier transform, over the entire configuration space $\mathcal{R}^3$, of the function $\hat{G} = \hat{G}(\boldsymbol{x},s)$ that satisfies the three-dimensional scalar Helmholtz equation with point-source excitation

$$(\partial_m\partial_m - \hat{\gamma}^2)\hat{G} = -\delta(\boldsymbol{x}) , \tag{5.2-4}$$

where $\delta(\boldsymbol{x})$ is the three-dimensional Dirac distribution (impulse function) operative at $\boldsymbol{x} = \boldsymbol{0}$.

The simplest way to evaluate the right-hand side of Equation (5.2-1) is to introduce spherical coordinates in $\boldsymbol{k}$ space with their centre at $\boldsymbol{k} = \boldsymbol{0}$ and the direction of $\boldsymbol{x}$ as the polar axis. Let θ be the angle between $\boldsymbol{k}$ and $\boldsymbol{x}$, and ϕ be the angle between the projection of $\boldsymbol{k}$ on the plane perpendicular to $\boldsymbol{x}$ and some fixed direction in this plane (Figure 5.2-1), then the range of integration is $0 \leqslant |\boldsymbol{k}| < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, while

$$k_s x_s = |\boldsymbol{k}||\boldsymbol{x}| \cos(\theta) , \tag{5.2-5}$$

$$k_m k_m = |\boldsymbol{k}|^2 , \tag{5.2-6}$$

and

$$\mathrm{d}V = |\boldsymbol{k}|^2 \sin(\theta)\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\theta\, \mathrm{d}\phi . \tag{5.2-7}$$

In the resulting right-hand side of Equation (5.2-1) we first carry out the integration with respect to ϕ. Since the integrand is independent of ϕ, this merely amounts to a multiplication by a factor of 2π. Next, we carry out the integration with respect to θ, which is elementary. After this we have, for $|\boldsymbol{x}| \neq 0$,

$$\hat{G}(\boldsymbol{x},s) = \frac{1}{4\pi^2} \int_{|\boldsymbol{k}|=0}^{\infty} \frac{|\boldsymbol{k}|^2}{|\boldsymbol{k}|^2 + \gamma^2} \left[\frac{\exp[-\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}| \cos(\theta)]}{\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|} \right]_{\theta=0}^{\pi} \mathrm{d}\,|\boldsymbol{k}|$$

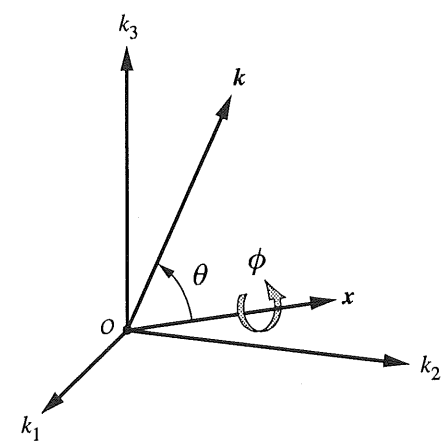

Figure 5.2-1 Integration in $\boldsymbol{k}$ space to evaluate $\hat{G}(\boldsymbol{x},s)$; $\{|\boldsymbol{k}|,\theta,\phi\}$ are the spherical polar coordinates, with $\boldsymbol{x}$ as polar axis and the ranges of integration $0 \leqslant |\boldsymbol{k}| < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$.

$$= \frac{1}{4\pi^2 \mathrm{j}|\boldsymbol{x}|} \int_{|\boldsymbol{k}|=0}^{\infty} \frac{\exp(\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|) - \exp(-\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|)}{|\boldsymbol{k}|^2 + \hat{\gamma}^2} |\boldsymbol{k}| \,\mathrm{d}|\boldsymbol{k}| . \tag{5.2-8}$$

Considering $|\boldsymbol{k}|$ as a variable that can take on arbitrary complex values and denoting this variable by k, Equation (5.2-8) can be rewritten as

$$\hat{G}(\boldsymbol{x},s) = -\frac{1}{4\pi^2 \mathrm{j}|\boldsymbol{x}|} \int_{k=-\infty}^{\infty} \frac{\exp(-\mathrm{j}k|\boldsymbol{x}|)}{k^2 + \hat{\gamma}^2} k \,\mathrm{d}k . \tag{5.2-9}$$

The integral on the right-hand side of Equation (5.2-9) is evaluated by continuing the integrand analytically into the complex k plane, supplementing the path of integration by a semi-circle situated in the lower half-plane $-\infty < \mathrm{Im}(k) \leqslant 0$ and of infinitely large radius, and applying the theorem of residues (Figure 5.2-2).

On account of Jordan's lemma, the contribution from the semi-circle at infinity vanishes. Furthermore, the only singularity of the integrand in the lower half of the complex k plane is the simple pole at $k = -\mathrm{j}\hat{\gamma}$ (note that $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$). Taking into account the residue of this pole and the fact that the contour integration is carried out clockwise instead of counter-clockwise, we arrive at

$$\hat{G}(\boldsymbol{x},s) = \exp(-\hat{\gamma}|\boldsymbol{x}|)/4\pi|\boldsymbol{x}| \qquad \text{for } |\boldsymbol{x}| \neq 0 . \tag{5.2-10}$$

This expression will be used in the process of inversely Fourier transforming the angular wave-vector domain wave quantities obtained in Section 5.1.

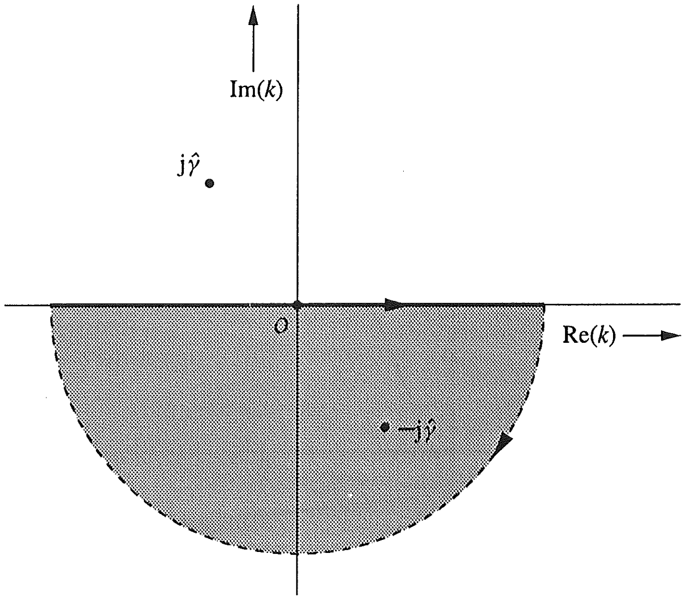

Figure 5.2-2 Integration in the complex k plane to evaluate $\hat{G}(\boldsymbol{x},s)$. Jordan's lemma and the theorem of residues are applied to the closed contour in the lower half-plane, where $\mathrm{Im}(k) < 0$. The simple pole in the lower half-plane is located at $k = -\mathrm{j}\gamma$, since $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$.

In the course of our further analysis we also need the first- and second-order spatial derivatives of $\hat{G}$. By straightforward differentiation these are obtained from Equation (5.2-10) as

$$\partial_m \hat{G}(\boldsymbol{x},s) = \frac{1}{4\pi}\left(-\frac{1}{|\boldsymbol{x}|^2} - \frac{\hat{\gamma}}{|\boldsymbol{x}|}\right)\frac{x_m}{|\boldsymbol{x}|}\exp(-\hat{\gamma}|\boldsymbol{x}|) \qquad \text{for } |\boldsymbol{x}| \neq 0\,, \tag{5.2-11}$$

and

$$\partial_k \partial_r \hat{G}(\boldsymbol{x},s) = \frac{1}{4\pi}\left[\frac{1}{|\boldsymbol{x}|^3}\left(\frac{3x_k x_r}{|\boldsymbol{x}|^2} - \delta_{k,r}\right) + \frac{\hat{\gamma}}{|\boldsymbol{x}|^2}\left(\frac{3x_k x_r}{|\boldsymbol{x}|^2} - \delta_{k,r}\right)\right.$$
$$\left. + \frac{\hat{\gamma}^2}{|\boldsymbol{x}|}\frac{x_k x_r}{|\boldsymbol{x}|^2}\right]\exp(-\hat{\gamma}|\boldsymbol{x}|) \qquad \text{for } |\boldsymbol{x}| \to 0\,. \tag{5.2-12}$$

These results will be used in our subsequent analysis.

Exercises

Exercise 5.2-1

Prove, by using Equations (5.2-1) and (5.2-2) and carrying out in the relevant Fourier integral a contour integration in the complex k_3 plane, that

$$\hat{G}(\boldsymbol{x},s) = \exp(-\hat{\gamma}|\boldsymbol{x}|)/4\pi|\boldsymbol{x}|$$
$$= \left(\frac{1}{2\pi}\right)^2 \int_{\{k_1,k_2\}\in\mathcal{R}^2} \frac{\exp\left[-\mathrm{j}(k_1x_1 + k_2x_2) - (k_1^2 + k_2^2 + \hat{\gamma}^2)^{1/2}|x_3|\right]}{2(k_1^2 + k_2^2 + \gamma^2)^{1/2}}\,\mathrm{d}k_1\mathrm{d}k_2\,. \tag{5.2-13}$$

(*Hint*: Observe that $k_3 = \pm\mathrm{j}(k_1^2 + k_2^2 + \hat{\gamma}^2)^{1/2}$ are simple poles of the analytically continued integrand (away from the real k_3 axis) in the upper and lower halves of the complex k_3 plane and that Jordan's lemma applies to a semi-circle in the lower half of the k_3 plane for $x_3 > 0$ and to a semi-circle in the upper half of the k_3 plane for $x_3 < 0$.) (The representation of Equation (12.2-13) plays a major role in the analysis of electromagnetic radiation problems in subdomains of $\mathcal{R}^3$ with parallel, planar, boundaries.)

5.3 The complex frequency-domain source-type integral representations for the acoustic pressure and the particle velocity

The complex frequency-domain source-type integral representations for the acoustic pressure and the particle velocity of the acoustic wave field radiated by the sources located in $\mathcal{D}^T$ are obtained by carrying out the inverse spatial Fourier transformation to the angular wave-vector domain expressions obtained in Equations (5.1-23) and (5.1-24). Using the rule that $-\mathrm{j}k_m$ corresponds to ∂_m (see Equation (5.1-12)), we obtain

$$\hat{p} = \hat{\zeta}\hat{\Phi}^q - \partial_r \hat{\Phi}_r^f \tag{5.3-1}$$

and

$$\hat{v}_k = \hat{\zeta}^{-1}\hat{f}_k - \partial_k \hat{\Phi}^q + \hat{\zeta}^{-1}\partial_k \partial_r \hat{\Phi}_r^f \, . \tag{5.3-2}$$

The expressions for the complex frequency-domain volume injection source scalar potential $\hat{\Phi}^q$ and the complex frequency-domain force source vector potential $\hat{\Phi}_r^f$ are obtained by carrying out the inverse spatial Fourier transformation of Equations (5.1-21) and (5.1-22), respectively. Since the product of two functions in angular wave-vector space corresponds to the convolution of these functions in configuration space (see Equations (B.2-11) and (B.2-12)), we obtain

$$\hat{\Phi}^q(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{G}(\boldsymbol{x}-\boldsymbol{x}',s)\hat{q}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{5.3-3}$$

and

$$\hat{\Phi}_k^f(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{G}(\boldsymbol{x}-\boldsymbol{x}',s)\hat{f}_k(\boldsymbol{x}',s)\,\mathrm{d}V \, , \tag{5.3-4}$$

in which $\hat{G}$ is given by (see Equation (5.2-10))

$$\hat{G}(\boldsymbol{x},s) = \exp(-\hat{\gamma}|\boldsymbol{x}|)/4\pi|\boldsymbol{x}| \qquad \text{for all } |\boldsymbol{x}| \neq 0 \, , \tag{5.3-5}$$

with (see Equation (5.2-3))

$$\hat{\gamma} = (\hat{\eta}\hat{\zeta})^{1/2} \, . \tag{5.3-6}$$

Note that on the right-hand side of Equations (5.3-3) and (5.3-4) we have taken care to distribute the arguments over the functions in the integrands such that the spatial integration is carried out over the fixed source domain $\mathcal{D}^T$ (see Figure 5.3-1). (If we had provided the source distributions with the argument $\boldsymbol{x}-\boldsymbol{x}'$ and the Green's functions with the argument $\boldsymbol{x}'$, we would have to integrate over a domain that varies with $\boldsymbol{x}$.)

Equations (5.3-1)–(5.3-6) constitute the solution to the complex frequency-domain acoustic radiation problem in an unbounded, homogeneous, isotropic fluid. The expressions are used in the calculation and computation of a great many acoustic radiation problems. In a number of simple cases, the integrals in Equations (5.3-3) and (5.3-4) can be calculated analytically, as can the subsequent differentiations in Equations (5.3-1) and (5.3-2). In more complicated cases, the integrals in Equations (5.3-3) and (5.3-4) must be computed with the aid of numerical methods. Since, however, numerical integration can be carried out with almost any desired degree of accuracy, such an evaluation presents, apart from the singularity in $\hat{G}$ at $\boldsymbol{x}' = \boldsymbol{x}$, no difficulties. Numerical differentiation, however, is much more difficult and inherently of a restricted accuracy. Therefore, it is in general advantageous to carry out all differentiations in Equations (5.3-1) and (5.3-2) analytically, which can be done since they act on the position vector $\boldsymbol{x}$ that occurs in the argument of $\hat{G}$ only (see Equations (5.3-3) and (5.3-4)). After having

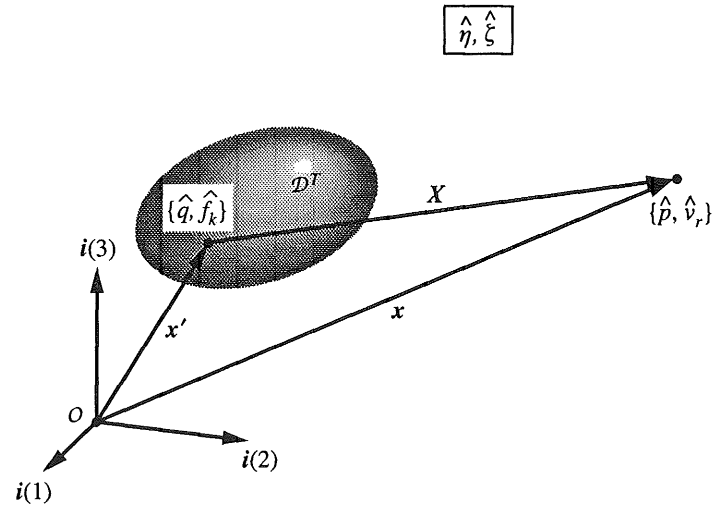

Figure 5.3-1 Complex frequency-domain source-type integral representations for the acoustic wave field $\{\hat{p},\hat{v}_r\}$ observed at position $x \in \mathcal{R}^3$, radiated by sources $\{\hat{q},\hat{f}_k\}$ at position $x' \in \mathcal{D}^T$ (bounded source domain) in an unbounded homogeneous, isotropic fluid with acoustic parameters $\{\hat{\eta},\hat{\zeta}\}$.

done this, only integrals remain to be evaluated numerically. The relevant derivatives have been evaluated already in Equations (5.2-11) and (5.2-12).

