8

Plane wave scattering by an object in an unbounded, homogeneous, isotropic, lossless embedding

In this chapter, the simplest scattering configuration is investigated in more detail. It consists of an unbounded, homogeneous, isotropic, lossless embedding in which a plane wave is incident upon a scattering object of bounded extent. First, the reciprocity properties of the amplitudes of the scattered wave in the far-field region are investigated. Next, an energy theorem ("extinction cross-section theorem") is derived that relates the sum of the energy carried by the scattered wave and the energy absorbed by the scattering object to the amplitude of the scattered wave in the far-field region when observed in the forward scattering direction. Finally, the first term in the Neumann solution to the relevant system of integral equations (the so-called "Rayleigh–Gans–Born approximation") is determined for penetrable, homogeneous scatterers of different shapes. The analysis is carried out in the time domain as well as in the complex frequency domain.

8.1 The scattering configuration, the incident plane wave and the far-field scattering amplitudes

The scattering configuration consists of a homogeneous, isotropic, lossless *embedding* that occupies all of $\mathcal{R}^3$. The acoustic properties of the embedding are characterised by its volume density of mass ρ and its compressibility κ, which are positive constants. The associated acoustic wave speed is $c = (\rho\kappa)^{-1/2}$, which is also a positive constant. In the embedding, an acoustic *scatterer* is present that occupies the bounded domain $\mathcal{D}^s$. The boundary surface of $\mathcal{D}^s$ is denoted by $\partial\mathcal{D}^s$ and ν is the unit vector along the normal to $\partial\mathcal{D}^s$ oriented away from $\mathcal{D}^s$. The complement of $\mathcal{D}^s \cup \partial\mathcal{D}^s$ in $\mathcal{R}^3$ is denoted by $\mathcal{D}^{s'}$ (Figure 8.1-1).

Time-domain analysis

In the time-domain analysis of the problem, the acoustic properties of the scatterer are, if the scatterer is an *acoustically penetrable object*, characterised by the relaxation functions

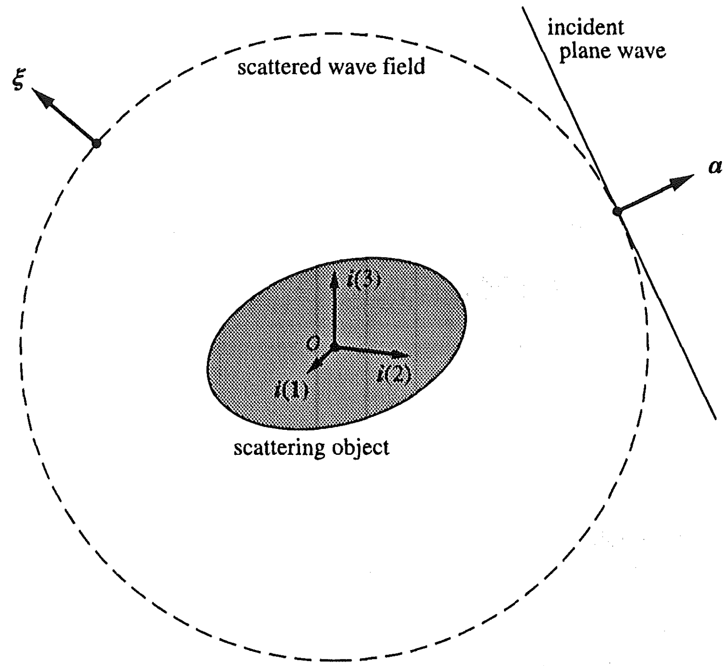

Figure 8.1-1 Scattering object occupying the bounded domain $\mathcal{D}^s$ in an unbounded acoustically homogeneous, isotropic, lossless embedding with volume density of mass ρ and compressibility κ.

$\{\mu^s_{k,r},\chi^s\} = \{\mu^s_{k,r},\chi^s\}(x,t)$, which are causal functions of time. The equivalent contrast volume source densities of injection rate and force are then given by (see Equations (7.9-18) and (7.9-19))

$$f^s_k = -\partial_t C_t(\mu^s_{k,r} - \rho\delta_{k,r}\delta(t), v_r; x; t) \qquad \text{for } x \in \mathcal{D}^s, \tag{8.1-1}$$

$$q^s = -\partial_t C_t(\chi^s - \kappa\delta(t), p; x; t) \qquad \text{for } x \in \mathcal{D}^s, \tag{8.1-2}$$

in which the *total acoustic wave field* $\{p,v_r\}$ is the sum of the *incident wave field* $\{p^i,v^i_r\}$ and the *scattered wave field* $\{p^s,v^s_r\}$ (see Equation (7.9-5)). If the scatterer is *acoustically impenetrable*, either of the two boundary conditions

$$\lim_{h \downarrow 0} p(x + h\nu, t) = 0 \qquad \text{for } x \in \partial\mathcal{D}^s \tag{8.1-3}$$

or

$$\lim_{h \downarrow 0} \nu_r v_r(x + h\nu, t) = 0 \qquad \text{for } x \in \partial\mathcal{D}^s \tag{8.1-4}$$

applies.

For the incident wave we now take the *uniform plane wave* (see Equations (6.4-7) and (6.4-13))

$$\{p^i,v^i_r\} = \{P,V_r\}a(t - \alpha_s x_s/c)\,, \tag{8.1-5}$$

that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ (i.e. $\alpha_s\alpha_s = 1$) and has the normalised pulse shape $a(t)$. Its acoustic pressure and particle velocity amplitudes are related through (see Equations (6.4-15) and (6.4-16))

$$V_r = YP\alpha_r \,, \tag{8.1-6}$$

in which (see Equation (6.4-17))

$$Y = (\kappa/\rho)^{1/2} = (\rho c)^{-1} \qquad \text{with } (...)^{1/2} > 0 \tag{8.1-7}$$

is the acoustic plane wave admittance of the wave.

For an *acoustically penetrable scatterer* we use for the scattered wave the constrast volume source integral representation (see Equations (7.9-20) and (7.9-21))

$$p^{s}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \left[\mathrm{C}_t(G^{pq},q^{s};\boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G_k^{pf},f_k^{s};\boldsymbol{x}',\boldsymbol{x},t)\right]\mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 \tag{8.1-8}$$

and

$$v_r^{s}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \left[\mathrm{C}_t(G_r^{vq},q^{s};\boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G_{r,k}^{vf},f_k^{s};\boldsymbol{x}',\boldsymbol{x},t)\right]\mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3, \tag{8.1-9}$$

in which (see Exercise 7.8-9 with $\boldsymbol{x}$ and $\boldsymbol{x}'$ interchanged)

$$G^{pq}(\boldsymbol{x}',\boldsymbol{x},t) = \rho\partial_t G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{8.1-10}$$

$$G_k^{pf}(\boldsymbol{x}',\boldsymbol{x},t) = -\partial_k' G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{8.1-11}$$

$$G_r^{vq}(\boldsymbol{x}',\boldsymbol{x},t) = -\partial_r' G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{8.1-12}$$

$$G_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},t) = \rho^{-1}\delta(\boldsymbol{x}'-\boldsymbol{x})H(t)\delta_{r,k} + \rho^{-1}\partial_r'\partial_k' \left[\mathrm{I}_t G(\boldsymbol{x}',\boldsymbol{x},t)\right], \tag{8.1-13}$$

where ∂_m' denotes differentiation with respect to x_m', and

$$G(\boldsymbol{x}',\boldsymbol{x},t) = \frac{\delta(t - |\boldsymbol{x}'-\boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}'-\boldsymbol{x}|} \qquad \text{for } \boldsymbol{x}' \neq \boldsymbol{x} \,. \tag{8.1-14}$$

In the *far-field region*, the expansion

$$\{p^{s},v_r^{s}\}(\boldsymbol{x}',t) = \frac{\{p^{s;\infty},v_r^{s;\infty}\}(\boldsymbol{\xi},t-|\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1+O(|\boldsymbol{x}'|^{-1})\right] \text{ as } |\boldsymbol{x}'|\to\infty \text{ with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi} \tag{8.1-15}$$

holds, where (see Equations (5.10-5)–(5.10-10))

$$p^{s;\infty} = \rho\partial_t\Phi^{q^{s};\infty} + c^{-1}\xi_k\partial_t\Phi_k^{f^{s};\infty} \,, \tag{8.1-16}$$

and

$$v_r^{s;\infty} = (\rho c)^{-1}\xi_r p^{s;\infty}, \tag{8.1-17}$$

in which

$$\Phi^{q^{s};\infty}(\boldsymbol{\xi},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} q^{s}(\boldsymbol{x},t+\xi_s x_s/c)\,\mathrm{d}V \tag{8.1-18}$$

and

$$\Phi_k^{f^{s;\infty}}(\xi,t) = \int_{x\in\mathcal{D}^s} f_k^s(x,t+\xi_s x_s/c)\,\mathrm{d}V\,. \tag{8.1-19}$$

For an *acoustically impenetrable scatterer* the acoustic wave field is not defined in the interior $\mathcal{D}^s$ of the scatterer and we have to resort to an equivalent surface source integral representation that expresses the scattered wave field in the exterior $\mathcal{D}^{s\prime}$ of the scatterer in terms of the wave field on the boundary surface $\partial\mathcal{D}^s$ of $\mathcal{D}^s$. This representation is, on account of Equations (7.12-38) and (7.12-39),

$$p^s(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G^{pq},\partial q^s;x',x,t) + \mathrm{C}_t(G_k^{pf},\partial f_k^s;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{8.1-20}$$

and

$$v_r^s(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G_r^{vq},\partial q^s;x',x,t) + \mathrm{C}_t(G_{r,k}^{vf},\partial f_k^s;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3, \tag{8.1-21}$$

in which (note the orientation of ν_m)

$$\partial q^s = \nu_r v_r^s \tag{8.1-22}$$

and

$$\partial f_k^s = \nu_k p^s. \tag{8.1-23}$$

In the *far-field region*, the expansion given in Equation (8.1-15) holds, where, based upon Equations (8.1-20) and (8.1-21), we have

$$p^{s;\infty} = \rho\partial_t\Phi^{\partial q^{s;\infty}} + c^{-1}\xi_k\partial_t\Phi_k^{\partial f^{s;\infty}} \tag{8.1-24}$$

and

$$v_r^{s;\infty} = (\rho c)^{-1}\xi_r p^{s;\infty}, \tag{8.1-25}$$

in which (note the orientation of ν_m)

$$\Phi^{\partial q^{s;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial q^s(x,t+\xi_s x_s/c)\,\mathrm{d}A \tag{8.1-26}$$

and

$$\Phi_k^{\partial f^{s;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial f_k^s(x,t+\xi_s x_s/c)\,\mathrm{d}A\,. \tag{8.1-27}$$

However, upon applying Equations (7.12-12) and (7.12-19) to the incident wave field $\{p^i,v_r^i\}$ and to the domain $\mathcal{D}^s$, we have (note that the incident wave field is source-free in $\mathcal{D}^s$)

$$p^i(x',t)\chi_{\mathcal{D}^s}(x') = \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G^{pq},\partial q^i;x',x,t) + \mathrm{C}_t(G_k^{pf},\partial f_k^i;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{8.1-28}$$

and

$$v_r^{\mathrm{i}}(x',t)\chi_{\mathcal{D}^{s}}(x') = \int_{x\in\partial\mathcal{D}^{s}} \left[\mathrm{C}_t(G_r^{vq},\partial q^{\mathrm{i}};x',x,t) + \mathrm{C}_t(G_{r,k}^{vf},\partial f_k^{\mathrm{i}};x',x,t)\right] \mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3, \tag{8.1-29}$$

in which (note the orientation of ν_m)

$$\partial q^{\mathrm{i}} = -\nu_r v_r^{\mathrm{i}} \tag{8.1-30}$$

and

$$\partial f_k^{\mathrm{i}} = -\nu_k p^{\mathrm{i}}. \tag{8.1-31}$$

Subtraction of Equation (8.1-28) from Equation (8.1-20) and of Equation (8.1-29) from Equation (8.1-21) leads to

$$p^{\mathrm{s}}(x',t)\chi_{\mathcal{D}^{s'}}(x') - p^{\mathrm{i}}(x',t)\chi_{\mathcal{D}^{s}}(x') = \int_{x\in\partial\mathcal{D}} \left[\mathrm{C}_t(G^{pq},\partial q;x',x,t) + \mathrm{C}_t(G_k^{pf},\partial f_k;x',x,t)\right] \mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{8.1-32}$$

and

$$v_r^{\mathrm{s}}(x',t)\chi_{\mathcal{D}^{s'}}(x') - v_r^{\mathrm{i}}(x',t)\chi_{\mathcal{D}^{s}}(x') = \int_{x\in\partial\mathcal{D}} \left[\mathrm{C}_t(G_r^{vq},\partial q;x',x,t) + \mathrm{C}_t(G_{r,k}^{vf},\partial f_k;x',x,t)\right] \mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 , \tag{8.1-33}$$

in which (note the orientation of ν_m)

$$\partial q = \nu_r v_r \tag{8.1-34}$$

and

$$\partial f_k = \nu_k p . \tag{8.1-35}$$

In the *far-field region*, again the expansion given in Equation (8.1-15) holds where, based upon Equations (8.1-32) and (8.1-33), we now have

$$p^{\mathrm{s};\infty} = \rho\partial_t\Phi^{\partial q;\infty} + c^{-1}\xi_k\partial_t\Phi_k^{\partial f;\infty} \tag{8.1-36}$$

and

$$v_r^{\mathrm{s};\infty} = (\rho c)^{-1}\xi_r p^{\mathrm{s};\infty}, \tag{8.1-37}$$

in which (note the orientation of ν_m)

$$\Phi^{\partial q;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^{s}} \partial q(x,t+\xi_s x_s/c)\,\mathrm{d}A \tag{8.1-38}$$

and

$$\Phi_k^{\partial f;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^{s}} \partial f_k(x,t+\xi_s x_s/c)\,\mathrm{d}A . \tag{8.1-39}$$

Of course, the equivalent surface source representations also apply to the case of an acoustically penetrable scatterer. For $x'\in\mathcal{D}^{s\,\prime}$, Equations (8.1-20) and (8.1-21), and Equations (8.1-32) and

(8.1-33) must then yield the same result as Equations (8.1-8) and (8.1-9). Similarly, in the far-field region, Equations (8.1-24)–(8.1-27) and (8.1-36)–(8.1-39) must yield the same result as Equations (8.1-16)–(8.1-19). Note, however, that the results for $\boldsymbol{x}'\in\mathcal{D}^{\mathrm{s}}$ differ.

Equations (8.1-8) and (8.1-9), when taken for $\boldsymbol{x}'\in\mathcal{D}^{\mathrm{s}}$, provide the basis for the *time-domain domain integral equation method* to solve problems of the scattering by penetrable objects. For solving problems of the scattering by impenetrable objects, Equations (8.1-32) and (8.1-33) provide, when taken for $\boldsymbol{x}'\in\partial\mathcal{D}^{\mathrm{s}}$, the basis for the *time-domain boundary integral equation method* and, when taken for $\boldsymbol{x}'\in\mathcal{D}^{\mathrm{s}}$, the basis for the *time-domain null-field method.* For general scatterers, all three methods need numerical implementation.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the acoustic properties of the scatterer are, if the scatterer is an acoustically penetrable object, characterised by the functions $\{\hat{\zeta}^{\mathrm{s}}_{k,r},\hat{\eta}^{\mathrm{s}}\} = \{\hat{\zeta}^{\mathrm{s}}_{k,r},\hat{\eta}^{\mathrm{s}}\}(\boldsymbol{x},s)$. The equivalent contrast volume source densities of injection rate and force are then given by (see Equations (7.9-41) and (7.9-42))

$$\hat{f}^{\mathrm{s}}_k = -(\hat{\zeta}^{\mathrm{s}}_{k,r} - s\rho\delta_{k,r})\hat{v}_r \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{8.1-40}$$

$$\hat{q}^{\mathrm{s}} = -(\hat{\eta}^{\mathrm{s}} - s\kappa)\hat{p} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{8.1-41}$$

in which the *total acoustic wave field* $\{\hat{p},\hat{v}_r\}$ is the sum of the *incident wave field* $\{\hat{p}^{\mathrm{i}},\hat{v}^{\mathrm{i}}_r\}$ and the *scattered wave field* $\{\hat{p}^{\mathrm{s}},\hat{v}^{\mathrm{s}}_r\}$ (see Equation (7.9-28)). If the scatterer is *acoustically impenetrable*, either of the two boundary conditions

$$\lim_{h\downarrow 0}\hat{p}(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0 \qquad \text{for } \boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}} \tag{8.1-42}$$

or

$$\lim_{h\downarrow 0}\nu_r\hat{v}_r(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0 \qquad \text{for } \boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}} \tag{8.1-43}$$

applies.

For the incident wave we now take the *uniform plane wave* (see Equations (6.1-3), (6.1-12), (6.2-3) and (6.2-12))

$$\{\hat{p}^{\mathrm{i}},\hat{v}^{\mathrm{i}}_r\} = \{P,V_r\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{8.1-44}$$

that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ (i.e. $\alpha_s\alpha_s = 1$) and has the complex frequency-domain normalised pulse shape $\hat{a}(s)$. Its acoustic pressure and particle velocity amplitudes are related through Equations (8.1-6) and (8.1-7).

For an *acoustically penetrable scatterer* we use for the scattered wave the contrast volume source representation (see Equations (7.9-43) and (7.9-44))

$$\hat{p}^{\mathrm{s}}(\boldsymbol{x}',s) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\hat{G}^{pq}(\boldsymbol{x}',\boldsymbol{x},s)\,\hat{q}^{\mathrm{s}}(\boldsymbol{x},s) + \hat{G}^{pf}_k(\boldsymbol{x}',\boldsymbol{x},s)\,\hat{f}^{\mathrm{s}}_k(\boldsymbol{x},s)\right]\mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 \tag{8.1-45}$$

and

$$\hat{v}^{\mathrm{s}}_r(\boldsymbol{x}',s) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\hat{G}^{vq}_r(\boldsymbol{x}',\boldsymbol{x},s)\,\hat{q}^{\mathrm{s}}(\boldsymbol{x},s) + \hat{G}^{vf}_{r,k}(\boldsymbol{x}',\boldsymbol{x},s)\,\hat{f}^{\mathrm{s}}_k(\boldsymbol{x},s)\right]\mathrm{d}V \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3, \tag{8.1-46}$$

in which (see Exercise 7.8-10 with x and x' interchanged)

$$\hat{G}^{pq}(x',x,s) = s\rho\hat{G}(x',x,s) , \tag{8.1-47}$$

$$\hat{G}_k^{pf}(x',x,s) = -\partial_k' \hat{G}(x',x,s) , \tag{8.1-48}$$

$$\hat{G}_r^{vq}(x',x,s) = -\partial_r' \hat{G}(x',x,s) , \tag{8.1-49}$$

$$\hat{G}_{r,k}^{vf}(x',x,s) = (s\rho)^{-1}\delta(x'-x)\delta_{r,k} + (s\rho)^{-1}\partial_r'\partial_k' \hat{G}(x',x,s) , \tag{8.1-50}$$

where ∂_m' means differentiation with respect to x_m', and

$$\hat{G}(x',x,s) = \frac{\exp(-s|x'-x|/c)}{4\pi|x'-x|} \qquad \text{for } x' \neq x . \tag{8.1-51}$$

In the *far-field region*, the expansion

$$\{\hat{p}^{s},\hat{v}_r^{s}\}(x',s) = \{\hat{p}^{s;\infty},\hat{v}_r^{s;\infty}\}(\xi,s)\,\frac{\exp(-s|x'|/c)}{4\pi|x'|}\left[1 + O(|x'|^{-1})\right]$$
$$\text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi \tag{8.1-52}$$

holds, where (see Equations (5.9-11)–(5.9-16))

$$\hat{p}^{s;\infty} = s\rho\hat{\Phi}^{q^s;\infty} + (s/c)\xi_k\hat{\Phi}_k^{f^s;\infty} \tag{8.1-53}$$

and

$$\hat{v}_r^{s;\infty} = (\rho c)^{-1}\xi_r\hat{p}^{s;\infty}, \tag{8.1-54}$$

in which

$$\hat{\Phi}^{q^s;\infty}(\xi,s) = \int_{x\in\mathcal{D}^s} \hat{q}^s(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}V , \tag{8.1-55}$$

and

$$\hat{\Phi}_k^{f^s;\infty}(\xi,s) = \int_{x\in\mathcal{D}^s} \hat{f}_k^{s}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}V . \tag{8.1-56}$$

