6

Plane acoustic waves in homogeneous fluids

In this chapter the notion of a plane acoustic wave that has turned up in the local description of the acoustic wave field in the far-field region of the acoustic radiation by extended sources, is generalised to cases where the wave amplitudes are arbitrary functions of the angular wave vector, not specifically the ones of the type that occurs in the far-field approximation. The concepts of dispersion equation and wave slowness are introduced for homogeneous, arbitrarily anisotropic fluids, with the homogeneous, isotropic fluid as a special case. The attenuation and the phase propagation of a plane wave are discussed for the real frequency domain.

6.1 Plane waves in the complex frequency domain

In the complex frequency domain, *plane waves* are solutions of the complex frequency-domain source-free acoustic wave equations (see Equations (4.4-1) and (4.4-2))

$$\partial_k \hat{p} + \hat{\zeta}_{k,r} \hat{v}_r = 0 \,, \qquad (6.1\text{-}1)$$

$$\partial_r \hat{v}_r + \hat{\eta} \hat{p} = 0 \,, \qquad (6.1\text{-}2)$$

of the form

$$\{\hat{p}, \hat{v}_r\} = \{\hat{P}, \hat{V}_r\} \exp(-\hat{\gamma}_s x_s) \,, \qquad (6.1\text{-}3)$$

in which the *amplitudes* $\{\hat{P}, \hat{V}_r\}$ and the *propagation vector* $\hat{\gamma}_s$ are independent of $\boldsymbol{x}$. The factor $\exp(-\hat{\gamma}_s x_s)$ is called the *propagation factor*. Substitution of Equation (6.1-3) in Equations (6.1-1) and (6.1-2) leads, in view of the relation

$$\partial_m [\exp(-\hat{\gamma}_s x_s)] = -\hat{\gamma}_m \exp(-\hat{\gamma}_s x_s) \,, \qquad (6.1\text{-}4)$$

to

$$-\hat{\gamma}_k \hat{P} + \hat{\zeta}_{k,r} \hat{V}_r = 0 \,, \qquad (6.1\text{-}5)$$

$$-\hat{\gamma}_r \hat{V}_r + \hat{\eta} \hat{P} = 0 \,. \qquad (6.1\text{-}6)$$

Equations (6.1-5) and (6.1-6) constitute a homogeneous system of linear algebraic equations in $\hat{P}$ and $\hat{V}_r$, which for arbitrary values of $\hat{\gamma}_s$ has only the trivial solution $\hat{P} = 0$ and $\hat{V}_r = 0$. For a non-trivial solution to exist, $\hat{\gamma}_s$ must be chosen appropriately. The condition to be put on $\hat{\gamma}_s$ in this respect could be found by setting equal to zero the determinant of the system of linear algebraic equations (Equations (6.1-5) and (6.1-6)), but an easier way to find the relevant relation is to solve $\hat{V}_r$ from Equation (6.1-5) and substitute the result in Equation (6.1-6). This yields

$$\hat{V}_r = \hat{\zeta}^{-1}_{r,k}\hat{\gamma}_k\hat{P} \tag{6.1-7}$$

and

$$(\hat{\gamma}_r\hat{\zeta}^{-1}_{r,k}\hat{\gamma}_k - \hat{\eta})\hat{P} = 0\,, \tag{6.1-8}$$

respectively. Obviously, for a non-zero solution of $\hat{P}$ to Equation (6.1-8) to exist, $\hat{\gamma}_s$ must satisfy the equation

$$\hat{\gamma}_r\hat{\zeta}^{-1}_{r,k}\hat{\gamma}_k - \hat{\eta} = 0\,. \tag{6.1-9}$$

Equation (6.1-9) is known as the complex frequency-domain plane wave *dispersion equation* for the propagation vector. Once a value for $\hat{\gamma}_s$ satisfying Equation (6.1-9) has been chosen, one is free to choose $\hat{P}$, while the corresponding value of $\hat{V}_r$ subsequently follows from Equation (6.1-7), which is then customarily rewritten as

$$\hat{V}_r = \hat{Y}_r\hat{P}\,, \tag{6.1-10}$$

where

$$\hat{Y}_r = \hat{\zeta}^{-1}_{r,k}\hat{\gamma}_k \tag{6.1-11}$$

is the vectorial *acoustic plane wave admittance* of the wave. For arbitrary values of $\hat{\gamma}_s$, satisfying Equation (6.1-9), the resulting expressions of the type of Equation (6.1-3) are called *non-uniform plane waves.*

Equation (6.1-9) shows that only the symmetric part of $\hat{\zeta}^{-1}_{r,k}$ contributes to the admissible values of $\hat{\gamma}_s$ and hence to the propagation properties of the plane wave. Furthermore, Equation (6.1-9) shows that with any value of $\hat{\gamma}_s$ satisfying the equation, $-\hat{\gamma}_s$ is also a solution. This property reduces the solution space in which admissible values of $\hat{\gamma}_s$ are to be sought.

Uniform plane waves

Uniform plane waves are a subset of the general class of non-uniform plane waves. For a *uniform plane wave* the propagation vector is of the special shape

$$\hat{\gamma}_s = \hat{\gamma}\xi_s\,, \tag{6.1-12}$$

where ξ_s is a real unit vector that specifies the *direction of propagation* of the wave. (Since ξ_s is a unit vector, we have $\xi_s\xi_s = 1$.) Now, $\hat{\gamma}$ is the (scalar) *propagation coefficient* of the uniform plane wave. Substitution of Equation (6.1-12) in Equation (6.1-9) yields

$$(\xi_r\hat{\zeta}^{-1}_{r,k}\xi_k)\hat{\gamma}^2 - \hat{\eta} = 0 \tag{6.1-13}$$

as the *dispersion equation for uniform plane waves.* Causality of the wave motion entails the condition that $\exp(-\hat{\gamma}\xi_s x_s)$ should remain bounded as $|\boldsymbol{x}|\to\infty$ in the half-space where $\xi_s x_s > 0$. This yields the condition $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$. In view of the properties that $\mathrm{Re}(\hat{\eta}) > 0$ for

$\mathrm{Re}(s) > 0$ and $\hat{\zeta}_{k,r}$ as well as $\hat{\zeta}_{k,r}^{-1}$ are positive definite for $\mathrm{Re}(s) > 0$, $\hat{\gamma}$ then follows from Equation (6.1-13) as

$$\hat{\gamma} = \left(\frac{\hat{\eta}}{\xi_r \hat{\zeta}_{r,k}^{-1} \xi_k} \right)^{1/2} \quad \text{with } \mathrm{Re}(...)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 . \tag{6.1-14}$$

Equation (6.1-14) clearly shows that the value of the propagation coefficient changes with the direction of propagation of the uniform plane wave, a property that is indicative of the presence of anisotropy.