The carrying out of the spatial differentiations has different consequences for the propagation factor $\exp(-\hat{\gamma}|x - x'|)$ from source point to observation point and the (inverse) powers of the distance decay from source point to observation point. As regards the latter, it will be shown below that the following four types of terms occur:

(a) Terms independent of distance; these terms represent the *direct source field.*

(b) Terms proportional to $(\text{distance})^{-3}$; these terms represent the *near field* (also called the quasi-static field).

(c) Terms proportional to $(\text{distance})^{-2}$; these terms represent the *intermediate field* (also called the induction field).

(d) Terms proportional to $(\text{distance})^{-1}$; these terms represent the *far field* (also called the radiation field).

In addition, each term in the expressions for the acoustic pressure and the particle velocity has its own directional characteristic in which only the unit vector in the direction of observation $(x_m - x_m')/|x - x'|$ as viewed from the source point occurs.

Finally, observe that, since $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$, $\hat{p}$ and $\hat{v}_r$ indeed show an exponential decay as $|x| \to \infty$, as has been assumed in Section 5.1.

In the following, the direct-source, near-field, intermediate-field, and far-field contributions are denoted by the double superscripts DS, NF, IF, and FF, respectively. Using Equations (5.2-11) and (5.2-12) in Equations (5.3-1) and (5.3-2), and employing the notation

$$X = x - x' \tag{5.3-7}$$

for the position vector from the source point $x' \in \mathcal{D}^T$ to the observation point $x \in \mathcal{R}^3$ and

$$\Xi_m = X_m/|X| \qquad \text{for } |X| \neq 0 , \tag{5.3-8}$$

for the unit vector along the direction of X (i.e. $\Xi_m \Xi_m = 1$), we arrive at the expressions

$$\hat{p} = \hat{p}^{\,\mathrm{DS}} + \hat{p}^{\,\mathrm{NF}} + \hat{p}^{\,\mathrm{IF}} + \hat{p}^{\,\mathrm{FF}} \tag{5.3-9}$$

and

$$\hat{v}_r = \hat{v}_r^{\,\mathrm{DS}} + \hat{v}_r^{\,\mathrm{NF}} + \hat{v}_r^{\,\mathrm{IF}} + \hat{v}_r^{\,\mathrm{FF}} , \tag{5.3-10}$$

where

$$\hat{p}^{\,\mathrm{DS}}(x,s) = 0 , \tag{5.3-11}$$

$$\hat{p}^{\,\mathrm{NF}}(x,s) = 0 , \tag{5.3-12}$$

$$\hat{p}^{\,\mathrm{IF}}(x,s) = \int_{x' \in \mathcal{D}^T} \Xi_k \hat{f}_k(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|^2} \,\mathrm{d}V , \tag{5.3-13}$$

$$\hat{p}^{\,\mathrm{FF}}(x,s) = \hat{\zeta} \int_{x' \in \mathcal{D}^T} \hat{q}(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|} \,\mathrm{d}V + \hat{\gamma} \int_{x' \in \mathcal{D}^T} \Xi_k \hat{f}_k(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|} \,\mathrm{d}V , \tag{5.3-14}$$

and

$$\hat{v}_r^{\,\mathrm{DS}}(x,s) = \hat{\zeta}^{-1} \hat{f}_r(x,s) , \tag{5.3-15}$$

$$\hat{v}_r^{\,\mathrm{NF}}(x,s) = \hat{\zeta}^{-1} \int_{x' \in \mathcal{D}^T} (3\Xi_r \Xi_k - \delta_{r,k}) \hat{f}_k(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|^3} \,\mathrm{d}V , \tag{5.3-16}$$

$$\begin{aligned} \hat{v}_r^{\,\mathrm{IF}}(x,s) = {} & \int_{x' \in \mathcal{D}^T} \Xi_r \hat{q}(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|^2} \,\mathrm{d}V \\ & + \hat{\gamma}\hat{\zeta}^{-1} \int_{x' \in \mathcal{D}^T} (3\Xi_r \Xi_k - \delta_{r,k}) \hat{f}_k(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|^2} \,\mathrm{d}V , \end{aligned} \tag{5.3-17}$$

$$\begin{aligned} \hat{v}_r^{\,\mathrm{FF}}(x,s) = {} & \hat{\gamma} \int_{x' \in \mathcal{D}^T} \Xi_r \hat{q}(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|} \,\mathrm{d}V \\ & + \hat{\gamma}^2 \hat{\zeta}^{-1} \int_{x' \in \mathcal{D}^T} \Xi_r \Xi_k \hat{f}_k(x',s) \frac{\exp(-\hat{\gamma}|X|)}{4\pi|X|} \,\mathrm{d}V . \end{aligned} \tag{5.3-18}$$

Equations (5.3-9)–(5.3-18) yield the complex frequency-domain acoustic pressure and particle velocity for the acoustic wave field radiated by an arbitrary distribution of volume sources of bounded extent immersed in a homogeneous, isotropic fluid. The special cases for the point source of volume injection (acoustic monopole transducer) and the point source of force (acoustic dipole transducer) will be discussed in Sections 5.7 and 5.8, respectively.

Exercises

Exercise 5.3-1

Employ three-dimensional spatial Fourier-transform methods to construct the solution of the "graddiv" vector Helmholtz equation

$$\partial_k \partial_r \hat{F}_r - \hat{\gamma}^2 \hat{F}_k = -\hat{Q}_k \tag{5.3-19}$$

that shows an exponential decay as $|x| \to \infty$ when $\mathrm{Re}(s) > 0$. In Equation (5.3-19), $\hat{\gamma} = (\hat{\eta}\hat{\gamma})^{1/2}$ with $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$, and $\hat{Q}_k$ differs from zero in a bounded subdomain $\mathcal{D}^T$ of $\mathcal{R}^3$ only. (a) Determine the equation that results upon Fourier transforming Equation (5.3-19) according to

$$\tilde{F}_k(\mathrm{j}k,s) = \int_{x \in \mathcal{R}^3} \exp(\mathrm{j}k_s x_s) \hat{F}_k(x,s)\, \mathrm{d}V\,. \tag{5.3-20}$$

(b) Derive an expression for $k_r \tilde{F}_r$ from the resulting equation. (c) Use the expression obtained under (b) to solve for $\tilde{F}_k$. (d) Transform the expression for $\tilde{F}_k$ back to the complex frequency-domain configuration space. (e) Write the result as

$$\hat{F}_k(x,s) = \int_{x' \in \mathcal{D}^T} \hat{\Gamma}_{k,r}(x - x', s) \hat{Q}_r(x',s)\, \mathrm{d}V \tag{5.3-21}$$

and determine $\hat{\Gamma}_{k,r}$.

Answers:

(a) $k_k k_r \tilde{F}_r + \hat{\gamma}^2 \tilde{F}_k = \tilde{Q}_k$,

(b) $k_r \tilde{F}_r = k_r \tilde{Q}_r \tilde{G}$, with $\tilde{G} = (k_m k_m + \hat{\gamma}^2)^{-1}$,

(c) $\tilde{F}_k = \hat{\gamma}^{-2}[\tilde{Q}_k - k_k k_r \tilde{Q}_r \tilde{G}]$.

(d) See Equation (5.3-21), with

(e) $$\hat{\Gamma}_{k,r} = \hat{\gamma}^{-2}\,[\delta_{k,r}\delta(x) + \partial_k \partial_r \hat{G}]\,, \tag{5.3-22}$$

in which $\hat{G}(x,s) = \exp(-\hat{\gamma}|x|)/4\pi|x|$ for $|x| \neq 0$.

5.4 The time-domain source-type integral representations for the acoustic pressure and the particle velocity in a lossless fluid

In this section we investigate the case where the homogeneous, isotropic fluid in which the sources radiate, is, in addition, lossless. Then, we have

$$\hat{\zeta} = s\rho\,, \tag{5.4-1}$$

$$\hat{\eta} = s\kappa\,, \tag{5.4-2}$$

and

$$\hat{\gamma} = s/c\,, \tag{5.4-3}$$

where

$$c = (\kappa\rho)^{-1/2}\,, \tag{5.4-4}$$

in which ρ, κ and c are real, positive constants. In view of Equation (5.4-3) we now have

$$\hat{G}(x,s) = \exp(-s|x|/c)/4\pi|x| \qquad \text{for } |x| \neq 0\,. \tag{5.4-5}$$

Owing to the simple way in which the Laplace-transform parameter s occurs in the complex frequency-domain expressions, the latter's inversion back to the time domain can now be carried out with the aid of some elementary rules. The time-domain equivalents of Equations (5.3-1) and (5.3-2) contain the time-differentiated and the time-integrated forms of the source scalar and vector potentials. For the latter, we employ the notation

$$\{\mathrm{I}_t\Phi^q, \mathrm{I}_t\Phi_k^f\}(x,t) = \left\{\int_{t'=t_0}^{t} \Phi^q(x,t')\,\mathrm{d}t', \int_{t'=t_0}^{t} \Phi_k^f(x,t')\,\mathrm{d}t'\right\}, \tag{5.4-6}$$

where t_0 is the instant at which the sources have been switched on. Equation (B.1-19) and Equations (5.3-1) and (5.3-2) lead to

$$\rho\partial_t\Phi^q - \partial_r\Phi_r^f = \chi_{\mathcal{T}}(t)p(x,t) \tag{5.4-7}$$

and

$$\rho^{-1}\mathrm{I}_t f_k - \partial_k\Phi^q + \rho^{-1}\partial_k\partial_r\mathrm{I}_t\Phi_r^f = \chi_{\mathcal{T}}(t)v_r(x,t)\,, \tag{5.4-8}$$

respectively. Here, $\chi_{\mathcal{T}}(t) = \{0,\tfrac{1}{2},1\}$ for $t \in \{\mathcal{T}', \partial\mathcal{T}, \mathcal{T}\}$ is the characteristic function of the time interval $\mathcal{T} = \{t \in \mathcal{R}; t > t_0\}$. Since the result only differs from zero in the time interval that succeeds the instant at which the sources have been switched on, Equations (5.4-7) and (5.4-8) satisfy the condition of causality. Furthermore, using the rule that the product of two functions in the complex frequency domain corresponds to a convolution in the time domain, the time-domain equivalents of Equations (5.3-3) and (5.3-4) follow as

$$\Phi^q(x,t) = \int_{t' \in \mathcal{T}} \mathrm{d}t' \int_{x' \in \mathcal{D}^T} G(x - x', t - t')q(x',t')\,\mathrm{d}t'\,\mathrm{d}V \tag{5.4-9}$$

and

$$\Phi_k^f(x,t) = \int_{t' \in \mathcal{T}} \mathrm{d}t' \int_{x' \in \mathcal{D}^T} G(x - x', t - t')f_k(x',t')\,\mathrm{d}t'\,\mathrm{d}V\,, \tag{5.4-10}$$

respectively, in which, by inversion of Equation (5.4-5),

$$G(x,t) = \delta(t - |x|/c)4\pi|x| \qquad \text{for } |x| \neq 0\,. \tag{5.4-11}$$

In view of the sifting property of the Dirac delta function in Equation (5.4-11), Equations (5.4-9) and (5.4-10) can be rewritten as

$$\Phi^q(x,t) = \int_{x' \in \mathcal{D}^T} \frac{q(x', t - |x - x'|/c)}{4\pi|x - x'|}\mathrm{d}V \tag{5.4-12}$$

and

$$\Phi_k^f(\boldsymbol{x},t) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \frac{f_k(\boldsymbol{x}', t - |\boldsymbol{x}-\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \mathrm{d}V, \tag{5.4-13}$$

respectively. Expressions of the type of Equations (5.4-12) and (5.4-13) are known as *retarded potentials*: the time argument in the integrands over the spatial supports of the source distributions is delayed by the travel time $|\boldsymbol{x}-\boldsymbol{x}'|/c$ that the acoustic wave needs to traverse the distance $|\boldsymbol{x}-\boldsymbol{x}'|$ from the source point $\boldsymbol{x}'$ to the observation point $\boldsymbol{x}$ with the speed c (Figure 5.4-1). From this it follows that c is the speed (*acoustic wave speed*) by which the acoustic disturbances travel from the point where they originate to the point where they are observed. In a simple fluid of the type under consideration this wave speed is expressed in terms of the constitutive parameters through Equation (5.4-4). Typical values of c are 330 m/s (air under atmospheric conditions) and 1440 m/s (water).

As to the evaluation of the expressions occurring in Equations (5.4-7)–(5.4-13), the same remarks as in Section 5.3 apply. Here, too, if numerical evaluations are necessary, it is advantageous to carry out the differentiations with respect to the spatial coordinates analytically. As in Section 5.3 this leads to expressions that can be arranged according to their behaviour as a function of distance between the source point and the observation point; and in the time domain the notions of direct source term, near-field, intermediate-field and far-field contributions apply. The simplest way to arrive at the relevant expressions is to use Equations (5.4-1)–(5.4-4) in the complex frequency-domain expressions that are given by Equations (5.3-9)–(5.3-18) and apply the elementary rules of the inverse time Laplace transformation that have been employed earlier in this section. Upon writing

$$p = p^{\mathrm{DS}} + p^{\mathrm{NF}} + p^{\mathrm{IF}} + p^{\mathrm{FF}} \tag{5.4-14}$$

and

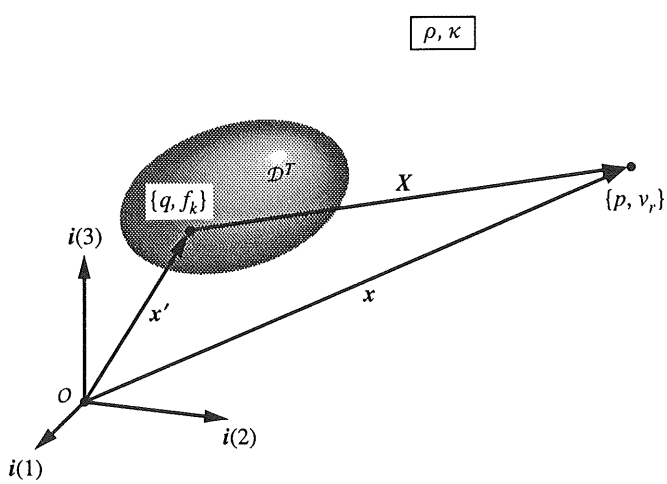

Figure 5.4-1 Time-domain source-type integral representations for the acoustic wave field $\{p,v_r\}$ observed at position $\boldsymbol{x}\in\mathcal{R}^3$, radiated by source distributions $\{q,f_k\}$ at position $\boldsymbol{x}'\in\mathcal{D}^T$ (bounded source domain) in an unbounded homogeneous, isotropic, lossless fluid with acoustic parameters $\{\rho,\kappa\}$ and wave speed $c = (\rho\kappa)^{-1/2}$.