For an *acoustically impenetrable scatterer* the acoustic wave field is not defined in the interior $\mathcal{D}^s$ of the scatterer and we have to resort to an equivalent surface source integral representation that expresses the scattered wave field in the exterior $\mathcal{D}^{s'}$ of the scatterer in terms of the wave field values on the boundary surface $\partial\mathcal{D}^s$ of $\mathcal{D}^s$. This representation is, on account of Equations (7.12-40) and (7.12-41),

$$\hat{p}^{s}(x',s)\chi_{\mathcal{D}^{s'}}(x') = \int_{x\in\partial\mathcal{D}^s}\left[\hat{G}^{pq}(x',x,s)\partial\hat{q}^{s}(x,s) + \hat{G}_m^{pf}(x',x,s)\partial\hat{f}_k^{s}(x,s)\right]\mathrm{d}A$$
$$\text{for } x'\in\mathcal{R}^3 \tag{8.1-57}$$

and

$$\hat{v}_r^{\,\mathrm{s}}(\boldsymbol{x}',s)\chi_{\mathcal{D}^{s'}}(\boldsymbol{x}') = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \left[\hat{G}_r^{vq}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{q}^{\,\mathrm{s}}(\boldsymbol{x},s) + \hat{G}_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{f}_k^{\,\mathrm{s}}(\boldsymbol{x},s)\right] \mathrm{d}A$$

$$\text{for } \boldsymbol{x}'\in\mathcal{R}^3 , \tag{8.1-58}$$

in which (note the orientation of ν_m)

$$\partial\hat{q}^{\,\mathrm{s}} = \nu_r\hat{v}_r^{\,\mathrm{s}} \tag{8.1-59}$$

and

$$\partial\hat{f}_k^{\,\mathrm{s}} = \nu_k\hat{p}^{\,\mathrm{s}}. \tag{8.1-60}$$

In the *far-field region*, the expansion given in Equation (8.1-15) holds, where, based upon Equations (8.1-57) and (8.1-58), we have

$$\hat{p}^{\,\mathrm{s};\infty} = s\rho\hat{\Phi}^{\partial q^{\mathrm{s}};\infty} + (s/c)\xi_k\hat{\Phi}_k^{\partial f^{\mathrm{s}};\infty} \tag{8.1-61}$$

and

$$\hat{v}_r^{\,\mathrm{s};\infty} = (\rho c)^{-1}\xi_r\,\hat{p}^{\,\mathrm{s};\infty}, \tag{8.1-62}$$

in which

$$\hat{\Phi}^{\partial q^{\mathrm{s}};\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{q}^{\,\mathrm{s}}(\boldsymbol{x},s)\exp(s\xi_s x_s/c)\,\mathrm{d}A \tag{8.1-63}$$

and

$$\hat{\Phi}_k^{\partial f^{\mathrm{s}};\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{f}_k^{\,\mathrm{s}}(\boldsymbol{x},s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,. \tag{8.1-64}$$

However, upon applying Equations (7.12-30) and (7.12-37) to the incident wave field $\{\hat{p}^{\,\mathrm{i}},\hat{v}_r^{\,\mathrm{i}}\}$ and to the domain $\mathcal{D}^{\mathrm{s}}$, we have (note that the incident wave field is source-free in $\mathcal{D}^{\mathrm{s}}$)

$$\hat{p}^{\,\mathrm{i}}(\boldsymbol{x}',s)\chi_{\mathcal{D}^{\mathrm{s}}}(\boldsymbol{x}') = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \left[\hat{G}^{pq}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{q}^{\,\mathrm{i}}(\boldsymbol{x},s) + \hat{G}_m^{pf}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{f}_k^{\,\mathrm{i}}(\boldsymbol{x},s)\right] \mathrm{d}A$$

$$\text{for } \boldsymbol{x}'\in\mathcal{R}^3 \tag{8.1-65}$$

and

$$\hat{v}_r^{\,\mathrm{i}}(\boldsymbol{x}',s)\chi_{\mathcal{D}^{\mathrm{s}}}(\boldsymbol{x}') = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \left[\hat{G}_r^{vq}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{q}^{\,\mathrm{i}}(\boldsymbol{x},s) + \hat{G}_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{f}_k^{\,\mathrm{i}}(\boldsymbol{x},s)\right] \mathrm{d}A$$

$$\text{for } \boldsymbol{x}'\in\mathcal{R}^3 , \tag{8.1-66}$$

in which (note the orientation of ν_m)

$$\partial\hat{q}^{\,\mathrm{i}} = -\nu_r\hat{v}_r^{\,\mathrm{i}} \tag{8.1-67}$$

and

$$\partial\hat{f}_k^{\,\mathrm{i}} = -\nu_k\hat{p}^{\,\mathrm{i}}. \tag{8.1-68}$$

Subtraction of Equation (8.1-65) from Equation (8.1-57) and of Equation (8.1-66) from Equation (8.1-58) leads to

$$\hat{p}(x',s)\chi_{\mathcal{D}^{s'}}(x') - \hat{p}^{\mathrm{i}}(x,s)\chi_{\mathcal{D}^s}(x')$$
$$= \int_{x\in\partial\mathcal{D}^s} \left[\hat{G}^{pq}(x',x,s)\partial\hat{q}(x,s) + \hat{G}^{pf}_k(x',x,s)\partial\hat{f}_k(x,s)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{R}^3 \tag{8.1-69}$$

and

$$\hat{v}_r(x',s)\chi_{\mathcal{D}^{s'}}(x') - \hat{v}^{\mathrm{i}}_r(x,s)\chi_{\mathcal{D}^s}(x')$$
$$= \int_{x\in\partial\mathcal{D}^s} \left[\hat{G}^{vq}_r(x',x,s)\partial\hat{q}(x,s) + \hat{G}^{vf}_{r,k}(x',x,s)\partial\hat{f}_k(x,s)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{R}^3, \tag{8.1-70}$$

in which (note the orientation of ν_m)

$$\partial\hat{q} = \nu_r\hat{v}_r \tag{8.1-71}$$

and

$$\partial\hat{f}_k = \nu_k\hat{p}\,. \tag{8.1-72}$$

In the *far-field region*, again the expansion given in Equation (8.1-52) holds, where, based upon Equations (8.1-69) and (8.1-70), we have

$$\hat{p}^{\,\mathrm{s};\infty} = s\rho\hat{\Phi}^{\partial q;\infty} + (s/c)\xi_k\hat{\Phi}^{\partial f;\infty}_k \tag{8.1-73}$$

and

$$\hat{v}^{\,\mathrm{s};\infty}_r = (\rho c)^{-1}\xi_r\hat{p}^{\,\mathrm{s};\infty}, \tag{8.1-74}$$

in which

$$\hat{\Phi}^{\partial q;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^s} \partial\hat{q}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A \tag{8.1-75}$$

and

$$\hat{\Phi}^{\partial f;\infty}_k(\xi,s) = \int_{x\in\partial\mathcal{D}^s} \partial\hat{f}_k(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,. \tag{8.1-76}$$

Of course, the equivalent surface source representations also apply to the case of an acoustically penetrable scatterer. For $x'\in\mathcal{D}^{s\,\prime}$, Equations (8.1-57) and (8.1-58) and Equations (8.1-69) and (8.1-70) must then yield the same result as Equations (8.1-45) and (8.1-46). Similarly, in the far-field region, Equations (8.1-61)–(8.1-64) and (8.1-73)–(8.1-76) must yield the same result as Equations (8.1-53)–(8.1-56). Note, however, that the results for $x'\in\mathcal{D}^s$ differ.

Equations (8.1-45) and (8.1-46), when taken for $x'\in\mathcal{D}^s$, provide the basis for the *complex frequency- domain domain integral equation method* to solve problems of the scattering by penetrable objects. For solving problems of the scattering by impenetrable objects, Equations (8.1-69) and (8.1-70) provide, when taken for $x'\in\partial\mathcal{D}^s$, the basis for the *complex frequency-domain boundary integral equation method* and, when taken for $x'\in\mathcal{D}^s$, the basis for the *complex* *frequency-domain null-field method.* For general scatterers, all three methods need numerical implementation.

The different representations in this section will be needed in the analysis in the remainder of this chapter.

Exercises

Exercise 8.1-1

Show that from Equations (8.1-32) and (8.1-33) it follows that

$$p(\boldsymbol{x}',t)\chi_{\mathcal{D}^{s\prime}}(\boldsymbol{x}') = p^{\mathrm{i}}(\boldsymbol{x}',t) + \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G^{pq},\partial q;\boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G_k^{pf},\partial f_k;\boldsymbol{x}',\boldsymbol{x},t)\right]\mathrm{d}A \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 \tag{8.1-77}$$

and

$$v_r(\boldsymbol{x}',t)\chi_{\mathcal{D}^{s\prime}}(\boldsymbol{x}') = v_r^{\mathrm{i}}(\boldsymbol{x}',t) + \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G_r^{vq},\partial q;\boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G_{r,k}^{vf},\partial f_k;\boldsymbol{x}',\boldsymbol{x},t)\right]\mathrm{d}A \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 , \tag{8.1-78}$$

in which (note the orientation of ν_m)

$$\partial q = \nu_r v_r \tag{8.1-79}$$

and

$$\partial f_k = \nu_k p . \tag{8.1-80}$$

(*Hint*: Consider the cases $\boldsymbol{x}'\in\mathcal{D}^{s\prime}$, $\boldsymbol{x}'\in\partial\mathcal{D}^s$ and $\boldsymbol{x}'\in\mathcal{D}^s$.)

Exercise 8.1-2

Show that from Equations (8.1-69) and (8.1-70) it follows that

$$\hat{p}(\boldsymbol{x}',s)\chi_{\mathcal{D}^{s\prime}}(\boldsymbol{x}') = \hat{p}^{\mathrm{i}}(\boldsymbol{x},s) + \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \left[\hat{G}^{pq}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{q}(\boldsymbol{x},s) + \hat{G}_k^{pf}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{f}_k(\boldsymbol{x},s)\right]\mathrm{d}A \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 \tag{8.1-81}$$

and

$$\hat{v}_r(\boldsymbol{x}',s)\chi_{\mathcal{D}^{s\prime}}(\boldsymbol{x}') = \hat{v}_r^{\mathrm{i}}(\boldsymbol{x},s) + \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \left[\hat{G}_r^{vq}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{q}(\boldsymbol{x},s) + \hat{G}_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},s)\partial\hat{f}_k(\boldsymbol{x},s)\right]\mathrm{d}A \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 , \tag{8.1-82}$$

in which (note the orientation of ν_m)

$$\partial\hat{q} = \nu_r \hat{v}_r \tag{8.1-83}$$

and

$$\partial\hat{f}_k = \nu_k \hat{p}\,. \tag{8.1-84}$$

(*Hint*: Consider the cases $x' \in \mathcal{D}^{s\,\prime}$, $x' \in \partial\mathcal{D}^s$ and $x' \in \mathcal{D}^s$.)

8.2 Far-field scattered wave amplitude reciprocity of the time convolution type

In this section we investigate the reciprocity relation of the time convolution type that applies to the far-field scattered wave amplitude at plane wave incidence upon an acoustically penetrable or impenetrable object. The scattering configuration shown in Figure 8.1-1 applies. Two states (A and B) in this configuration are considered. In state A, a uniform plane acoustic wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ is incident upon the scattering object; in state B, a uniform plane acoustic wave that propagates in the direction of the unit vector $\boldsymbol{\beta}$ is incident upon the scattering object. It will be shown that the far-field scattered wave amplitude in state A when observed in the direction of observation $\boldsymbol{\xi} = -\boldsymbol{\beta}$ is related, via reciprocity, to the far-field scattered wave amplitude in state B when observed in the direction of observation $\boldsymbol{\xi} = -\boldsymbol{\alpha}$ (Figure 8.2-1).

The corresponding relationships in the time domain and in the complex frequency domain are derived separately below.

Time-domain analysis

In the time-domain analysis, the incident uniform plane wave in state A is taken as

$$\{p^{\mathrm{i;A}}, v_r^{\mathrm{i;A}}\} = \{P^{\mathrm{A}}, V_r^{\mathrm{A}}\} a(t - \alpha_s x_s / c)\,, \tag{8.2-1}$$

with

$$V_r^{\mathrm{A}} = Y P^{\mathrm{A}} \alpha_r\,, \tag{8.2-2}$$

in which Y is given by Equation (8.1-7). In the far-field region, the scattered wave in state A is represented as

$$\{p^{\mathrm{s;A}}, v_r^{\mathrm{s;A}}\}(\boldsymbol{x}', t) = \frac{\{p^{\mathrm{s;A;\infty}}, v_r^{\mathrm{s;A;\infty}}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1 + O(|\boldsymbol{x}'|^{-1})\right]$$

$$\text{as } |\boldsymbol{x}'| \to \infty \text{ with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi}\,, \tag{8.2-3}$$

in which, on account of Equations (8.1-22)–(8.1-27),

$$p^{\mathrm{s;A;\infty}} = \rho\partial_t \Phi^{\partial q^{\mathrm{s;A;\infty}}} + c^{-1}\xi_k \partial_t \Phi_k^{\partial f^{\mathrm{s;A;\infty}}} \tag{8.2-4}$$

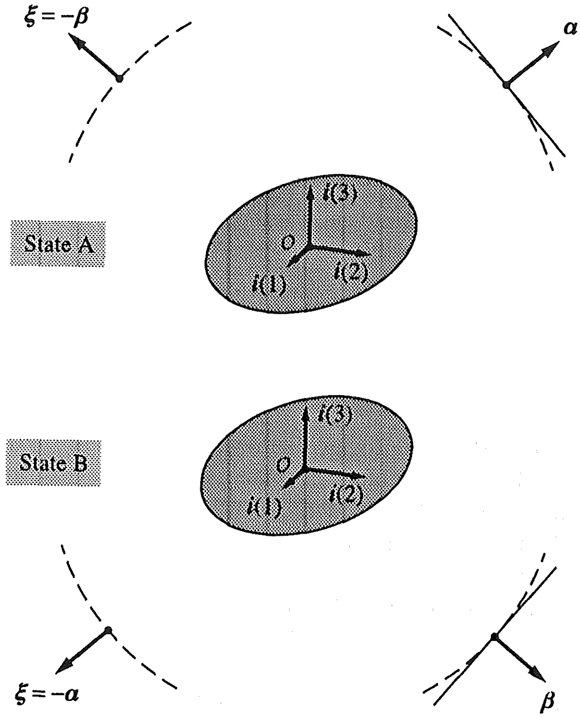

Figure 8.2-1 Configuration for the far-field scattered wave amplitude reciprocity of the time convolution type.

and

$$v_r^{s;A;\infty} = (\rho c)^{-1} \xi_r p^{s;A;\infty}, \tag{8.2-5}$$

with

$$\Phi^{\partial q^{s};A;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial q^{s;A}(x,t+\xi_s x_s/c)\,\mathrm{d}A \tag{8.2-6}$$

and

$$\Phi_k^{\partial f^{s};A;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial f_k^{s;A}(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{8.2-7}$$

in which (note the orientation of ν_m)

$$\partial q^{s;A} = \nu_r v_r^{s;A} \tag{8.2-8}$$

and

$$\partial f_k^{s;A} = \nu_k p^{s;A}. \tag{8.2-9}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{p^{\text{i;B}},v_r^{\text{i;B}}\} = \{P^{\text{B}},V_r^{\text{B}}\}b(t-\beta_s x_s/c)\,, \tag{8.2-10}$$

with

$$V_r^{\text{B}} = YP^{\text{B}}\beta_r\,. \tag{8.2-11}$$

In the far-field region, the scattered wave in state B is represented as

$$\{p^{\text{s;B}},v_r^{\text{s;B}}\}(x',t) = \frac{\{p^{\text{s;B;}\infty},v_r^{\text{s;B;}\infty}\}(\xi,t-|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$

$$\text{as } |x'|\to\infty \ \text{ with } x' = |x'|\xi\,, \tag{8.2-12}$$

in which, on account of Equations (8.1-22)–(8.1-27),

$$p^{\text{s;B;}\infty} = \rho\partial_t\Phi^{\partial q^{\text{s;B;}\infty}} + c^{-1}\xi_k\partial_t\Phi_k^{\partial f^{\text{s;B;}\infty}} \tag{8.2-13}$$

and

$$v_r^{\text{s;B;}\infty} = (\rho c)^{-1}\xi_r p^{\text{s;B;}\infty}, \tag{8.2-14}$$

with

$$\Phi^{\partial q^{\text{s;B;}\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^{\text{s}}} \partial q^{\text{s;B}}(x,t+\xi_s x_s/c)\,\mathrm{d}A \tag{8.2-15}$$

and

$$\Phi_k^{\partial f^{\text{s;B;}\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^{\text{s}}} \partial f_k^{\text{s;B}}(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{8.2-16}$$

in which (note the orientation of ν_m)

$$\partial q^{\text{s;B}} = \nu_r v_r^{\text{s;B}} \tag{8.2-17}$$

and

$$\partial f_k^{\text{s;B}} = \nu_k p^{\text{s;B}}. \tag{8.2-18}$$

If the scatterer is penetrable, its acoustic fluid properties in state B are assumed to be the adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (8.1-3) or (8.1-4) applies. These boundary conditions apply to both state A and state B, and are, therefore, self-adjoint.

To establish the desired reciprocity relation, we first apply the time-domain reciprocity theorem of the time convolution type (Equation (7.2-7)) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\text{s}}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\int_{x\in\partial\mathcal{D}^{\text{s}}} \nu_m\left[\mathrm{C}_t(p^{\text{A}},v_m^{\text{B}};x,t) - \mathrm{C}_t(p^{\text{B}}v_m^{\text{A}};x,t)\right]\mathrm{d}A = 0\,, \tag{8.2-19}$$

since in the interior of the scatterer the total wave fields are source-free. For an impenetrable scatterer, Equation (8.2-19) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\text{s}}$ via $\mathcal{D}^{\text{s}\prime}$. In Equation (8.2-19) we substitute

$$\{p^{A},v_r^{A}\} = \{p^{i;A} + p^{s;A}, v_r^{i;A} + v_r^{s;A}\} \tag{8.2-20}$$

and

$$\{p^{B},v_r^{B}\} = \{p^{i;B} + p^{s;B}, v_r^{i;B} + v_r^{s;B}\} . \tag{8.2-21}$$

Next, the time-domain reciprocity theorem of the time convolution type is applied to the incident wave fields in states A and B and to the domain $\mathcal{D}^s$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is self-adjoint in its acoustic properties, this leads to

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[\mathrm{C}_t(p^{i;A},v_m^{i;B};x,t) - \mathrm{C}_t(p^{i;B}v_m^{i;A};x,t)\right]\mathrm{d}A = 0 . \tag{8.2-22}$$

Finally, the time-domain reciprocity theorem of the time convolution type is applied to the scattered wave fields in states A and B and to the domain $\mathcal{D}^{s\prime}$. Since the embedding is self-adjoint in its acoustic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[\mathrm{C}_t(p^{s;A},v_m^{s;B};x,t) - \mathrm{C}_t(p^{s;B}v_m^{s;A};x,t)\right]\mathrm{d}A = 0 . \tag{8.2-23}$$

From Equations (8.2-19)–(8.2-23) we conclude that

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[\mathrm{C}_t(p^{i;A},v_m^{s;B};x,t) + \mathrm{C}_t(p^{s;A},v_m^{i;B};x,t)\right.$$

$$\left. - \mathrm{C}_t(p^{i;B}v_m^{s;A};x,t) - \mathrm{C}_t(p^{s;B}v_m^{i;A};x,t)\right]\mathrm{d}A = 0 . \tag{8.2-24}$$

However, on account of Equations (8.2-1) and (8.2-2), (8.2-4)–(8.2-9), (8.2-10) and (8.2-11), and (8.2-13)–(8.2-18) we have

