7

Acoustic reciprocity theorems and their applications

In this chapter we discuss the basic reciprocity theorems for acoustic wave fields in time-invariant configurations, together with a variety of their applications. The theorems will be presented both in the time domain and in the complex frequency domain. In view of the time invariance of the configurations to be considered, there are two versions of the theorems insofar as their operations on the time coordinate are concerned: a version that is denoted as the *time convolution type*, and a version that is denoted as the *time correlation type*. The two versions are related via a time inversion operation. Each of the two versions has its counterpart in the complex frequency domain.

The application of the theorems to the reciprocity in *transmitting/receiving properties of acoustic transducers*, and to the formulations of the *direct (forward) source* and *inverse source* and the *direct (forward) scattering* and *inverse scattering problems* will be discussed. Furthermore, it is indicated how the theorems lead, in a natural way, to the *integral equation formulation* of acoustic wave problems for numerical implementation. Finally, it is shown how the reciprocity theorems lead to a mathematical formulation of *Huygens' principle* and of *Oseen's extinction theorem.*

7.1 The nature of the reciprocity theorems and the scope of their consequences

A reciprocity theorem interrelates, in a specific manner, the field or wave quantities which characterise two admissible states that could occur in one and the same time-invariant domain $\mathcal{D}\subset\mathcal{R}^3$ in space. Each of the two states can be associated with its own set of time-invariant medium parameters and its own set of source distributions. It is assumed that the media in the two states are linear in their acoustic behaviour, i.e. the medium parameters are independent of the values of the field or wave quantities. The domain $\mathcal{D}$ to which the reciprocity theorems apply, may be bounded or unbounded. The application to unbounded domains will always be handled as a limiting case where the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ recedes (partially or entirely) to infinity.

From the pertaining acoustic wave equations, first the *local form* of a reciprocity theorem will be derived, which form applies to each point of any subdomain of $\mathcal{D}$ where the acoustic wave-field quantities are continuously differentiable. By integrating the local form over such subdomains and adding the results, the *global form* of the reciprocity theorem is arrived at. In it, a boundary integral over $\partial\mathcal{D}$ occurs, the integrand of which always contains the unit vector ν_m along the normal to $\partial\mathcal{D}$, oriented away from $\mathcal{D}$ (Figure 7.1-1).

The two states will be denoted by the superscripts A and B. The construction of the time-domain reciprocity theorems will be based on the acoustic wave equations (see Equations (2.7-22), (2.7-23), (2.7-26) and (2.7-27))

$$\partial_k p^{\mathrm{A}} + \partial_t C_t(\mu_{k,r}^{\mathrm{A}}, v_r^{\mathrm{A}}; \boldsymbol{x}, t) = f_k^{\mathrm{A}}, \tag{7.1-1}$$

$$\partial_r v_r^{\mathrm{A}} + \partial_t C_t(\chi^{\mathrm{A}}, p^{\mathrm{A}}; \boldsymbol{x}, t) = q^{\mathrm{A}}, \tag{7.1-2}$$

for state A, and

$$\partial_k p^{\mathrm{B}} + \partial_t C_t(\mu_{k,r}^{\mathrm{B}}, v_r^{\mathrm{B}}; \boldsymbol{x}, t) = f_k^{\mathrm{B}}, \tag{7.1-3}$$

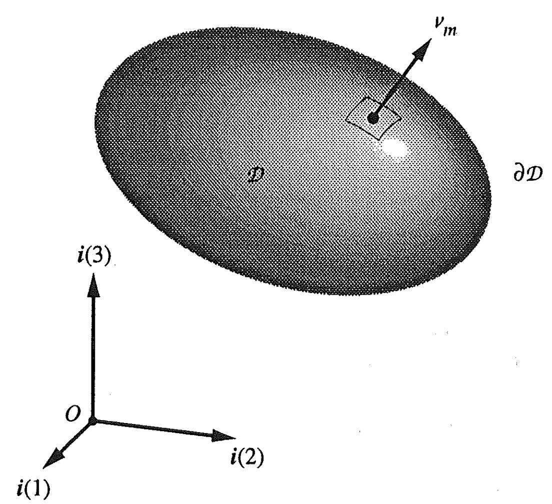
$$\partial_r v_r^{\mathrm{B}} + \partial_t C_t(\chi^{\mathrm{B}}, p^{\mathrm{B}}; \boldsymbol{x}, t) = q^{\mathrm{B}}, \tag{7.1-4}$$

for state B, where C_t denotes the time convolution operator (see Equation (B.1-11)) (Figure 7.1-2).

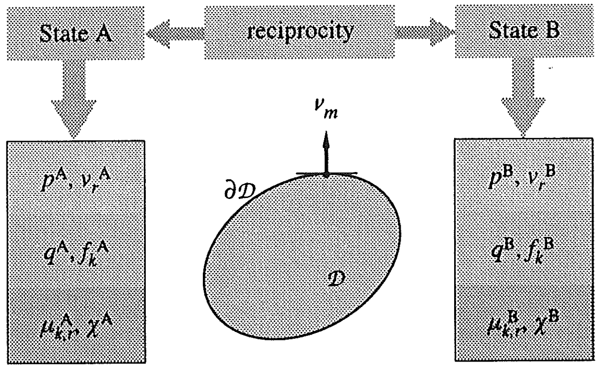
If, in $\mathcal{D}$, either surfaces of discontinuity in acoustic properties or acoustically impenetrable objects are present, Equations (7.1-1)–(7.1-4) are supplemented by boundary conditions of the type discussed in Section 2.6, both for state A and for state B. These are either (see Equations (2.6-2) and (2.6-3))

$$p^{\mathrm{A,B}} \quad \text{is continuous across any interface}, \tag{7.1-5}$$

and



7.1-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ and unit vector ν_m along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$, to which the reciprocity theorems apply.



7.1-2 Bounded domain $\mathcal{D}$ and states A and B to which the time-domain reciprocity theorems apply.

$$\nu_r v_r^{A,B} \quad \text{is continuous across any interface}\,, \tag{7.1-6}$$

where ν_r is the unit vector along the normal to the interface, or (see Equation (2.6-4))

$$\lim_{h\downarrow 0} p^{A,B}(\boldsymbol{x}+h\boldsymbol{\nu},t) = 0 \quad \text{on the boundary of a void}\,, \tag{7.1-7}$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to the boundary of the void, pointing away from the void, or (see Equation (2.6-8))

$$\lim_{h\downarrow 0} \nu_r v_r^{A,B}(\boldsymbol{x}+h\boldsymbol{\nu},t) = 0$$

$$\text{on the boundary of an immovable, perfectly rigid object}\,, \tag{7.1-8}$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to the boundary of the immovable, perfectly rigid object, pointing away from the object.

If $\mathcal{D}$ is a bounded domain, the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is assumed to be impenetrable (Figure 7.1-3).

To handle unbounded domains (Figure 7.1-4), we assume that outside some sphere $\mathcal{S}(\mathcal{O},\Delta_0)$, with its centre at the origin of the chosen reference frame and radius Δ_0, the fluid is homogeneous, isotropic and lossless, with the volume density of mass ρ_0 and the compressibility κ_0 as constitutive parameters, as well as source-free. In the domain outside the sphere, the so-called *embedding*, the asymptotic causal far-field representations are (see Equation (5.10-5))

$$\{p^{A,B}, v_r^{A,B}\} = \frac{\{p^{\infty;A,B}, v_r^{\infty;A,B}\}(\boldsymbol{\xi}, t-|\boldsymbol{x}|/c_0)}{4\pi|\boldsymbol{x}|} \quad \text{as } |\boldsymbol{x}|\to\infty\,, \tag{7.1-9}$$

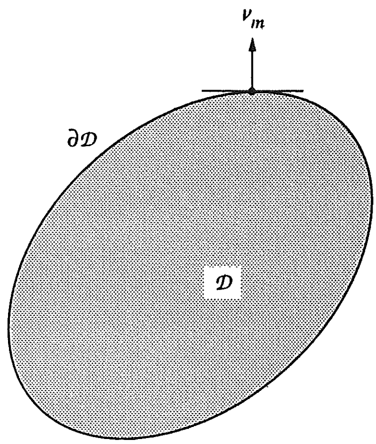
where $p^{\infty;A,B}$ and $v_r^{\infty;A,B}$ are interrelated through (see Equations (5.10-11) and (5.10-12))

$$-(\xi_k/c_0)p^{\infty;A,B} + \rho_0 v_k^{\infty;A,B} = 0\,, \tag{7.1-10}$$

$$-(\xi_r/c_0)v_r^{\infty;A,B} + \kappa_0 p^{\infty;A,B} = 0\,, \tag{7.1-11}$$

with

$$c_0 = (\rho_0\kappa_0)^{-1/2}\,. \tag{7.1-12}$$



7.1-3 Bounded domain $\mathcal{D}$ for the application of a reciprocity theorem. The boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is assumed to be impenetrable. In $\mathcal{D}$, interfaces between different fluids, voids and immovable perfectly rigid objects may be present.

The construction of the *complex frequency-domain* reciprocity theorems will be based on the complex frequency-domain acoustic wave equations (see Equations (4.5-1) and (4.5-2))

$$\partial_k \hat{p}^{\mathrm{A}} + \hat{\zeta}^{\mathrm{A}}_{k,r}\hat{v}^{\mathrm{A}}_r = \hat{f}^{\mathrm{A}}_k , \tag{7.1-13}$$

$$\partial_r \hat{v}^{\mathrm{A}}_r + \hat{\eta}^{\mathrm{A}}\hat{p}^{\mathrm{A}} = \hat{q}^{\mathrm{A}} , \tag{7.1-14}$$

for state A, and

$$\partial_k \hat{p}^{\mathrm{B}} + \hat{\zeta}^{\mathrm{B}}_{k,r}\hat{v}^{\mathrm{B}}_r = \hat{f}^{\mathrm{B}}_k , \tag{7.1-15}$$

$$\partial_r \hat{v}^{\mathrm{B}}_r + \hat{\eta}^{\mathrm{B}}\hat{p}^{\mathrm{B}} = \hat{q}^{\mathrm{B}} , \tag{7.1-16}$$

for state B. If, in $\mathcal{D}$, either surfaces of discontinuity in acoustic properties or acoustically impenetrable objects are present, Equations (7.1-13)–(7.1-16) are supplemented by boundary conditions of the type discussed in Section 4.3, both for state A and for state B. These are either (see Equations (4.3-1) and (4.3-2))

$$\hat{p}^{\mathrm{A,B}} \quad \text{is continuous across any interface} , \tag{7.1-17}$$

and

$$\nu_r \hat{v}^{\mathrm{A,B}}_r \quad \text{is continuous across any interface} , \tag{7.1-18}$$

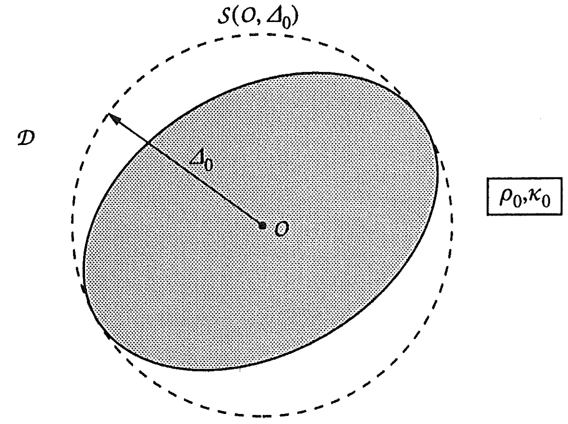
where ν_r is the unit vector along the normal to the interface, or (see Equation (4.3-3))

$$\lim_{h\downarrow 0} \hat{p}^{\mathrm{A,B}}(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0 \quad \text{on the boundary of a void} , \tag{7.1-19}$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to the boundary of the void, pointing away from the void, or (see Equation (4.3-7))

$$\lim_{h\downarrow 0} \nu_r \hat{v}^{\mathrm{A,B}}_r(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0$$

$$\text{on the boundary of an immovable, perfectly rigid object} , \tag{7.1-20}$$



7.1-4 Unbounded domain $\mathcal{D}$ for the application of a reciprocity theorem. Outside the sphere $S(O,\Delta_0)$, the fluid is homogeneous, isotropic and lossless with constitutive parameters $\{\rho_0,\kappa_0\}$. Inside $S(O,\Delta_0)$, interfaces between different fluids, voids and immovable perfectly rigid objects may be present.

where $\boldsymbol{\nu}$ is the unit vector along the normal to the boundary of the immovable, perfectly rigid object, pointing away from the object.

In the complex frequency domain, too, when handling unbounded domains we assume that outside some sphere $S(O,\Delta_0)$ with its centre at the origin of the chosen reference frame and radius Δ_0, the fluid is homogeneous, isotropic and lossless, with the volume density of mass ρ_0 and the compressibility κ_0 as constitutive parameters, as well as being source-free. In the domain outside that sphere, the *embedding*, the asymptotic causal far-field representations are (see Equation (5.9-11))

$$\{\hat{p}^{A,B},\hat{v}_r^{A,B}\} = \{\hat{p}^{\infty;A,B},\hat{v}_r^{\infty;A,B}\}(\boldsymbol{\xi},s)\frac{\exp(-\hat{\gamma}_0|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|} \quad \text{as } |\boldsymbol{x}|\to\infty\,, \tag{7.1-21}$$

where $\hat{p}^{\infty;A,B}$ and $\hat{v}_r^{\infty;A,B}$ are interrelated through (see Equations (5.9-17) and (5.9-18))

$$-\hat{\gamma}_0\xi_k\,\hat{p}^{\infty;A,B} + \hat{\zeta}_0\hat{v}_k^{\infty;A,B} = 0\,, \tag{7.1-22}$$

$$-\hat{\gamma}_0\xi_r\hat{v}_r^{\infty;A,B} + \hat{\eta}_0\,\hat{p}^{\infty;A,B} = 0\,, \tag{7.1-23}$$

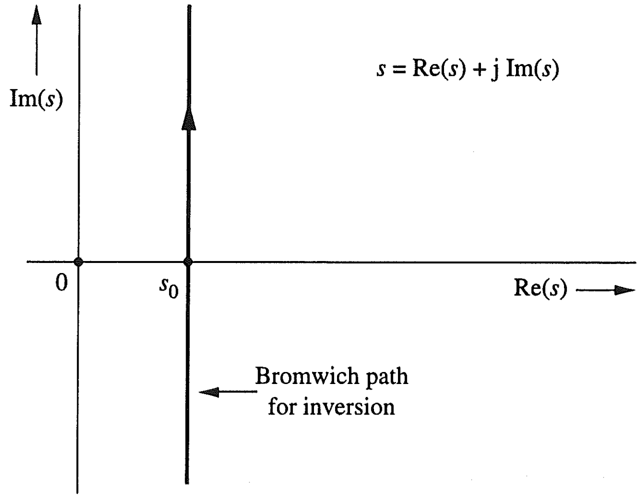
with

$$\hat{\zeta}_0 = s\rho_0\,, \tag{7.1-24}$$

$$\hat{\eta}_0 = s\kappa_0\,, \tag{7.1-25}$$

$$\hat{\gamma}_0 = (\hat{\eta}_0\hat{\zeta}_0)^{1/2} = s(\rho_0\kappa_0)^{-1/2} = s/c_0\,. \tag{7.1-26}$$

As a rule, state A will be chosen to correspond to the actual acoustic wave field in the configuration, or one of its constituents. This wave field will therefore satisfy the condition of causality, or, in other words, will be a *causal wave field.* If state B is another physical state, for example a state that corresponds to source distributions and/or acoustic medium parameters that differ from the ones in state A, state B will also be a causal wave field. If, however, state B is a computational state, i.e. a state that is representative of the manner in which the wave-field quantities in state A are computed, or a state that is representative of the manner in which the



7.1-5 Line $\mathrm{Re}(s) = s_0$, parallel to the imaginary axis of the complex s plane, at which the time Laplace transform of a wave field that is neither causal, nor anti-causal, but of a *transient* nature, exists.

acoustic wave-field data pertaining to state A are processed, there is no need to take state B to be a causal wave field as well, and it may, for example, be taken to be an anti-causal wave field (i.e. a wave field that is time reversed with respect to a causal wave field) or no wave field at all (which happens, for example, if one of the corresponding constitutive parameters is taken to be zero). No matter how the source distributions and the constitutive parameters are chosen, the wave-field quantities will always be assumed to satisfy the pertaining acoustic wave equations and the pertaining boundary conditions.

To accommodate causal, anti-causal and non-causal states in the complex frequency-domain analysis of reciprocity, the Laplace transform with respect to time of any *transient*, not necessarily causal or anti-causal, wave function $f = f(\boldsymbol{x},t)$ will always be taken as (see Equation (B.1-5))

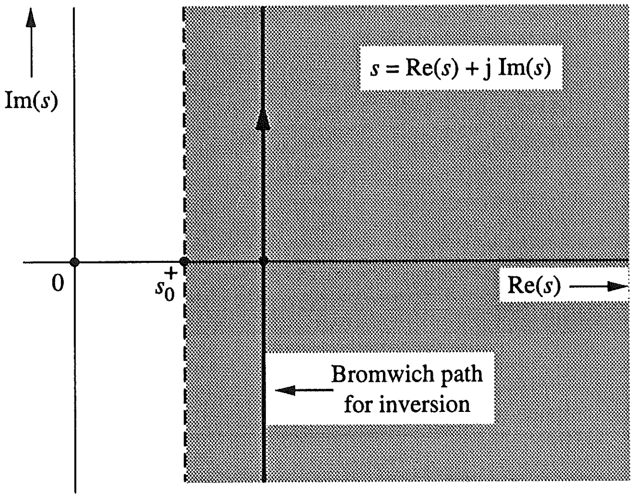
$$\hat{f}(\boldsymbol{x},s) = \int_{t\in\mathcal{R}} \exp(-st)\, f(\boldsymbol{x},t)\,\mathrm{d}t \qquad \text{for } \mathrm{Re}(s) = s_0\,, \tag{7.1-27}$$

i.e. the support of the wave function is, in principle, taken to be the entire interval of real values of time. Whenever appropriate, the support of the wave function will be indicated explicitly. For wave fields that are neither causal nor anti-causal (but are of a transient nature), for the transformation to make any sense at all, the right-hand side of Equation (7.1-27) should exist for some value $\mathrm{Re}(s) = s_0$ on a line parallel to the imaginary axis of the complex s plane (Figure 7.1-5).

For *causal wave functions* with support $\mathcal{T}^+ = \{t\in\mathcal{R}; t > t_0\}$, Equation (7.1-27) yields

$$\hat{f}(\boldsymbol{x},s) = \int_{t=t_0}^{\infty} \exp(-st)\, f(\boldsymbol{x},t)\,\mathrm{d}t \qquad \text{for } \mathrm{Re}(s) > s_0^+\,. \tag{7.1-28}$$

Here, the right-hand side is regular in some *right half* $\mathrm{Re}(s) > s_0^+$ of the complex s plane (Figure 7.1-6).



7.1-6 Right half $\mathrm{Re}(s) > s_0^+$ of the complex s plane, in which the time Laplace transform of a *causal* wave function exists.

For *anti-causal wave functions* with support $\mathcal{T}^- = \{t \in \mathcal{R}; t < t_0\}$, Equation (7.1-27) yields

$$\hat{f}(\boldsymbol{x},s) = \int_{t=-\infty}^{t_0} \exp(-st)\, f(\boldsymbol{x},t)\,\mathrm{d}t \qquad \text{for } \mathrm{Re}(s) < s_0^- \,. \tag{7.1-29}$$

Here, the right-hand side is regular in some *left half* $\mathrm{Re}(s) < s_0^-$ of the complex s plane (Figure 7.1-7).

A consequence of Equation (7.1-29) is that $\hat{f}(\boldsymbol{x},-s)$ is regular in the right half $\mathrm{Re}(s) > -s_0^-$ of the complex s plane if $\hat{f}(\boldsymbol{x},s)$ is regular in the left half $\mathrm{Re}(s) < s_0^-$. This result will be needed in reciprocity theorems of the time correlation type.

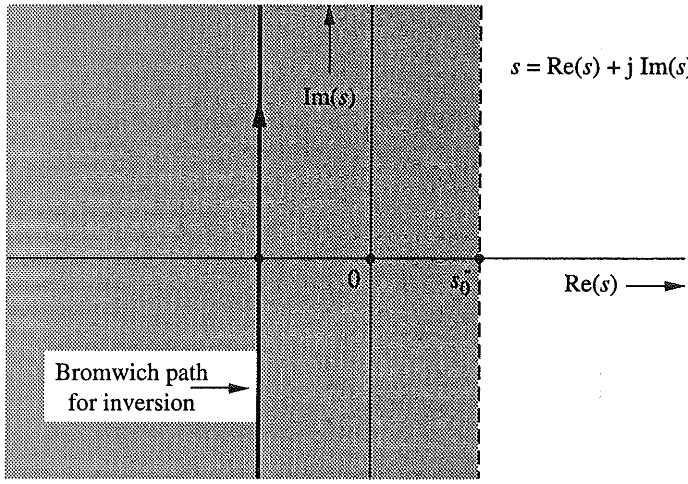
For the *time convolution* $\mathrm{C}_t\,(f_1, f_2; \boldsymbol{x},t)$ of any two transient wave functions we have (see Equations (B.1-11) and (B.1-12))

$$\hat{\mathrm{C}}_t(f_1, f_2; \boldsymbol{x},s) = \hat{f}_1(\boldsymbol{x},s)\,\hat{f}_2(\boldsymbol{x},s)\,. \tag{7.1-30}$$

This relation only holds in the common domain of regularity of $\hat{f}_1(\boldsymbol{x},s)$ and $\hat{f}_2(\boldsymbol{x},-s)$. If, in particular, both $f_1 = f_1(\boldsymbol{x},t)$ and $f_2 = f_2(\boldsymbol{x},t)$ are causal wave functions, they have a certain right half of the complex s plane as the domain of regularity in common. (Note that in this case $\hat{f}_1(\boldsymbol{x},s)$ is regular in some right half of the complex s plane, while $\hat{f}_2(\boldsymbol{x},s)$ is also regular in some right half of the complex s plane.) If, on the other hand, $f_1 = f_1(\boldsymbol{x},t)$ is a causal wave function and $f_2 = f_2(\boldsymbol{x},t)$ is an anti-causal wave function, the common domain of regularity where Equation (7.1-30) holds is at most a strip of finite width parallel to the imaginary axis of the complex s plane. (Note that in this case $\hat{f}_1(\boldsymbol{x},s)$ is regular in some right half of the complex s plane, while $\hat{f}_2(\boldsymbol{x},s)$ is regular in some left half of the complex s plane.)

For the *time correlation* $\mathrm{R}_t(f_1, f_2; \boldsymbol{x},s)$ of any two transient wave functions we have (see Equations (B.1-14) and (B.1-15))

$$\mathrm{R}_t(f_1, f_2; \boldsymbol{x},s) = \hat{f}_1(\boldsymbol{x},s)\,\hat{f}_2(\boldsymbol{x},-s)\,. \tag{7.1-31}$$



7.1-7 Left half $\mathrm{Re}(s) < s_0^-$ of the complex s plane, in which the time Laplace transform of an *anti-causal* wave function exists.

This relation only holds in the common domain of regularity of $\hat{f}_1(\boldsymbol{x},s)$ and $\hat{f}_2(\boldsymbol{x},-s)$. If, in particular, both $f_1 = f_1(\boldsymbol{x},t)$ and $f_2 = f_2(\boldsymbol{x},t)$ are causal wave functions, the common domain of regularity where Equation (7.1-31) holds, is at most a strip of finite width parallel to the imaginary axis of the complex s plane. (Note in this case that $\hat{f}_1(\boldsymbol{x},s)$ is regular in some right half of the complex s plane, and that $\hat{f}_2(\boldsymbol{x},-s)$ is regular in some left half of the complex s plane.) If, on the other hand, $f_1 = f_1(\boldsymbol{x},t)$ is a causal wave function and $f_2 = f_2(\boldsymbol{x},t)$ is an anti-causal wave function, the common domain of regularity where Equation (7.1-31) holds is some right half of the complex s plane. (Note that in this case $\hat{f}_1(\boldsymbol{x},s)$ is regular in some right half of the complex s plane, and that $\hat{f}_2(\boldsymbol{x},-s)$ is also regular in some right half of the complex s plane.)

In subsequent calculations the time correlation will, whenever appropriate, be replaced by (see Equation (B.1-18))

$$\mathrm{R}_t(f_1, f_2; \boldsymbol{x}, t) = \mathrm{C}_t(f_1, \mathrm{J}_t(f_2); \boldsymbol{x}, t)\,, \tag{7.1-32}$$

where J_t is the *time reversal* operator. The latter operator changes causal wave functions into anti-causal ones, and vice versa.

Exercises

Exercise 7.1-1

Of what type is the domain of regularity of the Laplace transform of the time convolution $\mathrm{C}_t(f_1, f_2; \boldsymbol{x}, t)$ of two wave functions $f_1 = f_1(\boldsymbol{x},t)$ and $f_2 = f_2(\boldsymbol{x},t)$ that are both anti-causal?

Answer: Some left half of the complex s plane.

Exercise 7.1-2

Of what type is the domain of regularity of the Laplace transform of the time correlation $R_t(f_1, f_2; x,t)$ of the anti-causal wave function $f_1 = f_1(x,t)$ and the causal wave function $f_2 = f_2(x,t)$?

Answer: Some left half of the complex s plane.

7.2 The time-domain reciprocity theorem of the time convolution type

The time-domain reciprocity theorem of the time convolution type follows upon considering the *local interaction quantity* $\partial_m [C_t(p^A, v_m^B; x,t) - C_t(p^B, v_m^A; x,t)]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\begin{aligned}
&\partial_m \left[C_t(p^A, v_m^B; x,t) - C_t(p^B, v_m^A; x,t) \right] \\
&= \partial_k C_t(p^A, v_k^B; x,t) - \partial_r C_t(p^B, v_r^A; x,t) \\
&= C_t(\partial_k p^A, v_k^B; x,t) + C_t(p^A, \partial_k v_k^B; x,t) - C_t(\partial_r p^B, v_r^A; x,t) - C_t(p^B, \partial_r v_r^A; x,t) \, . \qquad (7.2\text{-}1)
\end{aligned}$$

With the aid of Equations (7.1-1)–(7.1-4), the different terms on the right-hand side become

$$C_t(\partial_k p^A, v_k^B; x,t) = -\partial_t C_t(\mu_{k,r}^A, v_r^A, v_k^B; x,t) + C_t(f_k^A, v_k^B; x,t) \, , \qquad (7.2\text{-}2)$$

$$C_t(p^A, \partial_k v_k^B; x,t) = -\partial_t C_t(p^A, \chi^B, p^B; x,t) + C_t(p^A, q^B; x,t) \, , \qquad (7.2\text{-}3)$$

$$C_t(\partial_r p^B, v_r^A; x,t) = -\partial_t C_t(\mu_{r,k}^B, v_k^B, v_r^A; x,t) + C_t(f_r^B, v_r^A; x,t) \, , \qquad (7.2\text{-}4)$$

$$C_t(p^B, \partial_r v_r^A; x,t) = -\partial_t C_t(p^B, \chi^A, p^A; x,t) + C_t(p^B, q^A; x,t) \, , \qquad (7.2\text{-}5)$$

in which the convolution of three functions is a shorthand notation for the convolution of a function with the convolution of two other functions. (Note that in the convolution operation the order of the operators is immaterial.) Combining Equations (7.2-2)–(7.2-5) with Equation (7.2-1), it is found that

$$\begin{aligned}
&\partial_m \left[C_t(p^A, v_m^B; x,t) - C_t(p^B, v_m^A; x,t) \right] \\
&= \partial_t C_t(\mu_{r,k}^B - \mu_{k,r}^A, v_r^A, v_k^B; x,t) - \partial_t C_t(\chi^B - \chi^A, p^A, p^B; x,t) \\
&\quad + C_t(f_k^A, v_k^B; x,t) + C_t(p^A, q^B; x,t) - C_t(f_r^B, v_r^A; x,t) - C_t(p^B, q^A; x,t) \, . \qquad (7.2\text{-}6)
\end{aligned}$$

Equation (7.2-6) is the *local form of the acoustic reciprocity theorem of the time convolution type*. The first two terms on the right-hand side are representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; these terms vanish at those locations where $\mu_{r,k}^B(x,t) = \mu_{k,r}^A(x,t)$ and $\chi^B(x,t) = \chi^A(x,t)$ for all $t \in \mathcal{R}$. In case the latter conditions hold, the two media are referred to as each other's *adjoint*. Note is this respect that the adjoint of a causal (anti-causal) medium is causal (anti-causal) as well. The last four terms on the right-hand side of Equation (7.2-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (7.2-6) are continuous. Upon integrating Equation (7.2-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 7.2-1)

$$\int_{x\in\partial\mathcal{D}} \nu_m\left[C_t(p^{A},v_m^{B};x,t) - C_t(p^{B},v_m^{A};x,t)\right]\mathrm{d}A$$

$$= \int_{x\in\mathcal{D}} \left[\partial_t C_t(\mu_{r,k}^{B} - \mu_{k,r}^{A},v_r^{A},v_k^{B};x,t) - \partial_t C_t(\chi^{B} - \chi^{A},p^{A},p^{B};x,t)\right]\mathrm{d}V$$

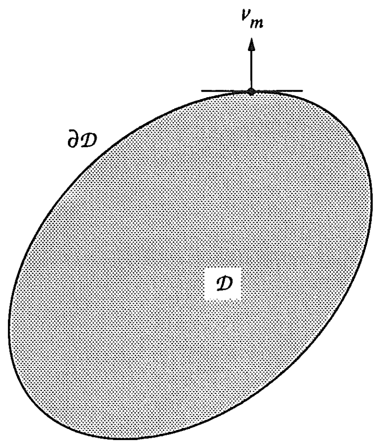
$$+ \int_{x\in\mathcal{D}} \left[C_t(f_k^{A},v_k^{B};x,t) + C_t(p^{A},q^{B};x,t) - C_t(f_r^{B},v_r^{A};x,t) - C_t(p^{B},q^{A};x,t)\right]\mathrm{d}V\,. \quad (7.2\text{-}7)$$

Equation (7.2-7) is the *global form, for the bounded domain $\mathcal{D}$, of the acoustic reciprocity theorem of the time convolution type.* Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (7.1-5) and (7.1-6)), and that the contributions from boundary surfaces of acoustically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (7.1-7) or (7.1-8)). On the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains, insofar as parts of this boundary do not coincide with the boundary surface of an acoustically impenetrable object. On the right-hand side, the first integral is representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's adjoint. The second integral on the right-hand side is representative of the action of the sources present in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem (Equation (7.2-7)) will be applied to an unbounded domain. To handle such cases, the embedding provisions described in Section 7.1 are made and Equation (7.2-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin and radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 7.2-2).