$$v_r = v_r^{\mathrm{DS}} + v_r^{\mathrm{NF}} + v_r^{\mathrm{IF}} + v_r^{\mathrm{FF}}, \tag{5.4-15}$$

and again using the notation

$$X = x - x' \tag{5.4-16}$$

for the position vector from a source point $x' \in \mathcal{D}^T$ to the observation point $x \in \mathcal{R}^3$ and

$$\Xi_m = X_m / |X| \qquad \text{for } |X| \neq 0 \tag{5.4-17}$$

for the unit vector along X_m, we obtain

$$p^{\mathrm{DS}}(x,t) = 0\,, \tag{5.4-18}$$

$$p^{\mathrm{NF}}(x,t) = 0\,, \tag{5.4-19}$$

$$p^{\mathrm{IF}}(x,t) = \int_{x' \in \mathcal{D}^T} \Xi_k \frac{f_k(x', t - |X|/c)}{4\pi |X|^2} \mathrm{d}V\,, \tag{5.4-20}$$

$$p^{\mathrm{FF}}(x,t) = \rho \int_{x' \in \mathcal{D}^T} \frac{\partial_t q(x', t - |X|/c)}{4\pi |X|} \mathrm{d}V + \frac{1}{c} \int_{x' \in \mathcal{D}^T} \Xi_k \frac{\partial_t f_k(x', t - |X|/c)}{4\pi |X|} \mathrm{d}V\,, \tag{5.4-21}$$

and

$$v_r^{\mathrm{DS}}(x,t) = \rho^{-1} \mathrm{I}_t f_r(x,t)\,, \tag{5.4-22}$$

$$v_r^{\mathrm{NF}}(x,t) = \rho^{-1} \int_{x' \in \mathcal{D}^T} (3\Xi_r \Xi_k - \delta_{r,k}) \frac{\mathrm{I}_t f_k(x', t - |X|/c)}{4\pi |X|^3} \mathrm{d}V\,, \tag{5.4-23}$$

$$v_r^{\mathrm{IF}}(x,t) = \int_{x' \in \mathcal{D}^T} \Xi_r \frac{q(x', t - |X|/c)}{4\pi |X|^2} \mathrm{d}V$$

$$+ \frac{1}{\rho c} \int_{x' \in \mathcal{D}^T} (3\Xi_r \Xi_k - \delta_{r,k}) \frac{f_k(x', t - |X|/c)}{4\pi |X|^2} \mathrm{d}V\,, \tag{5.4-24}$$

$$v_r^{\mathrm{FF}}(x,t) = \frac{1}{c} \int_{x' \in \mathcal{D}^T} \Xi_r \frac{\partial_t q(x', t - |X|/c)}{4\pi |X|} \mathrm{d}V$$

$$+ \frac{1}{\rho c^2} \int_{x' \in \mathcal{D}^T} \Xi_r \Xi_k \frac{\partial_t f_k(x', t - |X|/c)}{4\pi |X|} \mathrm{d}V\,. \tag{5.4-25}$$

Note that in the direct source term as well as in the near-field expressions the time-integrated pulse shapes of the source densities occur; the intermediate-field expressions involve the pulse shapes of the source densities themselves, and the far-field expressions involve their time-differentiated pulse shapes.

Exercises

Exercise 5.4-1

The problem in Exercise 5.3-1 is reconsidered for the case where $\hat{\gamma} = s/c$. Determine the time-domain result that follows from Equations (5.3-21) and (5.3-22).

Answer:

$$F_k(\boldsymbol{x},t) = c^2 \mathrm{I}_t^2 Q_k(\boldsymbol{x},t) + c^2 \partial_k \partial_r \int_{\boldsymbol{x}' \in \mathcal{D}^T} \frac{\mathrm{I}_t^2 Q_r(\boldsymbol{x}', t - |\boldsymbol{x}-\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \mathrm{d}V . \tag{5.4-26}$$

5.5 The Green's function of the dissipative scalar wave equation

The wave equation that corresponds to the complex frequency-domain scalar Helmholtz equations (see Equation (5.2-4))

$$(\partial_m \partial_m - \hat{\gamma}^2)\hat{G} = -\delta(\boldsymbol{x}) , \tag{5.5-1}$$

with

$$\hat{\gamma} = c^{-1} [(s+\alpha)(s+\beta)]^{1/2} , \tag{5.5-2}$$

where c is a real, positive constant and α and β are real, non-negative constants, is here called the dissipative Helmholtz equation for point-source excitation. It obviously applies to the fluid with acoustic frictional-force/bulk-viscosity losses (for which $\alpha = K/\rho$, $\beta = \Gamma/\kappa$, $c = (\kappa\rho)^{-1/2}$), but also occurs in other branches of mathematical physics. With the aid of some standard rules of the time Laplace transformation, the time-domain equivalent of Equation (5.5-1) (i.e. the *dissipative wave equation* with point source excitation) is

$$\partial_m \partial_m G - c^{-2}(\partial_t + \alpha)(\partial_t + \beta) G = -\delta(\boldsymbol{x},t) , \tag{5.5-3}$$

or

$$\partial_m \partial_m G - c^{-2}\partial_t^2 G - [(\alpha+\beta)/c^2]\, \partial_t G - (\alpha\beta/c^2) G = -\delta(\boldsymbol{x},t) . \tag{5.5-4}$$

The solution $G = G(\boldsymbol{x},t)$ of this equation is known as the Green's function of the dissipative wave equation. The ranges of the parameter values in the dissipative wave equation are: $0 < c < \infty$, $\alpha \leqslant 0 < \infty$, $\beta \leqslant 0 < \infty$. From the results it will be clear that c is the *wave speed*; α and β are *relaxation parameters*.

The solution of Equation (5.5-1) is still given by (see Equation (5.2-10))

$$\hat{G} = \exp(-\hat{\gamma}|\boldsymbol{x}|)/4\pi|\boldsymbol{x}| \qquad \text{for } |\boldsymbol{x}| \neq 0 . \tag{5.5-5}$$

The time-domain equivalent $G = G(\boldsymbol{x},t)$ of $\hat{G} = \hat{G}(\boldsymbol{x},s)$, as given by Equation (5.5-5) is obtained by first determining the time-domain equivalent of the function

$$\hat{U}_0 = \hat{U}_0(\alpha,\beta,T,s) = \frac{\exp(-\hat{\gamma}cT)}{\hat{\gamma}c} = \frac{\exp\{-[(s+\alpha)(s+\beta)]^{1/2}T\}}{[(s+\alpha)(s+\beta)]^{1/2}} . \tag{5.5-6}$$

The latter is done by evaluating the Bromwich inversion integral

$$U_0(\alpha,\beta,T,t) = \frac{1}{2\pi \mathrm{j}} \int_{s\in \mathrm{Br}} \exp(st)\hat{U}_0(\alpha,\beta,T,s)\,\mathrm{d}s\,, \tag{5.5-7}$$

where $\mathrm{Br} = \{s\in\mathcal{C};\ \mathrm{Re}(s) = s_0\}$ is the Bromwich path, in which, in view of the condition of causality, s_0 is chosen so large that $\hat{U}_0$ is analytic in the half-plane $\{s\in\mathcal{C};\ \mathrm{Re}(s) > s_0\}$ to the right of Br. As we will evaluate the integral on the right-hand side of Equation (5.5-7) by closing the contour to the left, it is necessary to identify the singularities of the integrand in the half-plane $\{s\in\mathcal{C};\ \mathrm{Re}(s) < s_0\}$ to the left of Br. As such we encounter the branch points $s = -\alpha$ and $s = -\beta$ on the negative real s axis, which are associated with the square-root expression. The corresponding branch cuts are chosen such that $\mathrm{Re}(s+\alpha)^{1/2} \geqslant 0$ and $\mathrm{Re}(s+\beta)^{1/2} \geqslant 0$ for all $s\in\mathcal{C}$; they run along $\{s\in\mathcal{C};\ -\infty<\mathrm{Re}(s)<-\alpha,\ \mathrm{Im}(s) = 0\}$ and $\{s\in\mathcal{C};\ -\infty<\mathrm{Re}(s)<-\beta,\ \mathrm{Im}(s) = 0\}$, respectively, i.e. along the negative real s axis from the pertaining branch points to infinity (Figure 5.5-1).

For the product $[(s+\alpha)(s+\beta)]^{1/2}$ this implies that only a branch cut remains along $\{s\in\mathcal{C};\ -\max(\alpha,\beta)<\mathrm{Re}(s)<-\min(\alpha,\beta), \mathrm{Im}(s) = 0\}$, i.e. along the finite portion of the negative real s axis between the two branch points. In accordance with this choice of the branch cuts we have the asymptotic relationship

$$[(s+\alpha)(s+\beta)]^{1/2} = s + O(1) \qquad \text{as } |s|\to\infty\,, \tag{5.5-8}$$

uniformly in $\arg(s)$. On account of this relation, Jordan's lemma can be used to supplement the Bromwich contour with a semi-circle to the right for $t < T$ and with a semi-circle to the left for $t > T$. (The contribution from both semi-circles then vanishes in the limit $|s|\to\infty$.) Now, in view of Cauchy's theorem, the integration along the resulting closed contour yields the value zero for $t < T$, while for $t > T$ the resulting integral is replaced by the one that is contracted along the branch cut (Figure 5.5-2).

In the latter integral, the variable of integration s is replaced by ψ according to

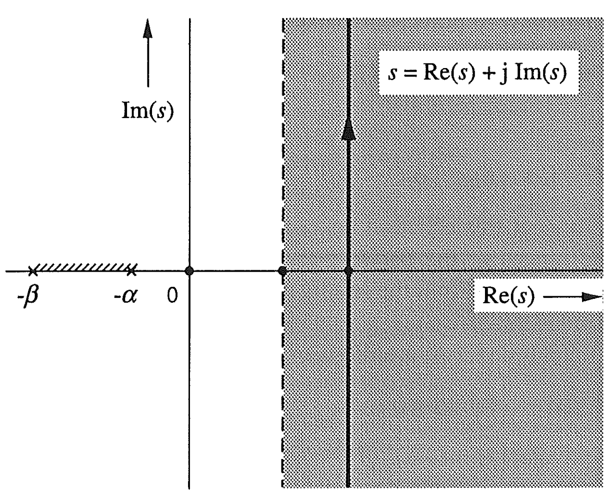

Figure 5.5-1 Bromwich contour in the complex s plane, and branch cuts, for the evaluation of the Green's function of the dissipative wave equation with relaxation parameters α and β.

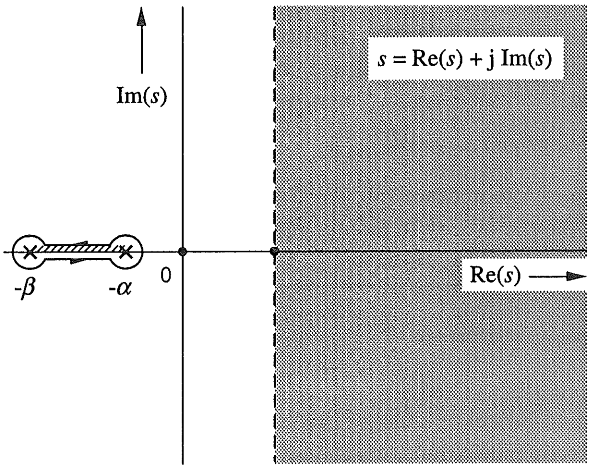

Figure 5.5-2 Contour around the branch cut of $[(s+\alpha)(s+\beta)]^{1/2}$ for the evaluation of the Green's function of the dissipative wave equation with relaxations parameters α and β.

$$s = -(\alpha+\beta)/2 + (|\beta-\alpha|/2)\cos(\psi) \qquad \text{with } 0 \leqslant \psi < 2\pi\,, \tag{5.5-9}$$

through which

$$ds = -(|\beta-\alpha|/2)\sin(\psi)\,d\psi \tag{5.5-10}$$

and, with the given definition of the square root,

$$[(s+\alpha)(s+\beta)]^{1/2} = j(|\beta-\alpha|/2)\sin(\psi)\,. \tag{5.5-11}$$

Using this in Equations (5.5-6) and (5.5-7), we arrive at

$$U_0(\alpha,\beta,T,t) = \exp\{-[(\alpha+\beta)/2]t\}\,\frac{1}{2\pi}\int_{\psi=0}^{2\pi} \exp\{(|\beta-\alpha|/2)\,[t\cos(\psi) - jT\sin(\psi)]\}\,d\psi$$

$$\text{for } t > T\,. \tag{5.5-12}$$

To reduce the integral on the right-hand side to a recognizable form, we introduce the parameter τ through

$$\cosh(\tau) = \frac{t}{(t^2-T^2)^{1/2}} \qquad \text{for } T<t<\infty\,, \tag{5.5-13}$$

which maps the interval $T<t<\infty$ onto $0<\tau<\infty$. Equation (5.5-13) implies that

$$\sinh(\tau) = \frac{T}{(t^2-T^2)^{1/2}} \qquad \text{for } T<t<\infty\,, \tag{5.5-14}$$

and, hence,

$$t\cos(\psi) - jT\sin(\psi) = (t^2-T^2)^{1/2}\cos(\psi + j\tau)\,. \tag{5.5-15}$$

The resulting integrand is continued analytically into the complex ψ plane away from the real interval $0 \leqslant \psi < 2\pi$, during which continuation it remains analytic and periodic in ψ with period 2π. Upon introducing

$$X = (|\beta - \alpha|/2)(t^2 - T^2)^{1/2} \qquad \text{for } T < t < \infty \,, \tag{5.5-16}$$

next shifting the path of integration from $\psi = 0$ to $\psi = 2\pi$, to $\psi = -\mathrm{j}\tau$ to $-\mathrm{j}\tau + 2\pi$, which is permitted in view of the periodicity of the integrand and Cauchy's theorem, and subsequently putting $\psi = \mathrm{j}\tau + \theta$ with $0 \leqslant \theta < 2\pi$, we have

$$\frac{1}{2\pi}\int_{\psi=0}^{2\pi} \exp[X\cos(\psi + \mathrm{j}\tau)]\,\mathrm{d}\psi = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \exp[X\cos(\theta)]\,\mathrm{d}\theta = I_0(X)\,, \tag{5.5-17}$$

where I_0 is the modified Bessel function of the first kind and order zero (see Abramowitz and Stegun, 1964a). Collecting the results, we end up with

$$U_0(\alpha,\beta,T,t) = \exp\{-[(\alpha+\beta)/2]t\} I_0\left[(|\beta-\alpha|/2)\,(t^2 - T^2)^{1/2}\right] H(t-T)\,, \tag{5.5-18}$$

where H denotes the Heaviside unit step function. Figure 5.5-3 shows U_0 as a function of t/T for different values of αT and βT.