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[\mathrm{C}_t(p^{s;A},v_m^{i;B};x,t) - \mathrm{C}_t(p^{i;B},v_m^{s;A};x,t)\right]\mathrm{d}A$$

$$= \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x\in\partial\mathcal{D}^s} \nu_m\left[p^{s;A}(x,t')V_m^{B} - P^{B}v_m^{s;A}(x,t')\right] b(t-\beta_s x_s/c - t')\,\mathrm{d}A$$

$$= \int_{t''\in\mathcal{R}} b(t-t'')\,\mathrm{d}t'' \int_{x\in\partial\mathcal{D}^s} \nu_m\left[p^{s;A}(x,t''-\beta_s x_s/c)V_m^{B} - P^{B}v_m^{s;A}(x,t''-\beta_s x_s/c)\right]\mathrm{d}A$$

$$= -\rho^{-1}P^{B}\int_{t''\in\mathcal{R}} b(t-t'')\mathrm{I}_t\left[p^{s;A;\infty}(-\beta,t'')\right]\mathrm{d}t'' \tag{8.2-25}$$

and

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[\mathrm{C}_t(p^{s;B},v_m^{i;A};x,t) - \mathrm{C}_t(p^{i;A},v_m^{s;B};x,t)\right]\mathrm{d}A$$

$$= \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{x \in \partial \mathcal{D}^s} \nu_m \left[p^{s;B}(x,t') V_m^{A} - P^{A} v_m^{s;B}(x,t') \right] a(t - \alpha_s x_s / c - t') \, \mathrm{d}A$$

$$= \int_{t'' \in \mathcal{R}} a(t - t'') \, \mathrm{d}t'' \int_{x \in \partial \mathcal{D}^s} \nu_m \left[p^{s;B}(x,t'' - \alpha_s x_s / c) V_m^{A} - P^{A} v_m^{s;B}(x,t'' - \alpha_s x_s / c) \right] \mathrm{d}A$$

$$= -\rho^{-1} P^{A} \int_{t'' \in \mathcal{R}} a(t - t'') \mathrm{I}_t \left[p^{s;B;\infty}(-\alpha,t'') \right] \mathrm{d}t''. \tag{8.2-26}$$

Equations (8.2-24)–(8.2-26) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$P^{B} \int_{t'' \in \mathcal{R}} b(t - t'') \mathrm{I}_t \left[p^{s;A;\infty}(-\beta,t'') \right] \mathrm{d}t''$$

$$= P^{A} \int_{t'' \in \mathcal{R}} a(t - t'') \mathrm{I}_t \left[p^{s;B;\infty}(-\alpha,t'') \right] \mathrm{d}t''. \tag{8.2-27}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, both in state A and in state B. To this end, we write

$$p^{s;A;\infty}(\xi,t) = P^{A} \int_{t' \in \mathcal{R}} a(t') S^{A}(\xi,\alpha,t - t') \, \mathrm{d}t' \tag{8.2-28}$$

and

$$p^{s;B;\infty}(\xi,t) = P^{B} \int_{t' \in \mathcal{R}} b(t') S^{B}(\xi,\beta,t - t') \, \mathrm{d}t', \tag{8.2-29}$$

where S^{A} and S^{B} are the configurational time-domain *acoustic pressure far-field scattering coefficients*. Substitution of Equations (8.2-28) and (8.2-29) in Equation (8.2-27) and rewriting the convolutions, we obtain

$$P^{B} P^{A} \mathrm{I}_t \int_{t'' \in \mathcal{R}} b(t'') \, \mathrm{d}t'' \int_{t' \in \mathcal{R}} a(t') S^{A}(-\beta,\alpha,t - t'' - t') \, \mathrm{d}t'$$

$$= P^{A} P^{B} \mathrm{I}_t \int_{t'' \in \mathcal{R}} a(t'') \, \mathrm{d}t'' \int_{t' \in \mathcal{R}} b(t') S^{B}(-\alpha,\beta,t - t'' - t') \, \mathrm{d}t', \tag{8.2-30}$$

where, in accordance with the rules applying to the time convolution, the operator I_t has been brought in front of the integral signs. Taking into account that Equation (8.2-30) has to hold for arbitrary values of P^{A}, P^{B}, $a(t)$ and $b(t)$, and using the causality of the scattered waves, we end up with

$$S^{A}(-\beta,\alpha,t) = S^{B}(-\alpha,\beta,t) \tag{8.2-31}$$

as the final expression of the time-domain reciprocity property under consideration.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the incident uniform plane wave in state A is taken as

$$\{\hat{p}^{\text{i;A}},\hat{v}_r^{\text{i;A}}\} = \{P^{\text{A}},V_r^{\text{A}}\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{8.2-32}$$

with

$$V_r^{\text{A}} = YP^{\text{A}}\alpha_r\,, \tag{8.2-33}$$

in which Y is given by Equation (8.1-7). In the far-field region, the scattered wave in state A is represented as

$$\{\hat{p}^{\text{s;A}},\hat{v}_r^{\text{s;A}}\}(\boldsymbol{x}',s) = \{\hat{p}^{\text{s;A;}\infty},\hat{v}_r^{\text{s;A;}\infty}\}(\boldsymbol{\xi},s)\,\frac{\exp(-s|\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1+O(|\boldsymbol{x}'|^{-1})\right]$$

$$\text{as } |\boldsymbol{x}'|\to\infty \text{ with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi}\,, \tag{8.2-34}$$

where, on account of Equations (8.1-59)–(8.1-64),

$$\hat{p}^{\text{s;A;}\infty} = s\rho\hat{\Phi}^{\partial q^{\text{s}};\text{A;}\infty} + (s/c)\xi_k\hat{\Phi}_k^{\partial f^{\text{s}};\text{A;}\infty}\,, \tag{8.2-35}$$

and

$$\hat{v}_r^{\text{s;A;}\infty} = (\rho c)^{-1}\xi_r\,\hat{p}^{\text{s;A;}\infty}, \tag{8.2-36}$$

with

$$\hat{\Phi}^{\partial q^{\text{s}};\text{A;}\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\text{s}}} \partial\hat{q}^{\text{s;A}}(\boldsymbol{x},s)\exp(s\xi_s x_s/c)\,\mathrm{d}A \tag{8.2-37}$$

and

$$\hat{\Phi}_k^{\partial f^{\text{s}};\text{A;}\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\text{s}}} \partial\hat{f}_k^{\text{s;A}}(\boldsymbol{x},s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{8.2-38}$$

in which (note the orientation of ν_m)

$$\partial\hat{q}^{\text{s;A}} = \nu_r\hat{v}_r^{\text{s;A}} \tag{8.2-39}$$

and

$$\partial\hat{f}_k^{\text{s;A}} = \nu_k\hat{p}^{\text{s;A}}. \tag{8.2-40}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{\hat{p}^{\text{i;B}},\hat{v}_r^{\text{i;B}}\} = \{P^{\text{B}},V_r^{\text{B}}\}\hat{b}(s)\exp(-s\beta_s x_s/c)\,, \tag{8.2-41}$$

with

$$V_r^{\text{B}} = YP^{\text{B}}\beta_r\,. \tag{8.2-42}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\hat{p}^{\mathrm{s;B}},\hat{v}_r^{\mathrm{s;B}}\}(x',s) = \{\hat{p}^{\mathrm{s;B;\infty}},\hat{v}_r^{\mathrm{s;B;\infty}}\}(\xi,s)\,\frac{\exp(-s|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$

$$\text{as } |x'|\to\infty \text{ with } x' = |x'|\xi, \tag{8.2-43}$$

where, on account of Equations (8.1-59)–(8.1-64),

$$\hat{p}^{\mathrm{s;B;\infty}} = s\rho\hat{\Phi}^{\partial q^{\mathrm{s}};\mathrm{B};\infty} + (s/c)\xi_k\hat{\Phi}_k^{\partial f^{\mathrm{s}};\mathrm{B};\infty} \tag{8.2-44}$$

and

$$\hat{v}_r^{\mathrm{s;B;\infty}} = (\rho c)^{-1}\xi_r\hat{p}^{\mathrm{s;B;\infty}}, \tag{8.2-45}$$

with

$$\hat{\Phi}^{\partial q^{\mathrm{s}};\mathrm{B};\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{q}^{\mathrm{s;B}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A \tag{8.2-46}$$

and

$$\hat{\Phi}_k^{\partial f^{\mathrm{s}};\mathrm{B};\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{f}_k^{\mathrm{s;B}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{8.2-47}$$

in which (note the orientation of ν_m)

$$\partial\hat{q}^{\mathrm{s;B}} = \nu_r\hat{v}_r^{\mathrm{s;B}} \tag{8.2-48}$$

and

$$\partial\hat{f}_k^{\mathrm{s;B}} = \nu_k\hat{p}^{\mathrm{s;B}}. \tag{8.2-49}$$

If the scatterer is penetrable, its acoustic fluid properties in state B are assumed to be the adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (8.1-42) or (8.1-43) applies. These boundary conditions apply to both state A and state B, and are, therefore, self-adjoint.

To establish the desired reciprocity relation, we first apply the complex frequency-domain reciprocity theorem of the time convolution type Equation (7.4-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\hat{p}^{\mathrm{A}}(x,s)\hat{v}_m^{\mathrm{B}}(x,s) - \hat{p}^{\mathrm{B}}(x,s)\hat{v}_m^{\mathrm{A}}(x,s)\right]\mathrm{d}A = 0\,, \tag{8.2-50}$$

since in the interior of the scatterer the total wave field is source-free. For an impenetrable scatterer, Equation (8.2-50) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\mathrm{s}}$ via $\mathcal{D}^{\mathrm{s}\prime}$. In Equation (8.2-50) we substitute

$$\{\hat{p}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{p}^{\mathrm{i;A}} + \hat{p}^{\mathrm{s;A}},\hat{v}_r^{\mathrm{i;A}} + \hat{v}_r^{\mathrm{s;A}}\} \tag{8.2-51}$$

and

$$\{\hat{p}^{\mathrm{B}},\hat{v}_r^{\mathrm{B}}\} = \{\hat{p}^{\mathrm{i;B}} + \hat{p}^{\mathrm{s;B}},\hat{v}_r^{\mathrm{i;B}} + \hat{v}_r^{\mathrm{s;B}}\}\,. \tag{8.2-52}$$

Next, the complex frequency-domain reciprocity theorem of the time convolution type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^{\mathrm{s}}$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is self-adjoint in its acoustic properties, this leads to

$$\int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,i;A}(x,s) \hat{v}_m^{i;B}(x,s) - \hat{p}^{\,i;B}(x,s) \hat{v}_m^{i;A}(x,s) \right] \mathrm{d}A = 0 . \tag{8.2-53}$$

Finally, the complex frequency-domain reciprocity theorem of the time convolution type is applied to the scattered wave fields in the states A and B, and to the domain $\mathcal{D}^{s'}$. Since the embedding is self-adjoint in its acoustic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,s;A}(x,s) \hat{v}_m^{s;B}(x,s) - \hat{p}^{\,s;B}(x,s) \hat{v}_m^{s;A}(x,s) \right] \mathrm{d}A = 0 . \tag{8.2-54}$$

From Equations (8.2-50)–(8.2-54) we conclude that

$$\int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,i;A}(x,s) \hat{v}_m^{s;B}(x,s) + \hat{p}^{\,s;A}(x,s) \hat{v}_m^{i;B}(x,s) \right.$$
$$\left. - \hat{p}^{\,i;B}(x,s) \hat{v}_m^{s;A}(x,s) - \hat{p}^{\,s;B}(x,s) \hat{v}_m^{i;A}(x,s) \right] \mathrm{d}A = 0 . \tag{8.2-55}$$

However, on account of Equations (8.2-32) and (8.2-33), (8.2-35)–(8.2-40), (8.2-41) and (8.2-42), and (8.2-44)–(8.2-49) we have

$$\int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,s;A}(x,s) \hat{v}_m^{i;B}(x,s) - \hat{p}^{\,i;B}(x,s) \hat{v}_m^{s;A}(x,s) \right] \mathrm{d}A$$
$$= \int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,s;A}(x,s) V_m^{B} - P^{B} \hat{v}_m^{s;A}(x,s) \right] \hat{b}(s) \exp(-s\beta_s x_s/c) \, \mathrm{d}A$$
$$= -(s\rho)^{-1} P^{B} \hat{b}(s) \hat{p}^{\,s;A;\infty}(-\beta,s) \tag{8.2-56}$$

and

$$\int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,s;B}(x,s) \hat{v}_m^{i;A}(x,s) - \hat{p}^{\,i;A}(x,s) \hat{v}_m^{s;B}(x,s) \right] \mathrm{d}A$$
$$= \int_{x \in \partial \mathcal{D}^s} \nu_m \left[\hat{p}^{\,s;B}(x,s) V_m^{A} - P^{A} \hat{v}_m^{s;B}(x,s) \right] \hat{a}(s) \exp(-s\alpha_s x_s/c) \, \mathrm{d}A$$
$$= -(s\rho)^{-1} P^{A} \hat{a}(s) \hat{p}^{\,s;B;\infty}(-\alpha,s) . \tag{8.2-57}$$

Equations (8.2-55)–(8.2-57) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$P^{B} \hat{b}(s) \hat{p}^{\,s;A;\infty}(-\beta,s) = P^{A} \hat{a}(s) \hat{p}^{\,s;B;\infty}(-\alpha,s) . \tag{8.2-58}$$

At this point it is, again, elegant to express the linear relationship that exists between the acoustic pressure far-field scattered wave amplitude and the incident wave field, both in state A and in state B. To this end, we write, in accordance with Equations (8.2-28) and (8.2-29)

$$\hat{p}^{\,s;A;\infty}(\xi,s) = P^{A}\hat{a}(s)\hat{S}^{A}(\xi,\alpha,s) \tag{8.2-59}$$

and

$$\hat{p}^{\,s;B;\infty}(\xi,s) = P^{B}\hat{b}(s)\hat{S}^{B}(\xi,\beta,s)\,, \tag{8.2-60}$$

where $\hat{S}^{A}$ and $\hat{S}^{B}$ are the configurational complex frequency-domain *acoustic pressure far-field scattering coefficients*. Substitution of Equations (8.2-59) and (8.2-60) in Equation (8.2-58) yields

$$P^{B}P^{A}\hat{b}(s)\hat{a}(s)\hat{S}^{A}(-\beta,\alpha,s) = P^{A}P^{B}\hat{a}(s)\hat{b}(s)\hat{S}^{B}(-\alpha,\beta,s)\,. \tag{8.2-61}$$

Taking into account that Equation (8.2-61) has to hold for arbitrary values of P^{A}, P^{B}, $\hat{a}(s)$ and $\hat{b}(s)$, we end up with

$$\hat{S}^{A}(-\beta,\alpha,s) = \hat{S}^{B}(-\alpha,\beta,s) \tag{8.2-62}$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

In a theoretical analysis, the reciprocity relations derived in this section serve as an important check on the correctness of the analytic solutions, as well as on the accuracy of numerical solutions to scattering problems. Note, however, that the reciprocity relations are necessary conditions to be satisfied by the scattered wave field (in the far-field region), but their satisfaction does not guarantee correctness of a total analytic solution or a certain accuracy of a total numerical solution. In a physical experiment, the redundancy induced by the reciprocity relations can be exploited to reduce the influence of noise on the quality of the observed data.

References to the earlier literature on the reciprocity relations of the type discussed in this section can be found in De Hoop (1960).

Exercises

Exercise 8.2-1

To what form does Equation (8.2-27) reduce if $a(t) = b(t)$ and $P^{A} = P^{B}$? (*Hint*: Use the fact that the resulting indentity has to hold for any pulse shape and employ the causality of the scattered wave.)

Answer:

$$p^{\,s;A;\infty}(-\beta,t) = p^{\,s;B;\infty}(-\alpha,t)\,.$$

Exercise 8.2-2

To what form does Equation (8.2-58) reduce if $\hat{a}(s) = \hat{b}(s)$ and $P^{A} = P^{B}$?

Answer:

$$\hat{p}^{\,s;A;\infty}(-\beta,s) = \hat{p}^{\,s;B;\infty}(-\alpha,s)\,.$$

Exercise 8.2-3

Show that Equation (8.2-62) follows from Equation (8.2-31) by taking the time Laplace transform.

8.3 Far-field scattered wave amplitude reciprocity of the time correlation type

In this section we investigate the reciprocity relation of the time correlation type that applies to the far-field scattered wave amplitude reciprocity for plane wave incidence upon an acoustically penetrable or impenetrable object. The scattering configuration shown in Figure 8.1-1 applies. Two states (A and B) in this configuration are considered. In state A, a uniform plane acoustic wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ is incident upon the scattering object; in state B, a uniform plane acoustic wave that propagates in the direction of the unit vector $\boldsymbol{\beta}$ is incident upon the scattering object. It will be shown that a certain relation exists between the far-field scattered wave amplitudes in states A and B (Figure 8.3-1).

The corresponding relationships in the time domain and in the complex frequency domain will be derived separately below.

Time-domain analysis

In the time-domain analysis, the incident uniform plane wave in state A is taken as

$$\{p^{\mathrm{i;A}}, v_r^{\mathrm{i;A}}\} = \{P^{\mathrm{A}}, V_r^{\mathrm{A}}\}a(t - \alpha_s x_s/c)\,, \tag{8.3-1}$$

with

$$V_r^{\mathrm{A}} = YP^{\mathrm{A}}\alpha_r\,, \tag{8.3-2}$$

in which Y is given by Equation (8.1-7). In the far-field region, the scattered wave in state A is represented as

$$\{p^{\mathrm{s;A}}, v_r^{\mathrm{s;A}}\}(\boldsymbol{x}',t) = \frac{\{p^{\mathrm{s;A;\infty}}, v_r^{\mathrm{s;A;\infty}}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1 + O(|\boldsymbol{x}'|^{-1})\right]$$

$$\text{as } |\boldsymbol{x}'| \to \infty \text{ with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi}\,, \tag{8.3-3}$$

in which, on account of Equations (8.1-22)–(8.1-27),

$$p^{\mathrm{s;A;\infty}} = \rho\partial_t\Phi^{\partial q^{\mathrm{s;A;\infty}}} + c^{-1}\xi_k\partial_t\Phi_k^{\partial f^{\mathrm{s;A;\infty}}} \tag{8.3-4}$$

and

$$v_r^{\mathrm{s;A;\infty}} = (\rho c)^{-1}\xi_r p^{\mathrm{s;A;\infty}}\,, \tag{8.3-5}$$

with

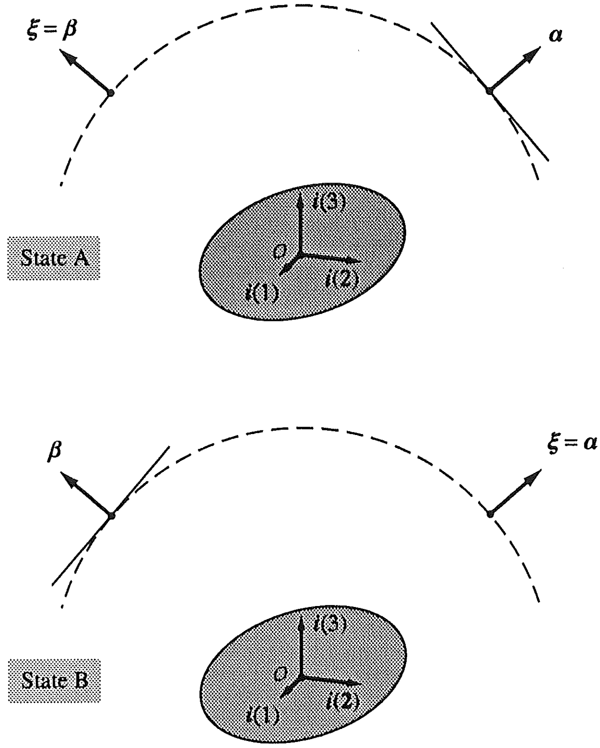

Figure 8.3-1 Configuration for the far-field scattered wave amplitude reciprocity of the time correlation type.

$$\Phi^{\partial q^{s};A;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^{s}} \partial q^{s;A}(x,t+\xi_s x_s/c)\,dA \tag{8.3-6}$$

and

$$\Phi_k^{\partial f^{s};A;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^{s}} \partial f_k^{s;A}(x,t+\xi_s x_s/c)\,dA\,, \tag{8.3-7}$$

in which (note the orientation of ν_m)

$$\partial q^{s;A} = \nu_r v_r^{s;A} \tag{8.3-8}$$

and

$$\partial f_k^{s;A} = \nu_k p^{s;A}. \tag{8.3-9}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{p^{i;B},v_r^{i;B}\} = \{P^{B},V_r^{B}\}b(t-\beta_s x_s/c)\,, \tag{8.3-10}$$

with

$$V_r^{\mathrm{B}} = YP^{\mathrm{B}}\beta_r . \tag{8.3-11}$$

In the far-field region, the scattered wave in state B is represented as

$$\{p^{\mathrm{s;B}}, v_r^{\mathrm{s;B}}\}(x',t) = \frac{\{p^{\mathrm{s;B;\infty}}, v_r^{\mathrm{s;B;\infty}}\}(\xi, t - |x'|/c)}{4\pi|x'|}\left[1 + O(|x'|^{-1})\right]$$

$$\text{as } |x'| \to \infty \quad \text{with } x' = |x'|\xi , \tag{8.3-12}$$

in which, on account of Equations (8.1-22)–(8.1-27),

$$p^{\mathrm{s;B;\infty}} = \rho\partial_t \Phi^{\partial q^{\mathrm{s;B;\infty}}} + c^{-1}\xi_k \partial_t \Phi_k^{\partial f^{\mathrm{s;B;\infty}}} \tag{8.3-13}$$

and

$$v_r^{\mathrm{s;B;\infty}} = (\rho c)^{-1}\xi_r p^{\mathrm{s;B;\infty}} , \tag{8.3-14}$$

with

$$\Phi^{\partial q^{\mathrm{s;B;\infty}}}(\xi,t) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial q^{\mathrm{s;B}}(x, t + \xi_s x_s/c)\,\mathrm{d}A \tag{8.3-15}$$

and

$$\Phi_k^{\partial f^{\mathrm{s;B;\infty}}}(\xi,t) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial f_k^{\mathrm{s;B}}(x, t + \xi_s x_s/c)\,\mathrm{d}A , \tag{8.3-16}$$

in which (note the orientation of ν_m)

$$\partial q^{\mathrm{s;B}} = \nu_r v_r^{\mathrm{s;B}} \tag{8.3-17}$$

and

$$\partial f_k^{\mathrm{s;B}} = \nu_k p^{\mathrm{s;B}} . \tag{8.3-18}$$

If the scatterer is penetrable, its acoustic fluid properties in state B are assumed to be the time-reverse adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (8.1-3) or (8.1-4) applies. These boundary conditions apply to both state A and state B, and are, therefore, time-reverse self-adjoint.