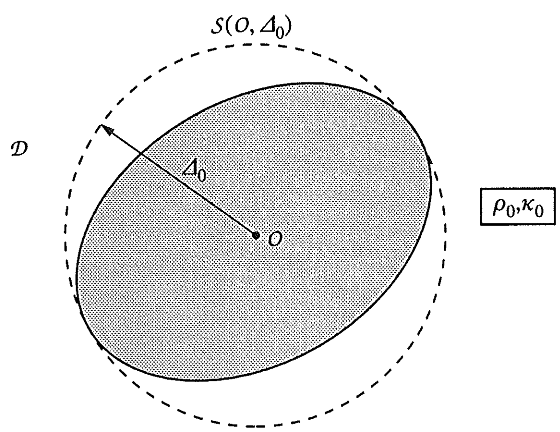
Whether or not the surface integral contribution over $\mathcal{S}(\mathcal{O},\Delta)$ does vanish as $\Delta\to\infty$ depends on the nature of the time behaviour of the wave fields in the two states. When the wave fields in states A and B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (7.1-9)–(7.1-11) apply for sufficiently large values of Δ. Then, the time convolutions occurring in the integrands of Equation (7.2-7) are also causal in time, and at any finite value of t, Δ can be chosen so large that on $\mathcal{S}(\mathcal{O},\Delta)$ the integrand vanishes. In this case, the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes. If, however, at least one of the two states is chosen to be non-causal (which, for example, can apply to the case where one of the two states is a computational one), the time convolutions occurring in Equation



7.2-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

(7.2-7) are non-causal as well and the contribution from $\mathcal{S}(O,\Delta)$ does not vanish, no matter how large a value of Δ is chosen. As outside the sphere $\mathcal{S}(O,\Delta_0)$ that is used to define the embedding (see Section 7.1) the media are each other's adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(O,\Delta)$ is, however, independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 7.2-2).

The time-domain reciprocity theorem of the convolution type is mainly used for investigating the transmission/reception reciprocity properties of acoustic transducers (see Sections 7.6 and 7.7) and for modelling direct (forward) source problems (see Section 7.8) and direct (forward) scattering problems (see Section 7.9). References to the earlier literature on the subject can be found in a paper by De Hoop (1988).



7.2-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the fluid is homogeneous, isotropic and lossless.

Exercises

Exercise 7.2-1

To what form do the contrast-in-media terms in the reciprocity theorems Equations (7.2-6) and (7.2-7) reduce if the fluids in states A and B are both instantaneously reacting?

Answer:

$$\mathrm{C}_t(\mu^{\mathrm{B}}_{r,k} - \mu^{\mathrm{A}}_{k,r}, v^{\mathrm{A}}_r, v^{\mathrm{B}}_k; \boldsymbol{x}, t) = \left[\rho^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \rho^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right] \mathrm{C}_t(v^{\mathrm{A}}_r, v^{\mathrm{B}}_k; \boldsymbol{x}, t)$$

and

$$\mathrm{C}_t(\chi^{\mathrm{B}} - \chi^{\mathrm{A}}, p^{\mathrm{A}}, p^{\mathrm{B}}; \boldsymbol{x}, t) = \left[\kappa^{\mathrm{B}}(\boldsymbol{x}) - \kappa^{\mathrm{A}}(\boldsymbol{x})\right] \mathrm{C}_t(p^{\mathrm{A}}, p^{\mathrm{B}}; \boldsymbol{x}, t) .$$

Exercise 7.2-2

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 7.2-3. The reciprocity theorem Equation (7.2-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, neither in state A nor in state B, and the fluid in $\mathcal{D}$ in state B is in its acoustic properties adjoint to the one in state A. Prove that

$$\int_{\boldsymbol{x} \in \mathcal{S}_1} \nu_m \left[\mathrm{C}_t(p^{\mathrm{A}}, v^{\mathrm{B}}_m; \boldsymbol{x}, t) - \mathrm{C}_t(p^{\mathrm{B}}, v^{\mathrm{A}}_m; \boldsymbol{x}, t)\right] \mathrm{d}A$$

$$= \int_{\boldsymbol{x} \in \mathcal{S}_2} \nu_m \left[\mathrm{C}_t(p^{\mathrm{A}}, v^{\mathrm{B}}_m; \boldsymbol{x}, t) - \mathrm{C}_t(p^{\mathrm{B}}, v^{\mathrm{A}}_m; \boldsymbol{x}, t)\right] \mathrm{d}A , \qquad (7.2\text{-}8)$$

i.e. that the surface integral is an invariant.

7.3 The time-domain reciprocity theorem of the time correlation type

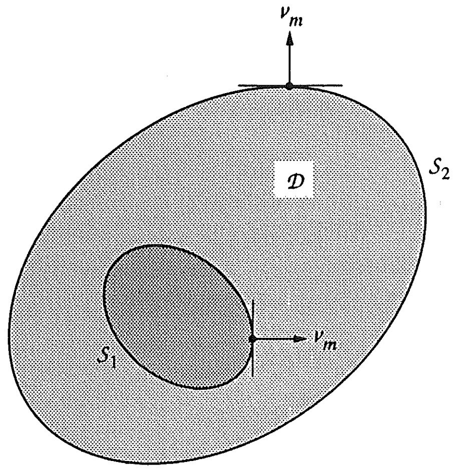
The time-domain reciprocity theorem of the time correlation type follows upon considering the *local interaction quantity* $\partial_m[\mathrm{R}_t(p^{\mathrm{A}}, v^{\mathrm{B}}_m; \boldsymbol{x}, t) + \mathrm{R}_t(p^{\mathrm{B}}, v^{\mathrm{A}}_m; \boldsymbol{x}, -t)]$. On account of Equations (B.1-14) and (B.1-18) and the symmetry of the convolution operator, this quantity can be rewritten as $\partial_m[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v^{\mathrm{B}}_m); \boldsymbol{x}, t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v^{\mathrm{A}}_m; \boldsymbol{x}, t)]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\partial_m\left[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v^{\mathrm{B}}_m); \boldsymbol{x}, t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v^{\mathrm{A}}_m; \boldsymbol{x}, t)\right]$$

$$= \partial_k \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v^{\mathrm{B}}_k); \boldsymbol{x}, t) + \partial_r \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v^{\mathrm{A}}_r; \boldsymbol{x}, t)$$

$$= \mathrm{C}_t(\partial_k p^{\mathrm{A}}, \mathrm{J}_t(v^{\mathrm{B}}_k); \boldsymbol{x}, t) + \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(\partial_k v^{\mathrm{B}}_k); \boldsymbol{x}, t)$$

$$+ \mathrm{C}_t(\mathrm{J}_t(\partial_r p^{\mathrm{B}}), v^{\mathrm{A}}_r; \boldsymbol{x}, t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), \partial_r v^{\mathrm{A}}_r; \boldsymbol{x}, t) . \qquad (7.3\text{-}1)$$



7.2-3 Domain $\mathcal{D}$ bounded internally by the closed surface S_1 and externally by the closed surface S_2.

With the aid of Equations (7.1-1)–(7.1-4) and the rule $\mathrm{J}_t(\partial_t f) = -\partial_t(\mathrm{J}_t(f))$, the different terms on the right-hand side become

$$\mathrm{C}_t(\partial_k p^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) = -\partial_t \mathrm{C}_t(\mu_{k,r}^{\mathrm{A}}, v_r^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) + \mathrm{C}_t(f_k^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) , \tag{7.3-2}$$

$$\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(\partial_k v_k^{\mathrm{B}});x,t) = +\partial_t \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(\chi^{\mathrm{B}}), \mathrm{J}_t(p^{\mathrm{B}});x,t) + \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(q^{\mathrm{B}});x,t) , \tag{7.3-3}$$

$$\mathrm{C}_t(\mathrm{J}_t(\partial_r p^{\mathrm{B}}), v_r^{\mathrm{A}};x,t) = +\partial_t \mathrm{C}_t(\mathrm{J}_t(\mu_{r,k}^{\mathrm{B}}), \mathrm{J}_t(v_k^{\mathrm{B}}), v_r^{\mathrm{A}};x,t) + \mathrm{C}_t(\mathrm{J}_t(f_r^{\mathrm{B}}), v_r^{\mathrm{A}};x,t) , \tag{7.3-4}$$

$$\mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), \partial_r v_r^{\mathrm{A}};x,t) = -\partial_t \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), \chi^{\mathrm{A}}, p^{\mathrm{A}};x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), q^{\mathrm{A}};x,t) , \tag{7.3-5}$$

in which the convolution of three functions is a shorthand notation for the convolution of a function with the convolution of two other functions. (Note that in the convolution operation the order of the operators is immaterial.) Combining Equations (7.3-2)–(7.3-5) with Equation (7.3-1), it is found that

$$\begin{aligned}
&\partial_m \left[\mathrm{R}_t(p^{\mathrm{A}}, v_m^{\mathrm{B}};x,t) + \mathrm{R}_t(p^{\mathrm{B}}, v_m^{\mathrm{A}};x,-t) \right] \\
&= \partial_m \left[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v_m^{\mathrm{B}});x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v_m^{\mathrm{A}};x,t) \right] \\
&= \partial_t \mathrm{C}_t(\mathrm{J}_t(\mu_{r,k}^{\mathrm{B}}) - \mu_{k,r}^{\mathrm{A}}, v_r^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) \\
&\quad + \partial_t \mathrm{C}_t(\mathrm{J}_t(\chi^{\mathrm{B}}) - \chi^{\mathrm{A}}, p^{\mathrm{A}}, \mathrm{J}_t(p^{\mathrm{B}});x,t) + \mathrm{C}_t(f_k^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) \\
&\quad + \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(q^{\mathrm{B}});x,t) + \mathrm{C}_t(\mathrm{J}_t(f_r^{\mathrm{B}}), v_r^{\mathrm{A}};x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), q^{\mathrm{A}};x,t) .
\end{aligned} \tag{7.3-6}$$

Equation (7.3-6) is the *local form of acoustic reciprocity theorem of the time correlation type.* The first two terms on the right-hand side are representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; these terms vanish at those locations where $\mathrm{J}_t(\mu_{r,k}^{\mathrm{B}})(x,t) = \mu_{k,r}^{\mathrm{A}}(x,t)$ and $\mathrm{J}_t(\chi^{\mathrm{B}})(x,t) = \chi^{\mathrm{A}}(x,t)$ for all $t \in \mathcal{R}$. When the latter conditions hold, the two media are referred to as each other's *time-reverse adjoint.* Note is this respect that the time-reverse adjoint of a causal (anti-causal) medium is an anti-causal (causal) medium. The last four terms on the right-hand side of Equation (7.3-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

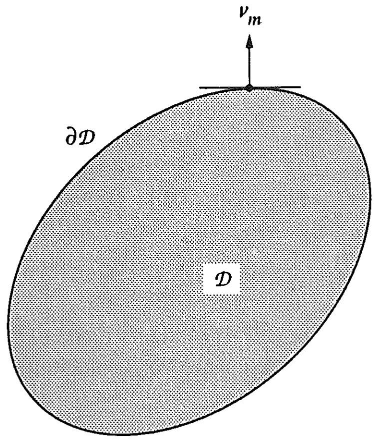
To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (7.3-6) are continuous. Upon integrating Equation (7.3-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 7.3-1)

$$
\begin{aligned}
&\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{R}_t(p^{\mathrm{A}}, v_m^{\mathrm{B}};x,t) + \mathrm{R}_t(p^{\mathrm{B}}, v_m^{\mathrm{A}};x,-t)\right]\mathrm{d}A \\
&= \int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v_m^{\mathrm{B}});x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v_m^{\mathrm{A}};x,t)\right]\mathrm{d}A \\
&= \int_{x\in\mathcal{D}} \left[\partial_t\mathrm{C}_t(\mathrm{J}_t(\mu_{r,k}^{\mathrm{B}}) - \mu_{k,r}^{\mathrm{A}}, v_r^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) + \partial_t\mathrm{C}_t(\mathrm{J}_t(\chi^{\mathrm{B}}) - \chi^{\mathrm{A}}, p^{\mathrm{A}}, \mathrm{J}_t(p^{\mathrm{B}});x,t)\right]\mathrm{d}V \\
&\quad + \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(f_k^{\mathrm{A}}, \mathrm{J}_t(v_k^{\mathrm{B}});x,t) + \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(q^{\mathrm{B}});x,t)\right. \\
&\quad \left. + \mathrm{C}_t(\mathrm{J}_t(f_r^{\mathrm{B}}), v_r^{\mathrm{A}};x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), q^{\mathrm{A}};x,t)\right]\mathrm{d}V \, .
\end{aligned}
\tag{7.3-7}
$$

Equation (7.3-7) is the *global form, for the bounded domain $\mathcal{D}$, of the acoustic reciprocity theorem of the time correlation type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (7.1-5) and (7.1-6)), and that the contributions from boundary surfaces of acoustically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (7.1-7) or (7.1-8)). On the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains, insofar as parts of this boundary do not coincide with the boundary surface of an acoustically impenetrable object. On the right-hand side, the first integral is representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's time-reverse adjoint. The second integral on the right-hand side is representative of the action of the sources present in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

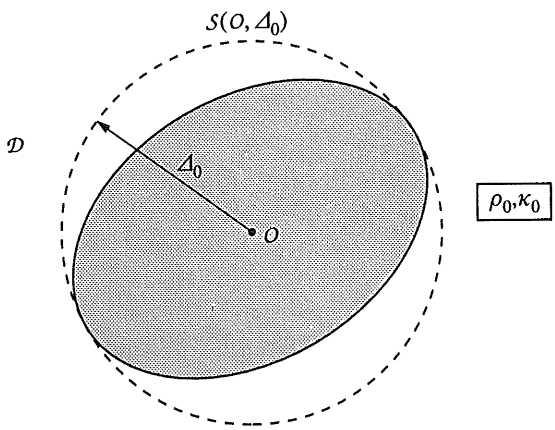
The limiting case of an unbounded domain

In a number of cases the reciprocity theorem (Equation (7.3-7)) will be applied to an unbounded domain. To handle such cases, the embedding provisions given in Section 7.1 are made and Equation (7.3-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin and of radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 7.3-2).



7.3-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

Since outside the sphere $\mathcal{S}(O,\Delta_0)$ that is used to define the embedding (see Section 7.1) the media are each other's time-reverse adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(O,\Delta)$ is, in any case, independent of the value of Δ for $\Delta > \Delta_0$ (see Exercise 7.3-2). Whether or not this contribution vanishes as $\Delta \to \infty$, depends on the nature of the time behaviour of the wave fields in the two states. When the wave fields in states A and B are both causal in time (which is the case if both states apply to physical wave fields), the contribution of $\mathcal{S}(O,\Delta)$ is a non-vanishing function that is independent of the value of Δ. If, however, state A is chosen to be causal and state B is chosen to be anti-causal (which, for example, can apply to the case where state B is a computational one), the contribution from $\mathcal{S}(O,\Delta)$ vanishes for sufficiently large values of Δ.



7.3-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the fluid is homogeneous, isotropic and lossless.

The time-domain reciprocity theorem of the correlation type is mainly used in modelling inverse source problems (see Section 7.10) and inverse scatttering problems (see Section 7.11). References to the earlier literature on the subject can be found in a paper by De Hoop (1988).

Exercises

Exercise 7.3-1

To what form do the contrast-in-media terms in the reciprocity theorems Equations (7.3-6) and (7.3-7) reduce if the fluids in states A and B are both instantaneously reacting?

Answer:

$$\partial_t \mathrm{C}_t(\mathrm{J}_t(\mu^{\mathrm{B}}_{r,k}) - \mu^{\mathrm{A}}_{k,r}, v^{\mathrm{A}}_r, \mathrm{J}_t(v^{\mathrm{B}}_k); \boldsymbol{x}, t) = \left[\rho^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \rho^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right] \partial_t \mathrm{C}_t(v^{\mathrm{A}}_r, \mathrm{J}_t(v^{\mathrm{B}}_k); \boldsymbol{x}, t)$$

and

$$\partial_t \mathrm{C}_t(\mathrm{J}_t(\chi^{\mathrm{B}}) - \chi^{\mathrm{A}}, p^{\mathrm{A}}, \mathrm{J}_t(p^{\mathrm{B}}); \boldsymbol{x}, t) = \left[\kappa^{\mathrm{B}}(\boldsymbol{x}) - \kappa^{\mathrm{A}}(\boldsymbol{x})\right] \partial_t \mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(p^{\mathrm{B}}); \boldsymbol{x}, t) .$$

Exercise 7.3-2

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 7.3-3. The reciprocity theorem Equation (7.3-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, in either state A or state B, and the fluid in $\mathcal{D}$ in state B is in its acoustic properties the time-reverse adjoint of the one in state A. Prove that

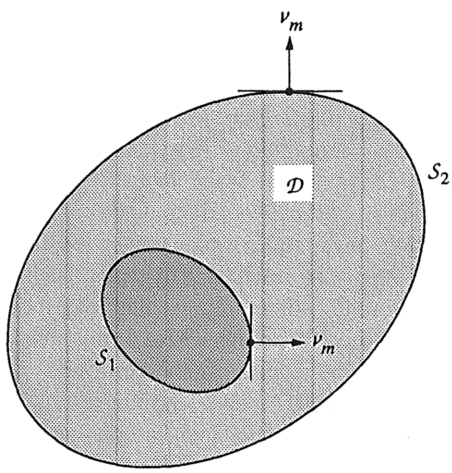
$$\int_{\boldsymbol{x}\in\mathcal{S}_1} \nu_m \left[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v^{\mathrm{B}}_m); \boldsymbol{x}, t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v^{\mathrm{A}}_m; \boldsymbol{x}, t)\right] \mathrm{d}A$$

$$= \int_{\boldsymbol{x}\in\mathcal{S}_2} \nu_m \left[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v^{\mathrm{B}}_m); \boldsymbol{x}, t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v^{\mathrm{A}}_m; \boldsymbol{x}, t)\right] \mathrm{d}A , \qquad (7.3\text{-}8)$$

i.e. that the surface integral is an invariant.

7.4 The complex frequency-domain reciprocity theorem of the time convolution type

The complex frequency-domain reciprocity theorem of the time convolution type follows upon considering the *local interaction* quantity $\partial_m[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_m(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}^{\mathrm{A}}_m(\boldsymbol{x},s)]$. Using standard rules for spatial differentiation and adjusting the subscripts for later convenience, we obtain



7.3-3 Domain $\mathcal{D}$ bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

$$
\begin{aligned}
&\partial_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right] \\
&= \partial_k\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)\right] - \partial_r\left[\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\right] \\
&= \left[\partial_k\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s) + \hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\left[\partial_k\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)\right] \\
&\quad - \left[\partial_r\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\left[\partial_r\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\right]. \qquad (7.4\text{-}1)
\end{aligned}
$$

With the aid of Equations (7.1-13)–(7.1-16), the different terms on the right-hand side become

$$\left[\partial_k\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s) = -\hat{\zeta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s) + \hat{f}_k^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)\,, \qquad (7.4\text{-}2)$$

$$\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\left[\partial_k\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)\right] = -\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{\eta}^{\mathrm{B}}(\boldsymbol{x},s)\hat{p}^{\mathrm{B}}(\boldsymbol{x},s) + \hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\mathrm{B}}(\boldsymbol{x},s)\,, \qquad (7.4\text{-}3)$$

$$\left[\partial_r\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s) = -\hat{\zeta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s) + \hat{f}_r^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\,, \qquad (7.4\text{-}4)$$

$$\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\left[\partial_r\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\right] = -\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{\eta}^{\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\mathrm{A}}(\boldsymbol{x},s) + \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{q}^{\mathrm{A}}(\boldsymbol{x},s)\,. \qquad (7.4\text{-}5)$$

Combining Equations (7.4-2)–(7.4-5) with Equation (7.4-1), it is found that

$$
\begin{aligned}
&\partial_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right] \\
&= \left[\hat{\zeta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},s) - \hat{\zeta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)
\end{aligned}
$$

$$-\left[\hat{\eta}^{\mathrm{B}}(\boldsymbol{x},s)-\hat{\eta}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)$$

$$+\hat{f}_k^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)+\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\mathrm{B}}(\boldsymbol{x},s)-\hat{f}_r^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)-\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{q}^{\mathrm{A}}(\boldsymbol{x},s)\,. \tag{7.4-6}$$

Equation (7.4-6) is the *local form of the complex frequency-domain counterpart of the acoustic reciprocity theorem of the time convolution type.* The first two terms on the right-hand side are representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; these terms vanish at those locations where $\hat{\zeta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},s)=\hat{\zeta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)$ and $\hat{\eta}^{\mathrm{B}}(\boldsymbol{x},s)=\hat{\eta}^{\mathrm{A}}(\boldsymbol{x},s)$ for all s in the domain in the complex s plane where Equation (7.4-6) holds. When the latter conditions hold, the two media are referred to as each other's *adjoint.* The last four terms on the right-hand side of Equation (7.4-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (7.4-6) are continuous. Upon integrating Equation (7.4-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 7.4-1)

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}}\nu_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},s)-\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right]\mathrm{d}A$$

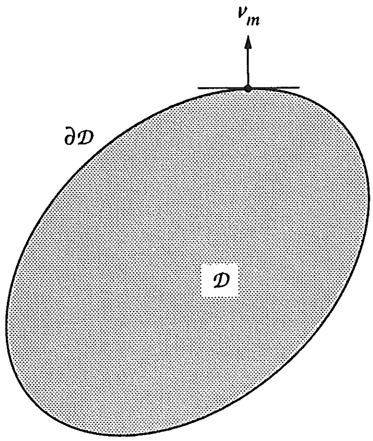
$$=\int_{\boldsymbol{x}\in\mathcal{D}}\Big\{\left[\hat{\zeta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},s)-\hat{\zeta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)$$

$$-\left[\hat{\eta}^{\mathrm{B}}(\boldsymbol{x},s)-\hat{\eta}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\Big\}\,\mathrm{d}V$$

$$+\int_{\boldsymbol{x}\in\mathcal{D}}\Big[\hat{f}_k^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},s)+\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\mathrm{B}}(\boldsymbol{x},s)$$

$$-\hat{f}_r^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)-\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{q}^{\mathrm{A}}(\boldsymbol{x},s)\Big]\,\mathrm{d}V\,. \tag{7.4-7}$$

Equation (7.4-7) is the *global form,* for the bounded domain $\mathcal{D}$, of the *complex frequency-domain counterpart of the acoustic reciprocity theorem of the time convolution type.* Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (7.1-17) and (7.1-18)), and that the contributions from boundary surfaces of acoustically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (7.1-19) or (7.1-20)). On the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an acoustically impenetrable object. On the right-hand side, the first integral is representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's adjoint. The second integral on the right-hand side is representative of the action of the sources in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.



7.4-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

The limiting case of an unbounded domain

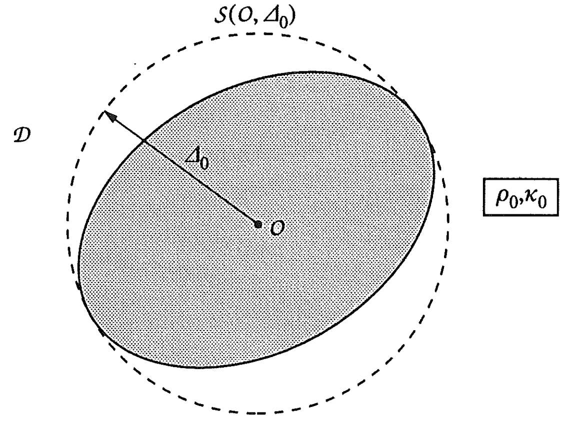
In quite a number of cases the reciprocity theorem (Equation (7.4-7)) will be applied to an unbounded domain. To handle such cases, the embedding provisions described in Section 7.1 are made and Equation (7.4-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin and of radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 7.4-2). Whether or not the surface integral contribution over $\mathcal{S}(\mathcal{O},\Delta)$ does vanish as $\Delta\to\infty$, depends on the nature of the time behaviour of the wave fields in the two states. When the wave fields in states A and B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (7.1-21)–(7.1-23) apply for sufficiently large values of Δ. Then, the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes in the limit $\Delta\to\infty$. If, however, at least one of the two states is chosen to be non-causal (which, for example, can apply to the case where one of the two states is a computational one), the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ does not vanish, no matter how large a value of Δ is chosen. However, since outside the sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ that is used to define the embedding (see Section 7.1) the media are each other's adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(\mathcal{O},\Delta)$ is independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 7.4-4).

The complex frequency-domain reciprocity theorem of the time convolution type is mainly used for investigating the transmission/reception reciprocity properties of acoustic transducers (see Sections 7.6 and 7.7) and for modelling direct (forward) source problems (see Section 7.8) and direct (forward) scattering problems (see Section 7.9).

Exercises

Exercise 7.4-1

Show, by taking the Laplace transform with respect to time, that Equation (7.4-6) follows from Equation (7.2-6).



7.4-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the fluid is homogeneous, isotropic and lossless.

Exercise 7.4-2

Show, by taking the Laplace transform with respect to time, that Equation (7.4-7) follows from Equation (7.2-7).

Exercise 7.4-3

To what form do the contrast-in-media terms in the reciprocity theorems Equations (7.4-6) and (7.4-7) reduce if the fluids in states A and B are both instantaneously reacting?

Answers:

$$\left[\hat{\zeta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},s) - \hat{\zeta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)\right]\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_k(\boldsymbol{x},s) = s\left[\rho^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \rho^{\mathrm{A}}_{r,k}(\boldsymbol{x})\right]\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_k(\boldsymbol{x},s)$$

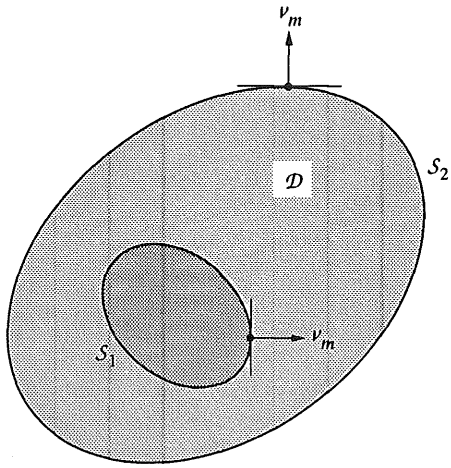
and

$$\left[\hat{\eta}^{\mathrm{B}}(\boldsymbol{x},s) - \hat{\eta}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\mathrm{B}}(\boldsymbol{x},s) = s\left[\kappa^{\mathrm{B}}(\boldsymbol{x}) - \kappa^{\mathrm{A}}(\boldsymbol{x})\right]\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\,.$$

Exercise 7.4-4

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 7.4-3. The reciprocity theorem Equation (7.4-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, in either state A or state B, and the fluid in $\mathcal{D}$ in state B is in its acoustic properties the adjoint of the one in state A. Prove that

$$\int_{\boldsymbol{x}\in\mathcal{S}_1} \nu_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_m(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}^{\mathrm{A}}_m(\boldsymbol{x},s)\right]\mathrm{d}A$$
$$= \int_{\boldsymbol{x}\in\mathcal{S}_2} \nu_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_m(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}^{\mathrm{A}}_m(\boldsymbol{x},s)\right]\mathrm{d}A\,, \qquad (7.4\text{-}8)$$



7.4-3 Domain $\mathcal{D}$ bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

i.e. that the surface integral is an invariant.