Since $I_0(0) = 1$, the initial value $U_0(\alpha,\beta,T,T)$ of U_0 is found to be

$$U_0(\alpha,\beta,T,T) = \exp\{-[(\alpha+\beta)/2]T\}\,, \tag{5.5-19}$$

while, asymptotically,

$$U_0(\alpha,\beta,T,t) \sim [2\pi(|\beta-\alpha|/2)t]^{-1/2} \exp[-\min(\alpha,\beta)t] \qquad \text{as } t \to \infty\,. \tag{5.5-20}$$

When $\beta = \alpha$, as $I_0(0) = 1$ we have,

$$U_0(\alpha,\alpha,T,t) = \exp(-\alpha t) H(t-T)\,. \tag{5.5-21}$$

Now that U_0 has been determined, we return to the evaluation of G, for which $\hat{G}$ is given by Equation (5.5-5). To this end, we first observe that

$$\hat{\gamma}|\boldsymbol{x}| = [(s+\alpha)(s+\beta)]^{1/2}|\boldsymbol{x}|/c\,. \tag{5.5-22}$$

Now, differentiation of Equation (5.5-6) with respect to T yields

$$-\partial_T \hat{U}_0(\alpha,\beta,T,s) = \exp\{-[(s+\alpha)(s+\beta)]^{1/2}T\} \tag{5.5-23}$$

and, hence,

$$\exp(-\hat{\gamma}|\boldsymbol{x}|) = [-\partial_T \hat{U}_0(\alpha,\beta,T,s)]_{T=|\boldsymbol{x}|/c}\,. \tag{5.5-24}$$

Upon introducing

$$\hat{U}_1(\alpha,\beta,T,s) = -\partial_T \hat{U}_0(\alpha,\beta,T,s) = \hat{\gamma} c \hat{U}_0(\alpha,\beta,T,s) \tag{5.5-25}$$

and, correspondingly,

$$U_1(\alpha,\beta,T,t) = -\partial_T U_0(\alpha,\beta,T,t)\,, \tag{5.5-26}$$

the expression for $\hat{G}$ can be written as

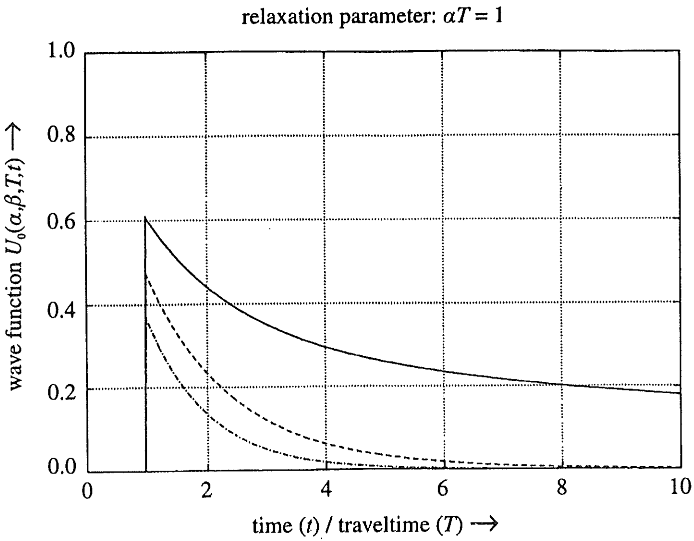

Figure 5.5-3 The wave function $U_0 = U_0(\alpha,\beta,T,t)$ as a function of normalised time t/T, with αT and βT as normalised relaxation parameters. (——): $\alpha T = 1.0$, $\beta T = 0.0$; (- - -): $\alpha T = 1.0$, $\beta T = 0.5$; (-·-·-): $\alpha T = 1.0$, $\beta T = 1.0$.

$$\hat{G} = \frac{\hat{U}_1(\alpha,\beta,|\boldsymbol{x}|/c,s)}{4\pi|\boldsymbol{x}|} \quad \text{for } |\boldsymbol{x}| \neq 0 . \tag{5.5-27}$$

Hence,

$$G = \frac{U_1(\alpha,\beta,|\boldsymbol{x}|/c,t)}{4\pi|\boldsymbol{x}|} \quad \text{for } |\boldsymbol{x}| \neq 0 . \tag{5.5-28}$$

Carrying out the differentiation with respect to T in Equation (5.5-18), U_1 is found to be

$$U_1(\alpha,\beta,T,t) = \exp\{-[(\alpha+\beta)/2]T\}\delta(t-T) + \exp\{-[(\alpha+\beta)/2]t\} \times \frac{(|\beta-\alpha|/2)T}{(t^2-T^2)^{1/2}} I_1\left[(|\beta-\alpha|/2)(t^2-T^2)^{1/2}\right] H(t-T) , \tag{5.5-29}$$

in which $\delta(t-T)$ is the Dirac distribution operative at $t = T$, and

$$I_1(X) = \partial_X I_0(X) \tag{5.5-30}$$

is the modified Bessel function of the first kind and order one (see Abramowitz and Stegun, 1964b).

For the special case $\beta = \alpha$ we have

$$U_1(\alpha,\alpha,T,t) = \exp(-\alpha T)\delta(t-T) . \tag{5.5-31}$$

As compared with the lossless case, for which

$$G = \frac{U_1(0,0,|\boldsymbol{x}|/c,t)}{4\pi|\boldsymbol{x}|} = \frac{\delta(t-|\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|} \quad \text{for } |\boldsymbol{x}| \neq 0 , \tag{5.5-32}$$

the first term on the right-hand side of Equation (5.5-29) is an attenuated delta pulse to the attenuation of which both relaxation parameters α and β contribute in an equal manner, while the second term represents a tail that is absent in the lossless case and that yields an asymptotic contribution

$$U_1(\alpha,\beta,T,t) \sim (|\beta-\alpha|T/2)[2\pi(|\beta-\alpha|/2)t^3]^{-1/2}\exp[-\min(\alpha,\beta)t] \qquad \text{as } t\to\infty , \qquad (5.5\text{-}33)$$

to the asymptotic decay of which only $\min(\alpha,\beta)$, i.e. the smaller of the two relaxation parameters, contributes and which has an amplitude decay proportional to $t^{-3/2}$.

The results of this section will be used to determine the time-domain source-type integral representations for the acoustic pressure and the particle velocity in a fluid with frictional-force/bulk-viscosity acoustic losses. In the relevant expressions we also need the first- and second-order spatial derivatives of $G = G(\alpha,\beta,|\boldsymbol{x}|/c,t)$, and hence of $U_1 = U_1(\alpha,\beta,|\boldsymbol{x}|/c,t)$. For the derivatives of the latter, denoting $|\boldsymbol{x}|/c$ by T, we employ the notation

$$U_2(\alpha,\beta,T,t) = -\partial_T U_1(\alpha,\beta,T,t) \qquad (5.5\text{-}34)$$

and

$$U_3(\alpha,\beta,T,t) = -\partial_T U_2(\alpha,\beta,T,t) . \qquad (5.5\text{-}35)$$

Although these derivatives can be expressed in terms of higher order modified Bessel functions of the first kind, the expressions become somewhat complicated, and they are probably not the most efficient ones for their numerical evaluation. With regard to the latter aspect, the integral representation that we started with (Equation (5.5-12)) seems a more promising point of departure. This will be discussed briefly at the end of this section.

By straightforward differentiation of Equation (5.5-27) the spatial derivatives of G needed in our further analysis are obtained as

$$\partial_m G(\boldsymbol{x},t) = \frac{1}{4\pi}\left(-\frac{U_1}{|\boldsymbol{x}|^2}-\frac{U_2}{c|\boldsymbol{x}|}\right)\frac{x_m}{|\boldsymbol{x}|} \qquad \text{for } |\boldsymbol{x}|\neq 0 \qquad (5.5\text{-}36)$$

and

$$\partial_r\partial_k G(\boldsymbol{x},t) = \frac{1}{4\pi}\left[\frac{U_1}{|\boldsymbol{x}|^3}\left(\frac{3x_r x_k}{|\boldsymbol{x}|^2}-\delta_{r,k}\right) + \frac{U_2}{c|\boldsymbol{x}|^2}\left(\frac{3x_r x_k}{|\boldsymbol{x}|^2}-\delta_{r,k}\right)+\frac{U_3}{c^2|\boldsymbol{x}|}\frac{x_r x_k}{|\boldsymbol{x}|^2}\right] \qquad \text{for } |\boldsymbol{x}|\neq 0 . \qquad (5.5\text{-}37)$$

These results will be used in Section 5.6.

Numerical evaluation of the Green's function and its derivatives

For the computation of the Green's function and its derivatives the integral representation Equation (5.5-12) for U_0 can profitably be taken as the point of departure. First of all, it is observed that

$$U_0(\alpha,\beta,T,t) = U_0(\beta,\alpha,T,t) , \qquad (5.5\text{-}38)$$

so that, as far as the parameters α and β are concerned, the range of computed values can be restricted to, for example, $\beta \geqslant \alpha$. Next, we shall show that

$$U_0(\alpha,\beta,T,T) = \exp\{\,-[(\alpha+\beta)/2]T\}\ . \tag{5.5-39}$$

To this end, it is observed that, in view of the analyticity of the integrand in the complex ψ plane and its periodicity in ψ with period 2π, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\psi=0}^{2\pi} \exp[(|\beta-\alpha|/2)T\exp(-\mathrm{j}\psi)]\,\mathrm{d}\psi \\
&= \lim_{\tau\to\infty}\frac{1}{2\pi}\int_{\psi=-\mathrm{j}\tau}^{-\mathrm{j}\tau+2\pi} \exp[(|\beta-\alpha|/2)T\exp(-\mathrm{j}\psi)]\,\mathrm{d}\psi \\
&= \lim_{\tau\to\infty}\frac{1}{2\pi}\int_{\psi'=0}^{2\pi} \exp[(|\beta-\alpha|/2)T\exp(-\tau-\mathrm{j}\psi')]\,\mathrm{d}\psi' \\
&= 1\ ,
\end{aligned} \tag{5.5-40}$$

by which Equation (5.5-39) follows. For $t > T$, and by subdividing the range of integration into four parts of $\pi/2$ each, Equation (5.5-12) is rewritten as

$$\begin{aligned}
U_0(\alpha,\beta,t,T) = \frac{1}{\pi}\int_{\psi=0}^{\pi/2} &\big[\exp\{-(\beta/2)t[1-\cos(\psi)] - (\alpha/2)t[1+\cos(\psi)]\} \\
&+ \exp\{-(\beta/2)t[1+\cos(\psi)] - (\alpha/2)t[1-\cos(\psi)]\}\big]\cos[(|\beta-\alpha|/2)T\sin(\psi)]\,\mathrm{d}\psi\ ,
\end{aligned} \tag{5.5-41}$$

in which we have taken care that all exponential functions have non-positive arguments to avoid the loss of significant figures. It is noted that Equation (5.5-41) shows symmetry in α and β. Equation (5.5-41) is suitable for numerical evaluation with the aid of any standard integration rule (for example, the trapezoidal rule). The corresponding representations for U_1, U_2 and U_3 directly follow from Equation (5.5-41) by differentiation with respect to T (see Equations (5.5-26), (5.5-34) and (5.5-35)).

Exercises

Exercise 5.5-1

What is the complex frequency-domain equivalent of $U_2 = U_2(\alpha,\beta,T,t)$ as given by Equation (5.5-34)?

Answer:

$$\hat{U}_2(\alpha,\beta,T,s) = (\hat{\gamma}c)^2\hat{U}_0(\alpha,\beta,T,s) = \hat{\gamma}c\,\exp(-\hat{\gamma}cT)\ .$$

Exercise 5.5-2

What is the complex frequency-domain equivalent of $U_3 = U_3(\alpha,\beta,T,t)$ as given by Equation (5.5-35)?

Answer:

$$\hat{U}_3(\alpha,\beta,T,s) = (\hat{\gamma}c)^3\hat{U}_0(\alpha,\beta,T,s) = (\hat{\gamma}c)^2\exp(-\hat{\gamma}cT)\,.$$

5.6 Time-domain source-type integral representations for the acoustic pressure and the particle velocity in a fluid with frictional-force/bulk-viscosity losses

In this section we investigate the case where the homogeneous, isotropic fluid in which the sources radiate has acoustic losses of the frictional-force/bulk-viscosity type. We have

$$\hat{\zeta} = (s+\alpha)\rho\,, \tag{5.6-1}$$

$$\hat{\eta} = (s+\beta)\kappa\,, \tag{5.6-2}$$

$$\hat{\gamma} = [(s+\alpha)(s+\beta)]^{1/2}/c\,, \tag{5.6-3}$$

with

$$\alpha = K/\rho,\ \beta = \Gamma/\kappa,\ c = (\kappa\rho)^{-1/2}\,, \tag{5.6-4}$$

in which ρ, κ and c are real positive constants and K, Γ, α and β are real, non-negative constants. In view of Equation (5.6-3), we now have

$$\hat{G}(\boldsymbol{x},s) = \frac{\exp\{-[(s+\alpha)(s+\beta)]^{1/2}\,|\boldsymbol{x}|/c\}}{4\pi|\boldsymbol{x}|} \qquad \text{for } |\boldsymbol{x}| \neq 0\,. \tag{5.6-5}$$

With the aid of the expressions derived in Section 5.5, Equation (5.6-5) leads to the time-domain result (see Equation (5.5-28))

$$G(\boldsymbol{x},t) = \frac{U_1(\alpha,\beta,|\boldsymbol{x}|/c,t)}{4\pi|\boldsymbol{x}|} \qquad \text{for } |\boldsymbol{x}| \neq 0\,, \tag{5.6-6}$$

in which U_1 is given by Equation (5.5-29). Applying standard rules of the time Laplace transformation, the time-domain expressions for the acoustic pressure and the particle velocity are obtained from Equations (5.3-1)–(5.3-6). In the result we use the property that the factor $s+\alpha$ corresponds to the operation $\partial_t + \alpha$ and that the factor $(s+\alpha)^{-1}$ corresponds to the operation of time convolution with the function $\exp(-\alpha t)H(t)$. For these operations we employ the notation

$$\partial_t^{\alpha}\{q,f_k\}(\boldsymbol{x},t) = (\partial_t + \alpha)\{q,f_k\}(\boldsymbol{x},t) \tag{5.6-7}$$

and

$$\mathrm{I}_t^{\alpha}\{q,f_k\}(\boldsymbol{x},t) = \left[\int_{t'=0}^{\infty} \exp(-\alpha t')\{q,f_k\}(\boldsymbol{x},t-t')\,\mathrm{d}t'\right]H(t)\,. \tag{5.6-8}$$

With this, we obtain (see Equations (5.3-1) and (5.3-2))

$$\rho\partial_t^{\alpha}\Phi^q - \partial_r\Phi_r^f = \chi_{\mathcal{T}}(t)p(\boldsymbol{x},t) \tag{5.6-9}$$

and

$$\rho^{-1}\mathrm{I}_t^{\alpha} f_k - \partial_k \Phi^q + \rho^{-1}\partial_k\partial_r \mathrm{I}_t^{\alpha}\Phi_r^f = \chi_{\mathcal{T}}(t) v_r(x,t)\,, \tag{5.6-10}$$

in which (see Equations (5.3-3) and (5.3-4))

$$\Phi^q(x,t) = \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} G(x-x',t-t') q(x',t') \mathrm{d}V\,, \tag{5.6-11}$$

and

$$\Phi_r^f(x,t) = \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} G(x-x',t-t') f_r(x',t') \mathrm{d}V\,. \tag{5.6-12}$$

(see Figure 5.6-1).