To establish the desired reciprocity relation, we first apply the time-domain reciprocity theorem of the time correlation type (Equation (7.3-7)) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(p^{\mathrm{A}}, \mathrm{J}_t(v_m^{\mathrm{B}}); x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{\mathrm{B}}), v_m^{\mathrm{A}}; x,t)\right]\mathrm{d}A = 0 , \tag{8.3-19}$$

since in the interior of the scatterer the total wave fields are source-free. For an impenetrable scatterer, Equation (8.3-19) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\mathrm{s}}$ via $\mathcal{D}^{\mathrm{s}\prime}$. In Equation (8.3-19) we substitute

$$\{p^{\mathrm{A}}, v_r^{\mathrm{A}}\} = \{p^{\mathrm{i;A}} + p^{\mathrm{s;A}}, v_r^{\mathrm{i;A}} + v_r^{\mathrm{s;A}}\} \tag{8.3-20}$$

and

$$\{p^{B},v_r^{B}\} = \{p^{i;B} + p^{s;B}, v_r^{i;B} + v_r^{s;B}\} . \tag{8.3-21}$$

Next, the time-domain reciprocity theorem of the time correlation type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^s$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is time reverse self-adjoint in its acoustic properties, this leads to

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[C_t(p^{i;A},J_t(v_m^{i;B});x,t) + C_t(J_t(p^{i;B}),v_m^{i;A};x,t)\right] dA = 0 . \tag{8.3-22}$$

Finally, the time-domain reciprocity theorem of the time correlation type is applied to the scattered wave fields in the states A and B and to the domain $\mathcal{D}^{s'}$. Since the embedding is time reverse self-adjoint in its acoustic properties, and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[C_t(p^{s;A}, J_t(v_m^{s;B});x,t) + C_t(J_t(p^{s;B}),v_m^{s;A};x,t)\right] dA$$

$$= \lim_{\Delta\to\infty} \int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[C_t(p^{s;A}, J_t(v_m^{s;B});x,t) + C_t(J_t(p^{s;B}),v_m^{s;A};x,t)\right] dA , \tag{8.3-23}$$

where $\mathcal{S}(O,\Delta)$ is the sphere of radius Δ with its centre at the origin O of the chosen reference frame. From Equations (8.3-19)–(8.3-23) we conclude that

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\Big[C_t(p^{i;A}, J_t(v_m^{s;B});x,t) + C_t(p^{s;A}, J_t(v_m^{i;B});x,t)$$

$$+ C_t(J_t(p^{i;B}),v_m^{s;A};x,t) + C_t(J_t(p^{s;B}),v_m^{i;A};x,t)\Big] dA$$

$$+ \lim_{\Delta\to\infty} \int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[C_t(p^{s;A}, J_t(v_m^{s;B});x,t) + C_t(J_t(p^{s;B}),v_m^{s;A};x,t)\right] dA = 0 . \tag{8.3-24}$$

However, on account of Equations (8.3-1) and (8.3-2), (8.3-4)–(8.3-9), (8.3-10) and (8.3-11), and (8.3-13)–(8.3-18) we have

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[C_t(p^{s;A}, J_t(v_m^{i;B});x,t) + C_t(J_t(p^{i;B}),v_m^{s;A};x,t)\right] dA$$

$$= \int_{t'\in\mathcal{R}} dt' \int_{x\in\partial\mathcal{D}^s} \nu_m\left[p^{s;A}(x,t')V_m^{B} + P^{B}v_m^{s;A}(x,t')\right] b(t' - \beta_s x_s/c - t)\, dA$$

$$= \int_{t''\in\mathcal{R}} b(t'' - t)\, dt'' \int_{x\in\partial\mathcal{D}^s} \nu_m\left[p^{s;A}(x,t'' + \beta_s x_s/c)V_m^{B} + P^{B}v_m^{s;A}(x,t'' + \beta_s x_s/c)\right] dA$$

$$= \rho^{-1}P^{B} \int_{t''\in\mathcal{R}} b(t'' - t) I_t\left[p^{s;A;\infty}(\beta,t'')\right] dt'' \tag{8.3-25}$$

and

$$\int_{x\in\partial\mathcal{D}^s} \nu_m\left[C_t(J_t(p^{s;B}),v_m^{i;A};x,t) + C_t(p^{i;A}, J_t(v_m^{s;B});x,t)\right] dA$$

$$= \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{x \in \partial \mathcal{D}^s} \nu_m \left[p^{s;B}(x,t'-t) V_m^A + P^A v_m^{s;B}(x,t'-t) \right] a(t' - \alpha_s x_s / c) \, \mathrm{d}A$$

$$= \int_{t'' \in \mathcal{R}} a(t'') \, \mathrm{d}t'' \int_{x \in \partial \mathcal{D}^s} \nu_m \left[p^{s;B}(x,t'' + \alpha_s x_s/c - t) V_m^A + P^A v_m^{s;B}(x,t'' + \alpha_s x_s/c - t) \right] \mathrm{d}A$$

$$= \rho^{-1} P^A \int_{t'' \in \mathcal{R}} a(t'') \mathrm{I}_t \left[p^{s;B;\infty}(\alpha, t'' - t) \right] \mathrm{d}t''. \tag{8.3-26}$$

Furthermore, we have

$$\lim_{\Delta \to \infty} \int_{x \in \mathcal{S}(O,\Delta)} \nu_m \left[\mathrm{C}_t(p^{s;A}, \mathrm{J}_t(v_m^{s;B}); x,t) + \mathrm{C}_t(\mathrm{J}_t(p^{s;B}), v_m^{s;A}; x,t) \right] \mathrm{d}A$$

$$= (4\pi)^{-2} \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{\xi \in \Omega} \xi_m \left[p^{s;A;\infty}(\xi,t') v_m^{s;B;\infty}(\xi,t'-t) + p^{s;B;\infty}(\xi,t'-t) v_m^{s;A;\infty}(\xi,t') \right] \mathrm{d}A$$

$$= (8\pi^2)^{-1} (\rho c)^{-1} \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{\xi \in \Omega} p^{s;A;\infty}(\xi,t') p^{s;B;\infty}(\xi,t'-t) \, \mathrm{d}A \,, \tag{8.3-27}$$

where Ω is the sphere of unit radius with its centre at O. Equations (8.3-24)–(8.3-27) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$\rho^{-1} P^B \int_{t'' \in \mathcal{R}} b(t'' - t) \mathrm{I}_t \left[p^{s;A;\infty}(\beta, t'') \right] \mathrm{d}t''$$

$$+ \rho^{-1} P^A \int_{t'' \in \mathcal{R}} a(t'') \mathrm{I}_t \left[p^{s;B;\infty}(\alpha, t'' - t) \right] \mathrm{d}t''$$

$$= -(8\pi^2 \rho c)^{-1} \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{\xi \in \Omega} p^{s;A;\infty}(\xi,t') p^{s;B;\infty}(\xi,t'-t) \, \mathrm{d}A \,. \tag{8.3-28}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, in both state A and state B. Substitution of Equations (8.2-28) and (8.2-29) in Equation (8.3-28) and rewriting the convolutions and the correlation, we obtain

$$P^B P^A \int_{t'' \in \mathcal{R}} b(t'' - t) \, \mathrm{d}t'' \int_{t' \in \mathcal{R}} a(t') \mathrm{I}_t S^A(\beta, \alpha, t'' - t') \, \mathrm{d}t'$$

$$+ P^A P^B \int_{t'' \in \mathcal{R}} a(t'') \, \mathrm{d}t'' \int_{t' \in \mathcal{R}} b(t') \mathrm{I}_t S^B(\alpha, t'' - t - t') \, \mathrm{d}t'$$

$$= -(8\pi^2 c)^{-1} P^A P^B \int_{\tau \in \mathcal{R}} \mathrm{d}\tau \int_{\xi \in \Omega} \left[\int_{t' \in \mathcal{R}} a(t') S^A(\xi, \tau - t') \, \mathrm{d}t' \right.$$

$$\left. \int_{t'' \in \mathcal{R}} b(t'') S^B(\xi, \tau - t - t'') \, \mathrm{d}t'' \right] \mathrm{d}A \,, \tag{8.3-29}$$

or

$$P^{\rm B}P^{\rm A}\int_{t'\in\mathcal{R}} a(t')\,{\rm d}t' \int_{t''\in\mathcal{R}} b(t'')\mathrm{I}_t S^{\rm A}(\beta,\alpha,t+t''-t')\,{\rm d}t''$$

$$+P^{\rm A}P^{\rm B}\int_{t'\in\mathcal{R}} a(t')\,{\rm d}t' \int_{t''\in\mathcal{R}} b(t'')\mathrm{I}_t S^{\rm B}(\alpha,\beta,t'-t''-t)\,{\rm d}t''$$

$$=-(8\pi^2 c)^{-1}P^{\rm A}P^{\rm B}\int_{t\in\mathcal{R}} a(t')\,{\rm d}t' \int_{t''\in\mathcal{R}} b(t'')\,{\rm d}t''$$

$$\int_{\xi\in\Omega}\left[\int_{\tau\in\mathcal{R}} S^{\rm A}(\xi,\tau-t')S^{\rm B}(\xi,\tau-t-t'')\,{\rm d}\tau\right]{\rm d}A\,. \tag{8.3-30}$$

Since Equation (8.3-30) has to hold for arbitrary values of $P^{\rm A}$, $P^{\rm B}$, $a(t)$ and $b(t)$, we end up with

$$\mathrm{I}_t S^{\rm A}(\beta,\alpha,t)+\mathrm{I}_t S^{\rm B}(\alpha,\beta,-t)=-(8\pi^2 c)^{-1}\int_{\xi\in\Omega}\left[\int_{\tau\in\mathcal{R}} S^{\rm A}(\xi,\tau)S^{\rm B}(\xi,\tau-t)\,{\rm d}\tau\right]{\rm d}A \tag{8.3-31}$$

as the final expression of the reciprocity relation under consideration.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the incident uniform plane wave in state A is taken as

$$\{\hat{p}^{\,\rm i;A},\hat{v}_r^{\,\rm i;A}\}=\{P^{\rm A},V_r^{\rm A}\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{8.3-32}$$

with

$$V_r^{\rm A}=YP^{\rm A}\alpha_r\,, \tag{8.3-33}$$

in which Y is given by Equation (8.1-7). In the far-field region, the scattered wave in state A is represented as

$$\{\hat{p}^{\,\rm s;A},\hat{v}_r^{\,\rm s;A}\}(x',s)=\{\hat{p}^{\,\rm s;A;\infty},\hat{v}_r^{\,\rm s;A;\infty}\}(\xi,s)\frac{\exp(-s|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$

$$\text{as } |x'|\to\infty \ \text{ with } x'=|x'|\xi\,, \tag{8.3-34}$$

where, on account of Equations (8.1-59)–(8.1-64),

$$\hat{p}^{\,\rm s;A;\infty}=s\rho\hat{\Phi}^{\partial q^{\rm s};{\rm A};\infty}+(s/c)\xi_k\hat{\Phi}_k^{\partial f^{\rm s};{\rm A};\infty} \tag{8.3-35}$$

and

$$\hat{v}_r^{\,\rm s;A;\infty}=(\rho c)^{-1}\xi_r\,\hat{p}^{\,\rm s;A;\infty}, \tag{8.3-36}$$

with

$$\hat{\Phi}^{\partial q^{s};A;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^s} \partial\hat{q}^{\,s;A}(x,s)\exp(s\xi_s x_s/c)\,dA \tag{8.3-37}$$

and

$$\hat{\Phi}_k^{\partial f^{s};A;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^s} \partial\hat{f}_k^{\,s;A}(x,s)\exp(s\xi_s x_s/c)\,dA\,, \tag{8.3-38}$$

in which (note the orientation of ν_m)

$$\partial\hat{q}^{\,s;A} = \nu_r\hat{v}_r^{\,s;A} \tag{8.3-39}$$

and

$$\partial\hat{f}_k^{\,s;A} = \nu_k\hat{p}^{\,s;A}. \tag{8.3-40}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{\hat{p}^{\,i;B},\hat{v}_r^{\,i;B}\} = \{P^B,V_r^B\}\hat{b}(s)\exp(-s\beta_s x_s/c)\,, \tag{8.3-41}$$

with

$$V_r^B = YP^B\beta_r\,. \tag{8.3-42}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\hat{p}^{\,s;B},\hat{v}_r^{\,s;B}\}(x',s) = \{\hat{p}^{\,s;B;\infty},\hat{v}_r^{\,s;B;\infty}\}(\xi,s)\,\frac{\exp(-s|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$

$$\text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi\,, \tag{8.3-43}$$

where, on account of Equations (8.1-59)–(8.1-64),

$$\hat{p}^{\,s;B;\infty} = s\rho\hat{\Phi}^{\partial q^{s};B;\infty} + (s/c)\xi_k\hat{\Phi}_k^{\partial f^{s};B;\infty} \tag{8.3-44}$$

and

$$\hat{v}_r^{\,s;B;\infty} = (\rho c)^{-1}\xi_r\,\hat{p}^{\,s;B;\infty}, \tag{8.3-45}$$

with

$$\hat{\Phi}^{\partial q^{s};B;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^s} \partial\hat{q}^{\,s;B}(x,s)\exp(s\xi_s x_s/c)\,dA \tag{8.3-46}$$

and

$$\hat{\Phi}_k^{\partial f^{s};B;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^s} \partial\hat{f}_k^{\,s;B}(x,s)\exp(s\xi_s x_s/c)\,dA\,, \tag{8.3-47}$$

in which (note the orientation of ν_m)

$$\partial\hat{q}^{\,s;B} = \nu_r\hat{v}_r^{\,s;B} \tag{8.3-48}$$

and

$$\partial \hat{f}_k^{s;B} = \nu_k \hat{p}^{s;B} . \tag{8.3-49}$$

If the scatterer is penetrable, its acoustic fluid properties in state B are assumed to be the time-reverse adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (8.1-42) or (8.1-43) applies. These boundary conditions apply to both state A and state B, and are, therefore, time reverse self-adjoint.

To establish the desired reciprocity relation, we first apply the complex frequency-domain reciprocity theorem of the time correlation type Equation (7.5-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^s$ occupied by the scatterer. For a penetrable scatterer this yields

$$\int_{x\in\partial\mathcal{D}^s} \nu_m \left[\hat{p}^A(x,s)\hat{v}_m^B(x,-s) + \hat{p}^B(x,-s)\hat{v}_m^A(x,s) \right] \mathrm{d}A = 0 , \tag{8.3-50}$$

since in the interior of the scatterer the total wave field is source-free. For an impenetrable scatterer, Equation (8.3-50) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^s$ via $\mathcal{D}^{s'}$. In Equation (8.3-50) we substitute

$$\{\hat{p}^A,\hat{v}_r^A\} = \{\hat{p}^{i;A} + \hat{p}^{s;A}, \hat{v}_r^{i;A} + \hat{v}_r^{s;A}\} \tag{8.3-51}$$

and

$$\{\hat{p}^B,\hat{v}_r^B\} = \{\hat{p}^{i;B} + \hat{p}^{s;B}, \hat{v}_r^{i;B} + \hat{v}_r^{s;B}\} . \tag{8.3-52}$$

Next, the complex frequency-domain reciprocity theorem of the time correlation type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^s$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is time reverse self-adjoint in its acoustic properties, this leads to

$$\int_{x\in\partial\mathcal{D}^s} \nu_m \left[\hat{p}^{i;A}(x,s)\hat{v}_m^{i;B}(x,-s) + \hat{p}^{i;B}(x,-s)\hat{v}_m^{i;A}(x,s) \right] \mathrm{d}A = 0 . \tag{8.3-53}$$

Finally, the complex frequency-domain reciprocity theorem of the time correlation type is applied to the scattered wave fields in the states A and B and to the domain $\mathcal{D}^{s'}$. Since the embedding is time reverse self-adjoint in its acoustic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\int_{x\in\partial\mathcal{D}^s} \nu_m \left[\hat{p}^{s;A}(x,s)\hat{v}_m^{s;B}(x,-s) + \hat{p}^{s;B}(x,-s)\hat{v}_m^{s;A}(x,s) \right] \mathrm{d}A$$

$$= \lim_{\Delta\to\infty} \int_{x\in\mathcal{S}(O,\Delta)} \nu_m \left[\hat{p}^{s;A}(x,s)\hat{v}_m^{s;B}(x,-s) + \hat{p}^{s;B}(x,-s)\hat{v}_m^{s;A}(x,s) \right] \mathrm{d}A , \tag{8.3-54}$$

where $\mathcal{S}(O,\Delta)$ is the sphere of radius Δ with its centre at the origin O of the chosen reference frame. From Equations (8.3-50)–(8.3-54) we conclude that

$$\int_{x\in\partial\mathcal{D}^s} \nu_m \left[\hat{p}^{i;A}(x,s)\hat{v}_m^{s;B})(x,-s) + \hat{p}^{s;A}(x,s)\hat{v}_m^{i;B}(x,-s) \right.$$

$$\left. + \hat{p}^{i;B}(x,-s)\hat{v}_m^{s;A}(x,s) + \hat{p}^{s;B}(x,-s)\hat{v}_m^{i;A}(x,s) \right] \mathrm{d}A$$

$$= \lim_{\Delta\to\infty} \int_{x\in\mathcal{S}(O,\Delta)} \nu_m \left[\hat{p}^{\,\mathrm{s;A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{s;B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{s;B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{s;A}}(\boldsymbol{x},s)\right] \mathrm{d}A \,. \tag{8.3-55}$$

However, on account of Equations (8.3-32) and (8.3-33), (8.3-35)–(8.3-40), (8.3-41) and (8.3-42), and (8.3-44)–(8.3-49) we have

$$\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[\hat{p}^{\,\mathrm{s;A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{i;B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{i;B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{s;A}}(\boldsymbol{x},s)\right] \mathrm{d}A$$

$$= \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[\hat{p}^{\,\mathrm{s;A}}(\boldsymbol{x},s)V_m^{\mathrm{B}} + P^{\,\mathrm{B}}\hat{v}_m^{\mathrm{s;A}}(\boldsymbol{x},s)\right] \hat{b}(-s)\exp(s\beta_s x_s/c)\,\mathrm{d}A$$

$$= (s\rho)^{-1} P^{\,\mathrm{B}}\hat{b}(-s)\hat{p}^{\,\mathrm{s;A;\infty}}(\boldsymbol{\beta},s) \tag{8.3-56}$$

and

$$\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[\hat{p}^{\,\mathrm{s;B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{i;A}}(\boldsymbol{x},s) + \hat{p}^{\,\mathrm{i;A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{s;B}}(\boldsymbol{x},-s)\right] \mathrm{d}A$$

$$= \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[\hat{p}^{\,\mathrm{s;B}}(\boldsymbol{x},-s)V_m^{\mathrm{A}} + P^{\,\mathrm{A}}\hat{v}_m^{\mathrm{s;B}}(\boldsymbol{x},-s)\right] \hat{a}(s)\exp(-s\alpha_s x_s/c)\,\mathrm{d}A$$

$$= (s\rho)^{-1} P^{\,\mathrm{A}}\hat{a}(s)\hat{p}^{\,\mathrm{s;B;\infty}}(\boldsymbol{\alpha},-s) \,. \tag{8.3-57}$$