The expression for the vectorial acoustic plane wave admittance reduces for a uniform plane wave to (see Equation (6.1-11))

$$\hat{Y}_r = (\zeta_{r,k}^{-1} \xi_k) \left(\frac{\hat{\eta}}{\xi_{r'} \hat{\zeta}_{r',k'}^{-1} \xi_{k'}} \right)^{1/2} \quad \text{with } \mathrm{Re}(...)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 . \tag{6.1-15}$$

Isotropic fluids

For an *isotropic fluid* we have

$$\hat{\zeta}_{k,r} = \hat{\zeta} \delta_{k,r} \tag{6.1-16}$$

and, as a consequence of the properties of the Kronecker tensor, the dispersion equation for non-uniform plane waves (Equation (6.1-9)) reduces to

$$\hat{\gamma}_r \hat{\gamma}_r = \hat{\eta} \hat{\zeta} , \tag{6.1-17}$$

while the dispersion equation for uniform plane waves (Equation (6.1-13)) reduces to

$$\hat{\gamma}^2 = \hat{\eta} \hat{\zeta} . \tag{6.1-18}$$

The solution of the latter equation is given by (see Equation (6.1-14))

$$\hat{\gamma} = (\hat{\eta} \hat{\zeta})^{1/2} \quad \text{with } \mathrm{Re}(...)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 . \tag{6.1-19}$$

Equation (6.1-19) clearly shows that for isotropic fluids the value of the propagation coefficient is independent of the direction of propagation of the uniform plane wave, a property that is indicative of the isotropy of the fluid and, hence, of the absence of anisotropy.

In the present case, the vectorial acoustic plane wave admittance reduces to (see Equation (6.1-11))

$$\hat{Y}_r = \hat{Y} \xi_r , \tag{6.1-20}$$

in which $\hat{Y}$, the scalar acoustic plane wave admittance, follows from Equation (6.1-15) as

$$Y = (\hat{\eta} / \hat{\zeta})^{1/2} \quad \text{with } \mathrm{Re}(...)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 . \tag{6.1-21}$$

Accordingly, Equations (6.1-5) and (6.1-6) can now be rewritten as

$$-\xi_k \hat{P} + \hat{Z} \hat{V}_k = 0 , \tag{6.1-22}$$

$$-\xi_r \hat{V}_r + \hat{Y} \hat{P} = 0 , \tag{6.1-23}$$

in which

$$\hat{Z} = \hat{Y}^{-1} = (\hat{\xi}/\hat{\eta})^{1/2} \qquad \text{with } \mathrm{Re}(\ldots)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 \tag{6.1-24}$$

is the scalar *acoustic plane wave impedance* of the uniform plane wave. From Equation (6.1-22) it is clear that in an isotropic fluid the particle velocity of a uniform plane wave is *longitudinal* (i.e. the particle velocity is oriented along the direction of propagation of the wave).

Exercises

Exercise 6.1-1

Construct the one-dimensional wave solutions of the source-free complex frequency-domain acoustic field equations in a homogeneous, isotropic fluid by taking a Euclidean reference frame such that the propagation takes place along the x_3 direction. What is the propagation factor for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3? For the two cases, express the non-vanishing components of $\hat{V}_r$ in terms of $\hat{P}$.

Answers:

(a) Propagation factor = $\exp(-\hat{\gamma}x_3)$, $\hat{V}_3 = (\hat{\eta}/\hat{\xi})^{1/2}\hat{P}$.

(b) Propagation factor = $\exp(\hat{\gamma}x_3)$, $\hat{V}_3 = -(\hat{\eta}/\hat{\xi})^{1/2}\hat{P}$.

Here, $\hat{\gamma} = (\hat{\xi}\hat{\eta})^{1/2}$, and $\mathrm{Re}(\ldots)^{1/2} > 0$ for $\mathrm{Re}(s) > 0$.

6.2 Plane waves in lossless fluids; the slowness surface

In a lossless fluid the complex frequency-domain longitudinal acoustic impedance per length $\hat{\xi}_{k,r}$ and the transverse acoustic admittance per length $\hat{\eta}$ reduce to

$$\hat{\xi}_{k,r} = s\rho_{k,r} \tag{6.2-1}$$

and

$$\hat{\eta} = s\kappa\,, \tag{6.2-2}$$

respectively, in which $\rho_{k,r}$ and κ are independent of s. Under these circumstances the complex propagation vector $\hat{\gamma}_s$ is written as

$$\hat{\gamma}_s = s\Lambda_s\,, \tag{6.2-3}$$

in which Λ_s is the *slowness vector*. Substitution of Equations (6.2-1)–(6.2-3) in the dispersion equation (Equation (6.1-9)) leads to

$$\Lambda_r\,\rho^{-1}_{r,k}\,\Lambda_k - \kappa = 0\,, \tag{6.2-4}$$

which is the equation to be satisfied by the slowness vector. Note that, although Equation (6.2-4) is independent of s, and $\rho_{k,r}$ (and, hence, $\rho^{-1}_{r,k}$) and κ are real-valued, Λ_s can still be complex-valued.

Uniform plane waves

For a uniform plane wave in a lossless fluid the propagation vector is written as

$$\hat{\gamma}_s = s\Lambda\xi_s \,, \tag{6.2-5}$$

in which ξ_s is the (real) unit vector in the direction of propagation of the plane wave (note that $\xi_s\xi_s = 1$) and Λ is the *scalar slowness*. Substitution of the corresponding

$$\Lambda_s = \Lambda\xi_s \tag{6.2-6}$$

in Equation (6.2-4) yields

$$(\xi_r \rho^{-1}_{r,k} \xi_\kappa)\Lambda^2 - \kappa = 0 \,. \tag{6.2-7}$$

In the three-dimensional Euclidean *slowness space* where $\Lambda_s = \Lambda\xi_s$ is the position vector, Equation (6.2-7) defines a surface that is known as the *slowness surface*. For the class of lossless fluids, the slowness surface characterises geometrically the propagation properties of uniform plane waves. In particular, the shape of the slowness surface is indicative of the presence of anisotropy in the acoustic properties of the fluid. As Equation (6.2-7) shows, the slowness surface for an anisotropic, lossless, fluid is, in general, a *tri-axial ellipsoid*, its principal axes coinciding with the principal axes of the positive definite tensor $\rho^{-1}_{r,k}$ (and, hence, of the positive definite tensor $\rho_{k,r}$) (Figure 6.2-1).

Isotropic fluids

For an isotropic, lossless fluid we have

$$\rho_{k,r} = \rho\delta_{k,r} \tag{6.2-8}$$

and, as a consequence of the properties of the Kronecker tensor, the equation for the complex slowness of a non-uniform plane wave (Equation (6.2-4)) reduces to

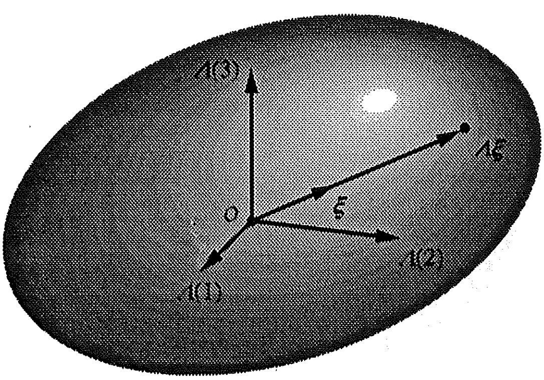

Figure 6.2-1 Slowness surface (tri-axial ellipsoid) for uniform plane waves in an anisotropic, lossless fluid.

$$\Lambda_r \Lambda_r = \rho\kappa \,, \tag{6.2-9}$$

while the equation for the slowness of a uniform plane wave (Equation (6.2-7)) reduces to

$$\Lambda^2 = \rho\kappa \,. \tag{6.2-10}$$

The solution of the latter equation is given by

$$\Lambda = (\rho\kappa)^{1/2} \qquad \text{with } (...)^{1/2} > 0 \,, \tag{6.2-11}$$

or

$$\Lambda = 1/c \,, \tag{6.2-12}$$

where

$$c = (\rho\kappa)^{-1/2} \qquad \text{with } (...)^{-1/2} > 0 \tag{6.2-13}$$

is the acoustic wave speed. As Equations (6.2-10) and (6.2-12) show, the slowness surface for an isotropic, lossless fluid is a *sphere* with radius $1/c$ (Figure 6.2-2).