As to the evaluation of the expressions occurring in Equations (5.6-9)–(5.6-12), the same remarks as in Section 5.4 apply. Here, too, if numerical evaluations are necessary, it is advantageous to carry out the differentiations with respect to the spatial coordinates analytically. This leads to expressions that can be arranged according to their behaviour as a function of the distance between the source point and the observation point, and the notions of direct source term, near-field, intermediate-field and far-field contributions apply. Upon writing

$$p = p^{\mathrm{DS}} + p^{\mathrm{NF}} + p^{\mathrm{IF}} + p^{\mathrm{FF}} \tag{5.6-13}$$

and

$$v_r = v_r^{\mathrm{DS}} + v_r^{\mathrm{NF}} + v_r^{\mathrm{IF}} + v_r^{\mathrm{FF}}\,, \tag{5.6-14}$$

and using again the notations

$$X = x - x' \tag{5.6-15}$$

for the position vector from a source point $x'\in\mathcal{D}^T$ to the observation point $x\in\mathcal{R}^3$ and

$$\Xi_m = X_m/|X| \qquad \text{for } |X| \neq 0 \tag{5.6-16}$$

for the unit vector along X_m, we obtain

$$p^{\mathrm{DS}}(x,t) = 0\,, \tag{5.6-17}$$

$$p^{\mathrm{NF}}(x,t) = 0\,, \tag{5.6-18}$$

$$p^{\mathrm{IF}}(x,t) = \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} \Xi_k f_k(x',t') \frac{U_1(\alpha,\beta,|X|/c,t-t')}{4\pi|X|^2} \mathrm{d}V\,, \tag{5.6-19}$$

$$\begin{aligned} p^{\mathrm{FF}}(x,t) = {} & \rho \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} \partial_{t'}^{\alpha} q(x',t') \frac{U_1(\alpha,\beta,|X|/c,t-t')}{4\pi|X|} \mathrm{d}V \\ & + \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} \Xi_k f_k(x',t') \frac{U_2(\alpha,\beta,|X|/c,t-t')}{4\pi c|X|} \mathrm{d}V\,, \end{aligned} \tag{5.6-20}$$

and

$$v_k^{\mathrm{DS}}(x,t) = \rho^{-1}\mathrm{I}_t^{\alpha} f_r(x,t)\,, \tag{5.6-21}$$

$$v_r^{\mathrm{NF}}(x,t) = \rho^{-1}\int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{x'\in\mathcal{D}^T} (3\Xi_r\Xi_k - \delta_{r,k})\, \mathrm{I}_{t'}^{\alpha} f_k(x',t') \frac{U_1(\alpha,\beta,|X|/c,t-t')}{4\pi|X|^3} \mathrm{d}V\,, \tag{5.6-22}$$

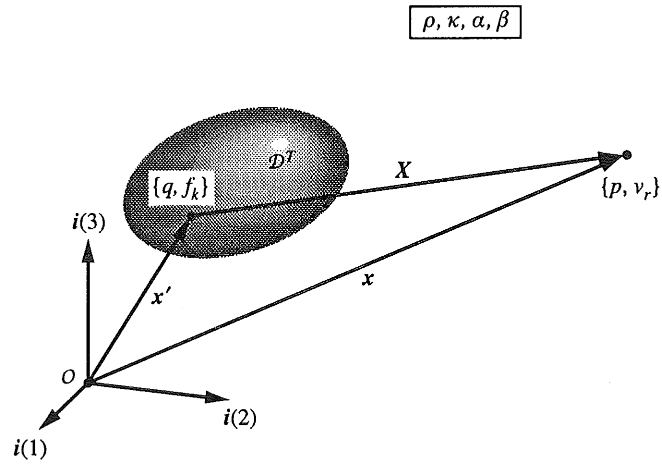

Figure 5.6-1 Time-domain source-type integral representations for the acoustic wave field $\{p,v_r\}$ observed at position $\boldsymbol{x}\in\mathcal{R}^3$, radiated by source distributions $\{q,f_k\}$ at position $\boldsymbol{x}\in\mathcal{D}^T$ (bounded source domain) in an unbounded, homogeneous, isotropic fluid with frictional-force/bulk-viscosity losses (acoustic parameters $\{\rho,\kappa,\alpha = K/\rho,\beta = \Gamma/\kappa\}$).

$$v_r^{\mathrm{IF}}(\boldsymbol{x},t) = \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{\boldsymbol{x}'\in\mathcal{D}^T} \Xi_r q(\boldsymbol{x}',t') \frac{U_1(\alpha,\beta,|\boldsymbol{X}|/c,t-t')}{4\pi|\boldsymbol{X}|^2}\mathrm{d}V$$

$$+\rho^{-1}\int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{\boldsymbol{x}'\in\mathcal{D}^T} (3\Xi_r\Xi_k - \delta_{r,k})\mathrm{I}_{t'}^{\alpha} f_k(\boldsymbol{x}',t') \frac{U_2(\alpha,\beta,|\boldsymbol{X}|/c,t-t')}{4\pi c|\boldsymbol{X}|^2}\mathrm{d}V, \quad (5.6\text{-}23)$$

$$v_r^{\mathrm{FF}}(\boldsymbol{x},t) = \int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{\boldsymbol{x}'\in\mathcal{D}^T} \Xi_r q(\boldsymbol{x}',t') \frac{U_2(\alpha,\beta,|\boldsymbol{X}|/c,t-t')}{4\pi c|\boldsymbol{X}|}\mathrm{d}V$$

$$+\rho^{-1}\int_{t'\in\mathcal{T}} \mathrm{d}t' \int_{\boldsymbol{x}'\in\mathcal{D}^T} \Xi_r\Xi_k\mathrm{I}_{t'}^{\alpha} f_k(\boldsymbol{x}',t') \frac{U_3(\alpha,\beta,|\boldsymbol{X}|/c,t-t')}{4\pi c^2|\boldsymbol{X}|}\mathrm{d}V. \quad (5.6\text{-}24)$$

Thus, the time-domain expressions for the acoustic pressure and the particle velocity radiated by sources of bounded extent immersed in a homogeneous, isotropic fluid with frictional-force/bulk-viscosity acoustic losses have been determined.

5.7 The acoustic wave field emitted by a monopole transducer

In this section we give the expressions for the acoustic pressure and the particle velocity of the acoustic wave field emitted by a monopole transducer. An acoustic *monopole transducer* is a *point source of volume injection*, i.e. a source of volume injection the maximum diameter of whose spatial support is negligibly small with respect to the distance from the location of the source to the point of observation and negligibly small with respect to the spatial extent of the pulsed emitted wave. The monopole point source is taken to be located at the point with position vector $\boldsymbol{b}$. The monopole transducer is a good model for a piezoelectric transducer vibrating in its dominant thickness mode (Figure 5.7-1).

Complex frequency-domain expressions for the emitted acoustic wave field

Let $\hat{q} = \hat{q}(\boldsymbol{x},s)$ denote the complex frequency-domain volume density of the time rate of volume injection of the source, then we have (see Equations (5.3-3) and (5.3-4))

$$\hat{\Phi}^q(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{G}(\boldsymbol{x}-\boldsymbol{x}',s)\hat{q}(\boldsymbol{x}',s)\,\mathrm{d}V \quad \text{and} \quad \hat{\Phi}^f_k(\boldsymbol{x},s) = 0\,, \tag{5.7-1}$$

while the expressions for the acoustic pressure and the particle velocity reduce to (see Equations (5.3-1) and (5.3-2))

$$\hat{p} = \zeta\hat{\Phi}^q \tag{5.7-2}$$

and

$$\hat{v}_k = -\partial_k\hat{\Phi}^q. \tag{5.7-3}$$

Let now the source domain $\mathcal{D}^T$ be of sufficiently small maximum diameter and let it be centred around the point $\boldsymbol{x}' = \boldsymbol{b}$ (for example, its barycentre). Then, Equation (5.7-1) reduces to

$$\hat{\Phi}^q(\boldsymbol{x},s) = \hat{Q}(s)\hat{G}(\boldsymbol{X},s)\,, \tag{5.7-4}$$

where

$$\hat{Q}(s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{q}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{5.7-5}$$

is the complex frequency-domain time rate of injected volume, and

$$\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{b} \tag{5.7-6}$$

is the position vector from the point $\boldsymbol{b}\in\mathcal{R}^3$ where the source is located to the point $\boldsymbol{x}\in\mathcal{R}^3$ of observation. Substitution of Equation (5.7-4) in Equations (5.7-2) and (5.7-3) leads to expressions for the acoustic pressure and the particle velocity that can be written as (see Equations (5.3-9)–(5.3-18))

$$\{\hat{p},\hat{v}_r\} = \{\hat{p},\hat{v}_r\}^{\mathrm{DS}} + \{\hat{p},\hat{v}_r\}^{\mathrm{NF}} + \{\hat{p},\hat{v}_r\}^{\mathrm{IF}} + \{\hat{p},\hat{v}_r\}^{\mathrm{FF}}, \tag{5.7-7}$$

in which the direct source contribution follows as

$$\hat{p}^{\mathrm{DS}}(\boldsymbol{x},s) = 0\,, \tag{5.7-8}$$

$$\hat{v}_r^{\mathrm{DS}}(\boldsymbol{x},s) = 0\,, \tag{5.7-9}$$

the near-field contribution as

$$\hat{p}^{\mathrm{NF}}(\boldsymbol{x},s) = 0\,, \tag{5.7-10}$$

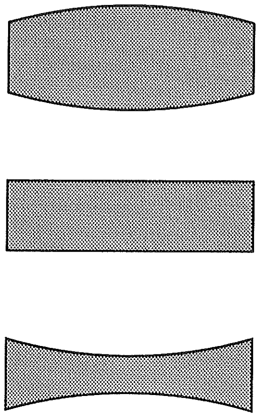

Figure 5.7-1 Acoustic monopole transducer: piezoelectric transducer vibrating in its dominant thickness mode (net volume injection, zero net force).

$$\hat{v}_r^{\mathrm{NF}}(\boldsymbol{x},s) = 0 \,, \tag{5.7-11}$$

the intermediate-field contribution as

$$\hat{p}^{\mathrm{IF}}(\boldsymbol{x},s) = 0 \,, \tag{5.7-12}$$

$$\hat{v}_r^{\mathrm{IF}}(\boldsymbol{x},s) = \hat{Q}(s)\Xi_r \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|^2} \,, \tag{5.7-13}$$

and the far-field contribution as

$$\hat{p}^{\mathrm{FF}}(\boldsymbol{x},s) = \hat{\zeta}\hat{Q}(s) \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|} \,, \tag{5.7-14}$$

$$\hat{v}_r^{\mathrm{FF}}(\boldsymbol{x},s) = \hat{\gamma}\hat{Q}(s)\Xi_r \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|} \,. \tag{5.7-15}$$

In these expressions,

$$\Xi_m = X_m/|\boldsymbol{X}| \qquad \text{for } |\boldsymbol{X}| \neq 0 \tag{5.7-16}$$

is the unit vector in the direction of $\boldsymbol{X}$.

All expressions have the complex frequency-domain propagation factor $\exp(-\hat{\gamma}|\boldsymbol{X}|)$ in common. As far as the distance from the location of the source to the point of observation is concerned, the direct source term and the near-field contribution are absent, while the intermediate-field contribution is proportional to $(\text{distance})^{-2}$, and the far-field contribution is proportional to $(\text{distance})^{-1}$. Furthermore, each acoustic wave constituent exhibits a particular directional pattern. The acoustic pressure is an omnidirectional scalar (Figure 5.7-2). The particle velocity is an omnidirectional vector which only has a component along Ξ_r (Figure 5.7-3), and hence is longitudinal in view of the fact that the direction of propagation of the wave is radially outward.

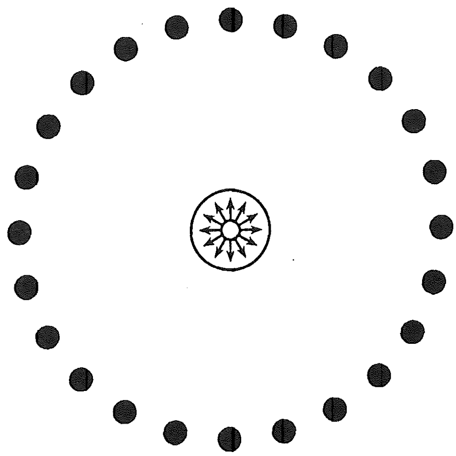

Figure 5.7-2 Acoustic monopole radiation characteristic $Q\Xi_m\Xi_m$ of the acoustic pressure in the far-field region.

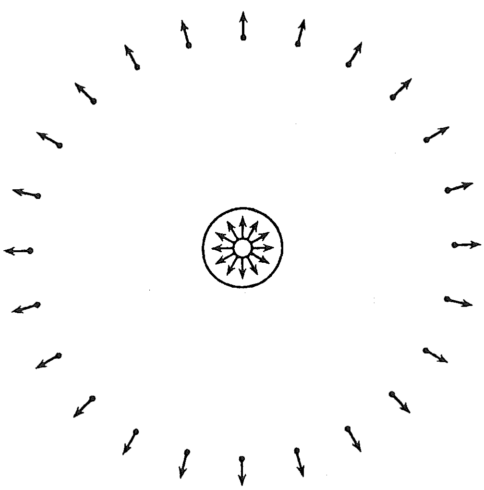

Figure 5.7-3 Acoustic monopole radiation characteristic $Q\Xi_r$ of the particle velocity in the intermediate-field and far-field regions.

Time-domain expressions for the emitted acoustic wave field in a lossless fluid

For the acoustic radiation in a homogeneous, isotropic, lossless fluid the corresponding time-domain expressions readily follow from the results of Section 5.4. We arrive at (see Equations (5.4-14) and (5.4-15) and Equations (5.4-18)–(5.4-25))

$$\{p,v_r\} = \{p,v_r\}^{\mathrm{DS}} + \{p,v_r\}^{\mathrm{NF}} + \{p,v_r\}^{\mathrm{IF}} + \{p,v_r\}^{\mathrm{FF}}, \tag{5.7-17}$$

in which the direct source contribution follows as

$$p^{\mathrm{DS}}(\boldsymbol{x},t) = 0\,, \tag{5.7-18}$$

$$v_r^{\mathrm{DS}}(\boldsymbol{x},t) = 0\,, \tag{5.7-19}$$

the near-field contribution as

$$p^{\mathrm{NF}}(\boldsymbol{x},t) = 0\,, \tag{5.7-20}$$

$$v_r^{\mathrm{NF}}(\boldsymbol{x},t) = 0\,, \tag{5.7-21}$$

the intermediate-field contribution as

$$p^{\mathrm{IF}}(\boldsymbol{x},t) = 0\,, \tag{5.7-22}$$

$$v_r^{\mathrm{IF}}(\boldsymbol{x},t) = \Xi_r \frac{Q(t-|\boldsymbol{X}|/c)}{4\pi|\boldsymbol{X}|^2}\,, \tag{5.7-23}$$

and the far-field contribution as

$$p^{\mathrm{FF}}(\boldsymbol{x},t) = \rho\,\frac{\partial_t Q(t-|\boldsymbol{X}|/c)}{4\pi|\boldsymbol{X}|}\,, \tag{5.7-24}$$

$$v_r^{\mathrm{FF}}(\boldsymbol{x},t) = \Xi_r \frac{\partial_t Q(t-|\boldsymbol{X}|/c)}{4\pi c|\boldsymbol{X}|}\,. \tag{5.7-25}$$

All time-domain expressions have the travel time delay $|\boldsymbol{X}|/c$ in common. As regards the dependence of the different expressions on the distance from the location of the source $\boldsymbol{b}$ to the point of observation $\boldsymbol{x}$ and the directional patterns of the different expressions, the same remarks as for the complex frequency-domain results apply.