7.5 The complex frequency-domain reciprocity theorem of the time correlation type

The complex frequency-domain reciprocity theorem of the time correlation type follows upon considering the *local interaction quantity* $\partial_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},-s)+\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\begin{aligned}
&\partial_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},-s)+\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right]\\
&=\partial_k\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)\right]+\partial_r\left[\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\right]\\
&=\left[\partial_k\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)+\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\left[\partial_k\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)\right]\\
&\quad+\left[\partial_r\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)+\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)\left[\partial_r\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\right].
\end{aligned}\tag{7.5-1}$$

With the aid of Equations (7.1-13)–(7.1-16), the different terms on the right-hand side become

$$\left[\partial_k\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)=-\hat{\zeta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)+\hat{f}_k^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)\,,\tag{7.5-2}$$

$$\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\left[\partial_k\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)\right]=-\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{\eta}^{\mathrm{B}}(\boldsymbol{x},-s)\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)+\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\mathrm{B}}(\boldsymbol{x},-s)\,,\tag{7.5-3}$$

$$\left[\partial_r\hat{p}^{\mathrm{B}}(\boldsymbol{x},-s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)=-\hat{\zeta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)+\hat{f}_r^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\,,\tag{7.5-4}$$

$$\hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\left[\partial_r \hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\right] = -\hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\hat{\eta}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s) + \hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\hat{q}^{\,\mathrm{A}}(\boldsymbol{x},s)\,. \tag{7.5-5}$$

Combining Equations (7.5-2)–(7.5-5) with Equation (7.5-1), it is found that

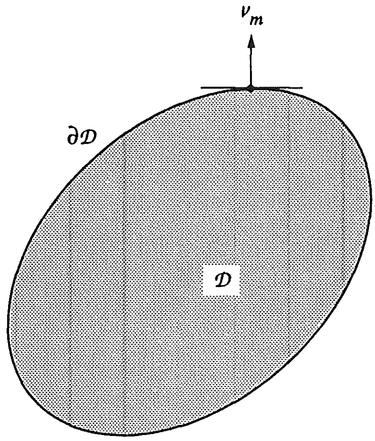
$$\begin{aligned}
&\partial_m\left[\hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right] \\
&= \left[-\hat{\zeta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},-s) - \hat{\zeta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s) \\
&\quad + \left[-\hat{\eta}^{\,\mathrm{B}}(\boldsymbol{x},-s) - \hat{\eta}^{\,\mathrm{A}}(\boldsymbol{x},s)\right]\hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s) \\
&\quad + \hat{f}_k^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\,\mathrm{B}}(\boldsymbol{x},-s) + \hat{f}_r^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s) + \hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\hat{q}^{\,\mathrm{A}}(\boldsymbol{x},s)\,.
\end{aligned} \tag{7.5-6}$$

Equation (7.5-6) is the *local form of the complex frequency-domain counterpart of the acoustic reciprocity theorem of the time correlation type.* The first two terms on the right-hand side are representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; these terms vanish at those locations where $\hat{\zeta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},-s) = -\hat{\zeta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)$ and $\hat{\eta}^{\,\mathrm{B}}(\boldsymbol{x},-s) = -\hat{\eta}^{\,\mathrm{A}}(\boldsymbol{x},s)$ for all s in the domain in the complex s plane where Equation (7.5-6) holds. When the latter conditions hold, the two media are referred to as each other's *time-reverse adjoint.* The last four terms on the right-hand side of Equation (7.5-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (7.5-6) are continuous. Upon integrating Equation (7.5-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 7.5-1)

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right]\mathrm{d}A \\
&= \int_{\boldsymbol{x}\in\mathcal{D}} \Big\{\left[-\hat{\zeta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},-s) - \hat{\zeta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s) \\
&\quad + \left[-\hat{\eta}^{\,\mathrm{B}}(\boldsymbol{x},-s) - \hat{\eta}^{\,\mathrm{A}}(\boldsymbol{x},s)\right]\hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\Big\}\,\mathrm{d}V \\
&\quad + \int_{\boldsymbol{x}\in\mathcal{D}} \Big[\hat{f}_k^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\,\mathrm{B}}(\boldsymbol{x},-s) \\
&\quad + \hat{f}_r^{\mathrm{B}}(\boldsymbol{x},-s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s) + \hat{p}^{\,\mathrm{B}}(\boldsymbol{x},-s)\hat{q}^{\,\mathrm{A}}(\boldsymbol{x},s)\Big]\,\mathrm{d}V\,.
\end{aligned} \tag{7.5-7}$$

Equation (7.5-7) is the *global form*, for the bounded domain $\mathcal{D}$, of the *complex frequency-domain counterpart of the acoustic reciprocity theorem of the time correlation type.* Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (7.1-17) and (7.1-18)), and that the contributions from boundary surfaces of acoustically impenetrable parts of the configuration have vanished in view of the pertaining



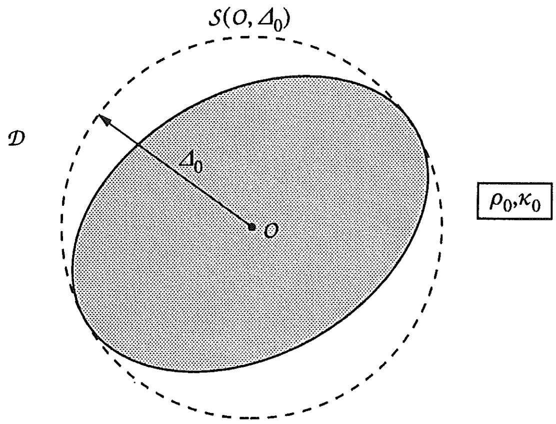
7.5-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

boundary conditions of the explicit type (Equation (7.1-19) or (7.1-20)). On the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains, insofar as parts of this boundary do not coincide with the boundary surface of an acoustically impenetrable object. On the right-hand side, the first integral is representative of the differences (contrasts) in the acoustic properties of the fluids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's time-reverse adjoint. The second integral on the right-hand side is representative of the action of the sources in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In a number of cases the reciprocity theorem (Equation (7.5-7)) will be applied to an unbounded domain. To handle such cases, the embedding provisions given in Section 7.1 are made and Equation (7.5-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin and of radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 7.5-2).

Since outside the sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ that is used to define the embedding (see Section 7.1) the media are each other's time-reverse adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(\mathcal{O},\Delta)$ is, in any case, independent of the value of Δ for $\Delta > \Delta_0$ (see Exercise 7.5-4). Whether or not this contribution vanishes as $\Delta\to\infty$ depends on the nature of the time behaviour of the wave fields in the two states. When the wave fields in states A and B are both causal in time (which is the case if both states apply to physical wave fields) the contribution of $\mathcal{S}(\mathcal{O},\Delta)$ is a non-vanishing function that is independent of the value of Δ. If, however, state A is chosen to be causal and state B is chosen to be anti-causal (which, for example, can apply to the case where state B is a computational one) the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes for sufficiently large values of Δ.



7.5-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the fluid is homogeneous, isotropic and lossless.

The complex frequency-domain reciprocity theorem of the time correlation type is mainly used in modelling inverse source problems (see Section 7.10) and inverse scattering problems (see Section 7.11).

Exercises

Exercise 7.5-1

Show, by taking the Laplace transform with respect to time, that Equation (7.5-6) follows from Equation (7.3-6).

Exercise 7.5-2

Show, by taking the Laplace transform with respect to time, that Equation (7.5-7) follows from Equation (7.3-7).

Exercise 7.5-3

To what form do the contrast-in-media terms in the reciprocity theorems Equations (7.5-6) and (7.5-7) reduce if the fluids in states A and B are both instantaneously reacting?

Answer:

$$\left[-\hat{\zeta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},-s) - \hat{\zeta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)\right]\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_k^{\mathrm{B}}(\boldsymbol{x},-s) = s\left[\hat{\rho}^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \hat{\rho}^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right]v_r^{\mathrm{A}}(\boldsymbol{x},s)v_k^{\mathrm{B}}(\boldsymbol{x},-s)$$

and

$$\left[-\hat{\eta}^{B}(x,-s)-\hat{\eta}^{A}(x,s)\right]\hat{p}^{A}(x,s)\hat{p}^{B}(x,-s)=s\left[\kappa^{B}(x)-\kappa^{A}(x)\right]\hat{p}^{A}(x,s)\hat{p}^{B}(x,-s).$$

Exercise 7.5-4

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 7.5-3. The reciprocity theorem (Equation (7.5-7)) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present either in state A or in state B, and the fluid in $\mathcal{D}$ in state B is in its acoustic properties the time-reverse adjoint of the one in state A. Prove that

$$\int_{x\in\mathcal{S}_1}\nu_m\left[\hat{p}^{A}(x,s)\hat{v}_m^{B}(x,-s)+\hat{p}^{B}(x,-s)\hat{v}_m^{A}(x,s)\right]\mathrm{d}A$$

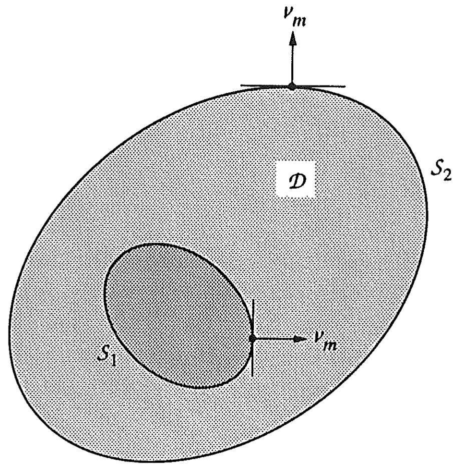
$$=\int_{x\in\mathcal{S}_2}\nu_m\left[\hat{p}^{A}(x,s)\hat{v}_m^{B}(x,-s)+\hat{p}^{B}(x,-s)\hat{v}_m^{A}(x,s)\right]\mathrm{d}A\,, \qquad (7.5\text{-}8)$$

i.e. that the surface integral is an invariant.

7.6 Transmission/reception reciprocity properties of a pair of acoustic transducers

The transmission and the reception of acoustic waves take place through the use of acoustic transducers (which can be, for example, of piezoelectric, magnetoelastic or electrodynamic type). Different representations exist to model the action of the transducers. In this respect we shall discuss the *volume action model* and the *surface action model*. To analyse the corresponding reciprocity properties, we consider the fundamental configuration of two transducers that are surrounded by a fluid. The entire configuration occupies a bounded or unbounded domain $\mathcal{D}$. Transducer A occupies the bounded domain Tr_A with boundary surface $\partial\mathrm{Tr}_A$, and a unit vector along the normal ν_m oriented away from Tr_A. Transducer B occupies the bounded domain Tr_B with boundary surface $\partial\mathrm{Tr}_B$, and a unit vector along the normal ν_m oriented away from Tr_B. The domain exterior to $\mathrm{Tr}_A\cup\partial\mathrm{Tr}_A$ is denoted by Tr_A'; the domain exterior to $\mathrm{Tr}_B\cup\partial\mathrm{Tr}_B$ is denoted by Tr_B'. The domains Tr_A and Tr_B are disjoint (Figure 7.6-1).

As to the boundary conditions across the interfaces between fluid parts with different acoustic properties and the boundary conditions at the boundary surfaces of acoustically impenetrable objects, the provisions necessary for the global reciprocity theorems to hold are made. If the immersing fluid occupies a bounded domain, the exterior of this domain is assumed to be acoustically impenetrable. If the immersing fluid occupies an unbounded domain, the standard limiting procedure described in Section 7.1 for handling an unbounded domain is applied. Since transmission and reception are both causal phenomena, the transmission/reception reciprocity properties are based on the reciprocity theorems (Equations (7.2-7) and (7.4-7)) of the time convolution type in which theorem causality is preserved.



7.5-3 Domain $\mathcal{D}$ bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

Volume action transducers

A *volume action transducer* is characterised by the property that in the transmitting mode its action can be accounted for by prescribed values of the volume source densities of volume injection rate and/or force, whose common support is the domain occupied by that transducer, while in the receiving mode it is sensitive to the acoustic pressure and/or the particle velocity over the domain it occupies. To investigate the transmission/reception reciprocity properties of a pair of such transducers, we take state A to be the causal acoustic state for which the volume source densities have the support Tr_A (i.e. in state A, transducer A is the transmitting transducer and transducer B is the receiving transducer). In addition, we take state B to be the causal acoustic state for which the volume source densities have the support Tr_B (i.e. in state B, transducer B is the transmitting transducer and transducer A is the receiving transducer). Application of the global time-domain reciprocity theorem of the time convolution type (Equation (7.2-7)), to the entire domain $\mathcal{D}$ occupied by the configuration, assuming the fluid to be self-adjoint, yields

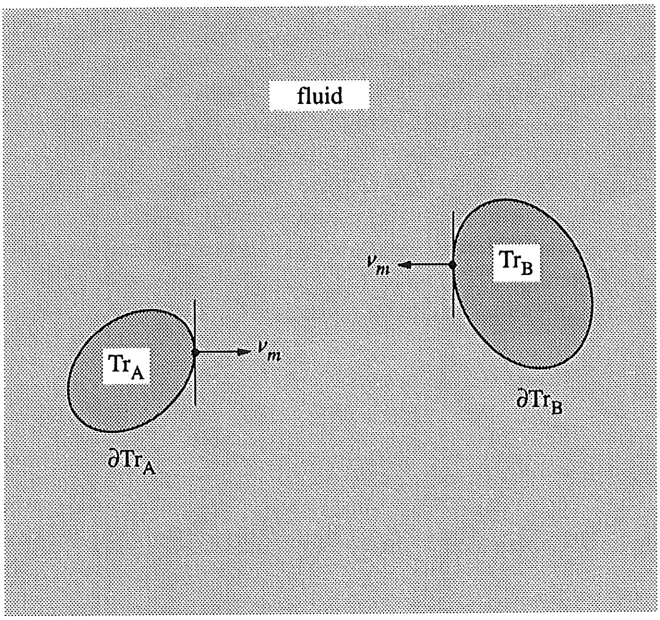
$$\int_{x\in\mathrm{Tr}_\mathrm{A}} \left[\mathrm{C}_t(f_k^\mathrm{A}, v_k^\mathrm{B}; x,t) - \mathrm{C}_t(p^\mathrm{B}, q^\mathrm{A}; x,t)\right] \mathrm{d}V$$

$$= \int_{x\in\mathrm{Tr}_\mathrm{B}} \left[\mathrm{C}_t(f_r^\mathrm{B}, v_r^\mathrm{A}; x,t) - \mathrm{C}_t(p^\mathrm{A}, q^\mathrm{B}; x,t)\right] \mathrm{d}V . \tag{7.6-1}$$

The complex frequency-domain counterpart of Equation (7.6-1) follows from Equation (7.4-7) as

$$\int_{x\in\mathrm{Tr}_\mathrm{A}} \left[\hat{f}_k^\mathrm{A}(x,s)\hat{v}_k^\mathrm{B}(x,s) - \hat{p}^\mathrm{B}(x,s)\hat{q}^\mathrm{A}(x,s)\right] \mathrm{d}V$$

$$= \int_{x\in\mathrm{Tr}_\mathrm{B}} \left[\hat{f}_r^\mathrm{B}(x,s)\hat{v}_r^\mathrm{A}(x,s) - \hat{p}^\mathrm{A}(x,s)\hat{q}^\mathrm{B}(x,s)\right] \mathrm{d}V . \tag{7.6-2}$$



7.6-1 Configuration of the transmission/reception reciprocity properties of a pair of acoustic transducers ($\mathrm{Tr_A}$ and $\mathrm{Tr_B}$) surrounded by a fluid.

In Equations (7.6-1) and (7.6-2), the terms containing the source densities of volume injection rate are representative of the action of the transducer as a (volume distributed) *transmitting monopole transducer*, while the terms containing the volume densities of force are representative of the action of the transducer as a (volume distributed) *transmitting dipole transducer*. Furthermore, the terms containing the acoustic pressure quantify the sensitivity of the transducer as a (volume distributed) *receiving monopole transducer*, while the terms containing the particle velocity quantify the sensitivity of the transducer as a (volume distributed) *receiving dipole transducer*. From Equations (7.6-1) and (7.6-2) it is concluded that a spatially distributed monopole transducer is only sensitive to the acoustic pressure (and insensitive to the particle velocity), while a spatially distributed dipole transducer is only sensitive to the particle velocity (and insensitive to the acoustic pressure). The reciprocity relations imply that the different sensitivities are related (i.e. through Equations (7.6-1) and (7.6-2)).

Surface action transducers

A *surface action transducer* is characterised by the property that in its transmitting mode its action can be accounted for by prescribed values of the normal component of the particle velocity and the acoustic pressure at its boundary surface, while in its receiving mode it is sensitive to the normal component of the particle velocity and the acoustic pressure at that surface. This description of the action of the transducer is employed when the description of its action by volume sources is either inapplicable or irrelevant. To investigate the reciprocity properties of a pair of such transducers, we take state A to be the causal acoustic state for which the prescribed surface source densities have the support $\partial\mathrm{Tr}_\mathrm{A}$ (i.e. in state A, transducer A is the transmitting transducer and transducer B is the receiving transducer). In addition, we take state B to be the causal acoustic state for which the prescribed surface source densities have the support $\partial\mathrm{Tr}_\mathrm{B}$ (i.e. in state B, transducer B is the transmitting transducer and transducer A is the receiving transducer). Application of the global time-domain reciprocity theorem of the time convolution type, (Equation (7.2-7)), to the entire domain $\mathcal{D}\cap\mathrm{Tr}'_\mathrm{A}\cap\mathrm{Tr}'_\mathrm{B}$ exterior to the transducers yields, assuming the fluid to be self-adjoint,

$$\int_{x\in\partial\mathrm{Tr}_\mathrm{A}} \nu_m\left[\mathrm{C}_t(p^\mathrm{A},v_m^\mathrm{B};x,t)-\mathrm{C}_t(p^\mathrm{B},v_m^\mathrm{A};x,t)\right]\mathrm{d}A$$

$$=\int_{x\in\partial\mathrm{Tr}_\mathrm{B}} \nu_m\left[\mathrm{C}_t(p^\mathrm{B},v_m^\mathrm{A};x,t)-\mathrm{C}_t(p^\mathrm{A},v_m^\mathrm{B};x,t)\right]\mathrm{d}A\,. \qquad (7.6\text{-}3)$$

The complex frequency-domain counterpart of Equation (7.6-3) follows from Equation (7.4-7) as

$$\int_{x\in\partial\mathrm{Tr}_\mathrm{A}} \nu_m\left[\hat{p}^\mathrm{A}(x,s)\hat{v}_m^\mathrm{B}(x,s)-\hat{p}^\mathrm{B}(x,s)\hat{v}_m^\mathrm{A}(x,s)\right]\mathrm{d}A$$

$$=\int_{x\in\partial\mathrm{Tr}_\mathrm{B}} \nu_m\left[\hat{p}^\mathrm{B}(x,s)\hat{v}_m^\mathrm{A}(x,s)-\hat{p}^\mathrm{A}(x,s)\hat{v}_m^\mathrm{B}(x,s)\right]\mathrm{d}A\,. \qquad (7.6\text{-}4)$$

In Equations (7.6-3) and (7.6-4), the terms containing the source densities of surface injection rate (i.e. $\nu_r\hat{v}_r$) are representative of the action of the transducer as a (surface distributed) *transmitting monopole transducer*, while the terms containing the surface densities of force (i.e. $\nu_k\hat{p}$) are representative of the action of the transducer as a (surface distributed) *transmitting dipole transducer*. Furthermore, the terms containing the acoustic pressure quantify the sensitivity of the transducer as a (surface distributed) *receiving monopole transducer*, while the terms containing the particle velocity quantify the sensitivity of the transducer as a (surface distributed) *receiving dipole transducer*. From Equations (7.6-3) and (7.6-4) it is concluded that a surface distributed monopole transducer is only sensitive to the acoustic pressure (and insensitive to the particle velocity), while a surface distributed dipole transducer is only sensitive to the (normal component of the) particle velocity (and insensitive to the acoustic pressure). The reciprocity relations imply that the different sensitivities are related (i.e. through Equations (7.6-3) and (7.6-4)).

Exercises

Exercise 7.6-1

Use Equation (7.2-7) to derive the time-domain transmission/reception reciprocity theorem for a pair of transducers (A and B) if transducer A is a volume action transducer and transducer B is a surface action transducer. (Note the orientation of the unit vector ν_m along the normal to $\partial\mathrm{Tr}_\mathrm{B}$.)

Answer:

$$\int_{x\in\mathrm{Tr}_\mathrm{A}} \left[\mathrm{C}_t(f_k^{\mathrm{A}}, v_k^{\mathrm{B}};x,t) - \mathrm{C}_t(p^{\mathrm{B}}, q^{\mathrm{A}};x,t)\right] \mathrm{d}V$$

$$= \int_{x\in\partial\mathrm{Tr}_\mathrm{B}} \nu_m \left[\mathrm{C}_t(p^{\mathrm{B}}, v_m^{\mathrm{A}};x,t) - \mathrm{C}_t(p^{\mathrm{A}}, v_m^{\mathrm{B}};x,t)\right] \mathrm{d}A\,. \tag{7.6-5}$$

Exercise 7.6-2

Use Equation (7.4-7) to derive the complex frequency-domain transmission/reception reciprocity theorem for a pair of transducers (A and B) if transducer A is a volume action transducer and transducer B is a surface action transducer. (Note the orientation of the unit vector ν_m along the normal to $\partial\mathrm{Tr}_\mathrm{B}$.)

Answer:

$$\int_{x\in\mathrm{Tr}_\mathrm{A}} \left[\hat{f}_k^{\mathrm{A}}(x,s)\hat{v}_k^{\mathrm{B}}(x,s) - \hat{p}^{\mathrm{B}}(x,s)\hat{q}^{\mathrm{A}}(x,s)\right] \mathrm{d}V$$

$$= \int_{x\in\partial\mathrm{Tr}_\mathrm{B}} \nu_m \left[\hat{p}^{\mathrm{B}}(x,s)\hat{v}_m^{\mathrm{A}}(x,s) - \hat{p}^{\mathrm{A}}(x,s)\hat{v}_m^{\mathrm{B}}(x,s)\right] \mathrm{d}A\,. \tag{7.6-6}$$

Exercise 7.6-3

If in the interior of Tr_A and Tr_B the acoustic wave-field quantities would be set equal to zero and these wave-field quantities would on $\partial\mathrm{Tr}_\mathrm{A}$ and $\partial\mathrm{Tr}_\mathrm{B}$ jump to their respective boundary values, the jumps would, on account of Equations (7.1-1)–(7.1-4), give rise to surface force sources with volume densities $f_k^{\mathrm{A,B}} = \nu_k p^{\mathrm{A,B}}\delta_{\partial\mathrm{Tr}_{\mathrm{A,B}}}(x)$ and surface injection rate sources with volume densities $q^{\mathrm{A,B}} = \nu_r v_r^{\mathrm{A,B}}\delta_{\partial\mathrm{Tr}_{\mathrm{A,B}}}(x)$, where $\delta_{\mathcal{S}}(x)$ is the surface Dirac delta distribution operative on the surface $\mathcal{S}$. Show, by taking the time convolution of the inner products of $f_k^{\mathrm{A,B}}$ with $v_k^{\mathrm{B,A}}$ and of $q^{\mathrm{A,B}}$ with $p^{\mathrm{B,A}}$, that in this physical picture Equation (7.6-3) is compatible with Equation (7.6-1).

Exercise 7.6-4

Show, in a manner similar to Exercise 7.6-3, that Equation (7.6-4) is compatible with Equation (7.6-2).

7.7 Transmission/reception reciprocity properties of a single acoustic transducer

To analyse the transmission/reception reciprocity properties of a single transducer, we consider the fundamental configuration of a single transducer surrounded by a fluid. The entire configuration occupies a, bounded or unbounded, domain $\mathcal{D}$. The transducer occupies the bounded domain Tr with boundary surface ∂Tr and a unit vector along the normal ν_m oriented away from Tr (Figure 7.7-1). The domain exterior to Tr$\cup\partial$Tr is denoted by Tr$'$.

As to the boundary conditions across interfaces between fluid parts with different acoustic properties and the boundary conditions at the boundary surfaces of acoustically impenetrable objects, the provisions necessary for the global reciprocity theorems to hold are made. If the immersing fluid occupies a bounded domain, the exterior of this domain is assumed to be acoustically impenetrable. If the immersing fluid occupies an unbounded domain, the standard limiting procedure described in Section 7.1 for handling an unbounded domain is applied. Since transmission and reception are both causal phenomena, the transmission/reception reciprocity properties are based on the reciprocity theorems (Equations (7.2-7) and (7.4-7)) of the time convolution type, in which causality is preserved.

Volume action transducer

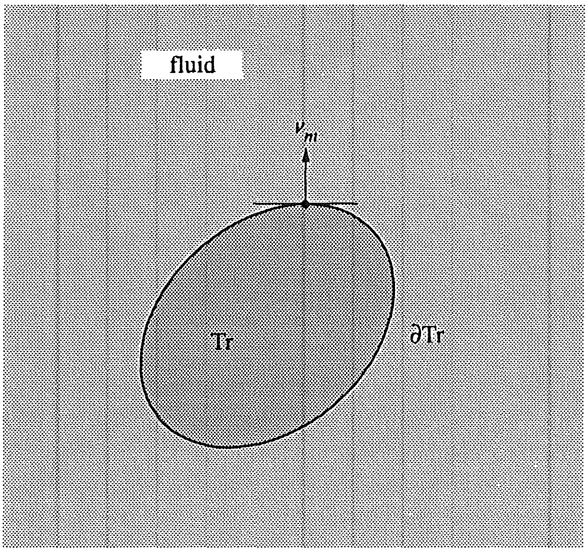
If Tr is a *volume action transducer*, its action in the transmitting mode is accounted for by prescribed values of the volume source densities of injection rate and/or force, whose support is the domain occupied by the transducer, while in the receiving mode it is sensitive to the acoustic pressure and/or the particle velocity over the domain it occupies. To investigate the transmission/reception reciprocity properties of a single transducer of this kind, state A is taken to be the causal state associated with the wave field $\{p^{\mathrm{T}},v_r^{\mathrm{T}}\}$ generated by the prescribed volume source densities $\{q^{\mathrm{T}},f_k^{\mathrm{T}}\}$ whose support is Tr. This state is referred to as the *transmitting state* and will be denoted by the superscript T. Next, state B is taken to be the causal state associated with the wave field that is generated by unspecified sources located in the domain Tr$'$ exterior to the transducer. In the surrounding fluid these sources would generate an *incident wave field* $\{p^{\mathrm{i}},v_r^{\mathrm{i}}\}$ if the transducer were not activated. The total wave field $\{p^{\mathrm{R}},v_r^{\mathrm{R}}\}$ in the presence of the transducer is then the superposition of the incident wave field and the *scattered wave field* $\{p^{\mathrm{s}},v_r^{\mathrm{s}}\}$, i.e.

$$\{p^{\mathrm{R}},v_r^{\mathrm{R}}\} = \{p^{\mathrm{i}}+p^{\mathrm{s}},v_r^{\mathrm{i}}+v_r^{\mathrm{s}}\} \, . \tag{7.7-1}$$

The relevant state is referred to as the *receiving state* and will be denoted by the superscript R. Note that in the receiving state the domain Tr occupied by the transducer is source-free and that the scattered wave field in this state is source-free in the domain Tr$'$ exterior to the domain occupied by the transducer. Application of the time-domain reciprocity theorem of the time convolution type (Equation (7.2-7)) to the transmitted and the scattered wave fields and to the domain Tr$'$ exterior to the transducer yields, assuming the fluid to be self-adjoint,

$$\int_{x\in\partial\mathrm{Tr}} \nu_m \left[\mathrm{C}_t(p^{\mathrm{T}},v_m^{\mathrm{s}};x,t) - \mathrm{C}_t(p^{\mathrm{s}},v_m^{\mathrm{T}};x,t)\right] \mathrm{d}A = 0 \, . \tag{7.7-2}$$

Here, we have used the property that the total wave field in the transmitting state and the scattered wave field in the receiving state are both source-free in the domain Tr$'$ exterior to the transducer and causally related to the action of (primary or secondary) source distributions with the domain Tr occupied by the transducer as their supports, on account of which both the volume



7.7-1 Configuration for the transmission/reception reciprocity properties of a single acoustic transducer Tr surrounded by a fluid.

integral over the domain exterior to the transducer and the surface integral over the outer boundary of the domain of application of Equation (7.2-7) vanish. Using Equation (7.7-1), it follows from Equation (7.7-2) that

$$\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{R}};x,t) - \mathrm{C}_t(p^{\mathrm{R}}, v_m^{\mathrm{T}};x,t)\right]\mathrm{d}A$$

$$= \int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{i}};x,t) - \mathrm{C}_t(p^{\mathrm{i}}, v_m^{\mathrm{T}};x,t)\right]\mathrm{d}A\,. \qquad (7.7\text{-}3)$$

Next, Equation (7.2-7) is applied to the total wave fields in the transmitting and the receiving states and to the domain Tr occupied by the transducer. This yields, again assuming the fluid to be self-adjoint,

$$\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{R}};x,t) - \mathrm{C}_t(p^{\mathrm{R}}, v_m^{\mathrm{T}};x,t)\right]\mathrm{d}A$$

$$= \int_{x\in\mathrm{Tr}} \left[\mathrm{C}_t(f_k^{\mathrm{T}}, v_k^{\mathrm{R}};x,t) - \mathrm{C}_t(p^{\mathrm{R}}, q^{\mathrm{T}};x,t)\right]\mathrm{d}V\,. \qquad (7.7\text{-}4)$$

Combining Equations (7.7-3) and (7.7-4), and using the continuity of the acoustic pressure and the normal component of the particle velocity across ∂Tr in both states, we arrive at

$$\int_{x\in\partial\mathrm{Tr}} \left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{i}};x,t) - \mathrm{C}_t(p^{\mathrm{i}}, v_m^{\mathrm{T}};x,t)\right]\mathrm{d}A$$

$$= \int_{x\in\mathrm{Tr}} \left[\mathrm{C}_t(f_k^{\mathrm{T}}, v_k^{\mathrm{R}};x,t) - \mathrm{C}_t(p^{\mathrm{R}}, q^{\mathrm{T}};x,t) \right] \mathrm{d}V . \tag{7.7-5}$$

The complex frequency-domain counterpart of Equation (7.7-5) follows, in a similar manner, from Equation (7.4-7) as

$$\int_{x\in\partial\mathrm{Tr}} \nu_m \left[\hat{p}^{\mathrm{T}}(x,s)\hat{v}_m^{\mathrm{i}}(x,s) - \hat{p}^{\mathrm{i}}(x,s)\hat{v}_m^{\mathrm{T}}(x,s) \right] \mathrm{d}A$$

$$= \int_{x\in\mathrm{Tr}} \left[\hat{f}_k^{\mathrm{T}}(x,s)\hat{v}_k^{\mathrm{R}}(x,s) - \hat{p}^{\mathrm{R}}(x,s)\hat{q}^{\mathrm{T}}(x,s) \right] \mathrm{d}V . \tag{7.7-6}$$

In view of what has been found in Section 7.6, the right-hand sides of Equations (7.7-5) and (7.7-6) are representative of the sensitivity of the transducer to a received acoustic wave field generated elsewhere in the domain exterior to the transducer. The left-hand sides express that the transducer can, in the receiving state, be conceived of as being excited, across its boundary surface, by the incident wave field. Equations (7.7-5) and (7.7-6) relate these two aspects quantitatively.