Exercises

Exercise 6.2-1

Let the orthogonal Cartesian reference frame $\{O; \Lambda_1, \Lambda_2, \Lambda_3\}$ in three-dimensional Euclidean slowness space be oriented along the principal axes of the positive definite tensor $\rho_{k,r}$, which then takes the form

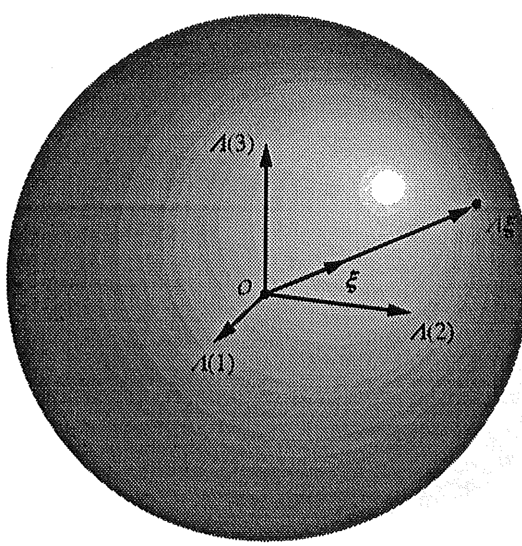

Figure 6.2-2 Slowness surface (sphere) for uniform plane waves in an isotropic, lossless fluid.

$$\rho_{1,1} = \rho(1),\ \rho_{2,2} = \rho(2),\ \rho_{3,3} = \rho(3),\quad \rho_{k,r} = 0 \quad \text{if } k \neq r\,. \tag{6.2-14}$$

Use Equation (6.2-7) to construct the equation of the tri-axial ellipsoidal slowness surface. (*Hint*: Observe that, in the given reference frame, $\rho^{-1}_{r,k}$ takes the form: $(\rho^{-1})_{1,1} = \rho(1)^{-1}$, $(\rho^{-1})_{2,2} = \rho(2)^{-1}$, $(\rho^{-1})_{3,3} = \rho(3)^{-1}$, $\rho^{-1}_{r,k} = 0$ if $r \neq k$.)

Answer:

$$\frac{\Lambda_1^2}{\rho(1)\kappa} + \frac{\Lambda_2^2}{\rho(2)\kappa} + \frac{\Lambda_3^2}{\rho(3)\kappa} = 1\,. \tag{6.2-15}$$

Exercise 6.2-2

Let $\{O;\Lambda_1,\Lambda_2,\Lambda_3\}$ be an orthogonal Cartesian reference frame in three-dimensional Euclidean slowness space. Use Equation (6.2-10) to construct the equation of the spherical slowness surface.

Answer:

$$\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 = \rho\kappa = 1/c^2\,. \tag{6.2-16}$$

6.3 Plane waves in the real frequency domain; attenuation vector and phase vector

In the signal processing of acoustic wave phenomena, extensive use is made of the highly efficient fast Fourier transform (FFT) algorithms that apply to the imaginary values $s = \mathrm{j}\omega$ (where j is the imaginary unit, and ω is the (real) angular frequency) of the complex frequency s to transform wave-field quantities from the time domain to the complex frequency domain, and vice versa. As a consequence, the corresponding imaginary values of s are of particular interest. Now, for imaginary values of s, the condition of causality can no longer be easily invoked on the frequency-domain wave quantities. To control the causality one must always consider the imaginary values of s as the limiting ones upon approaching, in the complex s plane, the imaginary axis via the right half $\mathrm{Re}(s) > 0$ of the complex s plane. For $s = \mathrm{j}\omega$ it is customary to decompose the complex propagation vector $\hat{\gamma}_s = \hat{\gamma}_s(\mathrm{j}\omega)$ into its real and imaginary parts according to

$$\hat{\gamma}_s(\mathrm{j}\omega) = \alpha_s(\omega) + \mathrm{j}\beta_s(\omega)\,, \tag{6.3-1}$$

where α_s is the attenuation vector (SI unit: neper/metre (Np/m)), and β_s is the phase vector (SI unit: radian/metre (rad/m)). In view of the property

$$|\exp(-\hat{\gamma}_s x_s)| = \exp(-\alpha_s x_s)\,, \tag{6.3-2}$$

which holds because $|\exp(-\mathrm{j}\beta_s x_s)| = 1$, the family of planes $\{\alpha\in\mathcal{R}^3, x\in\mathcal{R}^3; \alpha_s x_s = \text{constant}\}$ defines a set of *planes of equal amplitude*, while in view of the property

$$\arg[\exp(-\hat{\gamma}_s x_s)] = -\beta_s x_s\,, \tag{6.3-3}$$

which holds because $\arg[\exp(-\alpha_s x_s)] = 0$, the family of planes $\{\beta\in\mathcal{R}^3, x\in\mathcal{R}^3; \beta_s x_s = \text{constant}\}$ defines a set of *planes of equal phase*. These two properties elucidate the term "plane wave" for complex frequency-domain solutions of the acoustic wave equations of the type given in Equation (6.1-3).

Uniform plane waves

For a uniform plane wave propagating in the direction of the unit vector ξ_s we have (see Equations (6.1-12) and (6.3-1))

$$\alpha_s = \alpha \xi_s \tag{6.3-4}$$

and

$$\beta_s = \beta \xi_s \, , \tag{6.3-5}$$

where α is the (scalar) attenuation coefficient (Np/m), β is the (scalar) phase coefficient (rad/m) and

$$\hat{\gamma}(\mathrm{j}\omega) = \alpha(\omega) + \mathrm{j}\beta(\omega) \, . \tag{6.3-6}$$

For uniform plane waves, the set of planes of equal amplitude coincides with the set of planes of equal phase (Figure 6.3-1).