Furthermore, we have

$$\lim_{\Delta\to\infty} = \int_{x\in\mathcal{S}(O,\Delta)} \nu_m \left[\hat{p}^{\,\mathrm{s;A}}(\boldsymbol{x},s)\hat{v}_m^{\mathrm{s;B}}(\boldsymbol{x},-s) + \hat{p}^{\,\mathrm{s;B}}(\boldsymbol{x},-s)\hat{v}_m^{\mathrm{s;A}}(\boldsymbol{x},s)\right] \mathrm{d}A$$

$$= (4\pi)^{-2} \int_{\xi\in\Omega} \xi_m \left[\hat{p}^{\,\mathrm{s;\infty;A}}(\boldsymbol{\xi},s)\hat{v}_m^{\mathrm{s;\infty;B}}(\boldsymbol{\xi},-s) + \hat{p}^{\,\mathrm{s;\infty;B}}(\boldsymbol{\xi},-s)\hat{v}_m^{\mathrm{s;\infty;A}}(\boldsymbol{\xi},s)\right] \mathrm{d}A$$

$$= (8\pi^2)^{-1}(\rho c)^{-1} \int_{\xi\in\Omega} \hat{p}^{\,\mathrm{s;\infty;A}}(\boldsymbol{\xi},s)\hat{p}^{\,\mathrm{s;\infty;B}}(\boldsymbol{\xi},-s)\,\mathrm{d}A \,, \tag{8.3-58}$$

where Ω is the sphere of unit radius with its centre at O. Equations (8.3-55)–(8.3-58) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$P^{\,\mathrm{B}}\hat{b}(-s)\hat{p}^{\,\mathrm{s;A;\infty}}(\boldsymbol{\beta},s) + P^{\,\mathrm{A}}\hat{a}(s)\hat{p}^{\,\mathrm{s;B;\infty}}(\boldsymbol{\alpha},-s)$$

$$= -(8\pi^2)^{-1}(s/c) \int_{\xi\in\Omega} \hat{p}^{\,\mathrm{s;\infty;A}}(\boldsymbol{\xi},s)\hat{p}^{\,\mathrm{s;\infty;B}}(\boldsymbol{\xi},-s)\,\mathrm{d}A \,. \tag{8.3-59}$$

At this point it is, again, elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, in both state A and state B. Substitution of Equations (8.2-59) and (8.2-60) in Equation (8.3-59) yields

$$P^{\,\mathrm{B}}P^{\,\mathrm{A}}\hat{b}(-s)\hat{a}(s)\hat{S}^{\,\mathrm{A}}(\boldsymbol{\beta},\boldsymbol{\alpha},s) + P^{\,\mathrm{A}}P^{\,\mathrm{B}}\hat{a}(s)\hat{b}(-s)\hat{S}^{\,\mathrm{B}}(\boldsymbol{\alpha},\boldsymbol{\beta},-s)$$

$$= -(8\pi^2)^{-1}(s/c)\, P^{A}P^{B}\hat{a}(s)\hat{b}(-s) \int_{\xi\in\Omega} \hat{S}^{A}(\xi,\alpha,s)\hat{S}^{B}(\xi,\beta,-s)\, \mathrm{d}A \,. \tag{8.3-60}$$

Taking into account that Equation (8.3-60) has to hold for arbitrary values of P^{A}, P^{B}, $\hat{a}(s)$ and $\hat{b}(-s)$, we end up with

$$\hat{S}^{A}(\beta,\alpha,s) + \hat{S}^{B}(\alpha,\beta,-s) = -(s/8\pi^2 c) \int_{\xi\in\Omega} \hat{S}^{A}(\xi,\alpha,s)\hat{S}^{B}(\xi,\beta,-s)\, \mathrm{d}A \tag{8.3-61}$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

In a theoretical analysis the reciprocity relations derived in this section serve as an important check on the correctness of the analytic solutions as well as on the accuracy of numerical solutions to scattering problems. Note, however, that the reciprocity relations are necessary conditions to be satisfied by the scattered wave field (in the far-field region), but their satisfaction does not guarantee correctness of a total analytic solution or a certain accuracy of a total numerical solution. In a physical experiment, the redundancy induced by the reciprocity relations can be exploited to reduce the influence of noise on the quality of the observed data.

Exercises

Exercise 8.3-1

Show that Equation (8.3-61) follows from Equation (8.3-31) by taking the time Laplace transform.

8.4 An energy theorem about the far-field forward scattered wave amplitude

A special case arises when in the reciprocity relations of the time correlation type derived in Section 8.3, states A and B are taken to be identical. Since the superscripts A and B are then superfluous, they are omitted in the present section.

Time-domain version of the energy theorem

In the time-domain version of the theorem we start from Equation (8.3-19), take state A identical to state B, and consider the result at zero correlation time shift. Furthermore, for the case of an acoustically penetrable scatterer, the fluid occupying the scattering domain $\mathcal{D}^{s}$ is no longer assumed to be time reverse self-adjoint, i.e. it may have non-zero acoustic losses. Thus, we are led to consider the expression

$$\int_{x\in\partial\mathcal{D}^{s}} \nu_m \mathrm{C}_t(p,\mathrm{J}_t(v_m);x,0)\,\mathrm{d}A = \int_{x\in\partial\mathcal{D}^{s}} \nu_m \mathrm{C}_t(\mathrm{J}_t(p),v_m;x,0)\,\mathrm{d}A$$

$$= \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x\in\partial\mathcal{D}^{s}} \nu_m\left[p(x,t')v_m(x,t')\right]\mathrm{d}A = -W^{\mathrm{a}}, \tag{8.4-1}$$

where

$$W^{\mathrm{a}} = \int_{t'\in\mathcal{R}} P^{\mathrm{a}}(t')\mathrm{d}t' \tag{8.4-2}$$

is the *total acoustic energy absorbed by the scatterer* and

$$P^{\mathrm{a}}(t') = -\int_{x\in\partial\mathcal{D}^{s}} \nu_m\left[p(x,t')v_m(x,t')\right]\mathrm{d}A \tag{8.4-3}$$

is the instantaneous acoustic power absorbed by the scatterer. (Note that the minus sign in front of the integral sign on the right-hand side of Equation (8.4-3) is due to the fact that power absorption by the scatterer is effected by an inward power flow, while ν_m points away from the scatterer.)

Next, we substitute in the right-hand side of Equation (8.4-3) the relation

$$\{p,v_r\} = \{p^{\mathrm{i}}+p^{\mathrm{s}},v_r^{\mathrm{i}}+v_r^{\mathrm{s}}\}\ , \tag{8.4-4}$$

and observe that the incident wave dissipates no net energy upon traversing the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer when this domain has the acoustic fluid properties of the (lossless) embedding. Hence, with

$$P^{\mathrm{i}}(t') = -\int_{x\in\partial\mathcal{D}^{s}} \nu_m\left[p^{\mathrm{i}}(x,t')v_m^{\mathrm{i}}(x,t')\right]\mathrm{d}A \tag{8.4-5}$$

as the instantaneous acoustic power that the incident wave carries across $\partial\mathcal{D}^{\mathrm{s}}$ towards the domain $\mathcal{D}^{\mathrm{s}}$, we have

$$W^{\mathrm{i}} = \int_{t'\in\mathcal{R}} P^{\mathrm{i}}(t')\mathrm{d}t' = 0\,. \tag{8.4-6}$$

Furthermore, with the uniform incident plane wave

$$\{p^{\mathrm{i}},v_r^{\mathrm{i}}\}(x,t) = \{P,V_r\}a(t-\alpha_s x_s/c)\ , \tag{8.4-7}$$

for which

$$V_r = YP\alpha_r\,, \tag{8.4-8}$$

we have, upon using Equations (8.1-22)–(8.1-27),

$$\int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x\in\partial\mathcal{D}^{s}} \nu_m\left[p^{\mathrm{i}}(x,t')v_m^{\mathrm{s}}(x,t') + p^{\mathrm{s}}(x,t')v_m^{\mathrm{i}}(x,t')\right]\mathrm{d}A$$

$$= \rho^{-1}P\int_{t'\in\mathcal{R}} a(t')\mathrm{I}_t\,p^{\mathrm{s};\infty}(\alpha,t')\,\mathrm{d}t'. \tag{8.4-9}$$

Finally, the *total acoustic energy carried by the scattered wave* across $\partial\mathcal{D}^{\mathrm{s}}$ towards the embedding is introduced as

$$W^{\mathrm{s}} = \int_{t' \in \mathcal{R}} P^{\mathrm{s}}(t')\mathrm{d}t', \tag{8.4-10}$$

where

$$P^{\mathrm{s}}(t') = \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[p^{\mathrm{s}}(x,t') v_m^{\mathrm{s}}(x,t') \right] \mathrm{d}A \tag{8.4-11}$$

is the instantaneous acoustic power that the scattered wave carries across $\partial\mathcal{D}^{\mathrm{s}}$ towards the embedding.

Combining Equations (8.4-1)–(8.4-6) and (8.4-9)–(8.4-11) we end up with

$$W^{\mathrm{a}} + W^{\mathrm{s}} = -\rho^{-1} P \int_{t' \in \mathcal{R}} a(t') \mathrm{I}_t p^{\mathrm{s};\infty}(\boldsymbol{\alpha},t')\,\mathrm{d}t'. \tag{8.4-12}$$

Equation (8.4-12) is the desired *time-domain energy relation.* It relates the sum of the acoustic energies absorbed and scattered by the object to the scattered wave amplitude in the far-field region, for observation of this wave in the direction $\boldsymbol{\alpha}$ of propagation of the incident plane wave, i.e. in the "forward" direction, or "behind" the scatterer (Figure 8.4-1).

It is noted that for a lossless acoustically penetrable scatterer we have $W^{\mathrm{a}} = 0$. Also, $W^{\mathrm{a}} = 0$ for an impenetrable scatterer, since the right-hand side of Equation (8.4-3) then vanishes in view of the pertaining boundary conditions (Equation (8.1-3) or Equation (8.1-4)). Note also

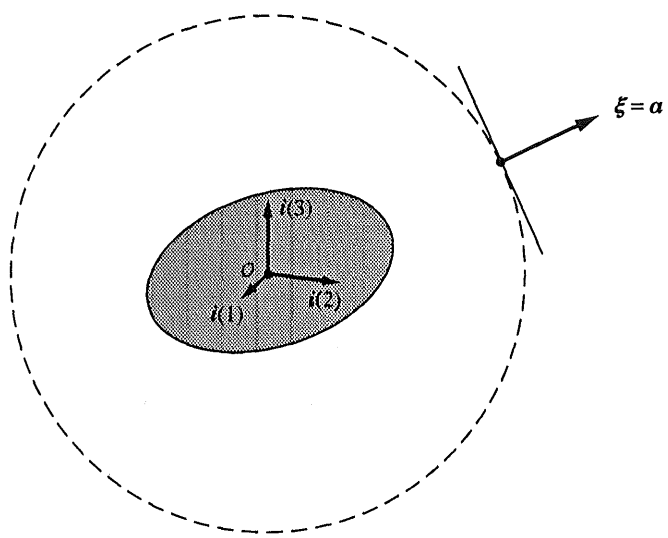

Figure 8.4-1 Acoustic scattering configuration for the energy theorem about the far-field forward scattered wave amplitude.

that in the derivation of the result we have nowhere used the linearity in the acoustic behaviour of the scatterer. Therefore, Equation (8.4-12) also holds for non-linear acoustic scatterers, subject to the condition, of course, that the embedding retains its linear properties.

Complex frequency-domain version of the energy theorem

In the complex frequency-domain version of the theorem we start from Equation (8.3-50) and take state A identical to state B. Furthermore, for the case of an acoustically penetrable scatterer the fluid occupying the scattering domain $\mathcal{D}^{\rm s}$ is no longer assumed to be time reverse self-adjoint, i.e. it may have non-zero acoustic losses. Thus, we are led to consider the expression

$$\frac{1}{4}\int_{x\in\partial\mathcal{D}^{\rm s}} \nu_m\left[\hat{p}(x,s)\hat{v}_m(x,-s)+\hat{p}(x,-s)\hat{v}_m(x,s)\right]{\rm d}A=-\hat{P}^{\rm a}(s)\,, \tag{8.4-13}$$

where the symbol on the right-hand side and the factor ¼ on the left-hand side have been chosen because of the equivalence with the time-averaged acoustic power flow for time-harmonic wave fields (for which $s={\rm j}\omega$, with $\omega\in\mathcal{R}$). It must be emphasised, however, that $\hat{P}^{\rm a}(s)$ is *not* the time Laplace transform of $P^{\rm a}(t)$ as given by Equation (8.4-3).

In the left-hand side of Equation (8.4-13) we now substitute the relation

$$\{\hat{p},\hat{v}_r\}=\{\hat{p}^{\rm i}+\hat{p}^{\rm s},\hat{v}_r^{\rm i}+\hat{v}_r^{\rm s}\}\,, \tag{8.4-14}$$

and observe that

$$\frac{1}{4}\int_{x\in\partial\mathcal{D}^{\rm s}} \nu_m\left[\hat{p}^{\rm i}(x,s)\hat{v}_m^{\rm i}(x,-s)+\hat{p}^{\rm i}(x,-s)\hat{v}_m^{\rm i}(x,s)\right]{\rm d}A=0\,, \tag{8.4-15}$$

since the fluid in the embedding has been assumed to be time-reverse self-adjoint.

Furthermore, with the uniform incident plane wave

$$\{\hat{p}^{\rm i},\hat{v}_r^{\rm i}\}(x,s)=\{P,V_r\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{8.4-16}$$

for which

$$V_r=YP\alpha_r\,, \tag{8.4-17}$$

we have, upon using Equations (8.3-59)–(8.3-64),

$$\frac{1}{4}\int_{x\in\partial\mathcal{D}^{\rm s}} \nu_m\left[\hat{p}^{\rm i}(x,s)\hat{v}_m^{\rm s}(x,-s)+\hat{p}^{\rm s}(x,-s)\hat{v}_m^{\rm i}(x,s)+\hat{p}^{\rm i}(x,-s)\hat{v}_m^{\rm s}(x,s)+\hat{p}^{\rm s}(x,s)\hat{v}_m^{\rm i}(x,-s)\right]{\rm d}A$$

$$=\frac{1}{4}(s\rho)^{-1}P\left[\hat{a}(s)\hat{p}^{\rm s;\infty}(\alpha,-s)-\hat{a}(-s)\hat{p}^{\rm s;\infty}(\alpha,s)\right]. \tag{8.4-18}$$

Finally, we introduce the quantity

$$\hat{P}^{\rm s}(s)=\frac{1}{4}\int_{x\in\partial\mathcal{D}^{\rm s}} \nu_m\left[\hat{p}^{\rm s}(x,s)\hat{v}_m^{\rm s}(x,-s)+\hat{p}^{\rm s}(x,-s)\hat{v}_m^{\rm s}(x,s)\right]{\rm d}A \tag{8.4-19}$$

that is, for time-harmonic waves, associated with the acoustic power carried across $\partial\mathcal{D}^{\mathrm{s}}$ by the scattered wave, where it must be emphasised that $\hat{P}^{\mathrm{s}}(s)$ is *not* the time Laplace transform of $P^{\mathrm{s}}(t)$ as given by Equation (8.4-11).

Combining Equations (8.4-13)–(8.4-15), and Equations (8.4-18) and (8.4-19), we end up with

$$\hat{P}^{\mathrm{a}}(s) + \hat{P}^{\mathrm{s}}(s) = -\frac{1}{4}(s\rho)^{-1}P\left[\hat{a}(s)\hat{p}^{\mathrm{s};\infty}(\boldsymbol{\alpha},-s) - \hat{a}(-s)\hat{p}^{\mathrm{s};\infty}(\boldsymbol{\alpha},s)\right]. \qquad (8.4\text{-}20)$$

Equation (8.4-20) is the desired complex frequency-domain energy relation. It relates the sum of the quantities $\hat{P}^{\mathrm{a}}(s)$ and $\hat{P}^{\mathrm{s}}(s)$ to the scattered wave amplitude in the far-field region for observation of this wave in the direction of propagation of the incident plane wave, i.e. in the "forward" direction, or "behind" the scatterer.

Equation (8.4-20) can be rewritten in a more elegant form by using the linear relationship between the far-field acoustic pressure scattered wave amplitude and the incident wave field. Using Equations (8.2-59) and (8.2-60), we have

$$\hat{P}^{\mathrm{a}}(s) + \hat{P}^{\mathrm{s}}(s) = -\frac{1}{4}(s\rho)^{-1}P^{2}\hat{a}(s)\hat{a}(-s)\left[\hat{S}(\boldsymbol{\alpha},\boldsymbol{\alpha},-s) - \hat{S}(\boldsymbol{\alpha},\boldsymbol{\alpha},s)\right]. \qquad (8.4\text{-}21)$$

Introducing the complex frequency-domain quantity

$$\hat{S}^{\mathrm{i}}(s) = \frac{1}{2}(\rho c)^{-1}P^{2}\hat{a}(s)\hat{a}(-s) \qquad (8.4\text{-}22)$$

that is associated with the acoustic power flow density in the incident plane wave, Equation (8.4-21) takes the form

$$\hat{P}^{\mathrm{a}}(s) + \hat{P}^{\mathrm{s}}(s) = -\frac{1}{2}\frac{c}{s}\hat{S}^{\mathrm{i}}(s)\left[\hat{S}(\boldsymbol{\alpha},\boldsymbol{\alpha},-s) - \hat{S}(\boldsymbol{\alpha},\boldsymbol{\alpha},s)\right]. \qquad (8.4\text{-}23)$$

Now, from Equations (8.4-13), (8.4-19) and (8.4-22) it is clear that $\hat{P}^{\mathrm{a}}(s) = \hat{P}^{\mathrm{a}}(-s)$, $\hat{P}^{\mathrm{s}}(s) = \hat{P}^{\mathrm{s}}(-s)$ and $\hat{S}^{\mathrm{i}}(s) = \hat{S}^{\mathrm{i}}(-s)$, respectively, which is in accordance with Equation (8.4-23).

It is noted that for a lossless acoustically penetrable scatterer we have $\hat{P}^{\mathrm{a}} = 0$. Also, $\hat{P}^{\mathrm{a}} = 0$ for an impenetrable scatterer, since the left-hand side of Equations (8.4-13) vanishes in view of the pertaining boundary conditions (Equations (8.1-42) or Equation (8.1-43)).

For imaginary values of s, i.e. for $s = \mathrm{j}\omega$, with $\omega \in \mathcal{R}$, Equation (8.4-23) is known as the *extinction cross-section theorem.* Note that in the complex frequency-domain result (contrary to the corresponding time-domain result) the linearity in the acoustic behaviour of the scatterer has implicitly been used since the space–time wave quantities have been represented, through the Bromwich integral, as a (linear) superposition of exponential time functions.

References to the earlier literature on the subject can be found in De Hoop (1959, 1985).