Surface action transducer

If Tr is a *surface action transducer*, its action in the transmitting mode is accounted for by prescribed values of the normal component of the particle velocity and the acoustic pressure at its boundary surface, while in the receiving mode it is sensitive to the normal component of the particle velocity and the acoustic pressure at that surface. This description of the action of the transducer is employed when the description of its action by volume sources is either inapplicable or irrelevant. To investigate the transmission/reception reciprocity properties of a single transducer of this kind, state A is, as above, taken to be the causal state associated with the wave field $\{p^{\mathrm{T}},v_r^{\mathrm{T}}\}$ generated by the prescribed surface source densities of injection rate (i.e. $\nu_r v_r^{\mathrm{T}}$) and force (i.e. $\nu_k p^{\mathrm{T}}$), whose support is ∂Tr. This state is referred to as the *transmitting state* and will be denoted by the superscript T. Next, state B is taken to be the causal state associated with the wave field that is generated by unspecified sources located in the domain Tr$'$ exterior to the transducer. In the surrounding fluid these sources would generate an *incident wave field* $\{p^{\mathrm{i}},v_r^{\mathrm{i}}\}$ if the transducer were not activated. The total wave field $\{p^{\mathrm{R}},v_r^{\mathrm{R}}\}$ in the presence of the transducer is again the superposition of the incident wave field and the *scattered wave field* $\{p^{\mathrm{s}},v_r^{\mathrm{s}}\}$, i.e.

$$\{p^{\mathrm{R}},v_r^{\mathrm{R}}\} = \{p^{\mathrm{i}}+p^{\mathrm{s}},v_r^{\mathrm{i}}+v_r^{\mathrm{s}}\} . \tag{7.7-7}$$

The relevant state is referred to as the *receiving state* and will be denoted by the superscript R. Note that in the receiving state the domain Tr occupied by the transducer is source-free insofar as the total field is concerned and that the scattered wave field in this state is source-free in the domain Tr$'$ exterior to the domain occupied by the transducer. Application of the time-domain reciprocity theorem of the time convolution type (Equation (7.2-7)), to the transmitted and the scattered wave fields and to the domain Tr′ exterior to the transducer, assuming the fluid to be self-adjoint, yields

$$\int_{x \in \partial \mathrm{Tr}} \nu_m \left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{s}}; x,t) - \mathrm{C}_t(p^{\mathrm{s}}, v_m^{\mathrm{T}}; x,t) \right] \mathrm{d}A = 0 . \tag{7.7-8}$$

Now, using Equation (7.7-7), it follows that

$$\int_{x \in \partial \mathrm{Tr}} \nu_m \left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{R}}; x,t) - \mathrm{C}_t(p^{\mathrm{R}}, v_m^{\mathrm{T}}; x,t) \right] \mathrm{d}A$$

$$= \int_{x \in \partial \mathrm{Tr}} \nu_m \left[\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\mathrm{i}}; x,t) - \mathrm{C}_t(p^{\mathrm{i}}, v_m^{\mathrm{T}}; x,t) \right] \mathrm{d}A . \tag{7.7-9}$$

The complex frequency-domain counterpart of Equation (7.7-9) follows, in a similar manner, from Equation (7.4-7) as

$$\int_{x \in \partial \mathrm{Tr}} \nu_m \left[\hat{p}^{\mathrm{T}}(x,s) \hat{v}_m^{\mathrm{R}}(x,s) - \hat{p}^{\mathrm{R}}(x,s) \hat{v}_m^{\mathrm{T}}(x,s) \right] \mathrm{d}A$$

$$= \int_{x \in \partial \mathrm{Tr}} \nu_m \left[\hat{p}^{\mathrm{T}}(x,s) \hat{v}_m^{\mathrm{i}}(x,s) - \hat{p}^{\mathrm{i}}(x,s) \hat{v}_m^{\mathrm{T}}(x,s) \right] \mathrm{d}A . \tag{7.7-10}$$

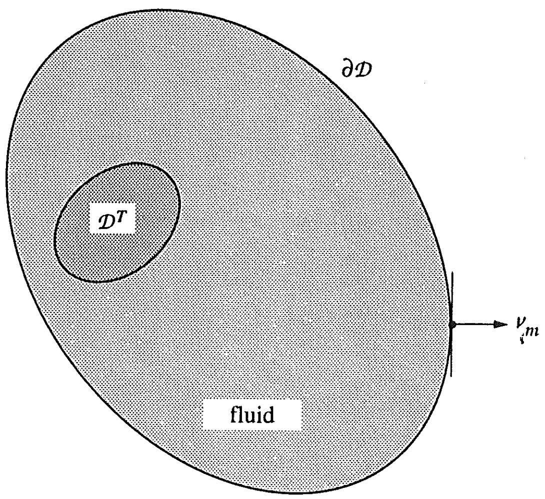
In view of what has been found in Section 7.6, the left-hand sides of Equations (7.7-9) and (7.7-10) are representative of the sensitivity of the transducer to a received acoustic wave field generated elsewhere in the domain exterior to the transducer. The right-hand sides express that the transducer can, in the receiving state, be conceived of as being excited, across its boundary surface, by the incident wave field. Equations (7.7-9) and (7.7-10) relate these two aspects quantitatively.

7.8 The direct (forward) source problem; point-source solutions and Green's functions

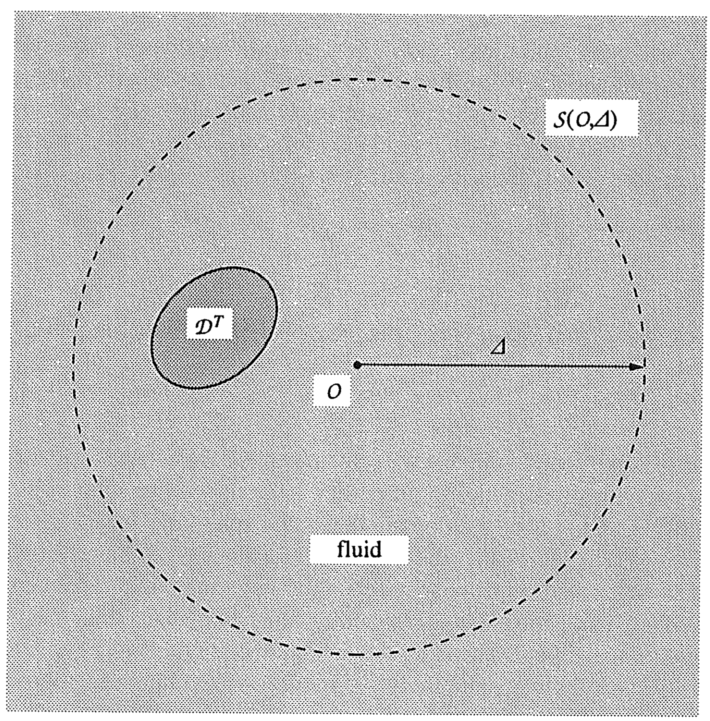
In the direct (or forward) source problem we want to express the acoustic wave-field quantities in a configuration with given acoustic properties in terms of the source distributions that generate the wave field. Let $\mathcal{D}$ be the domain in which expressions for the generated acoustic wave field $\{p^{\mathrm{T}}, v_r^{\mathrm{T}}\} = \{p^{\mathrm{T}}, v_r^{\mathrm{T}}\}(x,t)$ are to be found. If $\mathcal{D}$ is a bounded domain, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable. If $\mathcal{D}$ is an unbounded domain, the standard provisions described in Section 7.1 for handling an unbounded domain are made. Since $\{p^{\mathrm{T}}, v_r^{\mathrm{T}}\}$ is a physical wave field, it satisfies the condition of causality. The source distributions $\{q^{\mathrm{T}}, f_k^{\mathrm{T}}\} = \{q^{\mathrm{T}}, f_k^{\mathrm{T}}\}(x,t)$ that generate the wave field, have the bounded support $\mathcal{D}^T$ that is a proper subdomain of $\mathcal{D}$ (Figure 7.8-1).

The acoustic properties of the fluid present in $\mathcal{D}$ are characterised by the relaxation functions $\{\mu_{k,r}, \chi\} = \{\mu_{k,r}, \chi\}(x,t)$, which are causal functions of time. The case of an instantaneously reacting fluid easily follows from the more general case of a fluid with relaxation.

7.8-1(a)



7.8-1(b)



7.8-1 Configuration of the direct (forward) source problem. $\mathcal{D}^T$ is the bounded support of the source distributions. (a) The fluid occupies the bounded domain $\mathcal{D}$ with acoustically impenetrable boundary $\partial\mathcal{D}$. (b) The fluid occupies the unbounded domain $\mathcal{D}$; $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity.

Time-domain analysis

For the time-domain analysis of the problem the global reciprocity theorem of the time convolution type (Equation (7.2-7)) is taken as the point of departure. In it, state A is taken to be the generated acoustic wave field under consideration, i.e.

$$\{p^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{p^{\mathrm{T}},v_r^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}\,, \tag{7.8-1}$$

$$\{q^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{q^{\mathrm{T}},f_k^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{T}, \tag{7.8-2}$$

and

$$\{\mu_{k,r}^{\mathrm{A}},\chi^{\mathrm{A}}\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}\,. \tag{7.8-3}$$

Next, state B is chosen such that the application of Equation (7.2-7) to the domain $\mathcal{D}$ leads to the values of $\{p^{\mathrm{T}},v_r^{\mathrm{T}}\}$ at some arbitrary point $\boldsymbol{x}'\in\mathcal{D}$. Inspection of the right-hand side of Equation (7.2-7) reveals that this is accomplished if we take for the source distributions of state B a point source of volume injection rate at $\boldsymbol{x}'$ when we want an expression for the acoustic pressure at $\boldsymbol{x}'$, and a point source of force at $\boldsymbol{x}'$ when we want an expression for the particle velocity at $\boldsymbol{x}'$, while the fluid in state B must be taken to be the adjoint of the one in state A, i.e.

$$\{\mu_{r,k}^{\mathrm{B}},\chi^{\mathrm{B}}\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for all } \boldsymbol{x}\in\mathcal{D}\,. \tag{7.8-4}$$

Furthermore, if $\mathcal{D}$ is bounded, the acoustic wave field in state B must satisfy on $\partial\mathcal{D}$ the same boundary conditions for an acoustically impenetrable boundary as in state A, while if $\mathcal{D}$ is unbounded the acoustic wave field in state B must be causally related to the action of its (point) sources. The two possible choices of the source distributions will be discussed separately below.

First, we choose

$$q^{\mathrm{B}} = a\delta(\boldsymbol{x}-\boldsymbol{x}',t) \quad \text{and } f_r^{\mathrm{B}} = 0\,, \tag{7.8-5}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x}',t)$ represents the four-dimensional unit impulse (Dirac distribution) operative at the point $\boldsymbol{x}=\boldsymbol{x}'$ and at the instant $t=0$, while a is an arbitrary constant scalar. The acoustic wave field causally radiated by this source is given by

$$\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{q;\mathrm{B}},v_k^{q;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t)\,, \tag{7.8-6}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (7.2-7) is applied to the domain $\mathcal{D}$. When $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{A}},v_m^{\mathrm{B}};\boldsymbol{x},t) - \mathrm{C}_t(p^{\mathrm{B}},v_m^{\mathrm{A}};\boldsymbol{x},t)\right]\mathrm{d}A$$

$$= \int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}},v_m^{q;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t) - \mathrm{C}_t(p^{q;\mathrm{B}},v_m^{\mathrm{T}};\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}A = 0\,, \tag{7.8-7}$$

while if $\mathcal{D}$ is unbounded the standard provisions given in Section 7.1 for handling an unbounded domain yield

$$\int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)} \nu_m\left[\mathrm{C}_t(p^{\mathrm{A}},v_m^{\mathrm{B}};\boldsymbol{x},t) - \mathrm{C}_t(p^{\mathrm{B}},v_m^{\mathrm{A}};\boldsymbol{x},t)\right]\mathrm{d}A$$

$$= \int_{x \in \mathcal{S}(O,\Delta)} \nu_m \left[C_t(p^{\mathrm{T}}, v_m^{q;\mathrm{B}}; x, x', t) - C_t(p^{q;\mathrm{B}}, v_m^{\mathrm{T}}; x, x', t) \right] \mathrm{d}A \to 0 \quad \text{as } \Delta \to \infty . \tag{7.8-8}$$

Furthermore, in view of Equation (7.8-5) and the properties of $\delta(x - x', t)$,

$$\int_{x \in \mathcal{D}} \left[-C_t(f_r^{\mathrm{B}}, v_r^{\mathrm{A}}; x, t) + C_t(p^{\mathrm{A}}, q^{\mathrm{B}}; x, t) \right] \mathrm{d}V$$

$$= \int_{x \in \mathcal{D}} \left[C_t(p^{\mathrm{T}}, a\delta(x - x', t); x, t) \right] \mathrm{d}V = a p^{\mathrm{T}}(x', t) , \tag{7.8-9}$$

and, as the sources have the support $\mathcal{D}^{\mathrm{T}}$,

$$\int_{x \in \mathcal{D}} \left[C_t(f_k^{\mathrm{A}}, v_k^{\mathrm{B}}; x, t) - C_t(p^{\mathrm{B}}, q^{\mathrm{A}}; x, t) \right] \mathrm{d}V$$

$$= \int_{x \in \mathcal{D}^T} \left[C_t(f_k^{\mathrm{T}}, v_k^{q;\mathrm{B}}; x, x', t) - C_t(p^{q;\mathrm{B}}, q^{\mathrm{T}}; x, x', t) \right] \mathrm{d}V . \tag{7.8-10}$$

Collecting the results, we arrive at

$$a p^{\mathrm{T}}(x', t) = \int_{x \in \mathcal{D}^T} \left[C_t(p^{q;\mathrm{B}}, q^{\mathrm{T}}; x, x', t) - C_t(v_k^{q;\mathrm{B}}, f_k^{\mathrm{T}}; x, x', t) \right] \mathrm{d}V \quad \text{for } x' \in \mathcal{D} , \tag{7.8-11}$$

where, in the second term on the right-hand side, we have used the symmetry of the convolution in its functional arguments. From Equation (7.8-11) a representation for $p^{\mathrm{T}}(x', t)$ is obtained by taking into account that $p^{q;\mathrm{B}}$ and $v_k^{q;\mathrm{B}}$ are linearly related to a. The latter relationship is expressed as

$$\{p^{q;\mathrm{B}}, v_k^{q;\mathrm{B}}\}(x, x', t) = \{G^{pq;\mathrm{B}}, G_k^{vq;\mathrm{B}}\}(x, x', t) a . \tag{7.8-12}$$

However, as for the right-hand side the reciprocity relations (see Exercises 7.8-1 and 7.8-3)

$$\{G^{pq;\mathrm{B}}, G_k^{vq;\mathrm{B}}\}(x, x', t) = \{G^{pq}, -G_k^{pf}\}(x', x, t) \tag{7.8-13}$$

hold, Equation (7.8-11) leads, with Equations (7.8-12) and (7.8-13), and invoking the condition that the resulting equation has to hold for arbitrary values of a, to the final result

$$p^{\mathrm{T}}(x', t) = \int_{x \in \mathcal{D}^T} \left[C_t(G^{pq}, q^{\mathrm{T}}; x', x, t) + C_t(G_k^{pf}, f_k^{\mathrm{T}}; x', x, t) \right] \mathrm{d}V \quad \text{for } x' \in \mathcal{D} . \tag{7.8-14}$$

Equation (7.8-14) expresses the acoustic pressure p^{T} of the generated acoustic wave field at x' as the superposition of the contributions from the elementary distributed sources $q^{\mathrm{T}}\,\mathrm{d}V$ and $f_k^{\mathrm{T}}\,\mathrm{d}V$ at x. The intervening kernel functions are the *acoustic pressure/volume injection source Green's function* $G^{pq} = G^{pq}(x', x, t)$ and the *acoustic pressure/force source Green's function* $G_k^{pf} = G_k^{pf}(x', x, t)$. These Green's functions are the acoustic pressure at x', radiated in the actual fluid with constitutive parameters $\{\mu_{k,r}, \chi\} = \{\mu_{k,r}, \chi\}(x, t)$, by a point source of volume injection at x and a point source of force at x, respectively.

Secondly, we choose

$$q^{\mathrm{B}} = 0 \quad \text{and} \quad f_r^{\mathrm{B}} = b_r \delta(x - x', t) , \tag{7.8-15}$$

where b_r is an arbitrary constant vector. The acoustic wave field causally radiated by this source is given by

$$\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{f;\mathrm{B}},v_k^{f;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t)\,, \tag{7.8-16}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (7.2-7) is applied to the domain $\mathcal{D}$. In case $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{A}},v_m^{\mathrm{B}};\boldsymbol{x},t) - \mathrm{C}_t(p^{\mathrm{B}},v_m^{\mathrm{A}};\boldsymbol{x},t)\right]\mathrm{d}A$$

$$= \int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}},v_m^{f;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t) - \mathrm{C}_t(p^{f;\mathrm{B}},v_m^{\mathrm{T}};\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}A = 0\,, \tag{7.8-17}$$

while, if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain yield

$$\int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)} \nu_m\left[\mathrm{C}_t(p^{\mathrm{A}},v_m^{\mathrm{B}};\boldsymbol{x},t) - \mathrm{C}_t(p^{\mathrm{B}},v_m^{\mathrm{A}};\boldsymbol{x},t)\right]\mathrm{d}A$$

$$= \int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}},v_m^{f;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t) - \mathrm{C}_t(p^{f;\mathrm{B}},v_m^{\mathrm{T}};\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}A \to 0 \qquad \text{as } \Delta\to\infty\,. \tag{7.8-18}$$

Furthermore, in view of Equation (7.8-15) and the properties of $\delta(\boldsymbol{x}-\boldsymbol{x}',t)$,

$$\int_{\boldsymbol{x}\in\mathcal{D}} \left[-\mathrm{C}_t(f_r^{\mathrm{B}},v_r^{\mathrm{A}};\boldsymbol{x},t) + \mathrm{C}_t(p^{\mathrm{A}},q^{\mathrm{B}};\boldsymbol{x},t)\right]\mathrm{d}V$$

$$= -\int_{\boldsymbol{x}\in\mathcal{D}} \left[\mathrm{C}_t(b_r\delta(\boldsymbol{x}-\boldsymbol{x}',t),v_r^{\mathrm{T}};\boldsymbol{x};t)\right]\mathrm{d}V = -b_r v_r^{\mathrm{T}}(\boldsymbol{x}',t)\,. \tag{7.8-19}$$

Furthermore, as the sources have the support $\mathcal{D}^T$,

$$\int_{\boldsymbol{x}\in\mathcal{D}} \left[\mathrm{C}_t(f_k^{\mathrm{A}},v_k^{\mathrm{B}};\boldsymbol{x},t) - \mathrm{C}_t(p^{\mathrm{B}},q^{\mathrm{A}};\boldsymbol{x},t)\right]\mathrm{d}V$$

$$= \int_{\boldsymbol{x}\in\mathcal{D}^T} \left[\mathrm{C}_t(f_k^{\mathrm{T}},v_k^{f;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t) - \mathrm{C}_t(p^{f;\mathrm{B}},q^{\mathrm{T}};\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}V\,. \tag{7.8-20}$$

Collecting the results, we arrive at

$$b_r v_r^{\mathrm{T}}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^T} \left[-\mathrm{C}_t(p^{f;\mathrm{B}},q^{\mathrm{T}};\boldsymbol{x},\boldsymbol{x}',t) + \mathrm{C}_t(v_k^{f;\mathrm{B}},f_k^{\mathrm{T}};\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{D}\,, \tag{7.8-21}$$

where, in the second term on the right-hand side, we have used the symmetry of the convolution in its functional arguments. From Equation (7.8-21) a representation for $v_r^{\mathrm{T}}(\boldsymbol{x}',t)$ is obtained by taking into account that $p^{f;\mathrm{B}}$ and $v_k^{f;\mathrm{B}}$ are linearly related to b_r. The latter relationship is expressed by

$$\{p^{f;\mathrm{B}},v_k^{f;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t) = \{G_r^{pf;\mathrm{B}},G_{k,r}^{vf;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t)b_r\,. \tag{7.8-22}$$

However, as for the right-hand side the reciprocity relations (see Exercises 7.8-2 and 7.8-4)

$$\{G_r^{pf;\mathrm{B}}, G_{k,r}^{vf;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t) = \{-G_r^{vq}, G_{r,k}^{vf}\}(\boldsymbol{x}',\boldsymbol{x},t) \tag{7.8-23}$$

hold, Equation (7.8-21) leads with Equations (7.8-22) and (7.8-23), and invoking the condition that the resulting equation has to hold for arbitrary values of b_r, to the final result

$$v_r^{\mathrm{T}}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^T} \left[\mathrm{C}_t(G_r^{vq}, q^{\mathrm{T}}; \boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G_{r,k}^{vf}, f_k^{\mathrm{T}}; \boldsymbol{x}',\boldsymbol{x},t)\right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{D}. \tag{7.8-24}$$

Equation (7.8-24) expresses the particle velocity v_r^{T} of the generated acoustic wave field at $\boldsymbol{x}'$ as the superposition of the contributions from the elementary distributed sources $q^{\mathrm{T}}\,\mathrm{d}V$ and $f_k^{\mathrm{T}}\,\mathrm{d}V$ at $\boldsymbol{x}$. The intervening kernel functions are the *particle velocity/volume injection source Green's function* $G_r^{vq} = G_r^{vq}(\boldsymbol{x}',\boldsymbol{x},t)$ and the *particle velocity/force source Green's function* $G_{r,k}^{vf} = G_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},t)$. These Green's functions are the particle velocity at $\boldsymbol{x}'$, radiated in the actual fluid with constitutive parameters $\{\mu_{k,r},\chi\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t)$, by a point source of volume injection at $\boldsymbol{x}$ and a point source of force at $\boldsymbol{x}$, respectively.

Complex frequency-domain analysis

For the complex frequency-domain analysis of the problem the complex frequency-domain global reciprocity theorem of the time convolution type, Equation (7.4-7), is taken as the point of departure. In it, state A is taken to be the generated acoustic wave field under consideration, i.e.

$$\{\hat{p}^{\mathrm{A}}, \hat{v}_r^{\mathrm{A}}\} = \{\hat{p}^{\mathrm{T}}, \hat{v}_r^{\mathrm{T}}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.8-25}$$

$$\{\hat{q}^{\mathrm{A}}, \hat{f}_k^{\mathrm{A}}\} = \{\hat{q}^{\mathrm{T}}, \hat{f}_k^{\mathrm{T}}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^T, \tag{7.8-26}$$

and

$$\{\hat{\zeta}_{k,r}^{\mathrm{A}}, \hat{\eta}^{\mathrm{A}}\} = \{\hat{\zeta}_{k,r}, \hat{\eta}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.8-27}$$

Next, state B is chosen such that the application of Equation (7.4-7) to the domain $\mathcal{D}$ leads to the values of $\{\hat{p}^{\mathrm{T}}, \hat{v}_r^{\mathrm{T}}\}$ at some arbitrary point $\boldsymbol{x}'\in\mathcal{D}$. Inspection of the right-hand side of Equation (7.4-7) reveals that this is accomplished if we take for the source distributions of state B a point source of volume injection rate at $\boldsymbol{x}'$ when we want an expression for the acoustic pressure at $\boldsymbol{x}'$, and a point source of force at $\boldsymbol{x}'$ when we want an expression for the particle velocity at $\boldsymbol{x}'$, while the fluid in state B must be taken to be the adjoint of the one of state A, i.e.

$$\{\hat{\zeta}_{r,k}^{\mathrm{B}}, \hat{\eta}^{\mathrm{B}}\} = \{\hat{\zeta}_{k,r}, \hat{\eta}\}(\boldsymbol{x},s) \qquad \text{for all } \boldsymbol{x}\in\mathcal{D}. \tag{7.8-28}$$

Furthermore, if $\mathcal{D}$ is bounded, the acoustic wave field in state B must satisfy on $\partial\mathcal{D}$ the same boundary conditions for an acoustically impenetrable boundary as in state A, while if $\mathcal{D}$ is unbounded the acoustic wave field in state B must be causally related to the action of its (point) sources. The two choices for the source distributions will be discussed separately below.