Propagation in an isotropic fluid with frictional-force/bulk-viscosity acoustic losses

As a first example we shall discuss the propagation of plane waves in an isotropic fluid with frictional-force/bulk-viscosity acoustic losses. For such a fluid, we have (see Equations (6.4-9) and (6.4-10))

$$\hat{\zeta}(\mathrm{j}\omega) = K + \mathrm{j}\omega\rho \tag{6.3-7}$$

and

$$\hat{\eta}(\mathrm{j}\omega) = \Gamma + \mathrm{j}\omega\kappa \, . \tag{6.3-8}$$

Substitution of Equation (6.3-1) in the corresponding dispersion equation (see Equation (6.1-17)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = (K + \mathrm{j}\omega\rho)(\Gamma + \mathrm{j}\omega\kappa) \, . \tag{6.3-9}$$

Separation of Equation (6.3-9) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = K\Gamma - \omega^2\rho\kappa \tag{6.3-10}$$

and

$$2\alpha_s\beta_s = \omega(\Gamma\rho + K\kappa) \, . \tag{6.3-11}$$

From Equation (6.3-10) it is clear that for $K\Gamma < \omega^2\rho\kappa$ we have $\alpha_s\alpha_s < \beta_s\beta_s$, i.e. phase propagation is the predominant phenomenon, while for $K\Gamma > \omega^2\rho\kappa$ we have $\alpha_s\alpha_s > \beta_s\beta_s$, i.e. attenuation is

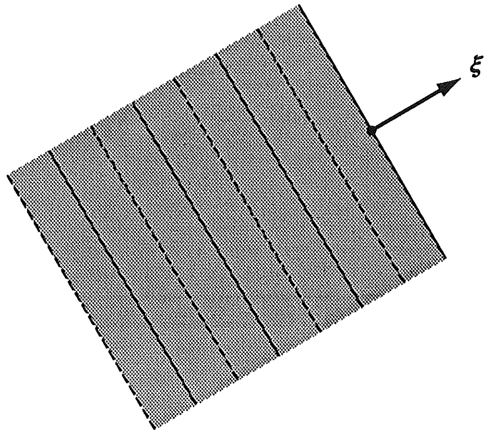

Figure 6.3-1 Planes of equal amplitude (———) and planes of equal phase (— — —) of a uniform plane wave in the real frequency domain.

the predominant phenomenon. Note that the condition about which of the two phenomena is predominant, is frequency dependent. Furthermore, Equation (6.3-11) indicates that $\alpha_s\beta_s \neq 0$ for non-zero frequency, since the right-hand side differs from zero for non-zero frequency. Hence, for the type of fluid under consideration, the set of planes of equal amplitude is, in general, inclined with respect to the set of planes of equal phase (Figure 6.3-2).

For a uniform plane wave in the type of fluid under consideration, substitution of Equations (6.3-4) and (6.3-5) in Equations (6.3-10) and (6.3-11) leads to

$$\alpha^2 - \beta^2 = K\Gamma - \omega^2\rho\kappa \tag{6.3-12}$$

and

$$2\alpha\beta = \omega(\Gamma\rho + K\kappa)\,, \tag{6.3-13}$$

where the property $\xi_s\xi_s = 1$ has been used. Since α must be non-negative in view of the condition of causality, Equation (6.3-13) indicates that $\beta \gtrless 0$ for $\omega \gtrless 0$. After some algebraic manipulations we obtain (see Exercise 6.3-1)

$$\alpha = \left[\frac{1}{2}\left(K\Gamma - \omega^2\rho\kappa + \left\{(K\Gamma)^2 + \omega^2\left[(\Gamma\rho)^2 + (K\kappa)^2\right] + \omega^4(\rho\kappa)^2\right\}^{1/2}\right)\right]^{1/2} \tag{6.3-14}$$

and

$$\beta = \pm\left[\frac{1}{2}\left(-K\Gamma + \omega^2\rho\kappa + \left\{(K\Gamma)^2 + \omega^2\left[(\Gamma\rho)^2 + (K\kappa)^2\right] + \omega^4(\rho\kappa)^2\right\}^{1/2}\right)\right]^{1/2} \quad \text{for } \omega \gtrless 0\,, \tag{6.3-15}$$

in which all square-root expressions are non-negative. For very low and very high frequencies these results yield the asymptotic representations

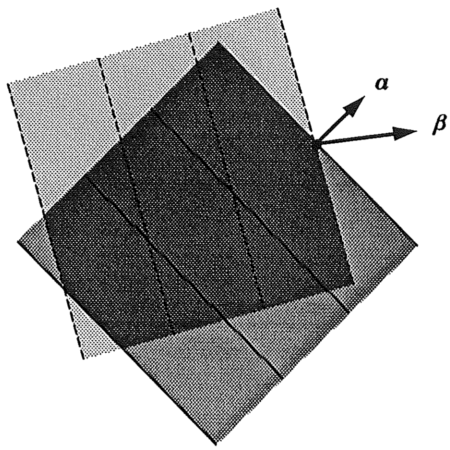

Figure 6.3-2 Planes of equal amplitude (————) and planes of equal phase (— — —) of a non-uniform plane wave in a fluid with frictional-force/bulk-viscosity acoustic losses in the real frequency domain.

$$\alpha \sim (K\Gamma)^{1/2} \quad \text{and} \quad \beta \sim \omega (K\Gamma)^{1/2}\, \frac{\rho/K + \kappa/\Gamma}{2} \quad \text{as } |\omega| \to 0 \tag{6.3-16}$$

and

$$\alpha \sim (\rho\kappa)^{1/2}\, \frac{\Gamma/\kappa + K/\rho}{2} \quad \text{and} \quad \beta \sim \omega (\rho\kappa)^{1/2} \quad \text{as } |\omega| \to \infty\,. \tag{6.3-17}$$

To put the results in a normalised form, we introduce the critical angular frequency of the frictional-force losses ω_{f} and the critical angular frequency of the bulk-viscosity losses ω_{b} via

$$\omega_{\mathrm{f}} = K/\rho \tag{6.3-18}$$

and

$$\omega_{\mathrm{b}} = \Gamma/\kappa\,, \tag{6.3-19}$$

and write

$$\alpha = (K\Gamma)^{1/2}\,\bar{\alpha}\,, \tag{6.3-20}$$

$$\beta = (K\Gamma)^{1/2}\,\bar{\beta} \tag{6.3-21}$$

and

$$\bar{\omega} = \omega/(\omega_{\mathrm{f}}\omega_{\mathrm{b}})^{1/2}\,, \tag{6.3-22}$$

where $\bar{\alpha}$ is the normalised attenuation coefficient, $\bar{\beta}$ is the normalised phase coefficient, and $\bar{\omega}$ is the normalised angular frequency. In terms of these quantities, Equations (6.3-12) and (6.3-13) become

$$\bar{\alpha}^2 - \bar{\beta}^2 = 1 - \bar{\omega}^2 \tag{6.3-23}$$

and

$$2\bar{\alpha}\bar{\beta} = \bar{\omega}\left[\left(\frac{\omega_{\mathrm{b}}}{\omega_{\mathrm{f}}}\right)^{1/2} + \left(\frac{\omega_{\mathrm{f}}}{\omega_{\mathrm{b}}}\right)^{1/2}\right], \tag{6.3-24}$$

with the solution (see Equations (6.3-14) and (6.3-15))

$$\bar{\alpha} = \left[\frac{1}{2}\left(1 - \bar{\omega}^2 + \left\{1 + \bar{\omega}^2\ (\omega_{\mathrm{b}}/\omega_{\mathrm{f}} + \omega_{\mathrm{f}}/\omega_{\mathrm{b}}) + \bar{\omega}^4\right\}^{1/2}\right)\right]^{1/2}, \tag{6.3-25}$$

and

$$\bar{\beta} = \pm\left[\frac{1}{2}\left(-1 + \bar{\omega}^2 + \left\{1 + \bar{\omega}^2\ (\omega_{\mathrm{b}}/\omega_{\mathrm{f}} + \omega_{\mathrm{f}}/\omega_{\mathrm{b}}) + \bar{\omega}^4\right\}^{1/2}\right)\right]^{1/2} \quad \text{for } \bar{\omega} \gtrless 0\,. \tag{6.3-26}$$