Time-domain expressions for the emitted acoustic wave field in a fluid with frictional-force/bulk-viscosity acoustic losses

For the acoustic radiation in a homogeneous, isotropic fluid with frictional-force/bulk-viscosity acoustic losses, the time-domain expressions for the acoustic pressure and the particle velocity readily follow from the results of Section 5.6. With (see Equations (5.6-13) and (5.6-14) and Equations (5.6-17)–(5.6-24))

$$\{p,v_r\} = \{p,v_r\}^{\mathrm{DS}} + \{p,v_r\}^{\mathrm{NF}} + \{p,v_r\}^{\mathrm{IF}} + \{p,v_r\}^{\mathrm{FF}}, \tag{5.7-26}$$

the direct source contribution follows as

$$p^{\mathrm{DS}}(\boldsymbol{x},t) = 0\,, \tag{5.7-27}$$

$$v_r^{\mathrm{DS}}(\boldsymbol{x},t) = 0\,, \tag{5.7-28}$$

the near-field contribution as

$$p^{\mathrm{NF}}(\boldsymbol{x},t) = 0\,, \tag{5.7-29}$$

$$v_r^{\mathrm{NF}}(\boldsymbol{x},t) = 0\,, \tag{5.7-30}$$

the intermediate-field contribution as

$$p^{\mathrm{IF}}(x,t) = 0 \,, \tag{5.7-31}$$

$$v_r^{\mathrm{IF}}(x,t) = \frac{\Xi_r}{4\pi|X|^2} \int_{t' \in \mathcal{T}} U_1(\alpha,\beta,|X|/c,t-t')Q(t')\,\mathrm{d}t' \,, \tag{5.7-32}$$

and the far-field contribution as

$$p^{\mathrm{FF}}(x,t) = \frac{\rho}{4\pi|X|} \int_{t' \in \mathcal{T}} U_1(\alpha,\beta,|X|/c,t-t')\partial_{t'}^{\alpha} Q(t')\,\mathrm{d}t' \,, \tag{5.7-33}$$

$$v_r^{\mathrm{FF}}(x,t) = \frac{\Xi_r}{4\pi c|X|} \int_{t' \in \mathcal{T}} U_2(\alpha,\beta,|X|/c,t-t')Q(t')\,\mathrm{d}t' \,. \tag{5.7-34}$$

In these expressions, the wave functions U_1 and U_2 are representative of the propagation along the straight path from the source point to the point of observation and for the attenuation due to the losses along this path. As regards the decay with distance and the directional patterns the same remarks apply as for the previous results in this section.

5.8 The acoustic wave field emitted by a dipole transducer

In this section we give the expressions for the acoustic pressure and the particle velocity of the acoustic wave field emitted by a dipole transducer. An acoustic *dipole transducer* is a *point source of volume force*, i.e. a source of volume force the maximum diameter of whose spatial support is negligibly small with respect to the distance from the location of the source to the point of observation and negligibly small with respect to the spatial extent of the pulsed emitted wave. The dipole source is taken to be located at the point with position vector $\boldsymbol{b}$. The dipole transducer is a good model for a piezoelectric transducer vibrating in its dominant flexural mode (Figure 5.8-1).

Complex frequency-domain expressions for the emitted acoustic wave field

Let $\hat{f}_k = \hat{f}_k(x,s)$ denote the complex frequency-domain volume source density of force, then we have (see Equations (5.3-3) and (5.3-4))

$$\hat{\Phi}_k^f(x,s) = \int_{x' \in \mathcal{D}^T} \hat{G}(x-x',s)\hat{f}_k(x',s)\,\mathrm{d}V \quad \text{and} \quad \hat{\Phi}^q = 0 \,, \tag{5.8-1}$$

while the expressions for the acoustic pressure and the particle velocity reduce to (see Equations (5.3-1) and (5.3-2))

$$\hat{p} = -\partial_r \hat{\Phi}_r^f \tag{5.8-2}$$

and

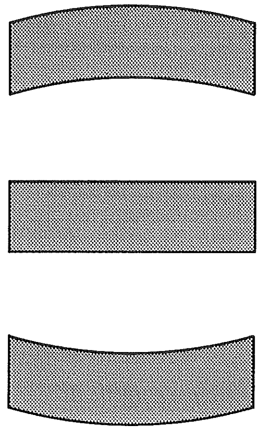

Figure 5.8-1 Acoustic dipole transducer: piezoelectric transducer vibrating in its dominant flexural mode (zero net volume injection, net force).

$$\hat{v}_k = \hat{\zeta}^{-1}\hat{f}_k + \hat{\zeta}^{-1}\partial_k\partial_r\hat{\Phi}_r^f \,. \tag{5.8-3}$$

Let now the source domain $\mathcal{D}^T$ be of sufficiently small maximum diameter and let it be centred around the point $\boldsymbol{x}' = \boldsymbol{b}$ (for example, its barycentre). Then, Equation (5.8-1) reduces to

$$\hat{\Phi}_k^f(\boldsymbol{x},s) = \hat{G}(\boldsymbol{X},s)\hat{F}_k(s) \,, \tag{5.8-4}$$

where

$$\hat{F}_k(s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \hat{f}_k(\boldsymbol{x}',s)\,\mathrm{d}V \tag{5.8-5}$$

is the complex frequency-domain source force, and

$$\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{b} \tag{5.8-6}$$

is the position vector from the point $\boldsymbol{b}\in\mathcal{R}^3$ where the source is located to the point $\boldsymbol{x}\in\mathcal{R}^3$ of observation. Substitution of Equation (5.8-4) in Equations (5.8-2) and (5.8-3) leads to expressions for the acoustic pressure and the particle velocity that can be written as (see Equations (5.3-9)–(5.3-18))

$$\{\hat{p},\hat{v}_r\} = \{\hat{p},\hat{v}_r\}^{\mathrm{DS}} + \{\hat{p},\hat{v}_r\}^{\mathrm{NF}} + \{\hat{p},\hat{v}_r\}^{\mathrm{IF}} + \{\hat{p},\hat{v}_r\}^{\mathrm{FF}} \,, \tag{5.8-7}$$

in which the direct source contribution follows as

$$\hat{p}^{\mathrm{DS}}(\boldsymbol{x},s) = 0 \,, \tag{5.8-8}$$

$$\hat{v}_r^{\mathrm{DS}}(\boldsymbol{x},s) = \hat{\zeta}^{-1}\hat{F}_r\delta(\boldsymbol{X}) \,, \tag{5.8-9}$$

the near-field contribution as

$$\hat{p}^{\mathrm{NF}}(\boldsymbol{x},s) = 0 \,, \tag{5.8-10}$$

$$\hat{v}_r^{\mathrm{NF}}(\boldsymbol{x},s) = \hat{\zeta}^{-1}(3\Xi_r\Xi_k - \delta_{r,k})\hat{F}_k(s)\,\frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|^3} \,, \tag{5.8-11}$$

the intermediate-field contribution as

$$\hat{p}^{\mathrm{IF}}(\boldsymbol{x},s) = \Xi_k \hat{F}_k(s) \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|^2} , \tag{5.8-12}$$

$$\hat{v}_r^{\mathrm{IF}}(\boldsymbol{x},s) = \hat{\gamma}\hat{\zeta}^{-1}(3\Xi_r \Xi_k - \delta_{r,k})\hat{F}_k(s) \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|^2} , \tag{5.8-13}$$

and the far-field contribution as

$$\hat{p}^{\mathrm{FF}}(\boldsymbol{x},s) = \hat{\gamma}\Xi_k \hat{F}_k(s) \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|} , \tag{5.8-14}$$

$$\hat{v}_r^{\mathrm{FF}}(\boldsymbol{x},s) = \hat{\gamma}^2\hat{\zeta}^{-1}\Xi_r \Xi_k \hat{F}_k(s) \frac{\exp(-\hat{\gamma}|\boldsymbol{X}|)}{4\pi|\boldsymbol{X}|} . \tag{5.8-15}$$

In these expressions,

$$\Xi_m = X_m/|\boldsymbol{X}| \qquad \text{for} \qquad |\boldsymbol{X}| \neq 0 \tag{5.8-16}$$

is the unit vector in the direction of $\boldsymbol{X}$.

All expressions have the complex frequency-domain propagation factor $\exp(-\hat{\gamma}|\boldsymbol{X}|)$ in common. As far as the distance from the location of the source to the point of observation is concerned, the particle velocity contains a direct source term, while the near-field contribution is proportional to $(\text{distance})^{-3}$, and both the acoustic pressure and the particle velocity contain an intermediate-field contribution proportional to $(\text{distance})^{-2}$ and a far-field contribution proportional to $(\text{distance})^{-1}$. Furthermore, each acoustic wave constituent exhibits a particular directional pattern (Figures 5.8-2–5.8-4). That of the acoustic pressure is characteristic for an acoustic dipole. Again, the far-field contribution of the particle velocity only has a radial component pointing away from the source.

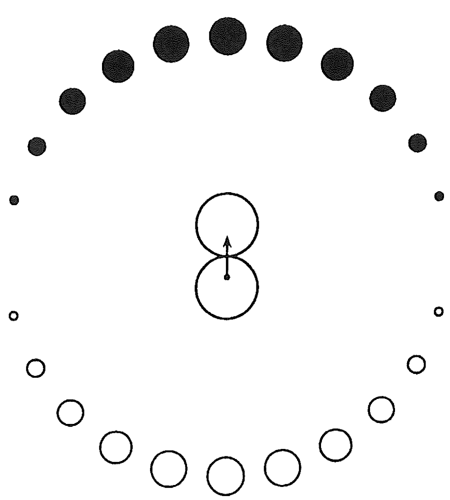

Figure 5.8-2 Acoustic dipole radiation characteristic $F_k \Xi_k$ of the acoustic pressure in the intermediate-field and far-field regions.

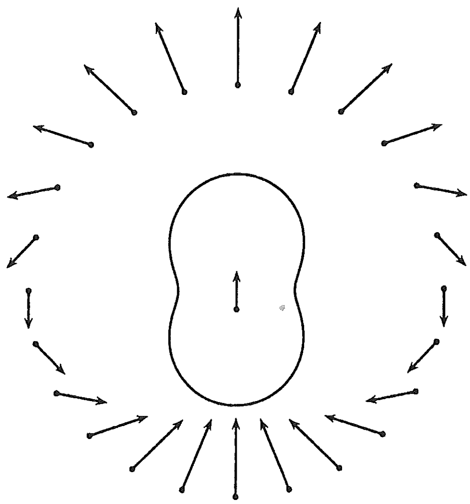

Figure 5.8-3 Acoustic dipole radiation characteristic $3F_k\Xi_k\Xi_r - F_r$ of the particle velocity in the near-field and intermediate-field regions.

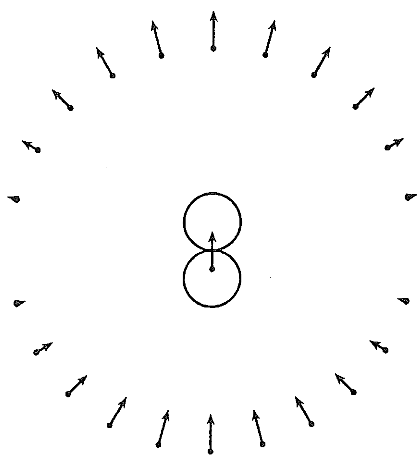

Figure 5.8-4 Acoustic dipole radiation characteristic $F_k\Xi_k\Xi_r$ of the particle velocity in the far-field region.

Time-domain expressions for the acoustic wave field emitted in a lossless fluid

For the acoustic radiation in a homogeneous, isotropic, lossless fluid, the corresponding time-domain expressions readily follow from the results of Section 5.4. Using for the time-integrated pulse shape of the force $F_k = F_k(t)$ the notation $\mathrm{I}_t F_k(t)$ (see Section 5.4), we arrive at (see Equations (5.4-14) and (5.4-15) and (5.4-18)–(5.4-25))

$$\{p,v_r\} = \{p,v_r\}^{\mathrm{DS}} + \{p,v_r\}^{\mathrm{NF}} + \{p,v_r\}^{\mathrm{IF}} + \{p,v_r\}^{\mathrm{FF}}, \tag{5.8-17}$$

in which the direct source contribution follows as

$$p^{\mathrm{DS}}(\boldsymbol{x},t) = 0\,, \tag{5.8-18}$$

$$v_r^{\mathrm{DS}}(\boldsymbol{x},t) = \rho^{-1}\mathrm{I}_t F_r(t)\delta(\boldsymbol{X})\,, \tag{5.8-19}$$

the near-field contribution as

$$p^{\mathrm{NF}}(\boldsymbol{x},t) = 0\,, \tag{5.8-20}$$

$$v_r^{\mathrm{NF}}(\boldsymbol{x},t) = \rho^{-1}(3\Xi_r\Xi_k - \delta_{r,k})\frac{\mathrm{I}_t F_k(t - |\boldsymbol{X}|/c)}{4\pi|\boldsymbol{X}|^3}\,, \tag{5.8-21}$$

the intermediate-field contribution as

$$p^{\mathrm{IF}}(\boldsymbol{x},t) = \Xi_k\frac{F_k(t - |\boldsymbol{X}|/c)}{4\pi|\boldsymbol{X}|^2}\,, \tag{5.8-22}$$

$$v_r^{\mathrm{IF}}(\boldsymbol{x},t) = \frac{1}{\rho c}(3\Xi_r\Xi_k - \delta_{r,k})\frac{F_k(t - |\boldsymbol{X}|/c)}{4\pi|\boldsymbol{X}|^2}\,, \tag{5.8-23}$$

and the far-field contribution as

$$p^{\mathrm{FF}}(\boldsymbol{x},t) = \Xi_k\frac{\partial_t F_k(t - |\boldsymbol{X}|/c)}{4\pi c|\boldsymbol{X}|}\,, \tag{5.8-24}$$

$$v_r^{\mathrm{FF}}(\boldsymbol{x},t) = \frac{1}{\rho c}\Xi_r\Xi_k\frac{\partial_t F_k(t - |\boldsymbol{X}|/c)}{4\pi c|\boldsymbol{X}|}\,. \tag{5.8-25}$$

All time-domain expressions have the travel time delay $|\boldsymbol{X}|/c$ in common. As regards the dependence of the different expressions on the distance from the location of the source $\boldsymbol{b}$ to the point of observation $\boldsymbol{x}$ and the directional patterns of the different contributions, the same remarks as for the complex frequency-domain results apply.