First, we choose

$$\hat{q}^{\mathrm{B}} = \hat{a}(s)\delta(\boldsymbol{x}-\boldsymbol{x}') \quad \text{and} \quad \hat{f}_r^{\mathrm{B}} = 0, \tag{7.8-29}$$

where $\delta(x-x')$ represents the three-dimensional unit impulse (Dirac distribution) operative at the point $x = x'$, while $\hat{a} = \hat{a}(s)$ is an arbitrary scalar function of s. The acoustic wave field causally related to this source is given by

$$\{\hat{p}^{\mathrm{B}}, \hat{f}_k^{\mathrm{B}}\} = \{\hat{p}^{q;\mathrm{B}}, \hat{v}_k^{q;\mathrm{B}}\}(x,x',s)\,, \tag{7.8-30}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (7.4-7) is applied to the domain $\mathcal{D}$. When $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{p}^{\mathrm{A}}(x,s)\hat{v}_m^{\mathrm{B}}(x,s) - \hat{p}^{\mathrm{B}}(x,s)\hat{v}_m^{\mathrm{A}}(x,s)\right]\mathrm{d}A$$

$$= \int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{p}^{\mathrm{T}}(x,s)\hat{v}_m^{q;\mathrm{B}}(x,x',s) - \hat{p}^{q;\mathrm{B}}(x,x',s)\hat{v}_m^{\mathrm{T}}(x,s)\right]\mathrm{d}A = 0\,, \tag{7.8-31}$$

while if $\mathcal{D}$ is unbounded the standard provisions given in Section 7.1 for handling an unbounded domain yield

$$\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[\hat{p}^{\mathrm{A}}(x,s)\hat{v}_m^{\mathrm{B}}(x,s) - \hat{p}^{\mathrm{B}}(x,s)\hat{v}_m^{\mathrm{A}}(x,s)\right]\mathrm{d}A$$

$$= \int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[\hat{p}^{\mathrm{T}}(x,s)\hat{v}_m^{q;\mathrm{B}}(x,x',s) - \hat{p}^{q;\mathrm{B}}(x,s)\hat{v}_m^{\mathrm{T}}(x,x',s)\right]\mathrm{d}A \to 0 \quad \text{as } \Delta\to\infty\,. \tag{7.8-32}$$

Furthermore, in view of Equation (7.8-29) and the properties of $\delta(x-x')$,

$$\int_{x\in\mathcal{D}} \left[-\hat{f}_r^{\mathrm{B}}(x,s)\hat{v}_r^{\mathrm{A}}(x,s) + \hat{p}^{\mathrm{A}}(x,s)\hat{q}^{\mathrm{B}}(x,s)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}} \hat{p}^{\mathrm{T}}(x,s)\hat{a}(s)\delta(x-x')\,\mathrm{d}V = \hat{a}(s)\hat{p}^{\mathrm{T}}(x',s)\,. \tag{7.8-33}$$

In addition, since the sources have the support $\mathcal{D}^T$,

$$\int_{x\in\mathcal{D}} \left[\hat{f}_k^{\mathrm{A}}(x,s)\hat{v}_k^{\mathrm{B}}(x,s) - \hat{p}^{\mathrm{B}}(x,s)\hat{q}^{\mathrm{A}}(x,s)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^T} \left[\hat{f}_k^{\mathrm{T}}(x,s)\hat{v}_k^{q;\mathrm{B}}(x,x',s) - \hat{p}^{q;\mathrm{B}}(x,x',s)\hat{q}^{\mathrm{T}}(x,s)\right]\mathrm{d}V\,. \tag{7.8-34}$$

Collecting the results, we arrive at

$$\hat{a}(s)\hat{p}^{\mathrm{T}}(x',s) = \int_{x\in\mathcal{D}^T} \left[\hat{p}^{q;\mathrm{B}}(x,x',s)\hat{q}^{\mathrm{T}}(x,s) - \hat{v}_k^{q;\mathrm{B}}(x,x',s)\hat{f}_k^{\mathrm{T}}(x,s)\right]\mathrm{d}V \quad \text{for } x'\in\mathcal{D}\,. \tag{7.8-35}$$

From Equation (7.8-35) a representation for $\hat{p}^{\mathrm{T}}(x',s)$ is obtained by taking into account that $\hat{p}^{q;\mathrm{B}}$ and $\hat{v}_k^{q;\mathrm{B}}$ are linearly related to $\hat{a}(s)$. The latter relationship is expressed by

$$\{\hat{p}^{q;\mathrm{B}},\hat{v}_k^{q;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s) = \{\hat{G}^{pq;\mathrm{B}},\hat{G}_k^{vq;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s)\hat{a}(s) \,. \tag{7.8-36}$$

However, since for the right-hand side the reciprocity relations (see Exercises 7.8-5 and 7.8-7)

$$\{\hat{G}^{pq;\mathrm{B}},\hat{G}_k^{vq;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s) = \{\hat{G}^{pq},-\hat{G}_k^{pf}\}(\boldsymbol{x}',\boldsymbol{x},s) \tag{7.8-37}$$

hold, Equation (7.8-35) leads, with Equations (7.8-36) and (7.8-37) and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{a}(s)$, to the final result

$$\hat{p}^{\mathrm{T}}(\boldsymbol{x}',s) = \int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\hat{G}^{pq}(\boldsymbol{x}',\boldsymbol{x},s)\hat{q}^{\mathrm{T}}(\boldsymbol{x},s) + \hat{G}_k^{pf}(\boldsymbol{x}',\boldsymbol{x},s)\hat{f}_k^{\mathrm{T}}(\boldsymbol{x},s)\right]\mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{D}\,. \tag{7.8-38}$$

Equation (7.8-38) expresses the acoustic pressure $\hat{p}^{\mathrm{T}}$ of the generated acoustic wave field at $\boldsymbol{x}'$ as the superposition of the elementary contributions $\hat{q}^{\mathrm{T}}\,\mathrm{d}V$ and $\hat{f}_k^{\mathrm{T}}\,\mathrm{d}V$ from the distributed sources at $\boldsymbol{x}$. The intervening kernel functions are the *acoustic pressure/volume injection source Green's function* $\hat{G}^{pq} = \hat{G}^{pq}(\boldsymbol{x}',\boldsymbol{x},s)$ and the *acoustic pressure/force source Green's function* $\hat{G}_k^{pf} = \hat{G}_k^{pf}(\boldsymbol{x}',\boldsymbol{x},s)$. These Green's functions are the acoustic pressure at $\boldsymbol{x}'$, radiated in the actual fluid with constitutive parameters $\{\hat{\zeta}_{k,r},\hat{\eta}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s)$, by a point source of volume injection at $\boldsymbol{x}$ and a point source of force at $\boldsymbol{x}$, respectively.

Secondly, we choose

$$\hat{q}^{\mathrm{B}} = 0 \quad \text{and} \quad \hat{f}_r^{\mathrm{B}} = \hat{b}_r(s)\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{7.8-39}$$

where $\hat{b}_r = \hat{b}_r(s)$ is an arbitrary vector function of s. The acoustic wave field causally radiated by this source is given by

$$\{\hat{p}^{\mathrm{B}},\hat{f}_k^{\mathrm{B}}\} = \{\hat{p}^{f;\mathrm{B}},\hat{v}_k^{f;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s)\,, \tag{7.8-40}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (7.4-7) is applied to the domain $\mathcal{D}$. When $\mathcal{D}$ is bounded we have for the integral over its boundary surface

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\hat{p}^{\mathrm{T}}(\boldsymbol{x},s)\hat{v}_m^{f;\mathrm{B}}(\boldsymbol{x},\boldsymbol{x}',s) - \hat{p}^{f;\mathrm{B}}(\boldsymbol{x},\boldsymbol{x}',s)\hat{v}_m^{\mathrm{T}}(\boldsymbol{x},s)\right]\mathrm{d}A = 0\,, \tag{7.8-41}$$

while if $\mathcal{D}$ is unbounded the standard provisions given in Section 7.1 for handling an unbounded domain yield

$$\int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)} \nu_m\left[\hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{B}}(\boldsymbol{x},s) - \hat{p}^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{A}}(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)} \nu_m\left[\hat{p}^{\mathrm{T}}(\boldsymbol{x},s)\hat{v}_m^{f;\mathrm{B}}(\boldsymbol{x},\boldsymbol{x}',s) - \hat{p}^{f;\mathrm{B}}(\boldsymbol{x},\boldsymbol{x}',s)\hat{v}_m^{\mathrm{T}}(\boldsymbol{x},s)\right]\mathrm{d}A \to 0 \quad \text{as } \Delta\to\infty\,. \tag{7.8-42}$$

Furthermore, in view of Equation (7.8-39) and the properties of $\delta(\boldsymbol{x}-\boldsymbol{x}')$,

$$\int_{\boldsymbol{x}\in\mathcal{D}}\left[-\hat{f}_r^{\mathrm{B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{A}}(\boldsymbol{x},s) + \hat{p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{q}^{\mathrm{B}}(\boldsymbol{x},s)\right]\mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \hat{b}_r(s)\delta(x-x')\,\hat{v}_r^{\mathrm{T}}(x,s)\,\mathrm{d}V = -\hat{b}_r(s)\hat{v}_r^{\mathrm{T}}(x',s)\,. \tag{7.8-43}$$

In addition, as the sources have the support $\mathcal{D}^T$,

$$\int_{x\in\mathcal{D}} \left[\hat{f}_k^{\mathrm{A}}(x,s)\hat{v}_k^{\mathrm{B}}(x,s) - \hat{p}^{\mathrm{B}}(x,s)\hat{q}^{\mathrm{A}}(x,s)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^T} \left[\hat{f}_k^{\mathrm{T}}(x,s)\hat{v}_k^{f;\mathrm{B}}(x,x',s) - \hat{p}^{f;\mathrm{B}}(x,x',s)\hat{q}^{\mathrm{T}}(x,s)\right]\mathrm{d}V\,. \tag{7.8-44}$$

Collecting the results, we arrive at

$$\hat{b}_r(s)v_r^{\mathrm{T}}(x',s) = \int_{x\in\mathcal{D}^T} \left[-\hat{p}^{f;\mathrm{B}}(x,x',s)\hat{q}^{\mathrm{T}}(x,s) + \hat{v}_k^{f;\mathrm{B}}(x,x's)\hat{f}_k^{\mathrm{T}}(x,s)\right]\mathrm{d}V \quad \text{for } x'\in\mathcal{D}\,. \tag{7.8-45}$$

From Equation (7.8-45) a representation for $\hat{v}_r^{\mathrm{T}}(x',s)$ is obtained by taking into account that $\hat{p}^{f;\mathrm{B}}$ and $\hat{v}_k^{f;\mathrm{B}}$ are linearly related to $\hat{b}_r(s)$. The latter relationship is expressed by

$$\{\hat{p}^{f;\mathrm{B}},\hat{v}_k^{f;\mathrm{B}}\}(x,x',s) = \{\hat{G}_r^{pf;\mathrm{B}},\hat{G}_{k,r}^{vf;\mathrm{B}}\}(x,x',s)\hat{b}_r(s)\,. \tag{7.8-46}$$

However, since for the right-hand side the reciprocity relations (see Exercises 7.8-6 and 7.8-8)

$$\{\hat{G}_r^{pf;\mathrm{B}},\hat{G}_{k,r}^{vf;\mathrm{B}}\}(x,x',s) = \{-\hat{G}_r^{vq},\hat{G}_{r,k}^{vf}\}(x',x,s) \tag{7.8-47}$$

hold, Equation (7.8-45) leads with Equations (7.8-46) and (7.8-47), and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{b}_r(s)$, to the final result

$$\hat{v}_r^{\mathrm{T}}(x',s) = \int_{x\in\mathcal{D}^T} \left[\hat{G}_r^{vq}(x',x,s)\hat{q}^{\mathrm{T}}(x,s) + \hat{G}_{r,k}^{vf}(x',x,s)\hat{f}_k^{\mathrm{T}}(x,s)\right]\mathrm{d}V \qquad \text{for } x'\in\mathcal{D}\,. \tag{7.8-48}$$

Equation (7.8-48) expresses the particle velocity $\hat{v}_r^{\mathrm{T}}$ of the generated acoustic wave field at x' as the superposition of the elementary contributions $\hat{q}^{\mathrm{T}}\,\mathrm{d}V$ and $\hat{f}_k^{\mathrm{T}}\,\mathrm{d}V$ from the distributed sources at x. The intervening kernel functions are the *particle velocity/volume injection source Green's function* $\hat{G}_r^{vq} = \hat{G}_r^{vq}(x',x,s)$ and the *particle velocity/force source Green's function* $\hat{G}_{r,k}^{vf} = \hat{G}_{r,k}^{vf}(x',x,s)$. These Green's functions are the particle velocity at x', radiated in the actual fluid with constitutive parameters $\{\hat{\zeta}_{k,r},\hat{\eta}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(x,s)$, by a point source of volume injection at x and a point source of force at x, respectively.

Exercises

Exercise 7.8-1

Let $\{p^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{p^{\mathrm{A}},v_r^{\mathrm{A}}\}(x,x',t)$ be the acoustic wave field at x that is causally radiated by the point source at x' with volume source density $\{q^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{a^{\mathrm{A}}\delta(x-x',t),0\}$ and let $\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{\mathrm{B}},v_k^{\mathrm{B}}\}(x,x'',t)$ be the acoustic wave field at x that is causally radiated by the point source at

x'' with volume source density $\{q^{\mathrm{B}},f_r^{\mathrm{B}}\} = \{a^{\mathrm{B}}\delta(x-x'',t),0\}$, with $x' \neq x''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.2-7)) to the domain $\mathcal{D}$. (b) Write

$$p^{\mathrm{A}} = G^{pq;\mathrm{A}}(x,x',t)a^{\mathrm{A}}, \qquad v_r^{\mathrm{A}} = G_r^{vq;\mathrm{A}}(x,x',t)a^{\mathrm{A}},$$
$$p^{\mathrm{B}} = G^{pq;\mathrm{B}}(x,x'',t)a^{\mathrm{B}}, \qquad v_k^{\mathrm{B}} = G_k^{vq;\mathrm{B}}(x,x'',t)a^{\mathrm{B}},$$

invoke the condition that the result should hold for arbitrary a^{A} and a^{B}, and show that $G^{pq;\mathrm{A}}(x'',x',t) = G^{pq;\mathrm{B}}(x',x'',t)$.

Exercise 7.8-2

Let $\{p^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{p^{\mathrm{A}},v_r^{\mathrm{A}}\}(x,x',t)$ be the acoustic wave field at x that is causally radiated by the point source at x' with volume source density $\{q^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{a^{\mathrm{A}}\delta(x-x',t),0\}$ and let $\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{\mathrm{B}},v_k^{\mathrm{B}}\}(x,x'',t)$ be the acoustic wave field at x that is causally radiated by the point source at x'' with volume source density $\{q^{\mathrm{B}},f_r^{\mathrm{B}}\} = \{0,b_r^{\mathrm{B}}\delta(x-x'',t)\}$ with $x' \neq x''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.2-7)) to the domain $\mathcal{D}$. (b) Write

$$p^{\mathrm{A}} = G^{pq;\mathrm{A}}(x,x',t)a^{\mathrm{A}}, \qquad v_r^{\mathrm{A}} = G_r^{vq;\mathrm{A}}(x,x',t)a^{\mathrm{A}},$$
$$p^{\mathrm{B}} = G_r^{pf;\mathrm{B}}(x,x'',t)b_r^{\mathrm{B}}, \qquad v_k^{\mathrm{B}} = G_{k,r}^{vf;\mathrm{B}}(x,x'',t)b_r^{\mathrm{B}},$$

invoke the condition that the result should hold for arbitrary a^{A} and b_r^{B}, and show that $G_r^{vq;\mathrm{A}}(x'',x',t) = -G_r^{pf;\mathrm{B}}(x',x'',t)$.

Exercise 7.8-3

Let $\{p^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{p^{\mathrm{A}},v_r^{\mathrm{A}}\}(x,x',t)$ be the acoustic wave field at x that is causally radiated by the point source at x' with volume source density $\{q^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{0,b_k^{\mathrm{A}}\delta(x-x',t)\}$ and let $\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{\mathrm{B}},v_k^{\mathrm{B}}\}(x,x'',t)$ be the acoustic wave field at x that is causally radiated by the point source at x'' with volume source density $\{q^{\mathrm{B}},f_r^{\mathrm{B}}\} = \{a^{\mathrm{B}}\delta(x-x'',t),0\}$ with $x' \neq x''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.2-7)) to the domain $\mathcal{D}$. (b) Write

$$p^{\mathrm{A}} = G_k^{pf;\mathrm{A}}(x,x',t)b_k^{\mathrm{A}}, \qquad v_r^{\mathrm{A}} = G_{r,k}^{vf;\mathrm{A}}(x,x',t)b_k^{\mathrm{A}},$$
$$p^{\mathrm{B}} = G^{pq;\mathrm{B}}(x,x'',t)a^{\mathrm{B}}, \qquad v_k^{\mathrm{B}} = G_k^{vq;\mathrm{B}}(x,x'',t)a^{\mathrm{B}},$$

invoke the condition that the result should hold for arbitrary b_k^{A} and a^{B}, and show that $G_k^{pf;\mathrm{A}}(x'',x',t) = -G_k^{vq;\mathrm{B}}(x',x'',t)$. (Note that this result is consistent with the result of Exercise 7.8-2.)

Exercise 7.8-4

Let $\{p^{\rm A},v_r^{\rm A}\} = \{p^{\rm A},v_r^{\rm A}\}(\boldsymbol{x},\boldsymbol{x}',t)$ be the acoustic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}'$ with volume source density $\{q^{\rm A},f_k^{\rm A}\} = \{0,b_k^{\rm A}\delta(\boldsymbol{x}-\boldsymbol{x}',t)\}$ and let $\{p^{\rm B},v_k^{\rm B}\} = \{p^{\rm B},v_k^{\rm B}\}(\boldsymbol{x},\boldsymbol{x}'',t)$ be the acoustic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}''$ with volume source density $\{q^{\rm B},f_r^{\rm B}\} = \{0,b_r^{\rm B}\delta(\boldsymbol{x}-\boldsymbol{x}'',t)\}$ with $\boldsymbol{x}' \neq \boldsymbol{x}''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.2-7)) to the domain $\mathcal{D}$. (b) Write

$$p^{\rm A} = G_k^{pf;{\rm A}}(\boldsymbol{x},\boldsymbol{x}',t)b_k^{\rm A}, \qquad v_r^{\rm A} = G_{r,k}^{vf;{\rm A}}(\boldsymbol{x},\boldsymbol{x}',t)b_k^{\rm A},$$
$$p^{\rm B} = G_r^{pf;{\rm B}}(\boldsymbol{x},\boldsymbol{x}'',t)b_r^{\rm B}, \qquad v_k^{\rm B} = G_{k,r}^{vf;{\rm B}}(\boldsymbol{x},\boldsymbol{x}'',t)b_r^{\rm B},$$

invoke the condition that the result should hold for arbitrary $b_k^{\rm A}$ and $b_r^{\rm B}$, and show that $G_{r,k}^{vf;{\rm A}}(\boldsymbol{x}'',\boldsymbol{x}',t) = G_{k,r}^{vf;{\rm B}}(\boldsymbol{x}',\boldsymbol{x}'',t)$.

Exercise 7.8-5

Let $\{\hat{p}^{\rm A},\hat{v}_r^{\rm A}\} = \{\hat{p}^{\rm A},\hat{v}_r^{\rm A}\}(\boldsymbol{x},\boldsymbol{x}',s)$ be the acoustic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}'$ with volume source density $\{\hat{q}^{\rm A},\hat{f}_k^{\rm A}\} = \{\hat{a}^{\rm A}(s)\delta(\boldsymbol{x}-\boldsymbol{x}'),0\}$ and let $\{\hat{p}^{\rm B},\hat{v}_k^{\rm B}\} = \{\hat{p}^{\rm B},\hat{v}_k^{\rm B}\}(\boldsymbol{x},\boldsymbol{x}'',s)$ be the acoustic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}''$ with volume source density $\{\hat{q}^{\rm B},\hat{f}_r^{\rm B}\} = \{\hat{a}^{\rm B}(s)\delta(\boldsymbol{x}-\boldsymbol{x}''),0\}$, with $\boldsymbol{x}' \neq \boldsymbol{x}''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.4-7)) to the domain $\mathcal{D}$. (b) Write

$$\hat{p}^{\rm A} = \hat{G}^{pq;{\rm A}}(\boldsymbol{x},\boldsymbol{x}',s)\hat{a}^{\rm A}(s), \qquad \hat{v}_r^{\rm A} = \hat{G}_r^{vq;{\rm A}}(\boldsymbol{x},\boldsymbol{x}',s)\hat{a}^{\rm A}(s),$$
$$\hat{p}^{\rm B} = \hat{G}^{pq;{\rm B}}(\boldsymbol{x},\boldsymbol{x}'',s)\hat{a}^{\rm B}(s), \qquad \hat{v}_k^{\rm B} = \hat{G}_k^{vq;{\rm B}}(\boldsymbol{x},\boldsymbol{x}'',s)\hat{a}^{\rm B}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}^{\rm A}(s)$ and $\hat{a}^{\rm B}(s)$, and show that $\hat{G}^{pq;{\rm A}}(\boldsymbol{x}'',\boldsymbol{x}',s) = \hat{G}^{pq;{\rm B}}(\boldsymbol{x}',\boldsymbol{x}'',s)$.

Exercise 7.8-6

Let $\{\hat{p}^{\rm A},\hat{v}_r^{\rm A}\} = \{\hat{p}^{\rm A},\hat{v}_r^{\rm A}\}(\boldsymbol{x},\boldsymbol{x}',s)$ be the acoustic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}'$ with volume source density $\{\hat{q}^{\rm A},\hat{f}_k^{\rm A}\} = \{\hat{a}^{\rm A}(s)\delta(\boldsymbol{x}-\boldsymbol{x}'),0\}$ and let $\{\hat{p}^{\rm B},\hat{v}_k^{\rm B}\} = \{\hat{p}^{\rm B},\hat{v}_k^{\rm B}\}(\boldsymbol{x},\boldsymbol{x}'',s)$ be the acoustic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}''$ with volume source density $\{\hat{q}^{\rm B},\hat{f}_r^{\rm B}\} = \{0,\hat{b}_r^{\rm B}(s)\delta(\boldsymbol{x}-\boldsymbol{x}'')\}$, with $\boldsymbol{x}' \neq \boldsymbol{x}''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.4-7)) to the domain $\mathcal{D}$. (b) Write

$$\hat{p}^{\rm A} = \hat{G}^{pq;{\rm A}}(\boldsymbol{x},\boldsymbol{x}',s)\hat{a}^{\rm A}(s), \qquad \hat{v}_r^{\rm A} = \hat{G}_r^{vq;{\rm A}}(\boldsymbol{x},\boldsymbol{x}',s)\hat{a}^{\rm A}(s),$$
$$\hat{p}^{\rm B} = \hat{G}_r^{pf;{\rm B}}(\boldsymbol{x},\boldsymbol{x}'',s)\hat{b}_r^{\rm B}(s), \qquad \hat{v}_k^{\rm B} = \hat{G}_{k,r}^{vf;{\rm B}}(\boldsymbol{x},\boldsymbol{x}'',s)\hat{b}_r^{\rm B}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}^{\mathrm{A}}(s)$ and $\hat{b}_r^{\mathrm{B}}(s)$, and show that $\hat{G}_r^{vq;\mathrm{A}}(x'',x',s) = -\hat{G}_r^{pf;\mathrm{B}}(x',x'',s)$.

Exercise 7.8-7

Let $\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(x,x',s)$ be the acoustic wave field at x that is causally radiated by the point source at x' with volume source density $\{\hat{q}^{\mathrm{A}}, \hat{f}_k^{\mathrm{A}}\} = \{0,\hat{b}_k^{\mathrm{A}}(s)\delta(x-x')\}$ and let $\{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\} = \{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(x,x'',s)$ be the acoustic wave field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{q}^{\mathrm{B}}, \hat{f}_r^{\mathrm{B}}\} = \{\hat{a}^{\mathrm{B}}(s)\delta(x-x''),0\}$, with $x' \neq x''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.4-7)) to the domain $\mathcal{D}$. (b) Write

$$\hat{p}^{\mathrm{A}} = \hat{G}_k^{pf;\mathrm{A}}(x,x',s)\hat{b}_k^{\mathrm{A}}(s), \qquad \hat{v}_r^{\mathrm{A}} = \hat{G}_{r,k}^{vf;\mathrm{A}}(x,x',s)\,\hat{b}_k^{\mathrm{A}}(s),$$

$$\hat{p}^{\mathrm{B}} = \hat{G}^{pq;\mathrm{B}}(x,x'',s)\hat{a}^{\mathrm{B}}(s), \qquad \hat{v}_k^{\mathrm{B}} = \hat{G}_k^{vq;\mathrm{B}}(x,x'',s)\hat{a}^{\mathrm{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{b}_k^{\mathrm{A}}(s)$ and $\hat{a}^{\mathrm{B}}(s)$, and show that $\hat{G}_k^{pf;\mathrm{A}}(x'',x',s) = -\hat{G}_k^{vq;\mathrm{B}}(x',x'',s)$. (Note that this result is consistent with the result of Exercise 7.8-6.)

Exercise 7.8-8

Let $\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(x,x',s)$ be the acoustic wave field at x that is causally radiated by the point source at x' with volume source density $\{\hat{q}^{\mathrm{A}}, \hat{f}_k^{\mathrm{A}}\} = \{0,\hat{b}_k^{\mathrm{A}}(s)\delta(x-x')\}$ and let $\{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\} = \{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(x,x'',s)$ be the acoustic wave field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{q}^{\mathrm{B}}, \hat{f}_r^{\mathrm{B}}\} = \{0,\hat{b}_r^{\mathrm{B}}(s)\delta(x-x'')\}$, with $x' \neq x''$. The two sources radiate in adjoint fluids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (7.4-7)) to the domain $\mathcal{D}$. (b) Write

$$\hat{p}^{\mathrm{A}} = \hat{G}_k^{pf;\mathrm{A}}(x,x',s)\hat{b}_k^{\mathrm{A}}(s), \qquad \hat{v}_r^{\mathrm{A}} = \hat{G}_{r,k}^{vf;\mathrm{A}}(x,x',s)\,\hat{b}_k^{\mathrm{A}}(s),$$

$$\hat{p}^{\mathrm{B}} = \hat{G}_r^{pf;\mathrm{B}}(x,x'',s)\hat{b}_r^{\mathrm{B}}(s), \qquad \hat{v}_k^{\mathrm{B}} = \hat{G}_{k,r}^{vf;\mathrm{B}}(x,x'',s)\hat{b}_r^{\mathrm{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{b}_k^{\mathrm{A}}(s)$ and $\hat{b}_r^{\mathrm{B}}(s)$, and show that $\hat{G}_{r,k}^{vf;\mathrm{A}}(x'',x',s) = \hat{G}_{k,r}^{vf;\mathrm{B}}(x',x'',s)$.

Exercise 7.8-9

Give the expressions for the time-domain Green's functions (a) $G^{pq}(x,x',t)$, (b) $G_k^{pf}(x,x',t)$, (c) $G_r^{vq}(x,x',t)$, (d) $G_{r,k}^{vf}(x,x',t)$ for a homogeneous isotropic, lossless fluid with constitutive parameters ρ and κ that occupies the entire $\mathcal{R}^3$. (*Hint*: Use Equations (5.4-7)–(5.4-13).)

Answers:

(a) $G^{pq} = \rho \partial_t G(x,x',t)$,

(b) $G_k^{pf} = -\partial_k G(x,x',t)$,

(c) $G_r^{vq} = -\partial_r G(x,x',t)$,

(d) $G_{r,k}^{vf} = \rho^{-1}\delta(x-x')H(t)\delta_{r,k} + \rho^{-1}\partial_r\partial_k[\mathrm{I}_t G(x,x',t)]$,

in which $G = 4\pi|x-x'|^{-1}\delta(t-|x-x'|/c)$ and $c = (\rho\kappa)^{-1/2}$.

Exercise 7.8-10

Give the expressions for the complex frequency-domain Green's functions (a) $\hat{G}^{pq}(x,x',s)$, (b) $\hat{G}_k^{pf}(x,x',s)$, (c) $\hat{G}_r^{vq}(x,x',s)$, (d) $\hat{G}_{r,k}^{vf}(x,x',s)$ for a homogeneous, isotropic fluid with constitutive parameters $\hat{\zeta}$ and $\hat{\eta}$ that occupies the entire $\mathcal{R}^3$. (*Hint*: Use Equations (5.3-1)–(5.3-6).)

Answers:

(a) $\hat{G}^{pq} = \hat{\zeta}\hat{G}(x,x',s)$,

(b) $\hat{G}_k^{pf} = -\partial_k \hat{G}(x,x',s)$,

(c) $\hat{G}_r^{vq} = -\partial_r \hat{G}(x,x',s)$,

(d) $\hat{G}_{r,k}^{vf} = \hat{\zeta}^{-1}\delta(x-x')\delta_{r,k} + \hat{\zeta}^{-1}\partial_r\partial_k \hat{G}(x,x',s)$,

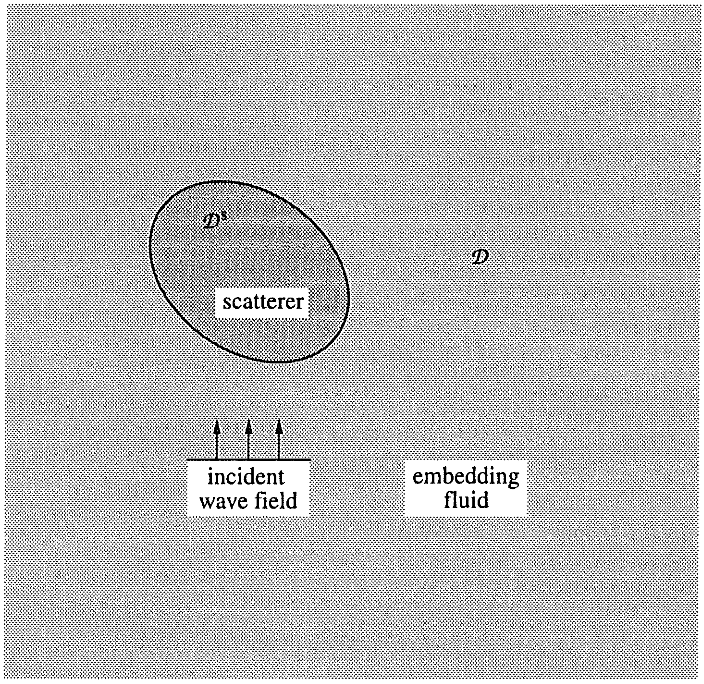
in which $\hat{G} = 4\pi|x-x'|^{-1}\exp(-\hat{\gamma}|x-x'|)$ and $\hat{\gamma} = (\hat{\eta}\hat{\zeta})^{1/2}$, with $\mathrm{Re}(\cdot)^{1/2} > 0$ for $\mathrm{Re}(s) > 0$.

Exercise 7.8-11

Show that Equation (7.8-38) follows from Equation (7.8-14) and Equation (7.8-48) from Equation (7.8-24) by taking the Laplace transform with respect to time.

7.9 The direct (forward) scattering problem

The configuration in an acoustic scattering problem generally consists of a background fluid with known acoustic properties, occupying the domain $\mathcal{D}$ (the "*embedding*"), in which, in principle, the radiation from given, arbitrarily distributed acoustic sources can be calculated with the aid of the theory developed in Section 7.8. In the embedding, an acoustically penetrable object of bounded support $\mathcal{D}^s$ (the "*scatterer*") is present, whose known acoustic properties differ from the ones of the embedding (Figure 7.9-1). The scatterer is acoustically irradiated by given sources located in the embedding, in a subdomain outside the scatterer. The problem is to determine the total acoustic wave field in the configuration. The standard procedure is first to calculate the so-called "*incident*" acoustic wave field, i.e. the wave field that would be present in the entire configuration if the object showed *no contrast* with respect to its embedding. (This can be done by employing the representations derived in Section 7.8.) Next, the total wave field



7.9-1 Scattering configuration with embedding $\mathcal{D}$ and scatterer $\mathcal{D}^s$.

is written as the superposition of the incident wave field and the "*scattered*" wave field, and, through a particular reasoning, the problem of determining the scattered wave field is reduced to calculating its *equivalent contrast source distributions*, whose common support will be shown to be the domain $\mathcal{D}^s$ occupied by the scatterer. In case the embedding $\mathcal{D}$ is a bounded domain, the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is assumed to be acoustically impenetrable. If $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. Both the incident wave field and the scattered wave field are causally related to the action of their respective sources.