The asymptotic representations (Equations (6.3-16) and (6.3-17)) become, in their normalised form,

$$\bar{\alpha} \sim 1 \quad \text{and} \quad \bar{\beta} \sim \bar{\omega}\,\frac{(\omega_{\mathrm{b}}/\omega_{\mathrm{f}})^{1/2} + (\omega_{\mathrm{f}}/\omega_{\mathrm{b}})^{1/2}}{2} \quad \text{as } |\bar{\omega}| \to 0 \tag{6.3-27}$$

and

$$\bar{\alpha} \sim \frac{1}{2}\,[(\omega_{\mathrm{b}}/\omega_{\mathrm{f}})^{1/2} + (\omega_{\mathrm{f}}/\omega_{\mathrm{b}})^{1/2}] \quad \text{and} \quad \bar{\beta} \sim \bar{\omega} \quad \text{as } |\bar{\omega}| \to \infty\,. \tag{6.3-28}$$

Figures 6.3-3(a) and 6.3-3(b) show $\bar{\alpha}$ and $\bar{\beta}$ as a function of $\bar{\omega}$ with $\omega_{\mathrm{f}}/\omega_{\mathrm{b}}$ as a parameter.

Propagation in an isotropic, lossless fluid

As a second example we shall discuss the propagation of plane waves in an isotropic, lossless fluid. For such a fluid, we have (see Equations (4.4-9) and (4.4-10))

$$\hat{\zeta} = \mathrm{j}\omega\rho \tag{6.3-29}$$

and

$$\hat{\eta} = \mathrm{j}\omega\kappa\,. \tag{6.3-30}$$

Substitution of Equation (6.3-1) in the corresponding dispersion equation (see Equation (6.1-17)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = -\omega^2\rho\kappa\,. \tag{6.3-31}$$

Separation of Equation (6.3-31) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2\rho\kappa \tag{6.3-32}$$

and

$$2\alpha_s\beta_s = 0\,. \tag{6.3-33}$$

From Equation (6.3-32) it is clear that we have $\alpha_s\alpha_s < \beta_s\beta_s$, i.e. phase propagation is, for all frequencies, the predominant phenomenon. Furthermore, Equation (6.3-33) indicates that, for

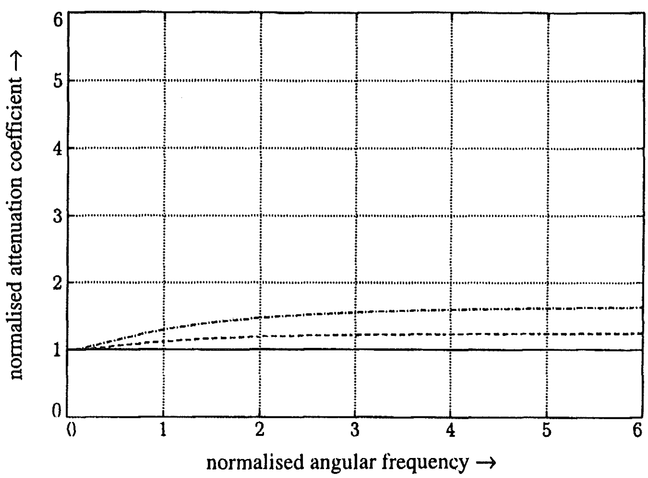

Figure 6.3-3(a) Normalised attenuation coefficient $\bar{\alpha} = \alpha/(K\Gamma)^{1/2}$ as a function of normalised angular frequency $\bar{\omega} = \omega/(\omega_f\omega_b)^{1/2}$ with ω_f/ω_b as a parameter for a uniform plane wave in an isotropic fluid with frictional-force/bulk-viscosity losses ($\omega_f = K/\rho, \omega_b = \Gamma/\kappa$). (—) $\omega_f/\omega_b = 1$; (- - -) $\omega_f/\omega_b = 4$; (-·-·-) $\omega_f/\omega_b = 9$.

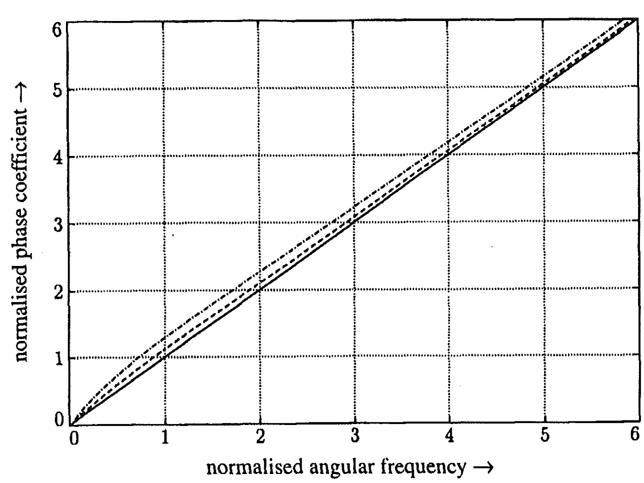

Figure 6.3-3(b) Normalised phase coefficient $\bar{\beta} = \beta/(K\Gamma)^{1/2}$ as a function of normalised angular frequency $\bar{\omega} = \omega/(\omega_f\omega_b)^{1/2}$ with ω_f/ω_b as a parameter for a uniform plane wave in an isotropic fluid with frictional-force/bulk-viscosity losses ($\omega_f = K/\rho, \omega_b = \Gamma/\kappa$). (—) $\omega_f/\omega_b = 1$; (- - -) $\omega_f/\omega_b = 4$; (-·-·-) $\omega_f/\omega_b = 9$.

all frequencies, $\alpha_s \beta_s = 0$. Hence, for the type of fluid under consideration, the set of planes of equal amplitude is, for a non-uniform plane wave, perpendicular to the set of planes of equal phase (Figure 6.3-4).

For a uniform plane wave in the type of fluid under consideration, substitution of Equations (6.3-4) and (6.3-5) in Equations (6.3-32) and (6.3-33) leads to

$$\alpha^2 - \beta^2 = -\omega^2 \rho \kappa \qquad (6.3\text{-}34)$$

and

$$2\alpha\beta = 0 \,, \qquad (6.3\text{-}35)$$

where the property $\xi_s \xi_s = 1$ has been used. These equations have as their only solution

$$\alpha = 0 \qquad (6.3\text{-}36)$$

and

$$\beta = \omega(\rho\kappa)^{1/2} \,, \qquad (6.3\text{-}37)$$

in which the square root expression is positive. Hence, in the lossless fluid there is no attenuation, and the phase coefficient varies linearly with the angular frequency. (Note that $\alpha_s \neq 0$ and $\beta_s = 0$ does not lead to a solution of Equation (6.3-31).)

Exercises

Exercise 6.3-1

Derive Equations (6.3-14) and (6.3-15) from Equations (6.3-12) and (6.3-13). (*Hint*: Derive expressions for α^2 and β^2 from Equation (6.3-12) and the sum of the squared versions of Equations (6.3-12) and (6.3-13).

Exercise 6.3-2

Derive the asymptotic representations given in Equations (6.3-16) and (6.3-17) from Equations (6.3-14) and (6.3-15) and check that the result is in accordance with Equations (6.3-12) and (6.3-13).