Exercises

Exercise 8.4-1

Consider in the complex frequency-domain energy relations (Equations (8.4-20) and (8.4-23)) the case $s = \mathrm{j}\omega$. Observe that the quantity $\hat{P}^{\mathrm{a}}(s)$ as introduced in Equation (8.4-13), the quantity

$\hat{P}^{s}(s)$ as introduced in Equation (8.4-19) and the quantity $\hat{S}^{i}(s)$ as introduced in Equation (8.4-22) satisfy the property $\hat{P}^{a}(s) = \hat{P}^{a}(-s)$, $\hat{P}^{s}(s) = \hat{P}^{s}(-s)$ and $\hat{S}^{i}(s) = \hat{S}^{i}(-s)$ in the common domain of regularity of both the left-hand and the right-hand sides. The latter property certainly holds for imaginary values of s and hence $\hat{P}^{a}(j\omega)$, $\hat{P}^{s}(j\omega)$ and $\hat{S}^{i}(j\omega)$ are real-valued for $\omega \in \mathcal{R}$. Furthermore, let

$$\hat{\sigma}^{a}(s) = \hat{P}^{a}(s)/S^{i}(s) \tag{8.4-24}$$

denote the complex frequency-domain *absorption cross-section* of the scattering object and

$$\hat{\sigma}^{s}(s) = \hat{P}^{s}(s)/S^{i}(s) \tag{8.4-25}$$

its complex frequency-domain *scattering cross-section.* Note that $\hat{\sigma}^{a}(s) = \hat{\sigma}^{a}(-s)$ and $\hat{\sigma}^{s}(s) = \hat{\sigma}^{s}(-s)$ in the common domain of regularity of both the left-hand and the right-hand sides. The latter property certainly holds for imaginary values of s and hence $\hat{\sigma}^{a}(j\omega)$ and $\hat{\sigma}^{a}(j\omega)$ are real-valued for $s = j\omega$, with $\omega \in \mathcal{R}$. Show that Equation (8.4-20) leads to

$$\hat{\sigma}^{a}(j\omega) + \hat{\sigma}^{s}(j\omega) = \frac{c}{\omega}\frac{\mathrm{Im}\left[P\hat{a}(-j\omega)\hat{p}^{s;\infty}(\boldsymbol{\alpha},j\omega)\right]}{P^{2}|\hat{a}(j\omega)|^{2}} \tag{8.4-26}$$

and Equation (8.4-23) to

$$\hat{\sigma}^{a}(j\omega) + \hat{\sigma}^{s}(j\omega) = \frac{c}{\omega}\,\mathrm{Im}\left[\hat{S}(\boldsymbol{\alpha},\boldsymbol{\alpha},j\omega)\right]. \tag{8.4-27}$$

Equations (8.4-26) and (8.4-27) are known as the *extinction cross-section theorem* (De Hoop 1959). (*Note*: Extinction cross-section = Absorption cross-section + Scattering cross-section.)

8.5 The Neumann expansion in the integral equation formulation of the scattering by a penetrable object

In this section we discuss the Neumann expansion in the integral equation formulation of the acoustic scattering problem. The expansion is an analytic procedure that applies to a *penetrable scatterer.* The procedure is *iterative* in nature and is expected to converge for sufficiently low contrast of the scatterer with respect to its embedding.

Time-domain analysis

In the time-domain presentation of the method we start from Equations (7.9-5) and (7.9-20)–(7.9-23), which, through combination of the time convolutions, we write for the present configuration as

$$p(\boldsymbol{x}',t) = p^{i}(\boldsymbol{x}',t) - \int_{\boldsymbol{x}\in\mathcal{D}^{s}} \Big[\partial_t C_t(G^{pq},\chi^{s} - \kappa\delta(t),p;\boldsymbol{x}',\boldsymbol{x},t) + \partial_t C_t(G^{pf}_{k'},\mu^{s}_{k',r'} - \rho\delta(t)\delta_{k',r'},v_{r'};\boldsymbol{x}',\boldsymbol{x},t)\Big]\,\mathrm{d}V \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^{3}, \tag{8.5-1}$$

and

$$v_r(x',t) = v_r^{\rm i}(x',t) - \int_{x\in\mathcal{D}^{\rm s}} \left[\partial_t C_t(G_r^{vq},\chi^{\rm s} - \kappa\delta(t),p;x',x,t)\right.$$

$$\left. + \partial_t C_t(G_{r,k'}^{vf},\mu_{k',r'}^{\rm s} - \rho\delta(t)\delta_{k',r'},v_{r'};x',x,t)\right] \mathrm{d}V \qquad \text{for } x'\in\mathcal{R}^3. \qquad (8.5\text{-}2)$$

For $x'\in\mathcal{D}^{\rm s}$, Equations (8.5-1) and (8.5-2) constitute a system of linear integral equations of the second kind to be solved for $\{p,v_r\}$ for $x\in\mathcal{D}^{\rm s}$ and $t\in\mathcal{R}$, and with $\{p^{\rm i},v_r^{\rm i}\}$ as forcing terms. To solve these equations analytically, an iterative procedure, known as the *Neumann expansion*, is set up. The successive steps in this procedure will be labelled by integer superscripts enclosed in brackets ([...]). The procedure is *initialised* by putting

$$p^{[0]} = p^{\rm i} \qquad \text{for } x'\in\mathcal{R}^3, \qquad (8.5\text{-}3)$$

$$v_r^{[0]} = v_r^{\rm i} \qquad \text{for } x'\in\mathcal{R}^3. \qquad (8.5\text{-}4)$$

Next, the procedure is *updated* through

$$p^{[n+1]}(x',t) = -\int_{x\in\mathcal{D}^{\rm s}} \left[\partial_t C_t(G^{pq},\chi^{\rm s} - \kappa\delta(t),p^{[n]};x',x,t)\right.$$

$$\left. + \partial_t C_t(G_{k'}^{pf},\mu_{k',r'}^{\rm s} - \rho\delta(t)\delta_{k',r'},v_{r'}^{[n]};x',x,t)\right] \mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{etc.} \qquad (8.5\text{-}5)$$

and

$$v_r^{[n+1]}(x',t) = -\int_{x\in\mathcal{D}^{\rm s}} \left[\partial_t C_t(G_r^{vq},\chi^{\rm s} - \kappa\delta(t),p^{[n]};x',x,t)\right.$$

$$\left. + \partial_t C_t(G_{r,k'}^{vf},\mu_{k',r'}^{\rm s} - \rho\delta(t)\delta_{k',r'},v_{r'}^{[n]};x',x,t)\right] \mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{etc.} \qquad (8.5\text{-}6)$$

As can be inferred from these updating equations, the terms of order $[n+1]$ can be expected to be "smaller" than their counterparts of order $[n]$, provided that the contrast quantities are "small enough". On account of this, it can be conjectured that for sufficiently small contrast of the scatterer with respect to its embedding the *procedure is convergent* and we can put

$$p = \sum_{n=0}^{\infty} p^{[n]} \qquad \text{for } x'\in\mathcal{R}^3, \qquad (8.5\text{-}7)$$

$$v_r = \sum_{n=0}^{\infty} v_r^{[n]} \qquad \text{for } x'\in\mathcal{R}^3. \qquad (8.5\text{-}8)$$

Assuming that the series on the right-hand sides of Equations (8.5-7) and (8.5-8) are uniformly convergent, it can easily be proved that $\{p,v_r\}$ as defined by these equations indeed satisfy Equations (8.5-1) and (8.5-2). To this end we observe that

$$-\int_{x\in\mathcal{D}^{\rm s}} \left[\partial_t C_t\left(G^{pq},\chi^{\rm s} - \kappa\delta(t),p;x',x,t\right) + \partial_t C_t\left(G_{k'}^{pf},\mu_{k',r'}^{\rm s} - \rho\delta(t)\delta_{k',r'},v_{r'};x',x,t\right)\right] \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}^s}\left[\partial_t C_t\left(G^{pq},\chi^s - \kappa\delta(t),\sum_{n=0}^{\infty} p^{[n]};x',x,t\right)\right.$$

$$\left. + \partial_t C_t\left(G^{pf}_{k'},\mu^s_{k',r'} - \rho\delta(t)\delta_{k',r'},\sum_{n=0}^{\infty} v^{[n]}_{r'};x',x,t\right)\right]\mathrm{d}V$$

$$= -\sum_{n=0}^{\infty}\int_{x\in\mathcal{D}^s}\left[\partial_t C_t\left(G^{pq},\chi^s - \kappa\delta(t),p^{[n]};x',x,t\right) + \partial_t C_t\left(G^{pf}_{k'},\mu^s_{k',r'} - \rho\delta(t)\delta_{k',r'},v^{[n]}_{r'};x',x,t\right)\right]\mathrm{d}V$$

$$= \sum_{n=0}^{\infty} p^{[n+1]}(x',t') = \sum_{m=0}^{\infty} p^{[m]}(x',t) - p^{[0]}(x',t) = p(x',t) - p^{\mathrm{i}}(x,t) \qquad \text{for } x'\in\mathcal{R}^3, \qquad (8.5\text{-}9)$$

and

$$-\int_{x\in\mathcal{D}^s}\left[\partial_t C_t\left(G^{vq}_r,\chi^s - \kappa\delta(t),p;x',x,t\right) + \partial_t C_t\left(G^{vf}_{r,k'},\mu^s_{k',r'} - \rho\delta(t)\delta_{k',r'},v_{r'};x',x,t\right)\right]\mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}^s}\left[\partial_t C_t\left(G^{vq}_r,\chi^s - \kappa\delta(t),\sum_{n=0}^{\infty} p^{[n]};x',x,t\right)\right.$$

$$\left. + \partial_t C_t\left(G^{vf}_{r,k'},\mu^s_{k',r'} - \rho\delta(t)\delta_{k',r'},\sum_{n=0}^{\infty} v^{[n]}_{r'};x',x,t\right)\right]\mathrm{d}V$$

$$= -\sum_{n=0}^{\infty}\int_{x\in\mathcal{D}^s}\left[\partial_t C_t\left(G^{vq}_r,\chi^s - \kappa\delta(t),p^{[n]};x',x,t\right) + \partial_t C_t\left(G^{vf}_{r,k'},\mu^s_{k',r'} - \kappa\delta(t)\delta_{k',r'},v^{[n]}_{r'};x',x,t\right)\right]\mathrm{d}V$$

$$= \sum_{n=0}^{\infty} v^{[n+1]}_r(x',t') = \sum_{m=0}^{\infty} v^{[m]}_r(x',t) - v^{[0]}_r(x',t) = v_r(x',t) - v^{\mathrm{i}}_r(x,t) \qquad \text{for } x'\in\mathcal{R}^3, \qquad (8.5\text{-}10)$$

where Equations (8.5-3)–(8.5-8) have been used and the interchange of the summations with respect to n and the integrations with respect to x is justified by the assumed uniform convergence of the series expansions. Equations (8.5-9) and (8.5-10) are evidently identical to Equations (8.5-1) and (8.5-2), and, hence, the expansions given in Equations (8.5-7) and (8.5-8) indeed solve the problem.

Complex frequency-domain analysis

In the complex frequency-domain presentation of the method we start from Equations (7.9-28) and (7.9-43)–(7.9-46), which are combined to

$$\hat{p}(x',s) = \hat{p}^{\mathrm{i}}(x',s) - \int_{x\in\mathcal{D}^{\mathrm{s}}} \left\{\hat{G}^{pq}(x',x,s)\left[\hat{\eta}^{\mathrm{s}}(x,s) - s\kappa\right]\hat{p}(x,s)\right.$$

$$\left. + \hat{G}^{pf}_{k'}(x',x,s)\left[\hat{\zeta}^{\mathrm{s}}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right]\hat{v}_{r'}(x,s)\right\} \mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3, \tag{8.5-11}$$

and

$$\hat{v}_r(x',s) = \hat{v}^{\mathrm{i}}_r(x,s) - \int_{x\in\mathcal{D}^{\mathrm{s}}} \left\{\hat{G}^{vq}_r(x',x,s)\left[\hat{\eta}^{\mathrm{s}}(x,s) - s\kappa\right]\hat{p}(x,s)\right.$$

$$\left. + \hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\zeta}^{\mathrm{s}}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right]\hat{v}_{r'}(x,s)\right\} \mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3. \tag{8.5-12}$$

For $x'\in\mathcal{D}^{\mathrm{s}}$, Equations (8.5-11) and (8.5-12) constitute a system of linear integral equations of the second kind to be solved for $\{\hat{p},\hat{v}_r\}$ for $x\in\mathcal{D}^{\mathrm{s}}$, and with $\{\hat{p}^{\mathrm{i}},\hat{v}^{\mathrm{i}}_r\}$ as forcing terms. The Neumann procedure to solve these equations in *initialised* by putting

$$\hat{p}^{[0]} = \hat{p}^{\mathrm{i}} \qquad \text{for } x'\in\mathcal{R}^3, \tag{8.5-13}$$

$$\hat{v}^{[0]}_r = \hat{v}^{\mathrm{i}}_r \qquad \text{for } x'\in\mathcal{R}^3. \tag{8.5-14}$$

Next, the procedure is *updated* through

$$\hat{p}^{[n+1]}(x',s) = -\int_{x\in\mathcal{D}^{\mathrm{s}}} \left\{\hat{G}^{pq}(x',x,s)\left[\hat{\eta}^{\mathrm{s}}(x,s) - s\kappa\right]\hat{p}^{[n]}(x,s)\right.$$

$$\left. + \hat{G}^{pf}_{k'}(x',x,s)\left[\hat{\zeta}^{\mathrm{s}}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right]\hat{v}^{[n]}_{r'}(x,s)\right\} \mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \text{ and n} = 0,1,2, \text{etc.} \tag{8.5-15}$$

and

$$\hat{v}^{[n+1]}_r(x',s) = -\int_{x\in\mathcal{D}^{\mathrm{s}}} \left\{\hat{G}^{vq}_r(x',x,s)\left[\hat{\eta}^{\mathrm{s}}(x,s) - s\kappa\right]\hat{p}^{[n]}(x,s)\right.$$

$$\left. + \hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\zeta}^{\mathrm{s}}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right]\hat{v}^{[n]}_{r'}(x,s)\right\} \mathrm{d}V \text{ for } x'\in\mathcal{R}^3 \text{ and n} = 0,1,2, \text{etc.} \tag{8.5-16}$$

Assuming that the *procedure is convergent*, we can put

$$\hat{p} = \sum_{n=0}^{\infty} \hat{p}^{[n]} \qquad \text{for } x'\in\mathcal{R}^3, \tag{8.5-17}$$

$$\hat{v}_r = \sum_{n=0}^{\infty} \hat{v}^{[n]} \qquad \text{for } x'\in\mathcal{R}^3. \tag{8.5-18}$$

Assuming that the series on the right-hand sides of Equations (8.5-17) and (8.5-18) are uniformly convergent, it can easily be proved that $\{\hat{p},\hat{v}_r\}$ as defined by these equations indeed satisfy Equations (8.5-11) and (8.5-12). To this end we observe that

$$-\int_{x\in\mathcal{D}^{\mathrm{s}}} \left\{\hat{G}^{pq}(x',x,s)\left[\hat{\eta}^{\mathrm{s}}(x,s) - s\kappa\right]\hat{p}(x,s) + \hat{G}^{pf}_{k'}(x',x,s)\left[\hat{\zeta}^{\mathrm{s}}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right]\hat{v}_{r'}(x,s)\right\} \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}^s} \left\{ \hat{G}^{pq}(x',x,s)\left[\hat{\eta}^{s}(x,s) - s\kappa\right] \sum_{n=0}^{\infty} \hat{p}^{[n]}(x,s) \right.$$

$$\left. + \hat{G}^{pf}_{k'}(x',x,s)\left[\hat{\zeta}^{s}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right] \sum_{n=0}^{\infty} \hat{v}^{[n]}_{r'}(x,s) \right\} \mathrm{d}V$$

$$= -\sum_{n=0}^{\infty} \int_{x\in\mathcal{D}^s} \left\{ \hat{G}^{pq}(x',x,s)\left[\hat{\eta}^{s}(x,s) - s\kappa\right] \hat{p}^{[n]}(x,s) \right.$$

$$\left. + \hat{G}^{pf}_{k'}(x',x,s)\left[\hat{\zeta}^{s}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right] \hat{v}^{[n]}_{r'}(x,s) \right\} \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \hat{p}^{[n+1]}(x,s) = \sum_{m=0}^{\infty} \hat{p}^{[m]}(x,s) - \hat{p}^{[0]}(x,s) = \hat{p}(x,s) - \hat{p}^{\mathrm{i}}(x,s) \quad \text{for } x'\in\mathcal{R}^3 \tag{8.5-19}$$

and

$$-\int_{x\in\mathcal{D}^s} \left\{ \hat{G}^{vq}_{r}(x',x,s)\left[\hat{\eta}^{s}(x,s) - s\kappa\right] \hat{p}(x,s) + \hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\zeta}^{s}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right] \hat{v}_{r'}(x,s) \right\} \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}^s} \left\{ \hat{G}^{vq}_{r}(x',x,s)\left[\hat{\eta}^{s}(x,s) - s\kappa\right] \sum_{n=0}^{\infty} \hat{p}^{[n]}(x,s) \right.$$

$$\left. + \hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\zeta}^{s}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right] \sum_{n=0}^{\infty} \hat{v}^{[n]}_{r'}(x,s) \right\} \mathrm{d}V$$

$$= -\sum_{n=0}^{\infty} \int_{x\in\mathcal{D}^s} \left\{ \hat{G}^{vq}_{r}(x',x,s)\left[\hat{\eta}^{s}(x,s) - s\kappa\right] \hat{p}^{[n]}(x,s) \right.$$

$$\left. + \hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\zeta}^{s}_{k',r'}(x,s) - s\rho\delta_{k',r'}\right] \hat{v}^{[n]}_{r'}(x,s) \right\} \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \hat{v}^{[n+1]}_{r}(x,s) = \sum_{m=0}^{\infty} \hat{v}^{[m]}_{r}(x,s) - \hat{v}^{[0]}_{r}(x,s) = \hat{v}_{r}(x,s) - \hat{v}^{\mathrm{i}}_{r}(x,s) \quad \text{for } x'\in\mathcal{R}^3, \tag{8.5-20}$$

where Equations (8.5-13)–(8.5-18) have been used and the interchange of the summations with respect to n and the integrations with respect to x is justified by the assumed uniform convergence of the series expansions. Equations (8.5-19) and (8.5-20) are evidently identical to Equations (8.5-11) and (8.5-12), and, hence, the expansions given in Equations (8.5-17) and (8.5-18) indeed solve the problem.

The construction of convergence criteria for the Neumann expansion is complicated by the singularities of the Green's functions. For the simpler case of the scattering problem associated with the scalar wave equation, a convergence criterion has been derived (De Hoop, 1991).

The nth term in the Neumann expansion is also known as the nth *Rayleigh–Gans–Born approximation.*

8.6 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation; time-domain analysis and complex frequency-domain analysis for canonical geometries of the scattering object

In this section the far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation is further investigated. In particular, closed-form analytic expressions are derived for the far-field scattered wave amplitude associated with the plane wave scattering by a *homogeneous* object in the shape of an ellipsoid, a rectangular block, an elliptical cylinder of finite height, an elliptical cone of finite height, or a tetrahedron. A structure consisting of the union of the listed objects can, in the first-order Rayleigh–Gans–Born approximation, be dealt with by superposition.

Time-domain analysis

In the time-domain analysis, the expressions for the scattered wave amplitude in the far-field region in the first-order Rayleigh–Gans–Born approximation follow, with the use of Equations (8.1-1) and (8.1-2), (8.1-5)–(8.1-7), (8.1-16)–(8.1-19), and (8.5-3) and (8.5-4) as (Figure 8.6-1)

$$p^{s;\infty}(\boldsymbol{\xi},t) = -Pc^{-2}\left[\mathcal{A}^{\chi}(\boldsymbol{\xi}/c - \boldsymbol{\alpha}/c,t) + \xi_k \mathcal{A}^{\mu}_{k,r}(\boldsymbol{\xi}/c - \boldsymbol{\alpha}/c,t)\alpha_r\right], \qquad (8.6\text{-}1)$$

and

$$v_r^{s;\infty}(\boldsymbol{\xi},t) = (\rho c)^{-1} p^{s;\infty}(\boldsymbol{\xi},t)\xi_r, \qquad (8.6\text{-}2)$$

where

$$\mathcal{A}^{\chi}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[\chi^{s}(\boldsymbol{x},t')/\kappa - \delta(t')\right] \partial_t^2 a(t - t' + u_s x_s)\,\mathrm{d}t' \qquad (8.6\text{-}3)$$

and

$$\mathcal{A}^{\mu}_{k,r}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[\mu^{s}_{k,r}(\boldsymbol{x},t')/\rho - \delta_{k,r}\delta(t')\right] \partial_t^2 a(t - t' + u_s x_s)\,\mathrm{d}t'. \qquad (8.6\text{-}4)$$

Note that these scattered wave amplitudes depend in their directional characteristics only on the difference $\boldsymbol{\xi}/c - \boldsymbol{\alpha}/c$ between the slowness $\boldsymbol{\xi}/c$ in the direction of observation $\boldsymbol{\xi}$ and the slowness $\boldsymbol{\alpha}/c$ in the direction of propagation $\boldsymbol{\alpha}$ of the incident uniform plane wave. This property only holds in the first-order Rayleigh–Gans–Born approximation and is not exact.