Time-domain analysis

In the time-domain analysis of the problem, the acoustic properties of the embedding fluid are characterised by the relaxation functions $\{\mu_{k,r},\chi\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t)$, which are causal functions of time. The acoustic properties of the scatterer are characterised by the relaxation functions $\{\mu^s_{k,r},\chi^s\} = \{\mu^s_{k,r},\chi^s\}(\boldsymbol{x},t)$, which are also causal functions of time. The cases of an instantaneously reacting embedding and/or an instantaneously reacting scatterer easily follow from the more general cases for fluids with relaxation. The contrast in the medium properties only differs from zero in $\mathcal{D}^s$, and hence

$$\{\mu^s_{k,r} - \mu_{k,r}, \chi^s - \chi\} = \{0,0\} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s\prime}, \tag{7.9-1}$$

where $\mathcal{D}^{s\prime}$ is the complement of $\mathcal{D}^s \cup \partial\mathcal{D}^s$ in $\mathcal{D}$, i.e. the part of $\mathcal{D}$ that is exterior to $\mathcal{D}^s$. The *incident wave field* is denoted by

$$\{p^{i},v_r^{i}\} = \{p^{i},v_r^{i}\}(x,t) \qquad \text{for } x\in\mathcal{D}, \tag{7.9-2}$$

and is considered to be known. (Once its generating sources are given, the expressions of the type derived in Section 7.8 yield the wave-field values at any $x\in\mathcal{D}$.) The *total wave field* is denoted by

$$\{p,v_r\} = \{p,v_r\}(x,t) \qquad \text{for } x\in\mathcal{D}, \tag{7.9-3}$$

and the *scattered wave field* by

$$\{p^{s},v_r^{s}\} = \{p^{s},v_r^{s}\}(x,t) \qquad \text{for } x\in\mathcal{D}. \tag{7.9-4}$$

Then,

$$\{p,v_r\} = \{p^{i}+p^{s},v_r+v_r^{s}\} \qquad \text{for } x\in\mathcal{D}. \tag{7.9-5}$$

First, we investigate the structure of the acoustic wave equations in the domain $\mathcal{D}^s$ occupied by the scatterer. Since the sources that generate the total wave field are located in the domain exterior to the scatterer, the total wave field is source-free in $\mathcal{D}^s$, and hence

$$\partial_k p + \partial_t C_t(\mu^{s}_{k,r},v_r;x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{s}, \tag{7.9-6}$$

$$\partial_r v_r + \partial_t C_t(\chi^{s},p;x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{s}. \tag{7.9-7}$$

Since the sources that generate the total wave field would also generate the incident wave field, this part of the wave field is also source-free in $\mathcal{D}^s$, and hence

$$\partial_k p^{i} + \partial_t C_t(\mu_{k,r},v_r^{i};x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{s}, \tag{7.9-8}$$

$$\partial_r v_r^{i} + \partial_t C_t(\chi,p^{i};x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{s}. \tag{7.9-9}$$

In view of Equation (7.9-5), Equations (7.9-6)–(7.9-9) lead to equations with the scattered wave field on the left-hand side and which can alternatively be written either as

$$\partial_k p^{s} + \partial_t C_t(\mu^{s}_{k,r},v_r^{s};x,t) = -\partial_t C_t(\mu^{s}_{k,r}-\mu_{k,r},v_r^{i};x,t) \qquad \text{for } x\in\mathcal{D}^{s}, \tag{7.9-10}$$

$$\partial_r v_r^{s} + \partial_t C_t(\chi^{s},p^{s};x,t) = -\partial_t C_t(\chi^{s}-\chi,p^{i};x,t) \qquad \text{for } x\in\mathcal{D}^{s}, \tag{7.9-11}$$

or as

$$\partial_k p^{s} + \partial_t C_t(\mu_{k,r},v_r^{s};x,t) = -\partial_t C_t(\mu^{s}_{k,r}-\mu_{k,r},v_r;x,t) \qquad \text{for } x\in\mathcal{D}^{s}, \tag{7.9-12}$$

$$\partial_r v_r^{s} + \partial_t C_t(\chi,p^{s};x,t) = -\partial_t C_t(\chi^{s}-\chi,p;x,t) \qquad \text{for } x\in\mathcal{D}^{s}. \tag{7.9-13}$$

Equations (7.9-10) and (7.9-11) express that the scattered wave field in $\mathcal{D}^s$ can be envisaged as being excited through both the presence of a contrast in the medium properties and the presence of an incident wave field. If either of these two factors is absent, the scattered wave field vanishes in $\mathcal{D}^s$. This system of equations customarily serves as the starting point for the computation of the wave field via a numerical discretisation procedure applied to the pertaining differential equations (finite-difference or finite-element techniques).

Equations (7.9-12) and (7.9-13) express that the scattered wave field can be envisaged as being generated by contrast sources (with support $\mathcal{D}^s$) radiating into the embedding. This system of equations customarily serves as the starting point for the field computation via an integral equation approach. This aspect, for which we also need the acoustic wave equations that govern the wave field in $\mathcal{D}^{s\prime}$, are discussed further below. Now, in $\mathcal{D}^{s\prime}$ the scattered wave field is source-free since the (actual) total wave field and the (calculated) incident wave field are assumed to be generated by the same source distributions. Consequently (note that in $\mathcal{D}^{s\prime}$ the medium parameters are the ones of the embedding),

$$\partial_k p^{s} + C_t(\mu_{k,r}, v_r^{s}; \boldsymbol{x}, t) = 0 \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{s\prime}, \tag{7.9-14}$$

$$\partial_r v_r^{s} + C_t(\chi, p^{s}; \boldsymbol{x}, t) = 0 \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{s\prime}. \tag{7.9-15}$$

Equations (7.9-12) and (7.9-13), and Equations (7.9-14) and (7.9-15) can be combined to give

$$\partial_k p^{s} + C_t(\mu_{k,r}, v_r^{s}; \boldsymbol{x}, t) = \{f_k^{s}, 0\} \qquad \text{for } \boldsymbol{x} \in \{\mathcal{D}^{s}, \mathcal{D}^{s\prime}\}, \tag{7.9-16}$$

$$\partial_r v_r^{s} + C_t(\chi, p^{s}; \boldsymbol{x}, t) = \{q^{s}, 0\} \qquad \text{for } \boldsymbol{x} \in \{\mathcal{D}^{s}, \mathcal{D}^{s\prime}\}, \tag{7.9-17}$$

where

$$f_k^{s} = -\partial_t C_t(\mu_{k,r}^{s} - \mu_{k,r}, v_r; \boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{s} \tag{7.9-18}$$

is the *equivalent contrast volume source density of force* and

$$q^{s} = -\partial_t C_t(\chi^{s} - \chi, p; \boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{s} \tag{7.9-19}$$

is the *equivalent contrast volume source density of injection rate*. If the contrast volume source densities f_k^{s} and q^{s} were known, Equations (7.9-16) and (7.9-17) would constitute a direct (forward) source problem in the embedding of the type discussed in Section 7.8. As yet, however, these contrast source densities are unknown.

To construct a system of equations from which the scattering problem can be solved, we employ the source type integral representations for the scattered wave field (see Equations (7.8-14) and (7.8-24)), i.e.

$$p^{s}(\boldsymbol{x}', t) = \int_{\boldsymbol{x} \in \mathcal{D}^{s}} \left[C_t(G^{pq}, q^{s}; \boldsymbol{x}', \boldsymbol{x}, t) + C_t(G_k^{pf}, f_k^{s}; \boldsymbol{x}', \boldsymbol{x}, t) \right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}' \in \mathcal{D}, \tag{7.9-20}$$

$$v_r^{s}(\boldsymbol{x}', t) = \int_{\boldsymbol{x} \in \mathcal{D}^{s}} \left[C_t(G_r^{vq}, q^{s}; \boldsymbol{x}', \boldsymbol{x}, t) + C_t(G_{r,k}^{vf}, f_k^{s}; \boldsymbol{x}', \boldsymbol{x}, t) \right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}' \in \mathcal{D}, \tag{7.9-21}$$

in which the Green's functions apply to a fluid with the same acoustic properties as the embedding. Writing Equations (7.9-18) and (7.9-19) with the aid of Equation (7.9-5) as

$$f_k^{s} = -\partial_t C_t(\mu_{k,r}^{s} - \mu_{k,r}, v_r^{i} + v_r^{s}; \boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{s}, \tag{7.9-22}$$

$$q^{s} = -\partial_t C_t(\chi^{s} - \chi, p^{i} + p^{s}; \boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{s}, \tag{7.9-23}$$

and invoking Equations (7.9-20) and (7.9-21) for $\boldsymbol{x}' \in \mathcal{D}^{s}$, a system of integral equations results from which f_k^{s} and q^{s} can be solved. Once these quantities have been determined, the scattered wave field can be calculated in the entire configuration by reusing Equations (7.9-20) and (7.9-21) for all $\boldsymbol{x} \in \mathcal{D}$, and since the incident wave field was presumably known already, the total wave field follows.

Except for some simple geometries (see, for example, Friedlander (1958)), where analytic methods can be employed, the integral equations for the scattering of acoustic waves have to be solved with the aid of numerical methods. The circumstance that the Green's tensors are singular when $\boldsymbol{x}' = \boldsymbol{x}$ presents difficulties, in the sense that in the neighbourhood of $\boldsymbol{x}'$ the integrations with respect to $\boldsymbol{x}$ cannot be evaluated by means of a simple numerical formula (such as the tetrahedral formula, which is the three-dimensional equivalent of the one-dimensional trapezoidal formula), but have to be evaluated by using a limiting analytic procedure. For the rest, the application of numerical methods to the relevant integral equations presents no essential difficulties.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the acoustic properties of the embedding fluid are characterised by the functions $\{\hat{\zeta}_{k,r},\hat{\eta}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s)$. The acoustic properties of the scatterer are characterised by the functions $\{\hat{\zeta}^{\mathrm{s}}_{k,r},\hat{\eta}^{\mathrm{s}}\} = \{\hat{\zeta}^{\mathrm{s}}_{k,r},\hat{\eta}^{\mathrm{s}}\}(\boldsymbol{x},s)$. The contrast in the medium properties only differs from zero in $\mathcal{D}^{\mathrm{s}}$, and hence

$$\{\hat{\zeta}^{\mathrm{s}}_{k,r} - \hat{\zeta}_{k,r},\hat{\eta}^{\mathrm{s}} - \hat{\eta}\} = \{0,0\} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}\,\prime}, \tag{7.9-24}$$

where $\mathcal{D}^{\mathrm{s}\,\prime}$ is the complement of $\mathcal{D}^{\mathrm{s}}\cup\partial\mathcal{D}^{\mathrm{s}}$ in $\mathcal{D}$, i.e. the part of $\mathcal{D}$ that is exterior to $\mathcal{D}^{\mathrm{s}}$. The *incident wave field* is denoted by

$$\{\hat{p}^{\,\mathrm{i}},\hat{v}^{\mathrm{i}}_r\} = \{\hat{p}^{\,\mathrm{i}},\hat{v}^{\mathrm{i}}_r\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.9-25}$$

and is considered to be known. (Once its generating sources are given, the expressions of the type derived in Section 7.8 yield the wave-field values at any $\boldsymbol{x}\in\mathcal{D}$.) The *total wave field* is denoted by

$$\{\hat{p},\hat{v}_r\} = \{\hat{p},\hat{v}_r\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.9-26}$$

and the *scattered wave field* by

$$\{\hat{p}^{\,\mathrm{s}},\hat{v}^{\mathrm{s}}_r\} = \{\hat{p}^{\,\mathrm{s}},\hat{v}^{\mathrm{s}}_r\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.9-27}$$

Then,

$$\{\hat{p},\hat{v}_r\} = \{\hat{p}^{\,\mathrm{i}} + \hat{p}^{\,\mathrm{s}},\hat{v}^{\mathrm{i}}_r + \hat{v}^{\mathrm{s}}_r\} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.9-28}$$

First, we investigate the structure of the complex frequency-domain acoustic wave equations in the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. Since the sources that generate the total wave field are located in the domain exterior to the scatterer, the total wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$\partial_k\hat{p} + \hat{\zeta}^{\mathrm{s}}_{k,r}\hat{v}_r = 0 \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.9-29}$$

$$\partial_r\hat{v}_r + \hat{\eta}^{\mathrm{s}}\hat{p} = 0 \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}. \tag{7.9-30}$$

Since the sources that generate the total wave field would also generate the incident wave field, this part of the wave field is also source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$\partial_k\hat{p}^{\,\mathrm{i}} + \hat{\zeta}_{k,r}\hat{v}^{\,\mathrm{i}} = 0 \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.9-31}$$

$$\partial_r\hat{v}^{\,\mathrm{i}} + \hat{\eta}\hat{p}^{\,\mathrm{i}} = 0 \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}. \tag{7.9-32}$$

In view of Equation (7.9-28), Equations (7.9-29)–(7.9-32) lead to equations with the scattered wave field on the left-hand side, which can be written either as

$$\partial_k\hat{p}^{\,\mathrm{s}} + \hat{\zeta}^{\mathrm{s}}_{k,r}\hat{v}^{\mathrm{s}}_r = -(\hat{\zeta}^{\mathrm{s}}_{k,r} - \hat{\zeta}_{k,r})\hat{v}^{\mathrm{i}}_r \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.9-33}$$

$$\partial_r\hat{v}^{\mathrm{s}}_r + \hat{\eta}^{\mathrm{s}}\hat{p}^{\,\mathrm{s}} = -(\hat{\eta}^{\mathrm{s}} - \hat{\eta})\hat{p}^{\,\mathrm{i}} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.9-34}$$

or as

$$\partial_k \hat{p}^{\mathrm{s}} + \hat{\zeta}_{k,r} \hat{v}_r^{\mathrm{s}} = -(\hat{\zeta}_{k,r}^{\mathrm{s}} - \hat{\zeta}_{k,r})\hat{v}_r \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}}, \tag{7.9-35}$$

$$\partial_r \hat{v}_r^{\mathrm{s}} + \hat{\eta} \hat{p}^{\mathrm{s}} = -(\hat{\eta}^{\mathrm{s}} - \hat{\eta})\hat{p} \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}}. \tag{7.9-36}$$

Equations (7.9-33) and (7.9-34) express the fact that the scattered wave field in $\mathcal{D}^{\mathrm{s}}$ can be envisaged as being excited through both the presence of a contrast in the medium properties and the presence of an incident wave field. If either of these two factors is absent, the scattered wave field vanishes in $\mathcal{D}^{\mathrm{s}}$. This system of equations customarily serves as the starting point for computation of the wave field via a numerical discretisation procedure applied to the pertaining differential equations (finite-difference or finite-element techniques).

Equations (7.9-35) and (7.9-36) express the fact that the scattered wave field can be envisaged as being generated by contrast sources (with support $\mathcal{D}^{\mathrm{s}}$) radiating into the embedding. This system of equations customarily serves as the starting point for computation of the wave field via an integral equation approach. This aspect, for which we also need the acoustic wave equations that govern the wave field in $\mathcal{D}^{\mathrm{s}\prime}$, are discussed further below. Now, in $\mathcal{D}^{\mathrm{s}\prime}$ the scattered wave field is source-free since the (actual) total wave field and the (calculated) incident wave field are assumed to be generated by the same source distributions. Consequently (note that in $\mathcal{D}^{\mathrm{s}\prime}$ the medium parameters are the ones of the embedding),

$$\partial_k \hat{p}^{\mathrm{s}} + \hat{\zeta}_{k,r} \hat{v}_r^{\mathrm{s}} = 0 \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}\prime}, \tag{7.9-37}$$

$$\partial_r \hat{v}_r^{\mathrm{s}} + \hat{\eta} \hat{p}^{\mathrm{s}} = 0 \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}\prime}. \tag{7.9-38}$$

Equations (7.9-35) and (7.9-36), and Equations (7.9-37) and (7.9-38) can be combined to give

$$\partial_k \hat{p}^{\mathrm{s}} + \hat{\zeta}_{k,r} \hat{v}_r^{\mathrm{s}} = \{\hat{f}_k^{\mathrm{s}}, 0\} \qquad \text{for } \boldsymbol{x} \in \{\mathcal{D}^{\mathrm{s}}, \mathcal{D}^{\mathrm{s}\prime}\}, \tag{7.9-39}$$

$$\partial_r \hat{v}_r^{\mathrm{s}} + \hat{\eta} \hat{p}^{\mathrm{s}} = \{\hat{q}^{\mathrm{s}}, 0\} \qquad \text{for } \boldsymbol{x} \in \{\mathcal{D}^{\mathrm{s}}, \mathcal{D}^{\mathrm{s}\prime}\}, \tag{7.9-40}$$

where

$$\hat{f}_k^{\mathrm{s}} = -(\hat{\zeta}_{k,r}^{\mathrm{s}} - \hat{\zeta}_{k,r})\hat{v}_r \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}} \tag{7.9-41}$$

is the *equivalent contrast volume source density of force* and

$$\hat{q}^{\mathrm{s}} = -(\hat{\eta}^{\mathrm{s}} - \hat{\eta})\hat{p} \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}} \tag{7.9-42}$$

is the *equivalent contrast volume source density of injection rate*. If the contrast source densities $\hat{f}_k^{\mathrm{s}}$ and $\hat{q}^{\mathrm{s}}$ were known, Equations (7.9-39) and (7.9-40) would constitute a direct (forward) source problem in the embedding of the type discussed in Section 7.8. As yet, however, these contrast source densities are unknown.

To construct a system of equations from which the scattering problem can be solved, we employ the source type integral representations for the scattered wave field (see Equations (7.8-38) and (7.8-48)), i.e.

$$\hat{p}^{\mathrm{s}}(\boldsymbol{x}', s) = \int_{\boldsymbol{x} \in \mathcal{D}^{\mathrm{s}}} \left[\hat{G}^{pq}(\boldsymbol{x}', \boldsymbol{x}, s)\hat{q}^{\mathrm{s}}(\boldsymbol{x}, s) + \hat{G}_k^{pf}(\boldsymbol{x}', \boldsymbol{x}, s)\, \hat{f}_k^{\mathrm{s}}(\boldsymbol{x}, s) \right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}' \in \mathcal{D}, \tag{7.9-43}$$

$$\hat{v}_r^{\mathrm{s}}(\boldsymbol{x}', s) = \int_{\boldsymbol{x} \in \mathcal{D}^{\mathrm{s}}} \left[\hat{G}_r^{vq}(\boldsymbol{x}', \boldsymbol{x}, s)\hat{q}^{\mathrm{s}}(\boldsymbol{x}, s) + \hat{G}_{r,k}^{vf}(\boldsymbol{x}', \boldsymbol{x}, s)\, \hat{f}_k^{\mathrm{s}}(\boldsymbol{x}, s) \right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}' \in \mathcal{D}, \tag{7.9-44}$$

in which the Green's functions apply to a fluid with the same acoustic properties as the embedding. Writing Equations (7.9-41) and (7.9-42) with the aid of Equation (7.9-28) as

$$\hat{f}_k^{\,\mathrm{s}} = -(\hat{\xi}^{\,\mathrm{s}}_{k,r} - \hat{\xi}_{k,r})(\hat{v}_r^{\,\mathrm{i}} + \hat{v}_r^{\,\mathrm{s}}) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.9-45}$$

$$\hat{q}^{\,\mathrm{s}} = -(\hat{\eta}^{\,\mathrm{s}} - \hat{\eta})(\hat{p}^{\,\mathrm{i}} + \hat{p}^{\,\mathrm{s}}) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.9-46}$$

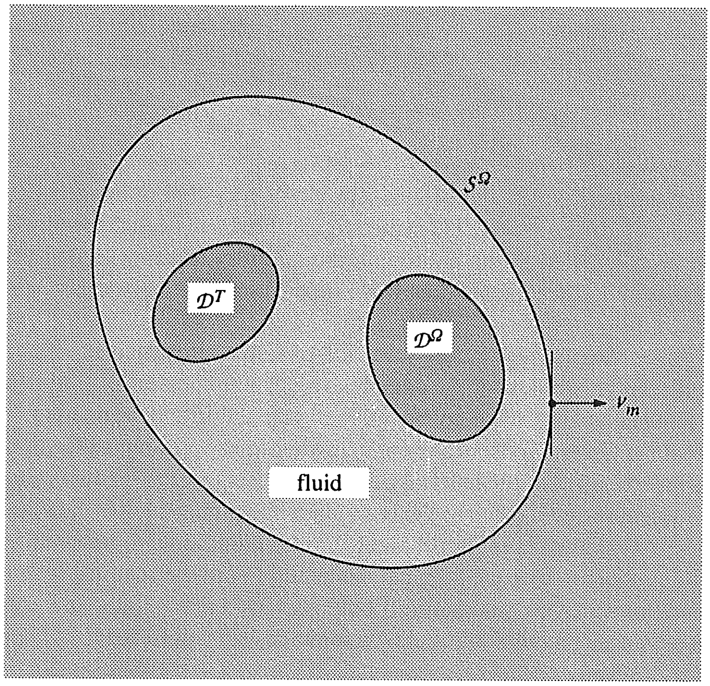
and invoking Equations (7.9-43) and (7.9-44) for $\boldsymbol{x}'\in\mathcal{D}^{\mathrm{s}}$, a system of integral equations results from which $\hat{f}_k^{\,\mathrm{s}}$ and $\hat{q}^{\,\mathrm{s}}$ can be solved. Once these quantities have been determined, the scattered wave field can be calculated in the entire configuration by re-using Equations (7.9-43) and (7.9-44) for all $\boldsymbol{x}\in\mathcal{D}$ and, since the incident wave field was presumably known already, the total wave field follows.

Except for some simple geometrics (see, for example, Bowman *et al.*, (1969)), where analytic methods can be employed, the complex frequency-domain integral equations for the scattering of acoustic waves have to be solved with the aid of numerical methods. The circumstance that the Green's tensors are singular when $\boldsymbol{x}'=\boldsymbol{x}$ presents difficulties, in the sense that in the neighbourhood of $\boldsymbol{x}'$ the integrations with respect to $\boldsymbol{x}$ cannot be evaluated by means of a simple numerical formula (such as the tetrahedral formula, which is the three-dimensional equivalent of the one-dimensional trapezoidal formula), but have to be evaluated by using a limiting analytic procedure. For the rest, the application of numerical methods to the relevant integral equations presents no essential difficulties. A useful reference is Poggio and Miller (1973). Recent advances on this subject can be found in Van den Berg (1991), and in Fokkema and Van den Berg (1993).

7.10 The inverse source problem

The configuration in an acoustic inverse source problem generally consists of a background fluid with known acoustic properties occupying the domain $\mathcal{D}$ (the "*embedding*") in which, in principle, the radiation from given, arbitrarily distributed acoustic sources can be calculated with the aid of the theory developed in Section 7.8. In the embedding a known or guessed bounded domain $\mathcal{D}^T$ is present in which acoustically radiating sources of unknown nature and unknown spatial distribution are present. The presence of these sources manifests itself in the entire embedding. In some bounded subdomain $\mathcal{D}^{\Omega}$ of $\mathcal{D}$, and exterior to $\mathcal{D}^T$, the radiated acoustic wave field is accessible to measurement (Figure 7.10-1).

We assume that the action of the radiating sources can be modelled by volume source densities of injection rate and force. The objective is to reconstruct these volume source densities with support $\mathcal{D}^T$ from (a set of) measured values of the acoustic pressure and/or the particle velocity in $\mathcal{D}^{\Omega}$. Since the inverse source problem is, by necessity, a remote sensing problem, the global reciprocity theorems given in Sections 7.2–7.5 can be expected to provide a means for interrelating the known, measured wave-field data with the unknown source distributions. In case the embedding $\mathcal{D}$ is a bounded domain, the boundary surface $\partial\mathcal{D}$ is assumed to be acoustically impenetrable. If $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. The radiated wave field is, by its nature, causally related to the sources by which it is generated. For gathering maximum information, the reciprocity theorems are applied to the domain interior to a closed surface $\mathcal{S}^{\Omega}$ that completely surrounds both $\mathcal{D}^T$ and $\mathcal{D}^{\Omega}$. If necessary, measurement on $\mathcal{S}^{\Omega}$ can also be carried out.



7.10-1 Configuration of the inverse source problem: $\mathcal{D}^T$ is the support of the unknown radiating sources; on $\mathcal{D}^{\Omega}$ and $\mathcal{S}^{\Omega}$ the transmitted wave field is accessible to measurement.

Time-domain analysis

In the time-domain analysis of the problem, the acoustic properties of the embedding fluid are characterised by the relaxation functions $\{\mu_{k,r},\chi\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t)$, which are causal functions of time. The causally radiated acoustic wave field is denoted by $\{p^{\mathrm{T}},v_r^{\mathrm{T}}\} = \{p^{\mathrm{T}},v_r^{\mathrm{T}}\}(\boldsymbol{x},t)$.

First, the measured acoustic wave-field data are interrelated with the unknown source distributions $\{q^{\mathrm{T}},f_k^{\mathrm{T}}\} = \{q^{\mathrm{T}},f_k^{\mathrm{T}}\}(\boldsymbol{x},t)$, via the global time-domain reciprocity theorem of the *convolution type* (Equation (7.2-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{p^{\mathrm{A}},v_r^{\mathrm{A}}\}(\boldsymbol{x},t) = \{p^{\mathrm{T}},v_r^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.10-1}$$

$$\{q^{\mathrm{A}},f_k^{\mathrm{A}}\}(\boldsymbol{x},t) = \{q^{\mathrm{T}},f_k^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{T}, \tag{7.10-2}$$

and

$$\{\mu_{k,r}^{\mathrm{A}},\chi^{\mathrm{A}}\}(\boldsymbol{x},t) = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.10-3}$$

For state B, we take a "computational" or "observational" state; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{p^{\mathrm{B}},v_k^{\mathrm{B}}\}(\boldsymbol{x},t) = \{p^{\Omega},v_k^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.10-4}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{q^{\mathrm{B}}, f_r^{\mathrm{B}}\}(\boldsymbol{x},t) = \{q^{\Omega}, f_r^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{7.10-5}$$

Furthermore, the fluid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\mu_{r,k}^{\mathrm{B}}, \chi^{\mathrm{B}}\}(\boldsymbol{x},t) = \{\mu_{k,r}, \chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.10-6}$$

Then, application of Equation (7.2-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\mathrm{C}_t(p^{\Omega}, q^{\mathrm{T}}; \boldsymbol{x},t) - \mathrm{C}_t(v_k^{\Omega}, f_k^{\mathrm{T}}; \boldsymbol{x},t)\right]\mathrm{d}V \\
&= \int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_t(p^{\mathrm{T}}, q^{\Omega}; \boldsymbol{x},t) - \mathrm{C}_t(v_r^{\mathrm{T}}, f_r^{\Omega}; \boldsymbol{x},t)\right]\mathrm{d}V \\
&+ \int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m\left[-\mathrm{C}_t(p^{\mathrm{T}}, v_m^{\Omega}; \boldsymbol{x},t) + \mathrm{C}_t(v_m^{\mathrm{T}}, p^{\Omega}; \boldsymbol{x},t)\right]\mathrm{d}A\,.
\end{aligned} \tag{7.10-7}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state T and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the integral over $\partial\mathcal{D}$ does and in between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ no sources of the radiated or the computational wave fields are present (see Exercise 7.2-2). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are also causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (because the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are not causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.2-2).

Secondly, the measured acoustic wave-field data are interrelated with the unknown source distributions $\{q^{\mathrm{T}}, f_k^{\mathrm{T}}\} = \{q^{\mathrm{T}}, f_k^{\mathrm{T}}\}(\boldsymbol{x},t)$, via the global time-domain reciprocity theorem of the *correlation type* (Equation (7.3-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{p^{\mathrm{A}}, v_r^{\mathrm{A}}\}(\boldsymbol{x},t) = \{p^{\mathrm{T}}, v_r^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.10-8}$$

$$\{q^{\mathrm{A}}, f_k^{\mathrm{A}}\}(\boldsymbol{x},t) = \{q^{\mathrm{T}}, f_k^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^T, \tag{7.10-9}$$

and

$$\{\mu_{k,r}^{\mathrm{A}}, \chi^{\mathrm{A}}\}(\boldsymbol{x},t) = \{\mu_{k,r}, \chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.10-10}$$

For state B, we take a "computational" or "observational" state; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{p^{\mathrm{B}}, v_k^{\mathrm{B}}\}(\boldsymbol{x},t) = \{p^{\Omega}, v_k^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.10-11}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{q^{\mathrm{B}},f_r^{\mathrm{B}}\}(\boldsymbol{x},t) = \{q^{\Omega},f_r^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{7.10-12}$$

Furthermore, the fluid properties in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\mu_{r,k}^{\mathrm{B}},\chi^{\mathrm{B}}\}(\boldsymbol{x},t) = \{\mathrm{J}_t(\mu_{k,r}),\mathrm{J}_t(\chi)\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.10-13}$$

Then, application of Equation (7.3-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{\boldsymbol{x}\in\mathcal{D}^{T}} \left[\mathrm{C}_t(\mathrm{J}_t(p^{\Omega}),q^{\mathrm{T}};\boldsymbol{x},t) + \mathrm{C}_t(\mathrm{J}_t(v_k^{\Omega}),f_k^{\mathrm{T}};\boldsymbol{x},t)\right]\mathrm{d}V$$

$$= -\int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[\mathrm{C}_t(p^{\mathrm{T}},\mathrm{J}_t(q^{\Omega});\boldsymbol{x},t) + \mathrm{C}_t(v_r^{\mathrm{T}},\mathrm{J}_t(f_r^{\Omega});\boldsymbol{x},t)\right]\mathrm{d}V$$

$$+ \int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{T}},\mathrm{J}_t(v_m^{\Omega});\boldsymbol{x},t) + \mathrm{C}_t(v_m^{\mathrm{T}},\mathrm{J}_t(p^{\Omega});\boldsymbol{x},t)\right]\mathrm{d}A\,. \tag{7.10-14}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.3-2)). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are non-causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.3-2). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are causal and the surface integral over $\mathcal{S}^{\Omega}$ does vanish (since the integral over a sphere with an infinitely large radius vanishes, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). For additional literature on the subject, see De Hoop (1988).