Exercise 6.3-3

Derive Equations (6.3-25) and (6.3-26) from Equations (6.3-14) and (6.3-15) by using Equations (6.3-20)–(6.3-22).

Exercise 6.3-4

Derive Equations (6.3-27) and (6.3-28) from Equations (6.3-16) and (6.3-17) by using Equations (6.3-20)–(6.3-22).

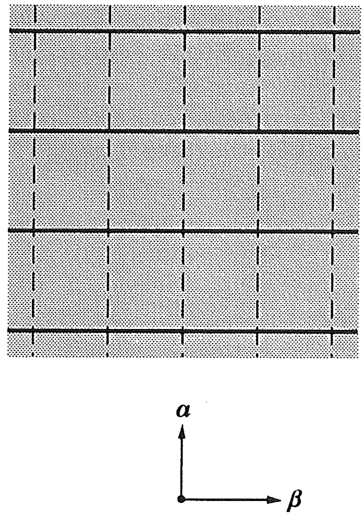

Figure 6.3-4 Planes of equal amplitude (———) and planes of equal phase (— — —) of a non-uniform plane wave in a lossless fluid in the real frequency domain.

6.4 Time-domain uniform plane waves in an isotropic, lossless fluid

Time-domain *uniform plane waves* in an isotropic, lossless fluid are solutions of the source-free acoustic wave equations (see Equations (2.7-22)–(2.7-25))

$$\partial_k p + \rho \partial_t v_k = 0 \,, \tag{6.4-1}$$

$$\partial_r v_r + \kappa \partial_t p = 0 \,, \tag{6.4-2}$$

of the form

$$\{p, v_r\} = \{P, V_r\}(t - \Lambda \xi_s x_s) \,, \tag{6.4-3}$$

in which ξ_s is the *unit vector in the direction of propagation* of the wave, Λ is its *slowness*, and $P(t)$ and $V_r(t)$ are the *pulse shapes* (*"signatures"*) of the acoustic pressure p and the particle velocity v_r, respectively. Substitution of Equation (6.4-3) in Equations (6.4-1) and (6.4-2) leads, in view of the relation

$$\partial_m \{P, V_r\}(t - \Lambda \xi_s x_s) = -\Lambda \xi_m \partial_t \{P, V_r\}(t - \Lambda \xi_s x_s) \,, \tag{6.4-4}$$

to

$$-\Lambda \xi_k \partial_t P + \rho \partial_t V_k = 0 \,, \tag{6.4-5}$$

$$-\Lambda \xi_r \partial_t V_r + \kappa \partial_t P = 0 \,. \tag{6.4-6}$$

Since Equations (6.4-5) and (6.4-6) have to be satisfied for all values of t at each position $\boldsymbol{x}$ of the source-free domain under consideration, $\partial_t P$ and all three components of $\partial_t V_r$ must have a common pulse shape. In view of this, and of the condition of causality by which P and V_r have the value zero prior to some instant in the finite past, P and all three components of V_r must have a common pulse shape. To meet this condition, Equation (6.4-3) is replaced by

$$\{p,v_r\} = \{P,V_r\}a(t - \Lambda\xi_s x_s) , \tag{6.4-7}$$

where $\{P,V_r\}$ are now the amplitudes of $\{p,v_r\}$, and $a(t)$ is the somehow normalised pulse shape of the wave motion. With Equation (6.4-7), Equations (6.4-5) and (6.4-6) reduce to

$$-\Lambda\xi_k P + \rho V_k = 0 , \tag{6.4-8}$$

$$-\Lambda\xi_r V_r + \kappa P = 0 . \tag{6.4-9}$$

Substitution of the expression for V_k resulting from Equation (6.4-8) in Equation (6.4-9) yields

$$(\Lambda^2 - \rho\kappa)P = 0 . \tag{6.4-10}$$

For a non-identically vanishing solution of Equation (6.4-10) to exist, the slowness Λ must satisfy the equation

$$\Lambda^2 = \rho\kappa . \tag{6.4-11}$$

This leads to

$$\Lambda = (\rho\kappa)^{1/2} \qquad \text{with } (\ldots)^{1/2} > 0 , \tag{6.4-12}$$

or

$$\Lambda = 1/c , \tag{6.4-13}$$

where

$$c = (\rho\kappa)^{-1/2} \qquad \text{with } (\ldots)^{-1/2} > 0 \tag{6.4-14}$$

is the acoustic wave speed for a wave propagating in the direction of ξ_s, i.e. for a wave whose spatial argument $\Lambda\xi_s x_s$ increases along ξ_s. For the value of Λ as given by Equation (6.4-12), Equation (6.4-8) can be rewritten as

$$V_r = Y_r P \tag{6.4-15}$$

in which

$$Y_r = Y\xi_r , \tag{6.4-16}$$

is the vectorial *acoustic plane wave admittance* of the wave and

$$Y = (\kappa/\rho)^{1/2} = (\rho c)^{-1} \qquad \text{with } (\ldots)^{1/2} > 0 \tag{6.4-17}$$

is the scalar acoustic plane wave admittance of the wave. With this, Equations (6.4-8) and (6.4-9) can be rewritten as

$$\xi_k P + Z V_k = 0 , \tag{6.4-18}$$

$$\xi_r V_r + Y P = 0 , \tag{6.4-19}$$

in which

$$Z = Y^{-1} = (\rho/\kappa)^{1/2} = \rho c \qquad \text{with } (\ldots)^{1/2} > 0 \tag{6.4-20}$$

is the *acoustic plane wave impedance* of the wave. From Equation (6.4-18) it is clear that in an isotropic lossless fluid the particle velocity of a uniform plane wave is *longitudinal* (i.e. the particle velocity is oriented along the direction of propagation).

For the acoustic Poynting vector (see Equation (2.8-7))

$$S_m^{\mathrm{a}} = p v_m \tag{6.4-21}$$

of the uniform plane wave, Equations (6.4-7), (6.4-18) and (6.4-19) lead to the following alternative expressions

$$S_m^{\mathrm{a}} = c\kappa P P \xi_m \,, \tag{6.4-22}$$

or

$$S_m^{\mathrm{a}} = c\rho V_r V_r \xi_m \,, \tag{6.4-23}$$

where we have used the identity

$$\xi_r V_r V_m = V_r V_r \xi_m \tag{6.4-24}$$

that follows from Equations (6.4-18) and (6.4-19). Upon comparing Equation (6.4-22) with Equation (6.4-23), we conclude that

$$\tfrac{1}{2}\kappa P P = \tfrac{1}{2}\rho V_r V_r \,, \tag{6.4-25}$$

i.e. the volume density of deformation energy is equal to the volume density of kinetic energy in the uniform plane wave. Using this in Equation (6.4-21), we can write

$$S_m^{\mathrm{a}} = c\left[\tfrac{1}{2}\kappa P P + \tfrac{1}{2}\rho V_r V_r\right]\xi_m \,. \tag{6.4-26}$$

This result leads to the picture that, for a uniform plane acoustic wave in an isotropic, lossless fluid, the Poynting vector carries the sum of the volume densities of deformation energy and kinetic energy, with the acoustic wave speed, in the direction of propagation of the wave.