For a *homogeneous* object, Equations (8.6-3) and (8.6-4) reduce to

$$\mathcal{A}^{\chi}(\boldsymbol{u},t) = \int_{t'=0}^{\infty} \left[\chi^{s}(t')/\kappa - \delta(t')\right] \Upsilon(\boldsymbol{u},t - t')\,\mathrm{d}t', \qquad (8.6\text{-}5)$$

and

$$\mathcal{A}^{\mu}_{k,r}(\boldsymbol{u},t) = \int_{t'=0}^{\infty} \left[\mu^{s}_{k,r}(t')/\rho - \delta_{k,r}\delta(t')\right] \Upsilon(\boldsymbol{u},t - t')\,\mathrm{d}t', \qquad (8.6\text{-}6)$$

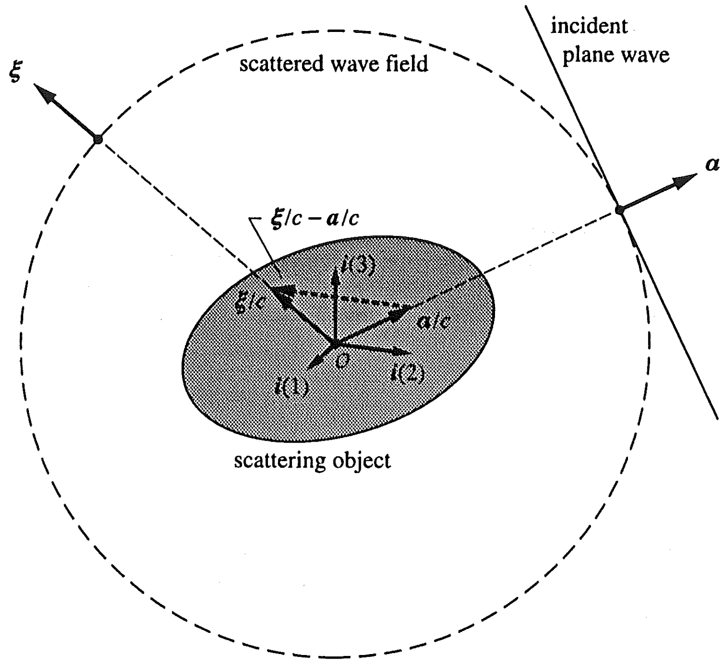

Figure 8.6-1 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation.

in which

$$\Upsilon(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \partial_t^2 a(t + u_s x_s)\,\mathrm{d}V \tag{8.6-7}$$

is the *time-domain shape factor* corresponding to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. From Equation (8.6-7) it immediately follows that for $\boldsymbol{\xi}/c = \boldsymbol{\alpha}/c$, i.e. for observation "behind" the scatterer or "forward scattering", we have

$$\Upsilon(\boldsymbol{0},t) = V^{\mathrm{s}}\partial_t^2 a(t)\,, \tag{8.6-8}$$

where V^{s} is the *volume of the scatterer*. Note, again, that Equation (8.6-8) only holds in the first-order Rayleigh–Gans–Born approximation, and is not exact.

Below, we derive closed-form analytic expressions for the shape factor $\Upsilon = \Upsilon(\boldsymbol{u},t)$ for a number of canonical geometries of the scatterer.

Ellipsoid

Let the scattering ellipsoid be defined by (see Equation (A.9-21) and Figure 8.6-2)

$$\mathcal{D}^{\mathrm{s}} = \left\{\boldsymbol{x}\in\mathcal{R}^3;0 \leqslant (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 < 1\right\}. \tag{8.6-9}$$

Its volume is

$$V^s = (4\pi/3)a_1a_2a_3 \,. \tag{8.6-10}$$

In the integral on the right-hand side of Equation (8.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{8.6-11}$$

as the variables of integration. In $\boldsymbol{y}$ space, the domain of integration is then the unit ball $\{\boldsymbol{y}\in\mathcal{R}^3;\ 0\leqslant y_1^2+y_2^2+y_3^2<1\}$. The integration over this unit ball is carried out with the aid of spherical polar coordinates $\{r,\theta,\phi\}$, with $0\leqslant r<1$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, about the vector $u_1a_1\boldsymbol{i}(1) + u_2a_2\boldsymbol{i}(2) + u_3a_3\boldsymbol{i}(3)$ as polar axis. Then

$$u_sx_s = u_1x_1 + u_2x_2 + u_3x_3 = (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3a_3)y_3 = Ur\cos(\theta)\,, \tag{8.6-12}$$

where

$$U = \left[(u_1a_1)^2 + (u_2a_2)^2 + (u_3a_3)^2\right]^{1/2} \geqslant 0\,, \tag{8.6-13}$$

while

$$\mathrm{d}V = a_1a_2a_3r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{8.6-14}$$

The integration then runs as follows:

$$\begin{aligned}\Upsilon(\boldsymbol{u},t) &= a_1a_2a_3\int_{r=0}^{1} r^2\,\mathrm{d}r\int_{\theta=0}^{\pi}\sin(\theta)\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\partial_t^2 a\left[t+Ur\cos(\theta)\right]\mathrm{d}\phi\\ &= 2\pi a_1a_2a_3\int_{r=0}^{1} r^2\,\mathrm{d}r\int_{\theta=0}^{\pi}\partial_t^2 a\left[t+Ur\cos(\theta)\right]\sin(\theta)\,\mathrm{d}\theta\\ &= 2\pi a_1a_2a_3U^{-1}\int_{r=0}^{1}\left[\partial_t^2 a\,(t+Ur) - \partial_t a(t-Ur)\right] r\,\mathrm{d}r\end{aligned}$$

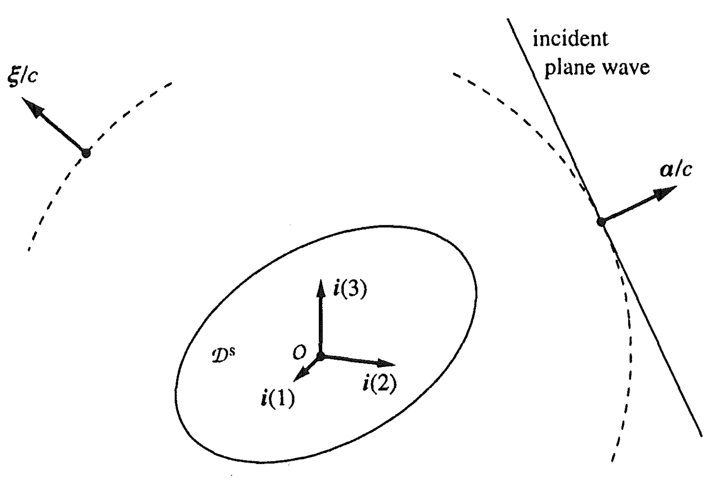

Figure 8.6-2 Scatterer in the shape of an ellipsoid.

$$= 2\pi a_1 a_2 a_3 \left\{ U^{-2} a(t+U) - U^{-3} \left[\mathrm{I}_t a(t+U) - \mathrm{I}_t a(t) \right] \right.$$

$$\left. + U^{-2} a(t-U) + U^{-3} \left[\mathrm{I}_t a(t-U) - \mathrm{I}_t a(t) \right] \right\}$$

$$= (3V^{\mathrm{s}}/2) \left\{ U^{-2} \left[a(t+U) + a(t-U) \right] - U^{-3} \left[\mathrm{I}_t a(t+U) - \mathrm{I}_t a(t-U) \right] \right\}. \qquad (8.6\text{-}15)$$

By using the Taylor expansion of the right-hand side about $U = 0$ and taking the limit $U \downarrow 0$, it can be verified that the result is in accordance with Equation (8.6-8).

Rectangular block

Let the scattering domain be the rectangular block defined by (see Equation (A.9-14) and Figure 8.6-3)

$$\mathcal{D}^{\mathrm{s}} = \left\{ x \in \mathcal{R}^3 ;\ -a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2,\ -a_3 < x_3 < a_3 \right\}. \qquad (8.6\text{-}16)$$

Its volume is given by

$$V^{\mathrm{s}} = 8 a_1 a_2 a_3 . \qquad (8.6\text{-}17)$$

In the integral on the right-hand side of Equation (8.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \qquad (8.6\text{-}18)$$

as the variables of integration. In $\boldsymbol{y}$ space the domain of integration is then the cube $\{y \in \mathcal{R}^3 ; -1 < y_1 < 1,\ -1 < y_2 < 1,\ -1 < y_3 < 1\}$ with edge lengths 2. With

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 a_3 , \qquad (8.6\text{-}19)$$

furthermore, we have

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3$$

$$= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 a_3) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3 , \qquad (8.6\text{-}20)$$

while

$$\mathrm{d}V = a_1 a_2 a_3 \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}y_3 . \qquad (8.6\text{-}21)$$

The integration then runs as follows:

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 a_3 \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \mathrm{d}y_2 \int_{y_1=-1}^{1} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3) \, \mathrm{d}y_1$$

$$= a_1 a_2 a_3 U_1^{-1} \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \left[\partial_t a(t + U_1 + U_2 y_2 + U_3 y_3) \right.$$

$$\left. - \partial_t a(t - U_1 + U_2 y_2 + U_3 y_3) \right] \mathrm{d}y_2$$

$$= a_1 a_2 a_3 (U_1 U_2)^{-1} \int_{y_3=-1}^{1} \left[a(t + U_1 + U_2 + U_3 y_3) - a(t + U_1 - U_2 + U_3 y_3) \right.$$

$$\left. - a(t - U_1 + U_2 + U_3 y_3) + a(t - U_1 - U_2 + U_3 y_3) \right] \mathrm{d}y_3$$

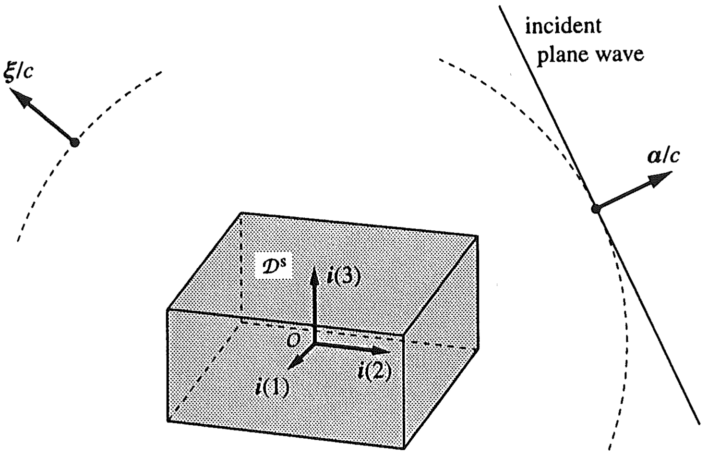

Figure 8.6-3 Scatterer in the shape of a rectangular block.

$$= a_1a_2a_3(U_1U_2U_3)^{-1}\,[\mathrm{I}_t a(t+U_1+U_2+U_3) - \mathrm{I}_t a(t+U_1+U_2-U_3)$$
$$- \mathrm{I}_t a(t+U_1-U_2+U_3) + \mathrm{I}_t a(t+U_1-U_2-U_3) - \mathrm{I}_t a(t-U_1+U_2+U_3)$$
$$+ \mathrm{I}_t a(t-U_1+U_2-U_3) + \mathrm{I}_t a(t-U_1-U_2+U_3) - \mathrm{I}_t a(t-U_1-U_2-U_3)]\,. \qquad (8.6\text{-}22)$$

Special cases occur for either $U_1 \to 0$, $U_2 \to 0$, and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (8.6-22) by using the pertaining Taylor expansions on the right-hand side. In particular, it can be verified that for $U_1 \to 0$ and $U_2 \to 0$ and $U_3 \to 0$ the result is in accordance with Equation (8.6-8).

Elliptical cylinder of finite height

Let the elliptical cylinder of finite height be defined by (Figure 8.6-4)

$$\mathcal{D}^s = \left\{x \in \mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\ -h < x_3 < h\right\}. \qquad (8.6\text{-}23)$$

Its volume is

$$V^s = 2\pi a_1 a_2 h\,. \qquad (8.6\text{-}24)$$

In the integral on the right-hand side of Equation (8.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \qquad (8.6\text{-}25)$$

as the variables of integration. In y space, the domain of integration is then the Cartesian product of the unit disk $\Delta^2 = \{(y_1,y_2) \in \mathcal{R}^2;\ 0 \leqslant y_1^2 + y_2^2 < 1\}$ and the interval $\{y_3 \in \mathcal{R};\ -1 < y_3 < 1\}$ along the axis of the cylinder. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \qquad (8.6\text{-}26)$$

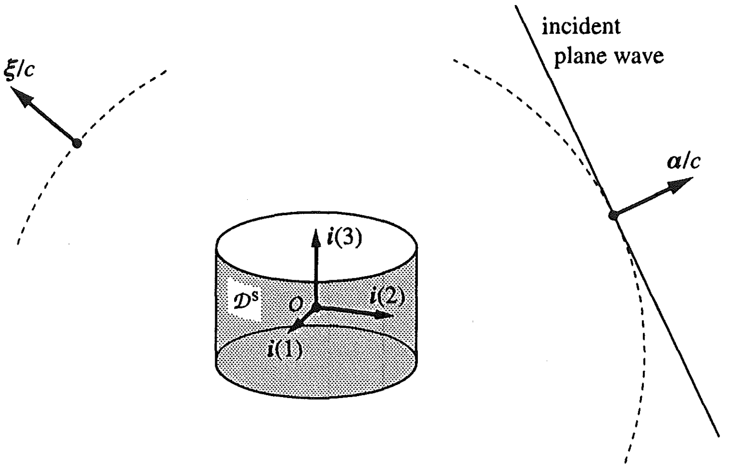

Figure 8.6-4 Scatterer in the shape of an elliptical cylinder of finite height.

we have

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3$$
$$= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3 \,, \qquad (8.6\text{-}27)$$

while

$$\mathrm{d}V = a_1 a_2 h \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}y_3 \,. \qquad (8.6\text{-}28)$$

The integration then runs as follows:

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} \mathrm{d}y_1 \, \mathrm{d}y_2 \int_{y_3=-1}^{1} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3) \, \mathrm{d}y_3$$
$$= a_1 a_2 h U_3^{-1} \int_{(y_1,y_2)\in\Delta^2} [\partial_t a(t + U_1 y_1 + U_2 y_2 + U_3)$$
$$- \partial_t a(t + U_1 y_1 + U_2 y_2 - U_3)] \, \mathrm{d}y_1 \, \mathrm{d}y_2 \,. \qquad (8.6\text{-}29)$$

Next, we observe that

$$\partial_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) = \partial_t^2 \mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)$$
$$= (U_1^2 + U_2^2)^{-1} (\partial_{y_1}^2 + \partial_{y_2}^2) \mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) \quad \text{for } U_1^2 + U_2^2 \neq 0 \,. \qquad (8.6\text{-}30)$$

Now, applying Gauss' divergence theorem to the integration over Δ^2, we obtain

$$\int_{(y_1,y_2)\in\Delta^2} (\partial_{y_1}^2 + \partial_{y_2}^2) \mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) \, \mathrm{d}y_1 \, \mathrm{d}y_2$$
$$= \int_{(y_1,y_2)\in C^2} (y_1 \partial_{y_1} + y_2 \partial_{y_2}) \mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) \, \mathrm{d}\sigma$$

$$= \int_{(y_1,y_2)\in C^2} (U_1 y_1 + U_2 y_2) a(t + U_1 y_1 + U_2 y_2 \pm U_3)\, \mathrm{d}\sigma, \tag{8.6-31}$$

where $\mathrm{d}\sigma$ is the elementary arc length along the unit circle C^2 that forms the closed boundary of the unit disk Δ^2, and where we have used the property that the unit vector along the normal to C^2 pointing away from Δ^2 is given by $\boldsymbol{\nu} = y_1\boldsymbol{i}(1) + y_2\boldsymbol{i}(2)$. In the integral on the right-hand side of Equation (8.6-31) we introduce the polar coordinates $\{r,\phi\}$, with $r = 1$ and $0 \leqslant \phi < 2\pi$, about the vector $U_1\boldsymbol{i}(1) + U_2\boldsymbol{i}(2)$ as polar axis, as the variables of integration. This yields

$$\int_{(y_1,y_2)\in C^2} (U_1 y_1 + U_2 y_2) a(t + U_1 y_1 + U_2 y_2 \pm U_3)\, \mathrm{d}\sigma$$

$$= \int_{\phi=0}^{2\pi} U\cos(\phi) a\left[t + U\cos(\phi) \pm U_3)\right] \mathrm{d}\phi\,, \tag{8.6-32}$$

where

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0\,. \tag{8.6-33}$$

Collecting the results, we end up with

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} U_3^{-1} \int_{\phi=0}^{2\pi} \cos(\phi) \Big\{ a\left[t + U\cos(\phi) + U_3\right] - a\left[t + U\cos(\phi) - U_3\right] \Big\}\, \mathrm{d}\phi\,. \tag{8.6-34}$$

Special cases occur for $U \downarrow 0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (8.6-34) by using the pertaining Taylor expansions on the right-hand side. In particular, it can be verified that for $U \downarrow 0$ and $U_3 \to 0$ the result is in accordance with Equation (8.6-8).

Elliptical cone of finite height

Let the elliptical cone of finite height be defined by (Figure 8.6-5)

$$\mathcal{D}^{\mathrm{s}} = \left\{ \boldsymbol{x} \in \mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < x_3^2/h^2,\ 0 < x_3 < h \right\}. \tag{8.6-35}$$

Its volume is

$$V^{\mathrm{s}} = \pi a_1 a_2 h/3\,. \tag{8.6-36}$$

In the integral on the right-hand side of Equation (8.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{8.6-37}$$

as the variables of integration. In $\boldsymbol{y}$ space, the domain of integration is then $\{\boldsymbol{y}\in\mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2,\ 0 < y_3 < 1\}$. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \tag{8.6-38}$$

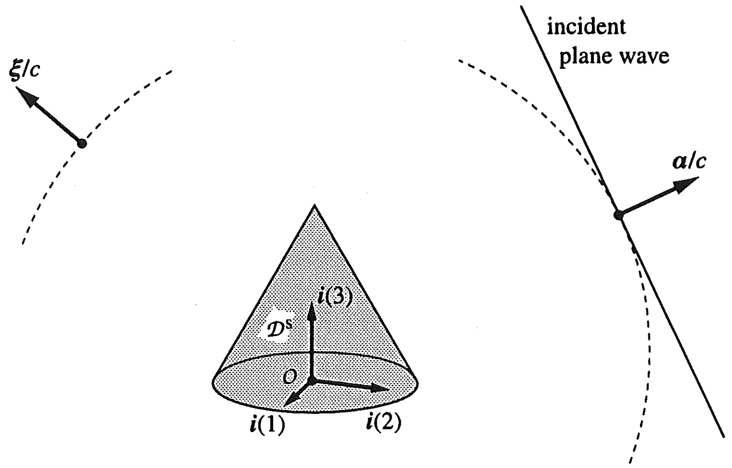

Figure 8.6-5 Scatterer in the shape of an elliptical cone of finite height.

we have

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3$$
$$= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3 \,, \tag{8.6-39}$$

while

$$\mathrm{d}V = a_1 a_2 h \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}y_3 \,. \tag{8.6-40}$$

The integration then runs as follows:

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h \int_{y_3=0}^{1} \mathrm{d}y_3 \int_{(y_1,y_2)\in\Delta^2(y_3)} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3) \, \mathrm{d}y_1 \, \mathrm{d}y_2 \,, \tag{8.6-41}$$

where $\Delta^2(y_3) = \{(y_1,y_2)\in\mathcal{R}^2; 0 \leqslant y_1^2 + y_2^2 < y_3^2\}$ is the circular disk of radius y_3. With a reasoning similar to that used in Equations (8.6-30)–(8.6-32), we obtain

$$\int_{(y_1,y_2)\in\Delta^2(y_3)} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3) \, \mathrm{d}y_1 \, \mathrm{d}y_2$$
$$= U^{-1} y_3 \int_{\phi=0}^{2\pi} \cos(\phi) \partial_t a \left[t + U y_3 \cos(\phi) + U_3 y_3\right] \mathrm{d}\phi \,, \tag{8.6-42}$$

in which

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0 \,. \tag{8.6-43}$$

Furthermore,

$$\int_{y_3=0}^{1} y_3 \partial_t a \left[t + U y_3 \cos(\phi) + U_3 y_3\right] \mathrm{d}y_3$$

$$= [U\cos(\phi) + U_3]^{-1} a\,[t + U\cos(\phi) + U_3]$$

$$- [U\cos(\phi) + U_3]^{-2} \left\{ \mathrm{I}_t a\,[t + U\cos(\phi) + U_3] - \mathrm{I}_t a(t) \right\}. \qquad (8.6\text{-}44)$$

Collecting the results, we end up with

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} \int_{\phi=0}^{2\pi} \cos(\phi) \Big\{ [U\cos(\phi) + U_3]^{-1} a\,[t + U\cos(\phi) + U_3]$$

$$- [U\cos(\phi) + U_3]^{-2} \left\{ \mathrm{I}_t a\,[t + U\cos(\phi) + U_3] - \mathrm{I}_t a(t) \right\}\Big\}\, \mathrm{d}\phi\,. \qquad (8.6\text{-}45)$$

Special cases occur for $U{\downarrow}0$ and/or $U_3{\to}0$. The corresponding limits easily follow from Equation (8.6-45) by using the pertaining Taylor expansions on the right-hand side. In particular, it can be verified that for $U{\downarrow}0$ and $U_3{\to}0$ the result is in accordance with Equation (8.6-8).