Time-domain expressions for the emitted acoustic wave field in a fluid with frictional-force/bulk-viscosity losses

For acoustic radiation in a homogeneous, isotropic fluid with frictional-force/bulk-viscosity losses, the time-domain expressions for the acoustic pressure and the particle velocity readily follow from the results of Section 5.6. With (see Equations (5.6-13) and (5.6-14) and Equations (5.6-17)–(5.6-24))

$$\{p,v_r\} = \{p,v_r\}^{\mathrm{DS}} + \{p,v_r\}^{\mathrm{NF}} + \{p,v_r\}^{\mathrm{IF}} + \{p,v_r\}^{\mathrm{FF}}, \tag{5.8-26}$$

the direct source contribution follows as

$$p^{\mathrm{DS}}(\boldsymbol{x},t) = 0\,, \tag{5.8-27}$$

$$v_r^{\mathrm{DS}}(\boldsymbol{x},t) = \rho^{-1}\mathrm{I}_t^{\alpha} F_r(t)\delta(\boldsymbol{X})\,, \tag{5.8-28}$$

the near-field contribution as

$$p^{\mathrm{NF}}(x,t) = 0 \,, \tag{5.8-29}$$

$$v_r^{\mathrm{NF}}(x,t) = \frac{\rho^{-1}}{4\pi|X|^3}(3\Xi_r\Xi_k - \delta_{r,k}) \int_{t'\in\mathcal{T}} U_1(\alpha,\beta,|X|/c,t-t')\mathrm{I}_{t'}^{\alpha}F_k(t')\,\mathrm{d}t' \,, \tag{5.8-30}$$

the intermediate-field contribution as

$$p^{\mathrm{IF}}(x,t) = \frac{\Xi_k}{4\pi|X|^2}\int_{t'\in\mathcal{T}} U_1(\alpha,\beta,|X|/c,t-t')F_k(t')\,\mathrm{d}t' \,, \tag{5.8-31}$$

$$v_r^{\mathrm{IF}}(x,t) = \frac{\rho^{-1}}{4\pi c|X|^2}(3\Xi_r\Xi_k - \delta_{r,k}) \int_{t'\in\mathcal{T}} U_2(\alpha,\beta,|X|/c,t-t')\mathrm{I}_{t'}^{\alpha}F_k(t')\,\mathrm{d}t' \,, \tag{5.8-32}$$

and the far-field contribution as

$$p^{\mathrm{FF}}(x,t) = \frac{\Xi_k}{4\pi c|X|}\int_{t'\in\mathcal{T}} U_2(\alpha,\beta,|X|/c,t-t')F_k(t')\,\mathrm{d}t' \,, \tag{5.8-33}$$

$$v_r^{\mathrm{FF}}(x,t) = \frac{\rho^{-1}}{4\pi c^2|X|}\Xi_r\Xi_k \int_{t'\in\mathcal{T}} U_3(\alpha,\beta,|X|/c,t-t')\mathrm{I}_{t'}^{\alpha}F_k(t')\,\mathrm{d}t' \,. \tag{5.8-34}$$

In these expressions, the wave functions U_1, U_2 and U_3 are representative of the propagation along the straight path from the source point to the point of observation and for the attenuation due to the losses along this path. As regards the decay with distance and the directional patterns, the same remarks apply as for the previous results of this section.

5.9 Far-field radiation characteristics of extended sources (complex frequency-domain analysis)

In many applications of acoustic radiation one is often particularly interested in the behaviour of the radiated field at large distances from the sound radiating structures. To investigate this behaviour, we consider the leading term in the expansion of the right-hand sides of Equations (5.3-1)–(5.3-4) as $|x| \to \infty$; this term is called the *far-field approximation* of the relevant acoustic wave field. The region in space where the far-field approximation represents the wave-field values with sufficient accuracy is called the *far-field region*. Since in the far-field region the mutual relationships between the acoustic pressure and the particle velocity prove to be the same for the wave-field constituents generated by volume injection sources and the wave-field constituents generated by force sources, it is advantageous to investigate those relationships for the total wave field. This is done below.

To construct the far-field approximation, we first observe that

$$|x - x'| = [(x_s - x_s')(x_s - x_s')]^{1/2} = |x|\,[1 - 2x_s x_s'/|x|^2 + |x'|^2/|x|^2]^{1/2} \,, \tag{5.9-1}$$

from which, by a Taylor expansion of the square-root expression about $|x| = \infty$, it follows that (Figure 5.9-1)

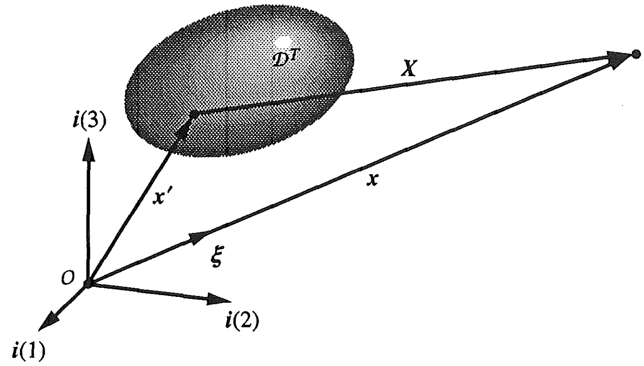

Figure 5.9-1 Far-field approximation to the distance function from source point $x' \in \mathcal{D}^T$ to observation point $x \in \mathcal{R}^3$: $|X| = |x - x'| = |x| - \xi_s x_s' + O(|x|^{-1})$ as $|x| \to \infty$.

$$|x - x'| = |x| - \xi_s x_s' + O(|x|^{-1}) \qquad \text{as } |x| \to \infty , \tag{5.9-2}$$

where

$$\xi_s = x_s / |x| \tag{5.9-3}$$

is the unit vector in the direction of observation. (Note that in the far-field region it is certain that $|x - x'| \neq 0$.) For the derivatives of $|x - x'|$, we have

$$\partial_m |x - x'| = (x_m - x_m')/|x - x'| . \tag{5.9-4}$$

This leads to

$$\partial_m |x - x'| = \xi_m + O(|x|^{-1}) \qquad \text{as } |x| \to \infty , \tag{5.9-5}$$

where the order term follows from a Taylor expansion of $|(x_m - x_m')/|x - x'| - \xi_m|$ about $|x| = \infty$. Using these results, the Green's function of the scalar Helmholtz equation (Equation (5.3-5)) can, in the far-field region, be approximated by

$$\hat{G}(x - x', s) = \frac{\exp(-\hat{\gamma}|x|)}{4\pi|x|} \exp(\hat{\gamma}\xi_s x_s') \, [1 + O(|x|^{-1})] \qquad \text{as } |x| \to \infty , \tag{5.9-6}$$

and its spatial derivatives by

$$\partial_m \hat{G}(x - x', s) = (-\hat{\gamma}\xi_m) \frac{\exp(-\hat{\gamma}|x|)}{4\pi|x|} \exp(\hat{\gamma}\xi_s x_s') \, [1 + O(|x|^{-1})] \qquad \text{as } |x| \to \infty . \tag{5.9-7}$$

Using Equation (5.9-6) in the expressions for the volume injection source scalar and force source vector potentials, Equations (5.3-3) and (5.3-4), we obtain their far-field approximations as

$$\{\hat{\Phi}^q, \hat{\Phi}_k^f\}(x, s) = \{\hat{\Phi}^{q;\infty}, \hat{\Phi}_k^{f;\infty}\}(\xi, s) \frac{\exp(-\hat{\gamma}|x|)}{4\pi|x|} [1 + O(|x|^{-1})] \qquad \text{as } |x| \to \infty , \tag{5.9-8}$$

in which

$$\hat{\Phi}^{q;\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \exp(\hat{\gamma}\xi_s x_s')\hat{q}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{5.9-9}$$

and

$$\hat{\Phi}_k^{f;\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \exp(\hat{\gamma}\xi_s x_s')\hat{f}_k(\boldsymbol{x}',s)\,\mathrm{d}V\,. \tag{5.9-10}$$

Using Equations (5.9-8)–(5.9-10) in the expressions Equations (5.3-1) and (5.3-2) for the acoustic pressure and the particle velocity, we obtain their far-field approximations as

$$\{\hat{p},\hat{v}_r\}(\boldsymbol{x},s) = \{\hat{p}^{\infty},\hat{v}_r^{\infty}\}(\boldsymbol{\xi},s)\,\frac{\exp(-\hat{\gamma}|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|}\,[1 + O(|\boldsymbol{x}|^{-1})] \quad \text{as } |\boldsymbol{x}|\to\infty\,, \tag{5.9-11}$$

in which

$$\hat{p}^{\infty} = \hat{\zeta}\hat{\Phi}^{q;\infty} + \hat{\gamma}\xi_k\hat{\Phi}_k^{f;\infty}\,, \tag{5.9-12}$$

$$\hat{v}_r^{\infty} = \hat{\gamma}\xi_r\hat{\Phi}^{q;\infty} + \hat{\eta}\xi_r\xi_k\hat{\Phi}_k^{f;\infty}\,. \tag{5.9-13}$$

In the expression for $\hat{v}_r^{\infty}$ it has been taken into account that the direct source term only differs from zero for points of observation in the source domain $\mathcal{D}^T$ and hence yields no contribution to the far-field approximation as the latter only holds at large distances from the source domain.

As Equation (5.9-11) shows, the acoustic pressure and the particle velocity have, in the far-field region, the structure of a spherical wave that expands radially from the origin of the coordinate system (which is also called the *phase centre* of the far-field approximation), the latter being chosen in the neighbourhood of the source domain, with an amplitude that depends on the direction of observation and that decreases in inverse proportion to the distance from the source domain. The amplitude radiation characteristics $\{\hat{p}^{\infty},\hat{v}_r^{\infty}\}$ depend only on the direction of observation $\boldsymbol{\xi}$, and on s. Their dependence on $\boldsymbol{\xi}$ is the resultant of the dependence on $\boldsymbol{\xi}$ of the integrals occurring on the right-hand sides of Equations (5.9-9) and (5.9-10), and the acoustic far-field radiation characteristics ξ_r and $\xi_r\xi_k$.

The far-field amplitude radiation characteristics of the acoustic pressure and the particle velocity are not independent of each other. It is easily verified that the right-hand sides of Equations (5.9-12) and (5.9-13) are interrelated in the following way:

$$-\hat{\gamma}\xi_k\hat{p}^{\infty} + \hat{\zeta}\hat{v}_k^{\infty} = 0\,, \tag{5.9-14}$$

$$-\hat{\gamma}\xi_r\hat{v}_r^{\infty} + \hat{\eta}\hat{p}^{\infty} = 0\,. \tag{5.9-15}$$

Now, relations of the kind given in Equations (5.9-14) and (5.9-15) would also have resulted if expressions of the type

$$\{\hat{p},\hat{v}_r\} = \{\hat{p}^{\infty},\hat{v}_r^{\infty}\}\exp(-\hat{\gamma}\xi_s x_s)\,, \tag{5.9-16}$$

where $\hat{p}^{\infty}$ and $\hat{v}_r^{\infty}$ only depend on the real unit vector $\boldsymbol{\xi}$ and the Laplace-transform parameter s and not on $\boldsymbol{x}$, had been substituted in the source-free acoustic wave-field equations pertaining to the homogeneous, isotropic fluid under consideration. Wave fields of the type given by Equation (5.9-16) are called complex frequency-domain acoustic *uniform plane waves*. Observing that $|\boldsymbol{x}| = \xi_s x_s$, we can therefore say that, after compensating for the $(\text{distance})^{-1}$ decay, the spherical-wave amplitudes in the far-field radiation pattern behave locally (i.e. for a fixed direction of observation $\boldsymbol{\xi}$) as if the wave were a uniform plane wave travelling in the radial direction away from the source.

To exhibit the further properties of $\{\hat{p}^{\infty}, \hat{v}_r^{\infty}\}$ we rewrite Equations (5.9-14) and (5.9-15) as

$$-\xi_k \hat{p}^{\infty} + \hat{Z}\hat{v}_k^{\infty} = 0 \,, \tag{5.9-17}$$

$$-\xi_r \hat{v}_r^{\infty} + \hat{Y}\hat{p}^{\infty} = 0 \,, \tag{5.9-18}$$

in which

$$\hat{Z} = \hat{\zeta}/\hat{\gamma} = (\hat{\zeta}/\hat{\eta})^{1/2} \tag{5.9-19}$$

is the *acoustic plane-wave impedance* and

$$\hat{Y} = \hat{\eta}/\hat{\gamma} = (\hat{\eta}/\hat{\zeta})^{1/2} \tag{5.9-20}$$

is the *acoustic plane-wave admittance* of the fluid under consideration. Furthermore, Equation (5.9-13) shows that, in the far-field region, the particle velocity is *longitudinal with respect to the radial direction of propagation*, while, since ξ_r has unit magnitude, Equations (5.9-17) and (5.9-18) show that the acoustic pressure and the particle velocity are proportional with proportionality factors $\hat{Z}$ or $\hat{Y}$.

Upon inspecting the dependence of the integrals occurring in the expressions for $\hat{\Phi}^{q;\infty}$ and $\hat{\Phi}_k^{f;\infty}$ on $\boldsymbol{\xi}$, comparison with Equation (5.1-15) shows that

$$\int_{\boldsymbol{x}' \in \mathcal{D}^T} \exp(\hat{\gamma}\xi_s x_s')\{\hat{q}, \hat{f}_k\}(\boldsymbol{x}', s)\,\mathrm{d}V = \{\tilde{q}, \tilde{f}_k\}(\hat{\gamma}\boldsymbol{\xi}, s) \,. \tag{5.9-21}$$

Consequently, in the far-field region only the spatial Fourier transforms of the source distribution at the subset of angular wave-vector values $\mathrm{j}\boldsymbol{k} = \hat{\gamma}\boldsymbol{\xi}$ are "visible".

Exercises

Exercise 5.9-1

Verify that Equations (5.9-12) and (5.9-13) indeed satisfy Equations (5.9-14) and (5.9-15).

5.10 Far-field radiation characteristics of extended sources (time-domain analysis for a lossless fluid)

In this section we investigate the time-domain far-field radiation characteristics of the acoustic radiation emitted by extended sources immersed in a homogeneous, isotropic, lossless fluid. For such a fluid,

$$\hat{\zeta} = s\rho \,, \tag{5.10-1}$$

$$\hat{\eta} = s\kappa \,, \tag{5.10-2}$$

and

$$\hat{\gamma} = s/c\,, \tag{5.10-3}$$

with

$$c = (\kappa\rho)^{-1/2}\,. \tag{5.10-4}$$

Since the factor $\exp(-\hat{\gamma}|\boldsymbol{x}|) = \exp(-s|\boldsymbol{x}|/c)$ in the complex frequency domain corresponds in the time domain to a time delay of $|\boldsymbol{x}|/c$, the time-domain equivalent of Equation (5.9-11) is (Figure 5.10-1)

$$\{p,v_r\}(\boldsymbol{x},t) = \frac{\{p^{\infty},v_r^{\infty}\}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}|}\,[1+O(|\boldsymbol{x}|^{-1})] \quad \text{as } |\boldsymbol{x}|\to\infty\,, \tag{5.10-5}$$

where p^{∞} and v_r^{∞} follow from Equations (5.9-12) and (5.9-13) as

$$p^{\infty} = \rho\partial_t\Phi^{q;\infty} + (\xi_k/c)\partial_t\Phi_k^{f;\infty}\,, \tag{5.10-6}$$

$$v_r^{\infty} = (\xi_r/c)\partial_t\Phi^{q;\infty} + \kappa\xi_r\xi_k\partial_t\Phi_k^{f;\infty}\,. \tag{5.10-7}$$

In view of the property that the factor $\exp(\hat{\gamma}\xi_s x_s') = \exp(s\xi_s x_s'/c)$ in the complex frequency domain corresponds in the time domain to a time advance by the amount of $\xi_s x_s'/c$, the time-domain equivalents of Equations (5.9-9) and (5.9-10) are

$$\Phi^{q;\infty}(\boldsymbol{\xi},t) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} q(\boldsymbol{x}',t+\xi_s x_s'/c)\,\mathrm{d}V\,, \tag{5.10-8}$$

$$\Phi_k^{f;\infty}(\boldsymbol{\xi},t) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} f_k(\boldsymbol{x}',t+\xi_s x_s'/c)\,\mathrm{d}V\,. \tag{5.10-9}$$

As Equation (5.10-5) shows, the acoustic pressure and the particle velocity have, in the far-field region, the shape of a spherical wave that expands radially from the chosen origin of the coordinate system (that is located in the neighbourhood of the source domain and which is called the time reference centre of the far-field approximation), with an amplitude that depends on the direction of observation and that decreases in inverse proportion to the distance from the source domain to the point of observation. The amplitude radiation characteristics $\{p^{\infty},v_r^{\infty}\}$ depend only on the direction of observation $\boldsymbol{\xi}$, and on the pulse shapes of the source distributions. The dependence on $\boldsymbol{\xi}$ is the resultant of the dependence on $\boldsymbol{\xi}$ of the integrals on the right-hand sides of Equations (5.10-8) and (5.10-9) and the acoustic far-field radiation characteristics ξ_r and $\xi_r\xi_k$. Note that on the right-hand sides of Equations (5.10-6) and (5.10-7), via Equations (5.10-8) and (5.10-9), only the time-differentiated pulse shapes of the volume source densities occur.