Exercises

Exercise 6.4-1

Construct the one-dimensional wave solutions of the source-free time-domain acoustic wave-field equations in a homogeneous, isotropic, lossless fluid by taking a Cartesian reference frame such that the propagation takes place along the x_3 direction. Write down the expression for p for: (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3. Express, for the two cases, the non-vanishing components of v_r in terms of p.

Answers:

(a) $p = P(t - x_3/c)$, $v_3 = (\kappa/\rho)^{1/2} p$.
(b) $p = P(t + x_3/c)$, $v_3 = -(\kappa/\rho)^{1/2} p$.
Here, $c = (\rho\kappa)^{-1/2}$.

Exercise 6.4-2

Determine the value of: (a) the acoustic plane wave impedance, (b) the acoustic plane wave admittance in water ($\rho = 10^3$ kg/m^3, $c = 1500$ m/s).

Answers:

(a) $Z = 1.5 \times 10^6$ kg/m^2 s.
(b) $Y = 0.667 \times 10^{-6}$ m^2 s/kg.

Exercise 6.4-3

Determine the value of: (a) the acoustic plane wave impedance, (b) the acoustic plane wave admittance in air ($\rho = 1.21$ kg/m^3, $c = 343$ m/s).

Answers:

(a) $Z = 145$ kg/m^2 s.
(b) $Y = 2.41 \times 10^{-3}$m^2 s/kg.

Exercise 6.4-4

Determine the ratio of the magnitudes of the acoustic Poynting vector of a plane acoustic wave in water ($\rho = 10^3$ kg/m, $c = 1500$ m/s) and a plane acoustic wave in air ($\rho = 1.21$ kg/m^3, $c =$ 343 m/s) with equal acoustic pressure amplitudes and wave shapes.

Answer: $S^{\mathrm{a}}_{\mathrm{water}}/S^{\mathrm{a}}_{\mathrm{air}} = (\rho c)_{\mathrm{water}}/(\rho c)_{\mathrm{air}} = 3614.$

Exercise 6.4-5

For a uniform plane wave, periodicity in time entails periodicity in space. Let T denote the time period of the wave and $f = 1/T$ be frequency (Hz). Show, with the aid of Equation (6.4-7), that the spatial period λ in the direction of propagation of the wave is related to T or f via

$$\lambda = cT = c/f. \tag{6.4-27}$$

(The quantity λ is the *wavelength* of the time-periodic plane wave.)

Exercise 6.4-6

Determine the wavelength (λ) of a time-periodic plane acoustic wave in water ($c = 1500$ m/s) if this wave has a frequency of (a) $f = 10$ Hz, (b) $f = 100$ Hz, (c) $f = 1000$ Hz, (d) $f = 10$ kHz, and (e) $f = 1$ MHz.

Answers: (a) $\lambda = 150$ m, (b) $\lambda = 15$ m, (c) $\lambda = 1.5$ m, (d) $\lambda = 0.15$ m, (e) $\lambda = 1.5$ mm.

Exercise 6.4-7

Determine the wavelength (λ) of a time-periodic plane acoustic wave in air ($c = 343$ m/s) if this wave has a frequency of (a) $f = 10$ Hz, (b) $f = 100$ Hz, (c) $f = 1000$ Hz, (d) $f = 10$ kHz, and (e) $f = 1$ MHz.

Answers: (a) $\lambda = 34.3$ m, (b) $\lambda = 3.43$ m, (c) $\lambda = 0.343$ m, (d) $\lambda = 34.3$ mm, (e) $\lambda = 0.343$ mm.

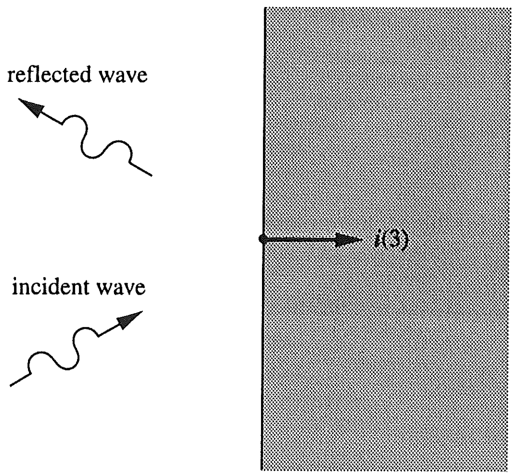

Figure 6.5-1 Reflection by an impenetrable object occupying the half-space $\{x_3 > 0\}$ on which a plane acoustic wave is incident.

6.5 Structure of the plane wave motion near the planar boundary of an acoustically impenetrable object

In a number of applications the structure of the plane wave motion near the planar boundary of an acoustically impenetrable object is needed. Two cases will be considered: the case where the impenetrable object is perfectly compliant (i.e. the acoustic pressure is zero on its boundary), and the case where the impenetrable object is immovably rigid (i.e. the normal component of the particle velocity is zero on its boundary). Since, for a single plane wave, neither the acoustic pressure nor the normal component of the particle velocity can be zero for all times at a particular plane, a single plane wave cannot satisfy the relevant boundary conditions. Physically this is immediately clear since, due to the presence of the discontinuity in acoustic properties across the boundary surface of the impenetrable object, a plane wave impinging on it will undergo a *reflection*, i.e. a wave travelling in the opposite direction, as far as the normal to the boundary is concerned, will be superimposed on the incident wave.

To construct the total wave motion, an orthogonal Cartesian reference frame is introduced such that the planar boundary of the impenetrable object coincides with the plane $\{x \in \mathcal{R}^3;\ -\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ x_3 = 0\}$, while the impenetrable object occupies the half-space $\{x \in \mathcal{R}^3;\ -\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ 0 < x_3 < \infty\}$ (Figure 6.5-1). The fluid in the half-space $\{x \in \mathcal{R}^3;\ -\infty < x_1 < \infty,\ -\infty < x_2 < \infty,\ -\infty < x_3 < 0\}$ is assumed to be homogeneous and isotropic. The boundary conditions to be satisfied upon approaching the boundary $\{x_3 = 0\}$ via the half-space $\{x_3 < 0\}$ are independent of x_1 and x_2 (and t or s). Therefore, we can attempt to solve the reflection problem by retaining for the dependence of the wave functions on x_1 and x_2 (and t or s) the dependence of the incident wave on x_1 and x_2 (and t or s), and taking for the reflected wave a plane wave whose component of the propagation vector along i_3 is the opposite of the one of the incident wave.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem we write

$$\{\hat{p},\hat{v}_r\} = \{\hat{p}^{\,i} + \hat{p}^{\,r} + \hat{v}_r^{i} + \hat{v}_r^{r}\}, \tag{6.5-1}$$

in which the *incident plane wave* $\{p^i,v_r^i\}$ is given by (see Equations (6.1-3), (6.1-10), (6.1-20) and (6.1-21))

$$\hat{p}^{\,i} = \hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(-\hat{\gamma}_3 x_3)\,, \tag{6.5-2}$$

$$\hat{v}_1^{i} = (\hat{\gamma}_1/\hat{\zeta})\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(-\hat{\gamma}_3 x_3)\,, \tag{6.5-3}$$

$$\hat{v}_2^{i} = (\hat{\gamma}_2/\hat{\zeta})\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(-\hat{\gamma}_3 x_3)\,, \tag{6.5-4}$$