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the acoustic properties of the embedding fluid are characterised by the functions $\{\hat{\zeta}_{k,r},\hat{\eta}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s)$. The causally radiated acoustic wave field is denoted by $\{\hat{p}^{\mathrm{T}},\hat{v}_r^{\mathrm{T}}\} = \{\hat{p}^{\mathrm{T}},\hat{v}_r^{\mathrm{T}}\}(\boldsymbol{x},s)$.

First, the measured acoustic wave-field data are interrelated with the unknown source distributions $\{\hat{q}^{\mathrm{T}},\hat{f}_k^{\mathrm{T}}\} = \{\hat{q}^{\mathrm{T}},\hat{f}_k^{\mathrm{T}}\}(\boldsymbol{x},s)$, via the global complex frequency-domain reciprocity theorem of the *time convolution type* (Equation (7.4-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(x,s) = \{\hat{p}^{\mathrm{T}},\hat{v}_r^{\mathrm{T}}\}(x,s) \qquad \text{for } x \in \mathcal{D}, \tag{7.10-15}$$

$$\{\hat{q}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\}(x,s) = \{\hat{q}^{\mathrm{T}},\hat{f}_k^{\mathrm{T}}\}(x,s) \qquad \text{for } x \in \mathcal{D}^T, \tag{7.10-16}$$

and

$$\{\hat{\xi}_{k,r}^{\mathrm{A}},\hat{\eta}^{\mathrm{A}}\}(x,s) = \{\hat{\xi}_{k,r},\hat{\eta}\}(x,s) \qquad \text{for } x \in \mathcal{D}. \tag{7.10-17}$$

For state B, we take a "computational" or "observational" state; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(x,s) = \{\hat{p}^{\Omega},\hat{v}_k^{\Omega}\}(x,s) \qquad \text{for } x \in \mathcal{D}, \tag{7.10-18}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{q}^{\mathrm{B}},\hat{f}_r^{\mathrm{B}}\}(x,s) = \{\hat{q}^{\Omega},\hat{f}_r^{\Omega}\}(x,s) \qquad \text{for } x \in \mathcal{D}^{\Omega}. \tag{7.10-19}$$

Furthermore, the fluid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\hat{\xi}_{r,k}^{\mathrm{B}},\hat{\eta}^{\mathrm{B}}\}(x,s) = \{\hat{\xi}_{k,r},\hat{\eta}\}(x,s) \qquad \text{for } x \in \mathcal{D}. \tag{7.10-20}$$

Then, application of Equation (7.4-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{x\in\mathcal{D}^T} \left[\hat{p}^{\Omega}(x,s)\hat{q}^{\mathrm{T}}(x,s) - \hat{v}_k^{\Omega}(x,s)\,\hat{f}_k^{\mathrm{T}}(x,s)\right]\mathrm{d}V \\
&= \int_{x\in\mathcal{D}^{\Omega}} \left[\hat{p}^{\mathrm{T}}(x,s)\hat{q}^{\Omega}(x,s) - \hat{v}_r^{\mathrm{T}}(x,s)\,\hat{f}_r^{\Omega}(x,s)\right]\mathrm{d}V \\
&\quad + \int_{x\in\mathcal{S}^{\Omega}} \nu_m\left[-\hat{p}^{\mathrm{T}}(x,s)\hat{v}_m^{\Omega}(x,s) + \hat{v}_m^{\mathrm{T}}(x,s)\hat{p}^{\Omega}(x,s)\right]\mathrm{d}A\,.
\end{aligned} \tag{7.10-21}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state T and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over $\partial\mathcal{D}$ does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.4-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.4-4).

Secondly, the measured acoustic wave-field data are interrelated with the unknown source distributions $\{\hat{q}^{\mathrm{T}},\hat{f}_k^{\mathrm{T}}\} = \{\hat{q}^{\mathrm{T}},\hat{f}_k^{\mathrm{T}}\}(x,s)$, via the global time-domain reciprocity theorem of the *time correlation type* (Equation (7.5-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\hat{p}^{A}, \hat{v}_r^{A}\}(x,s) = \{\hat{p}^{T}, \hat{v}_r^{T}\}(x,s) \quad \text{for } x \in \mathcal{D}, \tag{7.10-22}$$

$$\{\hat{q}^{A}, \hat{f}_k^{A}\}(x,s) = \{\hat{q}^{T}, \hat{f}_k^{T}\}(x,s) \quad \text{for } x \in \mathcal{D}^{T}, \tag{7.10-23}$$

and

$$\{\hat{\zeta}_{k,r}^{A}, \hat{\eta}^{A}\}(x,s) = \{\hat{\zeta}_{k,r}, \hat{\eta}\}(x,s) \quad \text{for } x \in \mathcal{D}. \tag{7.10-24}$$

For state B, we take a "computational" or "observational" state; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{p}^{B}, \hat{v}_k^{B}\}(x,s) = \{\hat{p}^{\Omega}, \hat{v}_k^{\Omega}\}(x,s) \quad \text{for } x \in \mathcal{D}, \tag{7.10-25}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{q}^{B}, \hat{f}_r^{B}\}(x,s) = \{\hat{q}^{\Omega}, \hat{f}_r^{\Omega}\}(x,s) \quad \text{for } x \in \mathcal{D}^{\Omega}. \tag{7.10-26}$$

Furthermore, the fluid properties in state B will be taken to be the time-reverse adjoint of the ones of state A, i.e.

$$\{\hat{\zeta}_{r,k}^{B}, \hat{\eta}^{B}\}(x,s) = \{-\hat{\zeta}_{k,r}, -\hat{\eta}\}(x,-s) \quad \text{for } x \in \mathcal{D}. \tag{7.10-27}$$

Then, application of Equation (7.5-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{x \in \mathcal{D}^{T}} \left[\hat{p}^{\Omega}(x,-s)\hat{q}^{T}(x,s) + \hat{v}_k^{\Omega}(x,-s)\hat{f}_k^{T}(x,s)\right] \mathrm{d}V$$
$$= -\int_{x \in \mathcal{D}^{\Omega}} \left[\hat{p}^{T}(x,s)\hat{q}^{\Omega}(x,-s) + \hat{v}_r^{T}(x,s)\hat{f}_r^{\Omega}(x,-s)\right] \mathrm{d}V$$
$$+ \int_{x \in \mathcal{S}^{\Omega}} \nu_m \left[\hat{p}^{T}(x,s)\hat{v}_m^{\Omega}(x,-s) + \hat{v}_m^{T}(x,s)\hat{p}^{\Omega}(x,-s)\right] \mathrm{d}A . \tag{7.10-28}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state T and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over $\partial\mathcal{D}$ does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.5-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.5-4). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere).

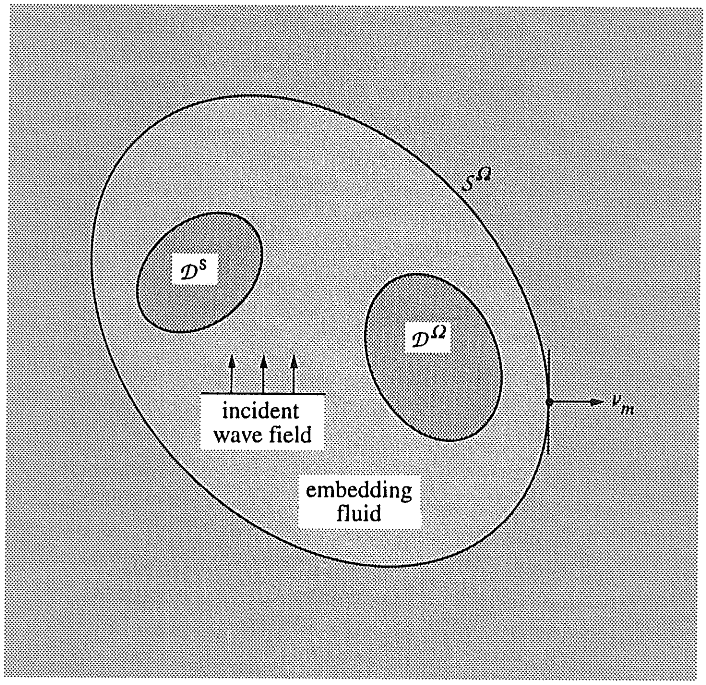
A solution to the inverse source problem is commonly constructed as follows. For the source distributions in the computational state Ω we take a sequence of M linearly independent spatial distributions with the common spatial support $\mathcal{D}^{\Omega}$. The corresponding sequence of acoustic wave-field distributions (in the medium adjoint, or time-reverse adjoint of the actual one) is computed. Next, the unknown source distributions are expanded into an appropriate sequence of N expansion functions with the common spatial support $\mathcal{D}^T$ or a subset of it; the corresponding expansion coefficients are unknown. Substitution of the results in Equations (7.10-7), (7.10-14), (7.10-21) or (7.10-28) and evaluation of the relevant integrals lead to a system of M linear algebraic equations with N unknowns. When $M < N$, the system is underdetermined and cannot be solved. When $M = N$, the system can be solved, unless the pertaining matrix of coefficients is singular. However, even if this matrix is non-singular, in most practical cases it turns out to be ill-conditioned. Therefore, one usually takes $M > N$, and a best fit of the expanded source distributions to the measured data is obtained by the application of minimisation techniques (for example, least-squares minimisation). Note that each of the Equations (7.10-7), (7.10-14), (7.10-21) and (7.10-28) leads to an associated inversion algorithm.

The computational state Ω is representative for the manner in which the measured data are processed in the inversion algorithms. Since a computational state does not have to meet the physical condition of causality, there is no objection against its being anti-causal. Which of the two possibilities (causal or anti-causal) leads to the best results as far as accuracy and amount of computational effort are concerned, is difficult to judge. Research on this aspect is still in full progress (see Fokkema and Van den Berg, 1993). It is to be noted that the solution to an inverse source problem is not unique because of the existence of non-radiating source distributions (i.e. non-zero source distributions with support $\mathcal{D}^T$ that yield a vanishing wave field in the domain exterior to $\mathcal{D}^T$). Therefore, a numerically constructed solution to an inverse source problem is always only *a solution* to the problem (and not *the solution*); which solution is obtained depends on the solution method employed. Examples of inverse source problems are found in the detection of (spontaneous) acoustic emission and the detection of industrial noise sources.

7.11 The inverse scattering problem

The configuration in an acoustic inverse scattering problem generally consists of a background fluid with known acoustic properties, occupying the domain $\mathcal{D}$ (the "*embedding*") in which, in principle, the radiation from given, arbitrarily distributed acoustic sources can be calculated with the aid of the theory developed in Section 7.8. In the embedding a known or guessed bounded domain $\mathcal{D}^s$ (the "scatterer") is present in which the fluid properties show an unknown contrast with the ones of the embedding. The contrasting domain is irradiated by an incident acoustic wave field that is generated by sources in some subdomain $\mathcal{D}^i$ of $\mathcal{D}$ and that propagates in the embedding. The presence of the contrasting domain manifests itself through the presence of a non-vanishing scattered wave field in the entire embedding. In some bounded subdomain $\mathcal{D}^\Omega$ of $\mathcal{D}$, and exterior to $\mathcal{D}^s$, the scattered acoustic wave field is accessible to measurement (Figure 7.11-1).

The objective is to reconstruct the medium parameters (or their contrasts with the ones of the embedding) from (a set of) measured values of the acoustic pressure and/or the particle velocity in $\mathcal{D}^\Omega$. Since the inverse scattering problem is, by necessity, a remote sensing problem, the global reciprocity theorems of Sections 7.2–7.5 can be expected to provide a means for interrelating the known, measured wave-field data with the unknown medium properties in the scattering region. When the embedding $\mathcal{D}$ is a bounded domain, the boundary surface is assumed to be acoustically impenetrable. If $\mathcal{D}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain are made. The scattered wave field is, by its nature,



7.11-1 Configuration of the inverse scattering problem: $\mathcal{D}^s$ is the support of the unknown contrast in fluid properties; on $\mathcal{D}^{\Omega}$ and $\mathcal{S}^{\Omega}$ the scattered wave field is accessible to measurement.

causally related to the contrast sources by which it is generated. For gathering maximum information, the reciprocity theorems are applied to the domain interior to a closed surface $\mathcal{S}^{\Omega}$ that completely surrounds both $\mathcal{D}^s$ and $\mathcal{D}^{\Omega}$. If necessary, measurements on $\mathcal{S}^{\Omega}$ can also be carried out. In general, $\mathcal{D}^s$ and $\mathcal{D}^i$ are disjoint, as are $\mathcal{D}^s$ and $\mathcal{D}^{\Omega}$. This need not be the case for $\mathcal{D}^i$ and $\mathcal{D}^{\Omega}$; these domains may have a non-empty cross-section.

The incident, scattered and total wave fields are introduced as in Section 7.9. Now, the easiest way to address the inverse scattering problem is to consider it to be partly an inverse source problem, with the contrast volume source densities as the unknowns. The non-uniqueness of these quantities is removed by invoking the remaining consistency relations of a constitutive nature. In the latter, the condition that the reconstructed contrast-in-medium parameters must be independent of the incident wave field plays a crucial role. Once the contrast volume source densities have been determined, the scattered wave field is (following the procedures of Section 7.9), calculated in the domain $\mathcal{D}^s$ and, since the incident wave field and the medium parameters of the embedding are known, the parameters of the fluid in $\mathcal{D}^s$ follow.

Time-domain analysis

In the time-domain analysis of the problem, the acoustic properties of the embedding fluid are characterised by the relaxation functions $\{\mu_{k,r},\chi\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t)$, which are causal functions of time. The case of an instantaneously reacting embedding easily follows from the more general case of an embedding fluid with relaxation. The unknown acoustic properties of the scatterer are characterised by the relaxation functions $\{\mu^{s}_{k,r},\chi^{s}\} = \{\mu^{s}_{k,r},\chi^{s}\}(\boldsymbol{x},t)$, which are also causal functions of time. The incident wave field is $\{p^{i},v^{i}_{r}\} = \{p^{i},v^{i}_{r}\}(\boldsymbol{x},t)$, the scattered wave field is $\{p^{s},v^{s}_{r}\} = \{p^{s},v^{s}_{r}\}(\boldsymbol{x},t)$, and the total wave field is $\{p,v_{r}\} = \{p,v_{r}\}(\boldsymbol{x},t)$ with $\{p,v_{r}\} = \{p^{i}+p^{s}, v^{i}_{r}+v^{s}_{r}\}$. The equivalent contrast volume source distributions that generate the scattered wave field are then (see Equations (7.9-18) and (7.9-19))

$$f^{s}_{k} = -\mathrm{C}_{t}(\mu^{s}_{k,r}-\mu_{k,r},v_{r};\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}, \tag{7.11-1}$$

$$q^{s} = -\mathrm{C}_{t}(\chi^{s}-\chi,p;\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}. \tag{7.11-2}$$

First, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{q^{s},f^{s}_{k}\}$, via the global time-domain reciprocity theorem of the *convolution type* (Equation (7.2-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{p^{A},v^{A}_{r}\}(\boldsymbol{x},t) = \{p^{s},v^{s}_{r}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-3}$$

$$\{q^{A},f^{A}_{k}\}(\boldsymbol{x},t) = \{q^{s},f^{s}_{k}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}, \tag{7.11-4}$$

and

$$\{\mu^{A}_{k,r},\chi^{A}\}(\boldsymbol{x},t) = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-5}$$

For state B, we take a "computational" or "observational" state; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{p^{B},v^{B}_{k}\}(\boldsymbol{x},t) = \{p^{\Omega},v^{\Omega}_{k}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-6}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{q^{B},f^{B}_{r}\}(\boldsymbol{x},t) = \{q^{\Omega},f^{\Omega}_{r}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{7.11-7}$$

Furthermore, the fluid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\mu^{B}_{r,k},\chi^{B}\}(\boldsymbol{x},t) = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-8}$$

Then, application of Equation (7.2-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{s}}\left[\mathrm{C}_{t}(p^{\Omega},q^{s};\boldsymbol{x},t)-\mathrm{C}_{t}(v^{\Omega}_{k},f^{s}_{k};\boldsymbol{x},t)\right]\mathrm{d}V\\
&=\int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_{t}(p^{s},q^{\Omega};\boldsymbol{x},t)-\mathrm{C}_{t}(v^{s}_{r},f^{\Omega}_{r};\boldsymbol{x},t)\right]\mathrm{d}V\\
&\quad+\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}}\nu_{m}\left[-\mathrm{C}_{t}(p^{s},v^{\Omega}_{m};\boldsymbol{x},t)+\mathrm{C}_{t}(v^{s}_{m},p^{\Omega};\boldsymbol{x},t)\right]\mathrm{d}A\,.
\end{aligned} \tag{7.11-9}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state s and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over $\partial\mathcal{D}$ does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.2-2)). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are also causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are not causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.2-2).

Secondly, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{q^{\mathrm{s}}, f_k^{\mathrm{s}}\}$, via the global time-domain reciprocity theorem of the *correlation type* (Equation (7.3-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{p^{\mathrm{A}}, v_r^{\mathrm{A}}\}(\boldsymbol{x},t) = \{p^{\mathrm{s}}, v_r^{\mathrm{s}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-10}$$

$$\{q^{\mathrm{A}}, f_k^{\mathrm{A}}\}(\boldsymbol{x},t) = \{q^{\mathrm{s}}, f_k^{\mathrm{s}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.11-11}$$

and

$$\{\mu_{k,r}^{\mathrm{A}}, \chi^{\mathrm{A}}\}(\boldsymbol{x},t) = \{\mu_{k,r}, \chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-12}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{p^{\mathrm{B}}, v_k^{\mathrm{B}}\}(\boldsymbol{x},t) = \{p^{\Omega}, v_k^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-13}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{q^{\mathrm{B}}, f_r^{\mathrm{B}}\}(\boldsymbol{x},t) = \{q^{\Omega}, f_r^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{7.11-14}$$

Furthermore, the fluid properties in state B will be taken to be the time-reverse adjoint of the ones of state A, i.e.

$$\{\mu_{r,k}^{\mathrm{B}}, \chi^{\mathrm{B}}\}(\boldsymbol{x},t) = \{\mathrm{J}_t(\mu_{k,r}), \mathrm{J}_t(\chi)\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-15}$$

Then, application of Equation (7.3-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(\mathrm{J}_t(p^{\Omega}), q^{\mathrm{s}}; \boldsymbol{x},t) + \mathrm{C}_t(\mathrm{J}_t(v_k^{\Omega}), f_k^{\mathrm{s}}; \boldsymbol{x},t)\right] \mathrm{d}V \\
&= -\int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[\mathrm{C}_t(p^{\mathrm{s}}, \mathrm{J}_t(q^{\Omega}); \boldsymbol{x},t) + \mathrm{C}_t(v_r^{\mathrm{s}}, \mathrm{J}_t(f_r^{\Omega}); \boldsymbol{x},t)\right] \mathrm{d}V \\
&\quad + \int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m \left[\mathrm{C}_t(p^{\mathrm{s}}, \mathrm{J}_t(v_m^{\Omega}); \boldsymbol{x},t) + \mathrm{C}_t(v_m^{\mathrm{s}}, \mathrm{J}_t(p^{\Omega}); \boldsymbol{x},t)\right] \mathrm{d}A\,.
\end{aligned} \tag{7.11-16}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.3-2). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. In case the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are non-causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.3-2). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere).

A solution to the time-domain inverse scattering problem is commonly constructed as follows. First, the contrast-in-medium parameters are discretised by writing them as linear combinations of M expansion functions with unknown expansion coefficients. Each of the expansion functions has $\mathcal{D}^{\rm s}$, or a subset of $\mathcal{D}^{\rm s}$, as its support. Next, for each given incident wave field, the scattered wave field is measured in N subdomains of $\mathcal{D}^{\Omega}$. The latter discretisation induces the choice of the N supports of the source distributions of the observational state. Finally, a number I of different incident wave fields is selected (where "different" may involve different choices in temporal behaviour, in location space, or in both). With $NI \geqslant M$, the non-linear problem of evaluating the M expansion coefficients of the contrast-in-medium parameter discretisation is solved by some iterative procedure (for example, by iterative minimisation of the global error over all domains in space and all time intervals involved where equality signs should hold pointwise). In this procedure, Equations (7.11-1), (7.11-2), (7.11-9) or (7.11-16), and the source type integral representations (Equations (7.9-20) and (7.9-21)) are used simultaneously.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the acoustic properties of the embedding fluid are characterised by the relaxation functions $\{\hat{\zeta}_{k,r},\hat{\eta}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s)$. The unknown acoustic properties of the scatterer are characterised by the relaxation functions $\{\hat{\zeta}^{\rm s}_{k,r},\hat{\eta}^{\rm s}\} = \{\hat{\zeta}^{\rm s}_{k,r},\hat{\eta}^{\rm s}\}(\boldsymbol{x},s)$. The incident wave field is $\{\hat{p}^{\rm i},\hat{v}^{\rm i}_r\} = \{\hat{p}^{\rm i},\hat{v}^{\rm i}_r\}(\boldsymbol{x},s)$, the scattered wave field is $\{\hat{p}^{\rm s},\hat{v}^{\rm s}_r\} = \{\hat{p}^{\rm s},\hat{v}^{\rm s}_r\}(\boldsymbol{x},s)$ and the total wave field is $\{\hat{p},\hat{v}_r\} = \{\hat{p},\hat{v}_r\}(\boldsymbol{x},s)$, with $\{\hat{p},\hat{v}_r\} = \{\hat{p}^{\rm i}+\hat{p}^{\rm s},\hat{v}^{\rm i}_r+\hat{v}^{\rm s}_r\}$. The equivalent contrast volume source distributions that generate the scattered wave field are then (see Equations (7.9-41) and (7.9-42))

$$\hat{f}^{\rm s}_k = -(\hat{\zeta}^{\rm s}_{k,r} - \hat{\zeta}_{k,r})\hat{v}_r \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\rm s}, \tag{7.11-17}$$

$$\hat{q}^{\rm s} = -(\hat{\eta}^{\rm s} - \hat{\eta})\hat{p} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\rm s}. \tag{7.11-18}$$

First, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{\hat{q}^{\rm s},\hat{f}^{\rm s}_k\}$, via the global complex frequency-domain reciprocity theorem of the *time convolution type* (Equation (7.4-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(\boldsymbol{x},s) = \{\hat{p}^{\mathrm{s}},\hat{v}_r^{\mathrm{s}}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-19}$$

$$\{\hat{q}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\}(\boldsymbol{x},s) = \{\hat{q}^{\mathrm{s}},\hat{f}_k^{\mathrm{s}}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.11-20}$$

and

$$\{\hat{\zeta}_{k,r}^{\mathrm{A}},\hat{\eta}^{\mathrm{A}}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-21}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(\boldsymbol{x},s) = \{\hat{p}^{\Omega},\hat{v}_k^{\Omega}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-22}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{q}^{\mathrm{B}},\hat{f}_r^{\mathrm{B}}\}(\boldsymbol{x},s) = \{\hat{q}^{\Omega},\hat{f}_r^{\Omega}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{7.11-23}$$

Furthermore, the fluid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\hat{\zeta}_{r,k}^{\mathrm{B}},\hat{\eta}^{\mathrm{B}}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-24}$$

Then, application of Equation (7.4-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\hat{p}^{\Omega}(\boldsymbol{x},s)\hat{q}^{\mathrm{s}}(\boldsymbol{x},s) - \hat{v}_k^{\Omega}(\boldsymbol{x},s)\,\hat{f}_k^{\mathrm{s}}(\boldsymbol{x},s)\right]\mathrm{d}V \\
&= \int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[\hat{p}^{\mathrm{s}}(\boldsymbol{x},s)\hat{q}^{\Omega}(\boldsymbol{x},s) - \hat{v}_r^{\mathrm{s}}(\boldsymbol{x},s)\,\hat{f}_r^{\Omega}(\boldsymbol{x},s)\right]\mathrm{d}V \\
&\quad + \int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m\left[-\hat{p}^{\mathrm{s}}(\boldsymbol{x},s)\hat{v}_m^{\Omega}(\boldsymbol{x},s) + \hat{v}_m^{\mathrm{s}}(\boldsymbol{x},s)\,\hat{p}^{\Omega}(\boldsymbol{x},s)\right]\mathrm{d}A\,.
\end{aligned} \tag{7.11-25}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state s and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over $\partial\mathcal{D}$ does, and no sources of the scattered or the computatinal wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.4-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.4-4).

Secondly, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{\hat{q}^{\mathrm{s}},\hat{f}_k^{\mathrm{s}}\}$, via the global complex frequency-domain reciprocity theorem of the *time correlation type* (Equation (7.5-7)). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(\boldsymbol{x},s) = \{\hat{p}^{\mathrm{s}},\hat{v}_r^{\mathrm{s}}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-26}$$

$$\{\hat{q}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\}(\boldsymbol{x},s) = \{\hat{q}^{\mathrm{s}},\hat{f}_k^{\mathrm{s}}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{7.11-27}$$

and

$$\{\hat{\zeta}_{k,r}^{\mathrm{A}},\hat{\eta}^{\mathrm{A}}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-28}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{p}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(\boldsymbol{x},s) = \{\hat{p}^{\Omega},\hat{v}_k^{\Omega}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.11-29}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{q}^{\mathrm{B}},\hat{f}_r^{\mathrm{B}}\}(\boldsymbol{x},s) = \{\hat{q}^{\Omega},\hat{f}_r^{\Omega}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{7.11-30}$$

Furthermore, the fluid properties in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\hat{\zeta}_{r,k}^{\mathrm{B}},\hat{\eta}^{\mathrm{B}}\}(\boldsymbol{x},s) = \{-\hat{\zeta}_{k,r},-\hat{\eta}\}(\boldsymbol{x},-s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}. \tag{7.11-31}$$

Then, application of Equation (7.5-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\hat{p}^{\Omega}(\boldsymbol{x},-s)\hat{q}^{\mathrm{s}}(\boldsymbol{x},s) + \hat{v}_k^{\Omega}(\boldsymbol{x},-s)\hat{f}_k^{\mathrm{s}}(\boldsymbol{x},s)\right]\mathrm{d}V \\
&= -\int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[\hat{p}^{\mathrm{s}}(\boldsymbol{x},s)\hat{q}^{\Omega}(\boldsymbol{x},-s) + \hat{v}_r^{\mathrm{s}}(\boldsymbol{x},s)\hat{f}_r^{\Omega}(\boldsymbol{x},-s)\right]\mathrm{d}V \\
&\quad + \int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m\left[\hat{p}^{\mathrm{s}}(\boldsymbol{x},s)\hat{v}_m^{\Omega}(\boldsymbol{x},-s) + \hat{v}_m^{\mathrm{s}}(\boldsymbol{x},s)\hat{p}^{\Omega}(\boldsymbol{x},-s)\right]\mathrm{d}A\,.
\end{aligned} \tag{7.11-32}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 7.5-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 7.5-4). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere).

A solution to the complex frequency-domain inverse scattering problem is commonly constructed as follows. First, the contrast-in-medium parameters are discretised by writing them as linear combinations of M expansion functions with unknown expansion coefficients. Each of the expansion functions has $\mathcal{D}^{\mathrm{s}}$, or a subset of $\mathcal{D}^{\mathrm{s}}$, as its support. Next, for each given incident wave field, the scattered wave field is measured in N subdomains of $\mathcal{D}^{\Omega}$. The latter discretisation induces the choice of the N supports of the source distributions of the observational state. Finally, a number I of different incident wave fields is selected (where "different" may involve different choices in frequency content, in location space, or in both). With $NI \geqslant M$, the non-linear problem of evaluating the M expansion coefficients of the contrast-in-medium parameter discretisation is solved by some iterative procedure (for example, by iterative minimisation of the global error over all domains in space and all frequency ranges involved where equality signs should hold pointwise). In this procedure, Equations (7.11-17), (7.11-18), (7.11-25) or (7.11-32), and the source type integral representations (7.9-43) and (7.9-44) are used simultaneously.