Equations (5.10-5)–(5.10-9) also follow directly from Equations (5.4-7) and (5.4-8), (5.4-12) and (5.4-13) with the use of Equations (5.9-2)–(5.9-5).

The far-field amplitude radiation characteristics of the acoustic pressure and the particle velocity are not independent of each other. It is easily verified that the right-hand sides of Equations (5.10-6) and (5.10-7) are interrelated in the following way:

$$-(\xi_k/c)p^{\infty} + \rho v_k^{\infty} = 0\,, \tag{5.10-10}$$

$$-(\xi_r/c)v_r^{\infty} + \kappa p^{\infty} = 0\,. \tag{5.10-11}$$

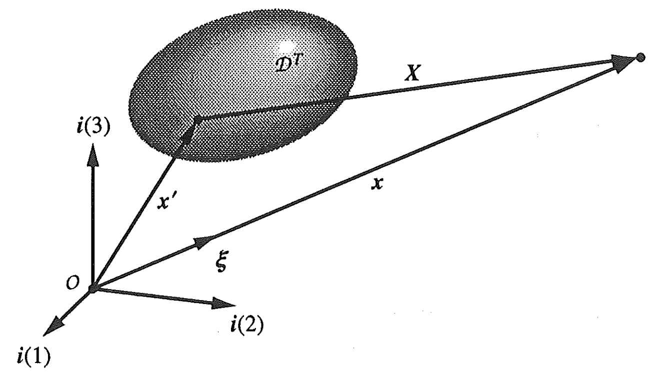

Figure 5.10-1 Far-field approximation to the distance function from source point $x' \in \mathcal{D}^T$ to observation point $x \in \mathcal{R}^3$: $|X| = |x - x'| = |x| - \xi_s x'_s + O(|x|^{-1})$ as $|x| \to \infty$.

Now, relations of the kind given by Equations (5.10-10) and (5.10-11) would also have resulted if expressions of the type

$$\{p, v_r\} = \{p^\infty, v_r^\infty\}(t - \xi_s x_s / c) \tag{5.10-12}$$

had been substituted in the source-free acoustic wave equations pertaining to the homogeneous, isotropic, lossless fluid under consideration and the causal relation between this wave field and its sources (that are located elsewhere in space), which entails zero initial values in time, had been used. Wave fields of the type given by Equation (5.10-12) are called *uniform acoustic plane waves*. Observing that $|x| = \xi_s x_s$, we can therefore say that, after compensating for the $(\text{distance})^{-1}$ decay, the spherical-wave amplitudes in the far-field radiation pattern behave locally as if the wave were a plane wave travelling in the radial direction away from the source.

To exhibit the further properties of $\{p^\infty, v_r^\infty\}$ we rewrite Equations (5.10-10) and (5.10-11) as

$$-\xi_k p^\infty + Z v_k^\infty = 0\,, \tag{5.10-13}$$

$$-\xi_r v_r^\infty + Y p^\infty = 0\,, \tag{5.10-14}$$

in which

$$Z = \rho c \tag{5.10-15}$$

is the *acoustic plane-wave impedance* and

$$Y = \kappa c \tag{5.10-16}$$

is the *acoustic plane-wave admittance* of the fluid under consideration. Equation (5.10-7) shows that, in the far-field region, the particle velocity is *longitudinal with respect to the direction of propagation*, while, since ξ_r has unit magnitude, Equations (5.10-13) and (5.10-14) show that the acoustic pressure and the particle velocity in the far-field region are proportional, with proportionality factors Z or Y.

Exercises

Exercise 5.10-1

Verify that Equations (5.10-6) and (5.10-7) indeed satisfy Equations (5.10-10) and (5.10-11).

Exercise 5.10-2

Let $F = F(x,t)$ be a tensor function of arbitrary rank, defined over some subdomain $\mathcal{D}$ of $\mathcal{R}^3$ and for all $t \in \mathcal{R}$. In addition, let $\partial_t F(x,t) = 0$ for all $x \in \mathcal{D}$ and all $t \in \mathcal{R}$, while $F(x,t_0) = 0$ for all $x \in \mathcal{D}$. Show that also $F(x,t) = 0$ for all $x \in \mathcal{D}$ and $t > t_0$. (*Hint*: Note that

$$0 = \int_{t'=t_0}^{t} \partial_{t'} F(x,t')\,\mathrm{d}t' = F(x,t) - F(x,t_0)\ .)$$

5.11 The time evolution of an acoustic wave field. The initial-value problem (Cauchy problem) for a homogeneous, isotropic, lossless fluid

In this section we present a solution for the initial-value problem (Cauchy problem) for acoustic waves in a homogeneous, isotropic, lossless fluid with volume density of mass ρ, compressibility κ and acoustic wave speed $c = (\rho\kappa)^{-1/2}$. From the given initial values $p(X,t_0)$ of the acoustic pressure and $v_k(x,t_0)$ of the particle velocity in all space at the instant t_0, the values of $p = p(x,t)$ and $v_r = v_r(x,t)$ for all succeeding instants $t \geqslant t_0$ need to be constructed for the case where nowhere in the fluid are sources active for $t \geqslant t_0$. Thus, we are looking for the pure time evolution for $t > t_0$ of the acoustic wave field, given its value at $t = t_0$. From Equations (4.1-3), (4.1-4), (4.2-18) and (4.2-19) we learn that this problem can be solved by transforming Equations (5.3-1)–(5.3-6) back to the time domain for the particular case where

$$\hat{q} = \kappa p(x,t_0)\exp(-st_0)\ , \tag{5.11-1}$$

$$\hat{f}_k = \rho v_k(x,t_0)\exp(-st_0)\ . \tag{5.11-2}$$

Substitution of Equations (5.11-1) and (5.11-2) in Equations (5.3-1)–(5.3-4) leads to

$$\hat{\Phi}^q(x,s) = \kappa \int_{x' \in \mathcal{R}^3} p(x',t_0)\,\frac{\exp[-s|x'-x|/c - st_0]}{4\pi|x'-x|}\,\mathrm{d}V \tag{5.11-3}$$

and

$$\hat{\Phi}_k^f(x,s) = \rho \int_{x' \in \mathcal{R}^3} v_k(x',t_0)\,\frac{\exp[-s|x'-x|/c - st_0]}{4\pi|x'-x|}\,\mathrm{d}V\ . \tag{5.11-4}$$

The integrals on the right-hand sides of Equations (5.11-3) and (5.11-4) are then rewritten such that their time-domain equivalents can be obtained by inspection. This is accomplished by introducing spherical polar coordinates about the observation point as the variables of integration. Consider to this end the generic expression

$$\hat{\Phi}(x,s) = \int_{x' \in \mathcal{R}^3} q(x',t_0) \frac{\exp[-s|x'-x|/c - st_0]}{4\pi|x'-x|} \mathrm{d}V . \tag{5.11-5}$$

On the right-hand side, $c(\tau - t_0)$, with $t_0 \leqslant \tau < \infty$, is taken as the radial coordinate and the unit vector θ with $\theta \in \Omega$, where Ω denotes the sphere of unit radius, as the angular coordinate. Then,

$$x' = x + c(\tau - t_0)\theta , \tag{5.11-6}$$

and, since $\theta_s \theta_s = 1$,

$$|x - x'| = c(\tau - t_0) \tag{5.11-7}$$

and hence

$$|x - x'|/c + t_0 = \tau , \tag{5.11-8}$$

while

$$\mathrm{d}V = c^3(\tau - t_0)^2 \, \mathrm{d}\tau \, \mathrm{d}\Omega , \tag{5.11-9}$$

where $\mathrm{d}\Omega$ is the elementary area on Ω. With this, Equation (5.11-5) is rewritten as

$$\hat{\Phi}(x,s) = \int_{\tau=t_0}^{\infty} \exp(-s\tau) c^2(\tau - t_0) \langle q(x',t_0) \rangle_{\mathcal{S}\,[x;c(\tau - t_0)]} \, \mathrm{d}\tau , \tag{5.11-10}$$

in which

$$\langle q(x',t_0) \rangle_{\mathcal{S}\,[x;c(\tau - t_0)]} = \frac{1}{4\pi} \int_{\theta \in \Omega} q\,[[x + c(\tau - t_0)\theta, t_0] \, \mathrm{d}\Omega \tag{5.11-11}$$

denotes the spherical mean over the sphere $\mathcal{S}\,[x;c(\tau - \tau_0)]$ with its centre at x and radius $c(\tau - t_0)$.

Now, the right-hand side of Equation (5.11-10) has the form of the Laplace transformation of a causal function of time whose support is $\{t \in \mathcal{R}; t > t_0\}$. In view of the uniqueness of the Laplace transformation with real, posititve transform parameter (see Section B.1), the time-domain counterpart $\Phi(x,t)$ of $\hat{\Phi}(x,s)$ is given by

$$\Phi(x,t) = c^2(t - t_0) \langle q(x',t_0) \rangle_{\mathcal{S}\,[x;c(t - t_0)]} \qquad \text{for } t \geqslant t_0 . \tag{5.11-12}$$

Using this generic result, the time-domain counterparts of Equations (5.11-3) and (5.11-4) are obtained as

$$\Phi^q(x,t) = \kappa c^2(t - t_0) \langle p(x',t_0) \rangle_{\mathcal{S}\,[x;c(t - t_0)]} \qquad \text{for } t \geqslant t_0 \tag{5.11-13}$$

and

$$\Phi_k^f(x,t) = \rho c^2(t - t_0) \langle v_k(x',t_0) \rangle_{\mathcal{S}\,[x;c(t - t_0)]} \qquad \text{for } t \geqslant t_0 . \tag{5.11-14}$$

In terms of these source potentials the expressions for the acoustic pressure and the particle velocity follow from Equations (5.4-7) and (5.4-8) as

$$p(x,t) = \rho \partial_t \Phi^q - \partial_k \Phi_k^f \qquad \text{for } t \geqslant t_0 , \tag{5.11-15}$$

and

$$v_r(x,t) = \rho^{-1} \mathrm{I}_t f_r - \partial_r \Phi^q + \rho^{-1} \partial_r \partial_k \mathrm{I}_t \Phi_k^f \qquad \text{for } t \geqslant t_0 . \tag{5.11-16}$$

Equations (5.11-15) and (5.11-16) constitute the solution to the acoustic initial-value problem and govern the time evolution of an acoustic wave in a homogeneous, isotropic, lossless medium.

Exercises

Exercise 5.11-1

Construct the solution to the initial-value problem (Cauchy problem) of the three-dimensional scalar wave equation

$$\partial_m \partial_m u - c^{-2} \partial_t^{\,2} u = 0 \tag{5.11-17}$$

for $t > t_0$ if $u(\boldsymbol{x},t_0) = u_0(\boldsymbol{x})$ and $\partial_t u(\boldsymbol{x},t_0) = v_0(\boldsymbol{x})$.

(a) Take the time Laplace transform of Equation (5.11-17) over the interval $t_0 < t < \infty$ and show that

$$\partial_m \partial_m \hat{u} - (s^2/c^2)\,\hat{u} = -c^{-2} v_0(\boldsymbol{x}) \exp(-st_0) - c^{-2} s u_0(\boldsymbol{x}) \exp(-st_0) \,. \tag{5.11-18}$$

(b) The solution to Equation (5.11-18) is given by

$$\hat{u}(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{R}^3} \hat{q}(\boldsymbol{x}') \frac{\exp\,[-s|\boldsymbol{x} - \boldsymbol{x}'|/c - st_0]}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} \,\mathrm{d}V \,, \tag{5.11-19}$$

in which

$$\hat{q}(\boldsymbol{x}') = c^{-2}\,[v_0(\boldsymbol{x}') + s u_0(\boldsymbol{x}')] \,. \tag{5.11-20}$$

Introduce spherical polar coordinates about the observation point $\boldsymbol{x}$ as the variables of integration and show that

$$\begin{aligned} \hat{u}(\boldsymbol{x},s) = &\int_{\tau=t_0}^{\infty} \exp(-s\tau)(\tau - t_0)\langle v_0(\boldsymbol{x}')\rangle_{\mathcal{S}\,[\boldsymbol{x};c(\tau-t_0)]}\,\mathrm{d}\tau \\ &+ s \int_{\tau=t_0}^{\infty} \exp(-s\tau)(\tau - t_0)\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}\,[\boldsymbol{x}';c(\tau-t_0)]}\,\mathrm{d}\tau \,, \end{aligned} \tag{5.11-21}$$

in which

$$\langle\{u_0,v_0\}(\boldsymbol{x}')\rangle_{\mathcal{S}\,[\boldsymbol{x};c(\tau-t_0)]} = \frac{1}{4\pi} \int_{\boldsymbol{\theta} \in \Omega} \{u_0,v_0\}\,[\boldsymbol{x}' + c(\tau - t_0)\boldsymbol{\theta}]\,\mathrm{d}\Omega \tag{5.11-22}$$

is the spherical mean over the sphere $\mathcal{S}\,[\boldsymbol{x};c(\tau - t_0)]$ with its centre at $\boldsymbol{x}$ and radius $c(\tau - t_0)$.

(c) Use the uniqueness of the time Laplace transformation to show that

$$u(\boldsymbol{x},t) = (t - t_0)\langle v_0(\boldsymbol{x}')\rangle_{\mathcal{S}\,[\boldsymbol{x};c(t-t_0)]} + \partial_t \left\{(t - t_0)\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}\,[\boldsymbol{x};c(t-t_0)]}\right\} \quad \text{for } t \geqslant t_0 \,. \tag{5.11-23}$$

Equation (5.11-23) is Poisson's solution to the initial-value problem of the three-dimensional scalar wave equation.

References

Abramowitz, M. and Stegun, I.E., 1964a, *Handbook of Mathematical Functions*, Washington DC: National Bureau of Standards, Formula 9.6.16, p. 376.

Abramowitz, M. and Stegun, I.E., 1964b, *Handbook of Mathematical Functions*, Washington, DC: National Bureau of Standards, Formula 9.6.27, p. 376.