$$\hat{v}_3^{i} = (\hat{\gamma}_3/\hat{\zeta})\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(-\hat{\gamma}_3 x_3)\,, \tag{6.5-5}$$

where (see Equation (6.1-17))

$$\hat{\gamma}_1^2 + \hat{\gamma}_2^2 + \hat{\gamma}_3^2 = \hat{\eta}\hat{\zeta} \tag{6.5-6}$$

and $\mathrm{Re}(\hat{\gamma}_3) > 0$ for $\mathrm{Re}(s) > 0$, and the *reflected plane wave* $\{p^r,v_r^r\}$ is written as

$$\hat{p}^{\,r} = \hat{R}\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(+\hat{\gamma}_3 x_3)\,, \tag{6.5-7}$$

$$\hat{v}_1^{r} = \hat{R}(\hat{\gamma}_1/\hat{\zeta})\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(+\hat{\gamma}_3 x_3)\,, \tag{6.5-8}$$

$$\hat{v}_2^{r} = \hat{R}(\hat{\gamma}_2/\hat{\zeta})\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(+\hat{\gamma}_3 x_3)\,, \tag{6.5-9}$$

$$\hat{v}_3^{r} = -\hat{R}(\hat{\gamma}_3/\hat{\zeta})\hat{P}\exp(-\hat{\gamma}_1 x_1 - \hat{\gamma}_2 x_2)\exp(+\hat{\gamma}_3 x_3)\,, \tag{6.5-10}$$

where $\hat{R}$ is the (acoustic-pressure) *reflection coefficient.* Note that the wave motion given by Equation (6.5-1) is defined in the half-space $\{x_3 < 0\}$ only and that in the half-space $\{x_3 > 0\}$ no wave motion is present. (The object in $\{x_3 > 0\}$ has been assumed to be impenetrable.)

Perfectly compliant object

For a perfectly compliant object, the boundary condition (see Equation (4.3-3))

$$\lim_{x_3 \uparrow 0} \hat{p} = 0 \qquad \text{for all } (x_1,x_2) \text{ and } s \tag{6.5-11}$$

must be satisfied. Substitution of Equations (6.5-2) and (6.5-7) in Equation (6.5-11) leads to

$$\hat{R} = -1\,. \tag{6.5-12}$$

In this case we have (see Equations (6.5-1), (6.5-5) and (6.5-10))

$$\lim_{x_3 \uparrow 0} \hat{v}_3 = 2\hat{v}_3^{i} \qquad \text{for all } (x_1,x_2) \text{ and } s\,, \tag{6.5-13}$$

i.e. the value of the normal component of the particle velocity at the boundary is twice that of the incident wave.

Immovable perfectly rigid object

For an immovable perfectly rigid object, the boundary condition (see Equation (6.3-7))

$$\lim_{x_3 \uparrow 0} \hat{v}_3 = 0 \qquad \text{for all } (x_1, x_2) \text{ and } s \tag{6.5-14}$$

must be satisfied. Substitution of Equations (6.5-5) and (6.5-9) in Equation (6.5-12) leads to

$$\hat{R} = 1 . \tag{6.5-15}$$

In this case we have (see Equation (6.5-1), (6.5-2) and (6.5-7))

$$\lim_{x_3 \uparrow 0} \hat{p} = 2\hat{p}^{\mathrm{i}} \qquad \text{for all } (x_1, x_2) \text{ and } s , \tag{6.5-16}$$

i.e. the value of the acoustic pressure at the boundary is twice that of the incident wave.

Time-domain analysis for a lossless fluid

In the time-domain analysis for a lossless fluid we write

$$\{p, v_r\} = \{p^{\mathrm{i}} + p^{\mathrm{r}}, v_r^{\mathrm{i}} + v_r^{\mathrm{r}}\} , \tag{6.5-17}$$

in which the *incident uniform plane wave* $\{p^{\mathrm{i}}, v_r^{\mathrm{i}}\}$ is given by (see Equations (6.4-7), and (6.4-15)–(6.4-17))

$$p^{\mathrm{i}} = Pa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)] , \tag{6.5-18}$$

$$v_1^{\mathrm{i}} = (\kappa/\rho)^{1/2} Pa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)] , \tag{6.5-19}$$

$$v_2^{\mathrm{i}} = (\kappa/\rho)^{1/2} Pa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)] , \tag{6.5-20}$$

$$v_3^{\mathrm{i}} = (\kappa/\rho)^{1/2} Pa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)] , \tag{6.5-21}$$

where (see Equation (6.4-12))

$$\Lambda = (\rho\kappa)^{-1/2} = 1/c \tag{6.5-22}$$

and $\xi_3 > 0$, and the *reflected uniform plane wave* $\{p^{\mathrm{r}}, v_r^{\mathrm{r}}\}$ is written as

$$p^{\mathrm{r}} = RPa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 - \xi_3 x_3)] , \tag{6.5-23}$$

$$v_1^{\mathrm{r}} = (\kappa/\rho)^{1/2} RPa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 - \xi_3 x_3)] , \tag{6.5-24}$$

$$v_2^{\mathrm{r}} = (\kappa/\rho)^{1/2} RPa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 - \xi_3 x_3)] , \tag{6.5-25}$$

$$v_3^{\mathrm{r}} = -(\kappa/\rho)^{1/2} RPa[t - \Lambda(\xi_1 x_1 + \xi_2 x_2 - \xi_3 x_3)] , \tag{6.5-26}$$

where R is the (acoustic-pressure) *reflection coefficient.* Note that the wave motion given by Equation (6.5-17) is defined in the half-space $\{x_3 < 0\}$ only and that in the half-space $\{x_3 > 0\}$ no wave motion is present. (The object in $\{x_3 > 0\}$ has been assumed to be impenetrable.)

Perfectly compliant object

For a perfectly compliant object, the boundary condition (see Equation (2.6-4))

$$\lim_{x_3 \uparrow 0} p = 0 \qquad \text{for all } (x_1,x_2) \text{ and } t \tag{6.5-27}$$

must be satisfied. Substitution of Equations (6.5-18) and (6.5-23) in Equation (6.5-27) leads to

$$R = -1 \, . \tag{6.5-28}$$

In this case we have (see Equations (6.5-17), (6.5-21) and (6.5-26))

$$\lim_{x_3 \uparrow 0} v_3 = 2v_3^{\mathrm{i}} \qquad \text{for all } (x_1,x_2) \text{ and } t \, , \tag{6.5-29}$$

i.e. the value of the normal component of the particle velocity at the boundary is twice that of the incident wave.

Immovable perfectly rigid object

For an immovable perfectly rigid object the boundary condition (see Equation (2.6-8))

$$\lim_{x_3 \uparrow 0} v_3 = 0 \qquad \text{for all } (x_1,x_2) \text{ and } t \tag{6.5-30}$$

must be satisfied. Substitution of Equations (6.5-21) and (6.5-26) in Equation (6.5-30) leads to

$$R = 1 \, . \tag{6.5-31}$$

In this case we have (see Equation (6.5-17), (6.5-18) and (6.5-23))

$$\lim_{x_3 \uparrow 0} p = 2p^{\mathrm{i}} \qquad \text{for all } (x_1,x_2) \text{ and } t \, , \tag{6.5-32}$$

i.e. the value of the acoustic pressure at the boundary is twice that of the incident wave.