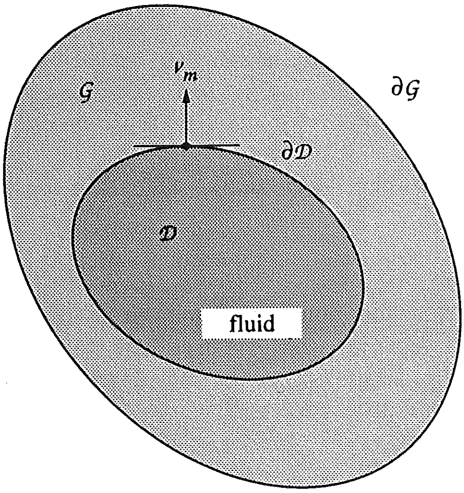
The computational state Ω is representative of the manner in which the measured data are processed in the inversion algorithms. Since a computational state does not have to meet the physical condition of causality, there is no objection against its being anti-causal. Which of the two possibilities (causal or anti-causal) leads to the best results as far as accuracy and amount of computational effort are concerned, is difficult to say. Research on this aspect is still in full progress (see Fokkema and Van den Berg, 1993).

Examples of inverse acoustic scattering problems are found in the non-destructive evaluation of mechanical structures, medical acoustic tomography and exploration geophysics.

7.12 Acoustic wave-field representations in a subdomain of the configuration space; equivalent surface sources; Huygens' principle and the Ewald–Oseen extinction theorem

In Section 7.8, wave-field representations have been derived that express the acoustic pressure and the particle velocity at any point of a configuration in terms of the volume source distributions of injection rate and force that generate the wave field. In them, the point-source solutions (Green's functions) to the radiation problem play a crucial role. In a number of cases we are, however, only interested in the values of the wave-field quantities in some subdomain of the configuration, and a wave-field representation pertaining to that subdomain would suffice. In the present section it is shown how the reciprocity theorem of the time convolution type leads to the desired expressions, albeit that now, in addition to the volume integrals over the volume source distributions (insofar as they are present in the subdomain of interest), surface integrals over the boundary surface of this subdomain occur. In these representations, the point-source solutions (Green's functions) are again the intervening kernels.

Let $\mathcal{G}$ be the domain for which the Green's functions introduced in Section 7.8 are defined. If $\mathcal{G}$ is bounded, its boundary surface $\partial\mathcal{G}$ is assumed to be acoustically impenetrable. If $\mathcal{G}$ is unbounded, the standard provisions given in Section 7.1 for handling an unbounded domain



7.12-1 Configuration for the wave-field representations in the subdomain $\mathcal{D}$ of the configuration space $\mathcal{G}$ for which the Green's functions are defined. $\partial\mathcal{D}$ is the (smooth) boundary surface of $\mathcal{D}$. (a) If $\mathcal{G}$ is bounded, $\partial\mathcal{G}$ is impenetrable; (b) for unbounded $\mathcal{G}$ the Green's functions satisfy causality conditions at infinity.

are made. Furthermore, let $\mathcal{D}$ be the subdomain of $\mathcal{G}$ in which expressions for the generated acoustic wave field are to be found. The boundary surface of $\mathcal{D}$ is $\partial\mathcal{D}$ and the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{G}$ is denoted by $\mathcal{D}'$ (Figure 7.12-1). In fact, the relevant wave field need only be defined in $\mathcal{D}$ and on $\partial\mathcal{D}$. The constitutive properties must, however, be defined in $\mathcal{G}$ in order that the necessary point-source solutions can be defined in $\mathcal{G}$. In this sense, $\mathcal{G}$ serves as an embedding of $\mathcal{D}$. Since the acoustic wave field generated is a physical wave field, it is causally related to its source distributions.

Time-domain analysis

For the time-domain analysis of the problem the acoustic properties of the fluid present in $\mathcal{G}$ are characterised by the relaxation functions $\{\mu_{k,r},\chi\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t)$, which are causal functions of time, and the global reciprocity theorem of the time convolution type (Equation (7.2-7)) is applied to the subdomain $\mathcal{D}$ of $\mathcal{G}$. In the theorem, state A is taken to be the acoustic wave field generated under consideration, i.e.

$$\{p^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{p,v_r\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.12-1}$$

$$\{q^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{q,f_k\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}, \tag{7.12-2}$$

and

$$\{\mu_{k,r}^{\mathrm{A}},\chi^{\mathrm{A}}\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{G}. \tag{7.12-3}$$

Next, state B is chosen such that the application of equation (7.2-7) to the subdomain $\mathcal{D}$ leads to the values of $\{p,v_r\}$ at some arbitrary point $\boldsymbol{x}'\in\mathcal{D}$. Inspection of the right-hand side of Equation (7.2-7) reveals that this is accomplished if we take for the source distributions of state

B a point source of volume injection rate at $\boldsymbol{x}'$ in case we want an expression for the acoustic pressure at $\boldsymbol{x}'$ and a point source of force at $\boldsymbol{x}'$ in case we want an expression for the particle velocity at $\boldsymbol{x}'$, while the fluid in state B must be taken to be the adjoint of the one of state A, i.e.

$$\{\mu_{r,k}^{\mathrm{B}},\chi^{\mathrm{B}}\} = \{\mu_{k,r},\chi\}(\boldsymbol{x},t) \qquad \text{for all } \boldsymbol{x}\in\mathcal{G}\,. \tag{7.12-4}$$

The two choices for the point-source distributions will be discussed separately below.

First, we choose

$$q^{\mathrm{B}} = a\delta(\boldsymbol{x}-\boldsymbol{x}',t) \quad \text{and } f_r^{\mathrm{B}} = 0\,, \tag{7.12-5}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x}',t)$ represents the four-dimensional unit impulse (Dirac distribution) operative at the point $\boldsymbol{x}=\boldsymbol{x}'$ and at the instant $t=0$, while a is an arbitrary constant scalar. The acoustic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is given by

$$\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{q;\mathrm{B}},v_k^{q;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t)\,, \tag{7.12-6}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (7.12-5) and the properties of $\delta(\boldsymbol{x}-\boldsymbol{x}',t)$ we have

$$\int_{\boldsymbol{x}\in\mathcal{D}}\left[-\mathrm{C}_t(f_r^{\mathrm{B}},v_r^{\mathrm{A}};\boldsymbol{x},t) + \mathrm{C}_t(p^{\mathrm{A}},q^{\mathrm{B}};\boldsymbol{x},t)\right]\mathrm{d}V$$

$$= \int_{\boldsymbol{x}\in\mathcal{D}} \mathrm{C}_t(p,a\delta(\boldsymbol{x}-\boldsymbol{x}',t);\boldsymbol{x},t)\,\mathrm{d}V = ap(\boldsymbol{x}',t)\chi_{\mathcal{D}}(\boldsymbol{x}') \qquad \text{for } \boldsymbol{x}'\in\mathcal{G}\,, \tag{7.12-7}$$

where

$$\chi_{\mathcal{D}}(\boldsymbol{x}') = \{1,½,0\} \qquad \text{for } \boldsymbol{x}'\in\{\mathcal{D},\partial\mathcal{D},\mathcal{D}'\} \tag{7.12-8}$$

is the characteristic function of the set $\mathcal{D}$, and $\mathcal{D}'$ is the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. With this, we arrive at

$$ap(\boldsymbol{x}',t)\chi_{\mathcal{D}}(\boldsymbol{x}') = \int_{\boldsymbol{x}\in\mathcal{D}}\left[\mathrm{C}_t(p^{q;\mathrm{B}},q;\boldsymbol{x},\boldsymbol{x}',t) - \mathrm{C}_t(v_k^{q;\mathrm{B}},f_k;\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}V$$

$$-\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(p^{q;\mathrm{B}},v_m;\boldsymbol{x},\boldsymbol{x}',t) - \mathrm{C}_t(v_m^{q;\mathrm{B}},p;\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}A \qquad \text{for } \boldsymbol{x}'\in\mathcal{G}\,, \tag{7.12-9}$$

where, in the second terms in the integrands, we have used the symmetry of the convolution in its functional arguments. From Equation (7.12-11) a representation for $p(\boldsymbol{x}',t)\chi_{\mathcal{D}}(\boldsymbol{x}')$ is obtained by taking into account that $p^{q;\mathrm{B}}$ and $v_k^{q;\mathrm{B}}$ are linearly related to a. Introducing the Green's functions through

$$\{p^{q;\mathrm{B}},v_k^{q;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t) = \{G^{pq;\mathrm{B}},G_k^{vq;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t)a\,, \tag{7.12-10}$$

using the reciprocity relations for these functions (see Exercises 7.8-1 and 7.8-3)

$$\{G^{pq;\mathrm{B}},G_k^{vq;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t) = \{G^{pq},-G_k^{pf}\}(\boldsymbol{x}',\boldsymbol{x},t)\,, \tag{7.12-11}$$

and invoking the condition that the resulting equation has to hold for arbitrary values of a, Equation (7.12-9) leads to the final result:

$$p(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(G^{pq},q;x',x,t) + \mathrm{C}_t(G_k^{pf},f_k;x',x,t)\right] \mathrm{d}V$$

$$-\int_{x\in\partial\mathcal{D}} \nu_m \left[\mathrm{C}_t(G^{pq},v_m;x',x,t) + \mathrm{C}_t(G_m^{pf},p;x',x,t)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{G}. \qquad (7.12\text{-}12)$$

Equation (7.12-12) expresses, for $x'\in\mathcal{D}$, the acoustic pressure p of the generated acoustic wave field at x' as the superposition of the contributions from the elementary volume sources $q\,\mathrm{d}V$ and $f_k\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the elementary contributions $-\nu_m v_m\,\mathrm{d}A$ and $-\nu_m p\,\mathrm{d}A$ from the equivalent surface sources at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Secondly, we choose

$$q^{\mathrm{B}} = 0 \quad \text{and} \quad f_r^{\mathrm{B}} = b_r\delta(x-x',t)\,, \qquad (7.12\text{-}13)$$

where b_r is an arbitrary constant vector. The acoustic wave field that in the present application is causally radiated by this source and satisfies the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is given by:

$$\{p^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{p^{f;\mathrm{B}},v_k^{f;\mathrm{B}}\}(x,x',t)\,, \qquad (7.12\text{-}14)$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (7.12-13) and the properties of $\delta(x-x',t)$, we have

$$\int_{x\in\mathcal{D}} \left[-\mathrm{C}_t(f_r^{\mathrm{B}},v_r^{\mathrm{A}};x,t) + \mathrm{C}_t(p^{\mathrm{A}},q^{\mathrm{B}};x,t)\right] \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \mathrm{C}_t(b_r\delta(x-x',t),v_r;x,t)\,\mathrm{d}V = -b_r v_r(x',t)\chi_{\mathcal{D}}(x') \qquad \text{for } x'\in\mathcal{G}. \qquad (7.12\text{-}15)$$

With this, we arrive at

$$b_r v_r(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[-\mathrm{C}_t(p^{f;\mathrm{B}},q;x,x',t) + \mathrm{C}_t(v_k^{f;\mathrm{B}},f_k;x,x',t)\right] \mathrm{d}V$$

$$-\int_{x\in\partial\mathcal{D}} \nu_m \left[-\mathrm{C}_t(p^{f;\mathrm{B}},v_m;x,x',t) + \mathrm{C}_t(v_m^{f;\mathrm{B}},p;x,x',t)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{G}, \qquad (7.12\text{-}16)$$

where, in the second terms in the integrands, we have used the symmetry of the convolution in its functional arguments. From Equation (7.12-16) a representation for $v_r(x',t)\chi_{\mathcal{D}}(x')$ is obtained by taking into account the fact that $p^{f;\mathrm{B}}$ and $v_k^{f;\mathrm{B}}$ are linearly related to b_r. Introducing the Green's functions through

$$\{p^{f;\mathrm{B}},v_k^{f;\mathrm{B}}\}(x,x',t) = \{G_r^{pf;\mathrm{B}},G_{k,r}^{vf;\mathrm{B}}\}(x,x',t)b_r\,, \qquad (7.12\text{-}17)$$

using the reciprocity relations for these functions (see Exercises 7.8-2 and 7.8-4)

$$\{G_r^{pf;\mathrm{B}},G_{k,r}^{vf;\mathrm{B}}\}(x,x',t) = \{-G_r^{vq},G_{r,k}^{vf}\}(x',x,t)\,, \qquad (7.12\text{-}18)$$

and invoking the condition the resulting equation has to hold for arbitrary values of b_r, Equation (7.12-16) leads to the final result

$$v_r(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(G_r^{vq},q;x',x,t) + \mathrm{C}_t(G_{r,k}^{vf},f_k;x',x,t)\right] \mathrm{d}V$$

$$-\int_{x\in\partial\mathcal{D}} \nu_m \left[\mathrm{C}_t(G_r^{vq},v_m;x',x,t) + \mathrm{C}_t(G_{r,m}^{vf},p;x',x,t)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{G}. \tag{7.12-19}$$

Equation (7.12-19) expresses, for $x'\in\mathcal{D}$, the particle velocity v_r of the generated acoustic wave field at x' as the superposition of the contributions from the elementary volume sources $q\,\mathrm{d}V$ and $f_k\,\mathrm{d}V$ at x insofar as they are present in $\mathcal{D}$, and the elementary equivalent surface sources $-\nu_m v_m\,\mathrm{d}A$ and $-\nu_m p\,\mathrm{d}A$ at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Complex frequency-domain analysis

For the complex frequency-domain analysis of the problem the acoustic properties of the fluid present in $\mathcal{G}$ are characterised by $\{\hat{\zeta}_{k,r},\hat{\eta}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(x,s)$ and the global complex frequency-domain reciprocity theorem of the time convolution type (Equation (7.4-7)), is applied to the subdomain $\mathcal{D}$ of $\mathcal{G}$. In the theorem, state A is taken to be the generated acoustic wave field under consideration, i.e.

$$\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{p},\hat{v}_r\}(x,s) \qquad \text{for } x\in\mathcal{D}, \tag{7.12-20}$$

$$\{\hat{q}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\} = \{\hat{q},\hat{f}_k\}(x,s) \qquad \text{for } x\in\mathcal{D}, \tag{7.12-21}$$

and

$$\{\hat{\zeta}_{k,r}^{\mathrm{A}},\hat{\eta}^{\mathrm{A}}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(x,s) \qquad \text{for } x\in\mathcal{G}. \tag{7.12-22}$$

Next, state B is chosen such that the application of Equation (7.4-7) to the subdomain $\mathcal{D}$ leads to the values of $\{\hat{p},\hat{v}_r\}$ at some arbitrary point $x'\in\mathcal{D}$. Inspection of the right-hand side of Equation (7.4-7) reveals that this is accomplished if we take for the source distributions of state B a point source of the volume injection rate at x' when we want an expression for the acoustic pressure at x', and a point source of force at x' when we want an expression for the particle velocity at x', while the fluid in state B must be taken to be the adjoint of the one of state A, i.e.

$$\{\hat{\zeta}_{r,k}^{\mathrm{B}},\hat{\eta}^{\mathrm{B}}\} = \{\hat{\zeta}_{k,r},\hat{\eta}\}(x,s) \qquad \text{for all } x\in\mathcal{G}. \tag{7.12-23}$$

The two choices for the point source distributions will be discussed separately below.

First, we choose

$$\hat{q}^{\mathrm{B}} = \hat{a}(s)\delta(x-x') \quad \text{and} \quad \hat{f}_r^{\mathrm{B}} = 0, \tag{7.12-24}$$

where $\delta(x-x')$ represents the three-dimensional unit impulse (Dirac distribution) operative at the point $x=x'$, while $\hat{a}=\hat{a}(s)$ is an arbitrary scalar function of s. The acoustic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is given by

$$\{\hat{p}^{\mathrm{B}},\hat{f}_k^{\mathrm{B}}\} = \{\hat{p}^{q;\mathrm{B}},\hat{v}_k^{q;\mathrm{B}}\}(x,x',s), \tag{7.12-25}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (7.12-24) and the properties of $\delta(x-x')$, we have

$$\int_{x\in\mathcal{D}} \left[-\hat{f}_r^{\mathrm{B}}(x,s)\hat{v}_r^{\mathrm{A}}(x,s) + \hat{p}^{\mathrm{A}}(x,s)\hat{q}^{\mathrm{B}}(x,s)\right] \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}} \hat{p}(x,s)\hat{a}(s)\delta(x-x')\,\mathrm{d}V = \hat{a}(s)\hat{p}(x',s)\chi_{\mathcal{D}}(x') \qquad \text{for } x'\in\mathcal{G}. \tag{7.12-26}$$

With this, we arrive at

$$\hat{a}(s)\,\hat{p}(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{p}^{\,q;\mathrm{B}}(x,x',s)\hat{q}(x,s) - \hat{v}_k^{q;\mathrm{B}}(x,x's)\hat{f}_k(x,s)\right]\mathrm{d}V$$

$$-\int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{p}^{\,q;\mathrm{B}}(x,x',s)\hat{v}_m(x,s) - \hat{v}_m^{q;\mathrm{B}}(x,x's)\hat{p}(x,s)\right]\mathrm{d}A \qquad \text{for } x'\in\mathcal{G}. \tag{7.12-27}$$

From Equation (7.12-27) a representation for $\hat{p}(x',s)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\hat{p}^{\,q;\mathrm{B}}$ and $\hat{v}_k^{q;\mathrm{B}}$ are linearly related to $\hat{a}(s)$. Introducing the Green's functions through

$$\{\hat{p}^{\,q;\mathrm{B}}, \hat{v}_k^{q;\mathrm{B}}\}(x,x',s) = \{\hat{G}^{pq;\mathrm{B}}, \hat{G}_k^{vq;\mathrm{B}}\}(x,x',s)\hat{a}(s)\,, \tag{7.12-28}$$

using the reciprocity relations for these functions (see Exercises 7.8-5 and 7.8-7)

$$\{\hat{G}^{pq;\mathrm{B}}, \hat{G}_k^{vq;\mathrm{B}}\}(x,x',s) = \{\hat{G}^{pq}, -\hat{G}_k^{pf}\}(x',x,s)\,, \tag{7.12-29}$$

and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{a}(s)$, Equation (7.12-27) leads to the final result:

$$\hat{p}(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{G}^{pq}(x',x,s)\hat{q}(x,s) + \hat{G}_k^{pf}(x',x,s)\hat{f}_k(x,s)\right]\mathrm{d}V$$

$$+\int_{x\in\partial\mathcal{D}} \left\{\hat{G}^{pq}(x',x,s)[-\nu_m\hat{v}_m(x,s)] + \hat{G}_m^{pf}(x',x,s)[-\nu_m\hat{p}(x,s)]\right\}\mathrm{d}A \quad \text{for } x'\in\mathcal{G}. \tag{7.12-30}$$

Equation (7.12-30) expresses, for $x'\in\mathcal{D}$, the acoustic pressure $\hat{p}$ of the generated acoustic wave field at x' as the superposition of the contributions from the elementary volume sources $\hat{q}^{\,\mathrm{T}}\,\mathrm{d}V$ and $\hat{f}_k^{\mathrm{T}}\,\mathrm{d}V$ at x, insofar as they are present in $\mathcal{D}$, and the elementary equivalent surface sources $-\nu_m\hat{v}_m\,\mathrm{d}A$ and $-\nu_m\hat{p}\,\mathrm{d}A$ at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Secondly, we choose

$$\hat{q}^{\,\mathrm{B}} = 0 \quad \text{and} \quad \hat{f}_r^{\mathrm{B}} = \hat{b}_r(s)\delta(x-x')\,, \tag{7.12-31}$$

where $\hat{b}_r = \hat{b}_r(s)$ is an arbitrary vector function of s. The acoustic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary condition at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is given by

$$\{\hat{p}^{\,\mathrm{B}}, \hat{f}_k^{\mathrm{B}}\} = \{\hat{p}^{\,f;\mathrm{B}}, \hat{v}_k^{f;\mathrm{B}}\}(x,x',s)\,, \tag{7.12-32}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (7.12-31) and the properties of $\delta(x-x')$, we have

$$\int_{x\in\mathcal{D}} \left[-\hat{f}_r^{\mathrm{B}}(x,s)\hat{v}_r^{\mathrm{A}}(x,s) + \hat{p}^{\,\mathrm{A}}(x,s)\hat{q}^{\,\mathrm{B}}(x,s)\right]\mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \hat{b}_r(s)\delta(x-x')\hat{v}_r(x,s)\,\mathrm{d}V = -\hat{b}_r(s)\hat{v}_r(x',s)\chi_{\mathcal{D}}(x') \qquad \text{for } x'\in\mathcal{G}. \tag{7.12-33}$$

With this, we arrive at

$$\hat{b}_r(s)v_r(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[-\hat{p}^{f;\mathrm{B}}(x,x',s)\hat{q}(x,s) + \hat{v}_k^{f;\mathrm{B}}(x,x's)\hat{f}_k(x,s)\right] \mathrm{d}V$$

$$-\int_{x\in\partial\mathcal{D}} \nu_m \left[-\hat{p}^{f;\mathrm{B}}(x,x',s)\hat{v}_m(x,s) + \hat{v}_m^{f;\mathrm{B}}(x,x's)\,\hat{p}(x,s)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{G}. \qquad (7.12\text{-}34)$$

From Equation (7.12-34) a representation for $\hat{v}_r(x',s)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\hat{p}^{f;\mathrm{B}}$ and $\hat{v}_k^{f;\mathrm{B}}$ are linearly related to $\hat{b}_r(s)$. Introducing the Green's functions through

$$\{\hat{p}^{f;\mathrm{B}}, \hat{v}_k^{f;\mathrm{B}}\}(x,x',s) = \{\hat{G}_r^{pf;\mathrm{B}}, \hat{G}_{k,r}^{vf;\mathrm{B}}\}(x,x',s)\hat{b}_r(s), \qquad (7.12\text{-}35)$$

using the reciprocity relations for these functions (see Exercises 7.8-6 and 7.8-8)

$$\{\hat{G}_r^{pf;\mathrm{B}}, \hat{G}_{k,r}^{vf;\mathrm{B}}\}(x,x',s) = \{-\hat{G}_r^{vq}, \hat{G}_{r,k}^{vf}\}(x',x,s), \qquad (7.12\text{-}36)$$

and invoking the condition the resulting equation has to hold for arbitrary values of $\hat{b}_r(s)$, Equation (7.12-34) leads to the final result:

$$\hat{v}_r(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{G}_r^{vq}(x',x,s)\hat{q}(x,s) + \hat{G}_{r,k}^{vf}(x',x,s)\hat{f}_k(x,s)\right] \mathrm{d}V$$

$$+\int_{x\in\partial\mathcal{D}} \nu_m \left\{\hat{G}_r^{vq}(x',x,s)[-\nu_m\hat{v}_m(x,s)] + \hat{G}_{r,m}^{vf}(x',x,s)[-\nu_m\hat{p}(x,s)]\right\} \mathrm{d}A \quad \text{for } x'\in\mathcal{G}. \qquad (7.12\text{-}37)$$

Equation (7.12-37) expresses, for $x'\in\mathcal{D}$, the particle velocity $\hat{v}_r$ of the generated acoustic wave field at x' as the superposition of the contributions from the elementary volume sources $\hat{q}\,\mathrm{d}V$ and $\hat{f}_k\,\mathrm{d}V$ at x, insofar as they are present in $\mathcal{D}$, and the elementary equivalent surface sources $-\nu_m\hat{v}_m\,\mathrm{d}A$ and $-\nu_m\hat{p}\,\mathrm{d}A$ at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

For $x'\in\mathcal{D}$, Equations (7.12-12), (7.12-19), (7.12-30) and (7.12-37) express the values of the acoustic pressure and the particle velocity in some point of $\mathcal{D}$ as the sum of the contributions from the volume sources of injection rate and force, insofar as these are present in $\mathcal{D}$, and the equivalent surface sources on $\partial\mathcal{D}$. Evidently, the equivalent surface sources yield, in the interior of $\mathcal{D}$, the contribution to the wave field that arises from (unspecified) sources located in $\mathcal{D}'$, i.e. in the exterior of $\mathcal{D}$. In particular, the surface integrals in these expressions vanish when the wave field is not only defined in $\mathcal{D}$ and on $\partial\mathcal{D}$, but also in $\mathcal{D}'$ and no sources are located in between $\partial\mathcal{D}$ and $\partial\mathcal{G}$, and the wave fields in state B either satisfy the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded (see Exercises 7.2-2, 7.3-2, 7.4-4 and 7.5-4), or are causally related to their point source excitations if $\mathcal{G}$ is unbounded. In the latter case, Equations (7.12-12), (7.12-19), (7.12-30) and (7.12-37) reduce to Equations (7.8-14), (7.8-24), (7.8-38) and (7.8-48), respectively.

Another property of Equations (7.12-12), (7.12-19), (7.12-30) and (7.12-37) is that the wave field emitted by the volume sources in $\mathcal{D}$ and the wave field emitted by the equivalent surface sources on $\partial\mathcal{D}$ apparently cancel each other when $x'\in\mathcal{D}'$. This property is known as the Ewald–Oseen *extinction theorem* (Oseen, 1915; Ewald, 1916).

Another special case arises when Equations (7.12-12), (7.12-19), (7.12-30) and (7.12-37) are used in a domain in which no volume source distributions are present. Then, they express

Huygens' principle (Huygens, 1690) which states that an acoustic wave field arising from sources "behind" a closed surface that divides the configuration space into two disjoint regions and "in front" of which no volume sources are present, can be represented as being due to equivalent surface sources located on that surface, while that representation yields the value zero "behind" that surface. In particular, Huygens stated his principle for the case where the relevant surface is a wave front of the wave motion in space–time. A number of historical details about the development of the mathematical theory of Huygens' principle can be found in Baker and Copson (1950). Additional literature on the subject can be found in Blok *et al.* (1992), and De Hoop (1992).

Applications of the wave-field representations in a subdomain of space are found in the integral equation formulation of scattering problems, while the Ewald–Oseen extinction theorem forms the basis of the so-called "null-field method" for solving such problems.

Exercises

Exercise 7.12-1

Let $\mathcal{D}$ be a bounded subdomain of three-dimensional Euclidean space $\mathcal{R}^3$. Let $\partial\mathcal{D}$ be the closed boundary surface of $\mathcal{D}$ and denote by $\mathcal{D}'$ the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$ (i.e. towards $\mathcal{D}'$), is denoted by ν (Figure 7.12-2). In the domain $\mathcal{D}'$ an acoustic wave field $\{p,v_r\}$ is present whose sources are located in $\mathcal{D}$. Use Equations (7.12-12) and (7.12-19) to arrive at the equivalent surface source time-domain integral representation

$$p(x',t)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(G^{pq},v_m;x',x,t) + \mathrm{C}_t(G_m^{pf},p;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \quad (7.12\text{-}38)$$

and

$$v_r(x',t)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(G_r^{vq},v_m;x',x,t) + \mathrm{C}_t(G_{r,m}^{vf},p;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3. \quad (7.12\text{-}39)$$

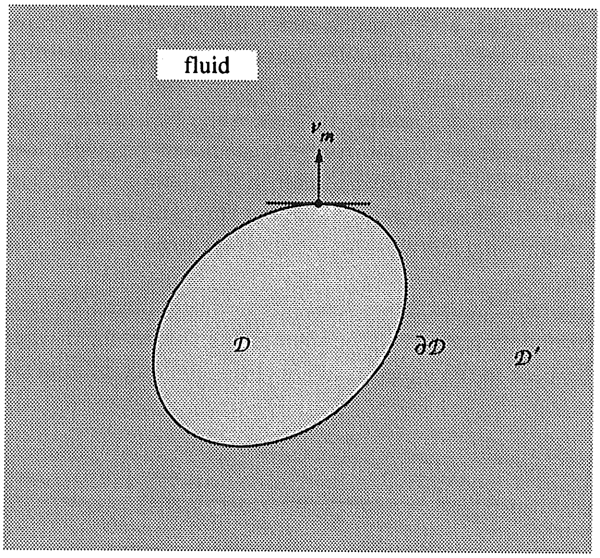
Exercise 7.12-2

Let $\mathcal{D}$ be a bounded subdomain of three-dimensional Euclidean space $\mathcal{R}^3$. Let $\partial\mathcal{D}$ be the closed boundary surface of $\mathcal{D}$ and denote by $\mathcal{D}'$ the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$ (i.e. towards $\mathcal{D}'$), is denoted by ν (Figure 7.12-2). In the domain $\mathcal{D}'$ an acoustic wave field $\{\hat{p},\hat{v}_r\}$ is present whose sources are located in $\mathcal{D}$. Use Equations (7.12-30) and (7.12-37) to arrive at the equivalent surface source complex frequency-domain integral representations

$$\hat{p}(x',s)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{G}^{pq}(x',x,s)\hat{v}_m(x,s) + \hat{G}_m^{pf}(x',x,s)\,\hat{p}(x,s)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \quad (7.12\text{-}40)$$

and

$$\hat{v}_r(x',s)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{G}_r^{vq}(x',x,s)\hat{v}_m(x,s) + \hat{G}_{r,m}^{vf}(x',x,s)\,\hat{p}(x,s)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3. \quad (7.12\text{-}41)$$



7.12-2 Configuration for the equivalent surface source integral representation for an acoustic wave field in the source free domain $\mathcal{D}'$ exterior to a bounded subdomain $\mathcal{D}$ of $\mathcal{R}^3$.

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