Tetrahedron

Let the tetrahedron be defined by (see Equation (A.9-17) and Figure 8.6-6)

$$\mathcal{D}^{\mathrm{s}} = \left\{ \boldsymbol{x} \in \mathcal{R}^3 \,;\, \boldsymbol{x} = \sum_{I=0}^{3} \lambda(I)\boldsymbol{x}(I),\quad 0 < \lambda(I) < 1,\ \sum_{I=0}^{3} \lambda(I) = 1 \right\}, \qquad (8.6\text{-}46)$$

in which $\{\boldsymbol{x}(0),\boldsymbol{x}(1),\boldsymbol{x}(2),\boldsymbol{x}_3(3)\}$ are the position vectors of the vertices and $\{\lambda(0),\ \lambda(1),\ \lambda(2),\lambda(3)\}$ are the barycentric coordinates. Its volume is (see Equations (A.10-29) and (A.10-33))

$$V^{\mathrm{s}} = \det\,[\boldsymbol{x}(1) - \boldsymbol{x}(0), \boldsymbol{x}(2) - \boldsymbol{x}(1), \boldsymbol{x}(3) - \boldsymbol{x}(2)]\,/6\,. \qquad (8.6\text{-}47)$$

In the integral on the right-hand side of Equation (8.6-7) we replace $\lambda(0)$ by $1 - \lambda(1) - \lambda(2) - \lambda(3)$ and introduce $\{\lambda(1),\lambda(2),\lambda(3)\}$ as the (dimensionless) variables of integration. In $\{\lambda(1),\lambda(2),\lambda(3)\}$ space the domain of integration is then $\{0 < \lambda(1) < 1,\ 0 < \lambda(2) < 1 - \lambda(1),\ 0 < \lambda(3) < 1 - \lambda(1) - \lambda(2)\}$. Then, with

$$U(I) = u_s x_s(I) \qquad \text{for } I = 0, 1, 2, 3\,, \qquad (8.6\text{-}48)$$

we have

$$u_s x_s = \lambda(0)U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3)$$

$$= [1 - \lambda(1) - \lambda(2) - \lambda(3)]\ U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3)$$

$$= U(0) + [U(1) - U(0)]\ \lambda(1) + [U(2) - U(0)]\ \lambda(2) + [U(3) - U(0)]\ \lambda(3)\,, \qquad (8.6\text{-}49)$$

while, with the Jacobian (see Equation (A.10-31))

$$\frac{\partial(x_1 x_2 x_3)}{\partial[\lambda(1),\lambda(2),\lambda(3)]} = 6V^{\mathrm{s}}, \qquad (8.6\text{-}50)$$

the elementary volume is expressed as

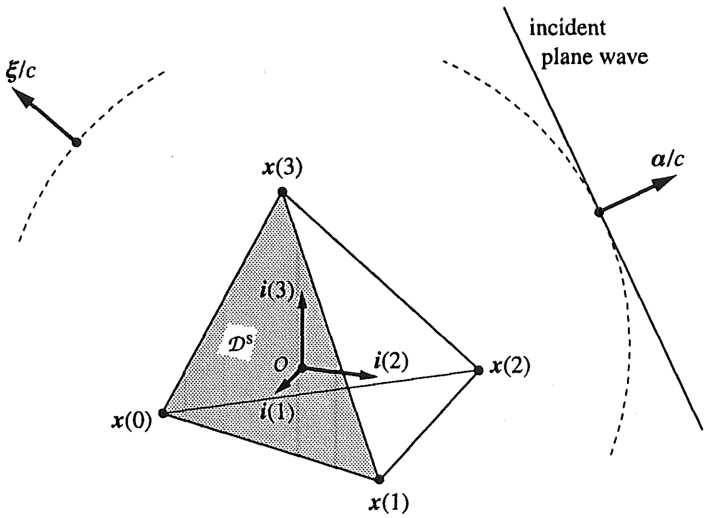

Figure 8.6-6 Scatterer in the shape of a tetrahedron (3-simplex).

$$dV = 6V^s \, d\lambda(1)\, d\lambda(2)\, d\lambda(3) \,. \tag{8.6-51}$$

After some lengthy, but elementary, calculations it is found that

$$\begin{aligned} \Upsilon(\boldsymbol{u},t) = 6V^s \Bigg\{ & \frac{1}{U(0)-U(1)} \frac{1}{U(0)-U(2)} \frac{1}{U(0)-U(3)} \mathrm{I}_t a\,[t+U(0)] \\ & + \frac{1}{U(1)-U(0)} \frac{1}{U(1)-U(2)} \frac{1}{U(1)-U(3)} \mathrm{I}_t a\,[t+U(1)] \\ & + \frac{1}{U(2)-U(0)} \frac{1}{U(2)-U(1)} \frac{1}{U(2)-U(3)} \mathrm{I}_t a\,[t+U(2)] \\ & + \frac{1}{U(3)-U(0)} \frac{1}{U(3)-U(1)} \frac{1}{U(3)-U(2)} \mathrm{I}_t a\,[t+U(3)] \Bigg\} \,. \end{aligned} \tag{8.6-52}$$

In a symmetrical fashion, this result can be written as

$$T(\boldsymbol{u},t) = 6V^s \sum_{I=0}^{3} \frac{1}{U(I)-U(J)} \frac{1}{U(I)-U(K)} \frac{1}{U(I)-U(L)} \mathrm{I}_t a\,[t+U(I)] \,, \tag{8.6-53}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$.

Special cases occur for $U(I) = U(J)$ and/or $U(I) = U(K)$ and/or $U(I) = U(L)$. The easiest way to arrive at the expressions for the relevant cases is to redo the integrations that need modifications.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the expressions for the scattered wave amplitude in the far-field region in the first-order Rayleigh–Gans–Born approximation follow, with the use of Equations (8.1-45)–(8.1-48), and Equations (8.5-13) and (8.5-14) as (Figure 8.6-7)

$$\hat{p}^{\mathrm{S};\infty}(\boldsymbol{\xi},s) = -Pc^{-2}\left[\hat{\Lambda}^{\chi}(\boldsymbol{\xi}/c - \boldsymbol{\alpha}/c,s) + \xi_k \hat{\Lambda}^{\mu}_{k,r}(\boldsymbol{\xi}/c - \boldsymbol{\alpha}/c,s)\alpha_r\right], \tag{8.6-54}$$

and

$$\hat{v}_r^{\mathrm{S};\infty}(\boldsymbol{\xi},s) = (\rho c)^{-1}\hat{p}^{\mathrm{S};\infty}(\boldsymbol{\xi},s)\xi_r, \tag{8.6-55}$$

where

$$\hat{\Lambda}^{\chi}(\boldsymbol{u},s) = s^2\hat{a}(s)\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{S}}}\left[\hat{\chi}^{\mathrm{S}}(\boldsymbol{x},s)/\kappa - 1\right]\exp(su_s x_s)\,\mathrm{d}V, \tag{8.6-56}$$

and

$$\hat{\Lambda}^{\mu}_{k,r}(\boldsymbol{u},s) = s^2\hat{a}(s)\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{S}}}\left[\hat{\mu}^{\mathrm{S}}_{k,r}(\boldsymbol{x},s)/\rho - \delta_{k,r}\right]\exp(su_s x_s)\,\mathrm{d}V. \tag{8.6-57}$$

Note that these scattered wave amplitudes depend in their directional characteristics only on the difference $\boldsymbol{\xi}/c - \boldsymbol{\alpha}/c$ between the slowness $\boldsymbol{\xi}/c$ in the direction of observation $\boldsymbol{\xi}$ and the slowness $\boldsymbol{\alpha}/c$ in the direction of propagation $\boldsymbol{\alpha}$ of the incident uniform plane wave. This property only holds in the Rayleigh–Gans–Born approximation and is not exact.

For a homogeneous object, Equations (8.6-56) and (8.6-57) reduce to

$$\hat{\Lambda}^{\chi}(\boldsymbol{u},s) = s^2\hat{a}(s)\left[\hat{\chi}^{\mathrm{S}}(s)/\kappa - 1\right]\hat{\Upsilon}(\boldsymbol{u},s), \tag{8.6-58}$$

and

$$\hat{\Lambda}^{\mu}_{k,r}(\boldsymbol{u},s) = s^2\hat{a}(s)\left[\hat{\mu}^{\mathrm{S}}_{k,r}(s)/\rho - \delta_{k,r}\right]\hat{\Upsilon}(\boldsymbol{u},s), \tag{8.6-59}$$

in which

$$\hat{\Upsilon}(\boldsymbol{u},s) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{S}}}\exp(su_s x_s)\,\mathrm{d}V \tag{8.6-60}$$

is the *complex frequency-domain shape factor* corresponding to the domain $\mathcal{D}^{\mathrm{S}}$ occupied by the scatterer. From Equation (8.6-60) it immediately follows that for $\boldsymbol{\xi}/c = \boldsymbol{\alpha}/c$, i.e. for observation "behind" the scatterer or "forward scattering", we have

$$\hat{\Upsilon}(\boldsymbol{0},s) = V^{\mathrm{S}}, \tag{8.6-61}$$

where V^{S} is the *volume of the scatterer*. Note, again, that Equation (8.6-61) only holds in the first-order Rayleigh–Gans–Born approximation, and is not exact.

Below, we shall derive for a number of canonical geometries of the scatterer, closed-form analytic expressions for the shape factor $\hat{\Upsilon}(\boldsymbol{u},s)$.

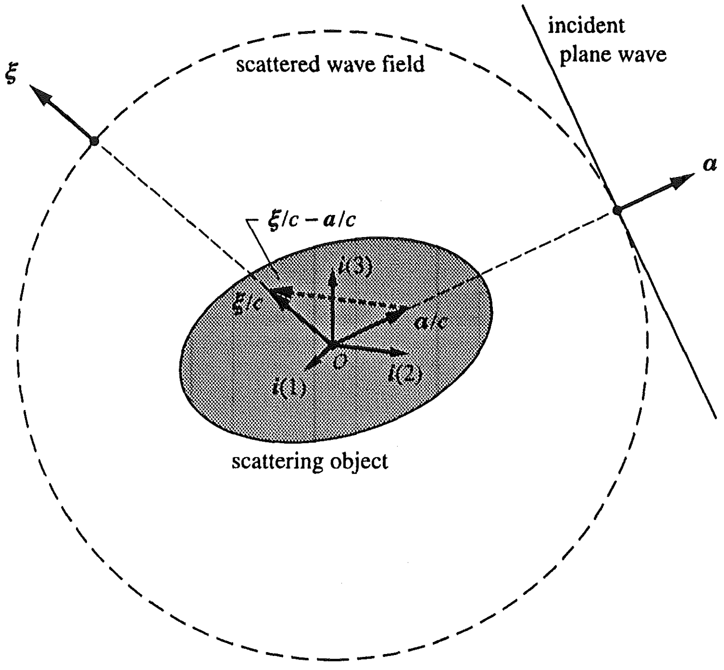

Figure 8.6-7 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation.

Ellipsoid

Let the scattering ellipsoid be defined by (see Equation (A.9-21) and Figure 8.6-8

$$\mathcal{D}^{\mathrm{s}} = \left\{x \in \mathcal{R}^3 ;\ 0 \leqslant (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 < 1\right\}. \qquad (8.6\text{-}62)$$

Its volume is

$$V^{\mathrm{s}} = (4\pi/3)a_1 a_2 a_3 . \qquad (8.6\text{-}63)$$

In the integral on the right-hand side of Equation (8.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \qquad (8.6\text{-}64)$$

as the variables of integration. In $\boldsymbol{y}$ space, the domain of integration is then the unit ball $\{y \in \mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 + y_3^2 < 1\}$. The integration over this unit ball is carried out with the aid of spherical polar coordinates $\{r,\theta,\phi\}$, with $0 \leqslant r < 1$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, about the vector $u_1 a_1 \boldsymbol{i}(1) + u_2 a_2 \boldsymbol{i}(2) + u_3 a_3 \boldsymbol{i}(3)$ as the polar axis. Then

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3 = (u_1 a_1)y_1 + (u_2 a_2)y_2 + (u_3 a_3)y_3 = Ur\cos(\theta) , \qquad (8.6\text{-}65)$$

where

$$U = \left[(u_1 a_1)^2 + (u_2 a_2)^2 + (u_3 a_3)^2\right]^{1/2} \geqslant 0 , \qquad (8.6\text{-}66)$$

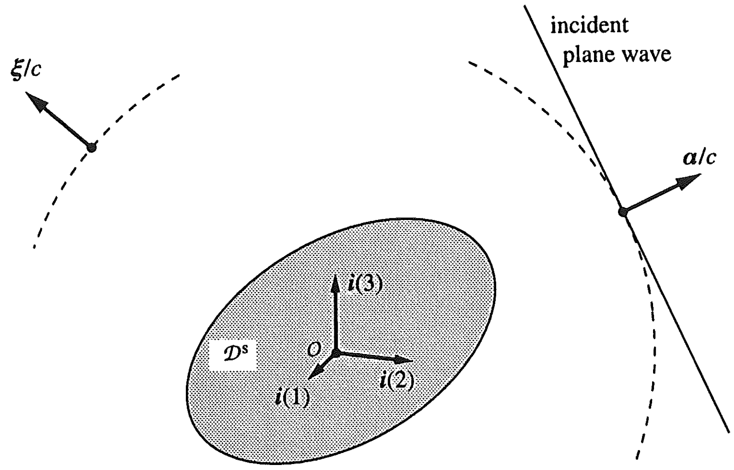

Figure 8.6-8 Scatterer in the shape of an ellipsoid.

while

$$\mathrm{d}V = a_1 a_2 a_3 r^2 \sin(\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi\,. \tag{8.6-67}$$

The integration then runs as follows:

$$\hat{\Upsilon}(\boldsymbol{u},s) = a_1 a_2 a_3 \int_{r=0}^{1} r^2\, \mathrm{d}r \int_{\theta=0}^{\pi} \sin(\theta)\, \mathrm{d}\theta \int_{\phi=0}^{2\pi} \exp\,[sUr\cos(\theta)]\, \mathrm{d}\phi$$

$$= 2\pi a_1 a_2 a_3 \int_{r=0}^{1} r^2\, \mathrm{d}r \int_{\theta=0}^{\pi} \exp\,[sUr\cos(\theta)] \sin(\theta)\, \mathrm{d}\theta$$

$$= 2\pi a_1 a_2 a_3 (sU)^{-1} \int_{r=0}^{1} [\exp(sUr) - \exp(-sUr)]\, r\, \mathrm{d}r$$

$$= 2\pi a_1 a_2 a_3 (sU)^{-2} \left\{ \exp(sU) + \exp(-sU) - \int_{r=0}^{1} [\exp(sUr) + \exp(-sUr)]\, \mathrm{d}r \right\}$$

$$= 2\pi a_1 a_2 a_3 (sU)^{-2} \left\{ \exp(sU) + \exp(-sU) - (sU)^{-1} [\exp(sU) - \exp(-sU)] \right\}$$

$$= 3V^{\mathrm{s}} \frac{sU\cosh(sU) - \sinh(sU)}{(sU)^3}\,. \tag{8.6-68}$$

By using the Taylor expansion of the right-hand side about $U = 0$ and taking the limit $U \downarrow 0$, it can be verified that the result is in accordance with Equation (8.6-61).

Rectangular block

Let the scattering domain be the rectangular block defined by (see Equation (A.9-14) and Figure 8.6-9)

$$\mathcal{D}^s = \left\{x \in \mathcal{R}^3;\ -a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2,\ -a_3 < x_3 < a_3\right\}. \tag{8.6-69}$$

Its volume is given by

$$V^s = 8a_1 a_2 a_3\,. \tag{8.6-70}$$

In the integral on the right-hand side of Equations (8.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{8.6-71}$$

as the variables of integration. In y space the domain of integration is then the cube $\{y \in \mathcal{R}^3;\ -1 < y_1 < 1,\ -1 < y_2 < 1,\ -1 < y_3 < 1\}$ with edge lengths 2. With

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 a_3\,, \tag{8.6-72}$$

furthermore, we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 a_3) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3\,, \end{aligned} \tag{8.6-73}$$

while

$$\mathrm{d}V = a_1 a_2 a_3\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3\,. \tag{8.6-74}$$

The integration then runs as follows:

$$\begin{aligned} \hat{\Upsilon}(u,s) &= a_1 a_2 a_3 \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \mathrm{d}y_2 \int_{y_1=-1}^{1} \exp\,[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\,\mathrm{d}y_1 \\ &= a_1 a_2 a_3 \int_{y_3=-1}^{1} \exp(sU_3 y_3)\,\mathrm{d}y_3 \int_{y_2=-1}^{1} \exp(sU_2 y_2)\,\mathrm{d}y_2 \int_{y_1=-1}^{1} \exp(sU_1 y_1)\,\mathrm{d}y_1 \\ &= a_1 a_2 a_3 \frac{\exp(sU_3) - \exp(-sU_3)}{sU_3} \frac{\exp(sU_2) - \exp(-sU_2)}{sU_2} \frac{\exp(sU_1) - \exp(-sU_1)}{sU_1} \\ &= V^s \frac{\sinh(sU_3)}{sU_3} \frac{\sinh(sU_2)}{sU_2} \frac{\sinh(sU_1)}{sU_1}\,. \end{aligned} \tag{8.6-75}$$

Special cases occur for either $U_1 \to 0$, $U_2 \to 0$, and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (8.6-75) by using the pertaining Taylor expansions on the right-hand side. In particular, it can be verified that for $U_1 \to 0$ and $U_2 \to 0$ and $U_3 \to 0$ the result is in accordance with Equation (8.6-61).

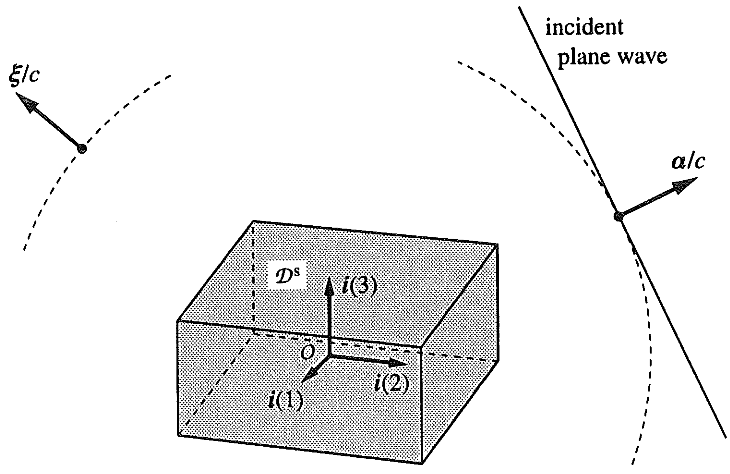

Figure 8.6-9 Scatterer in the shape of a rectangular block.

Elliptical cylinder of finite height

Let the elliptical cylinder of finite height be defined by (Figure 8.6-10)

$$\mathcal{D}^s = \left\{x \in \mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\ -h < x_3 < h\right\}. \tag{8.6-76}$$

Its volume is

$$V^s = 2\pi a_1 a_2 h\,. \tag{8.6-77}$$

In the integral on the right-hand side of Equation (8.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{8.6-78}$$

as the variables of integration. In y space, the domain of integration is then the Cartesian product of the unit disk $\Delta^2 = \{(y_1,y_2) \in \mathcal{R}^2;\ 0 \leqslant y_1^2 + y_2^2 < 1\}$ and the interval $\{y_3 \in \mathcal{R};\ -1 < y_3 < 1\}$ along the axis of the cylinder. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \tag{8.6-79}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3\,, \end{aligned} \tag{8.6-80}$$

while

$$\mathrm{d}V = a_1 a_2 h\ \mathrm{d}y_1\ \mathrm{d}y_2\ \mathrm{d}y_3\,. \tag{8.6-81}$$

The integration then runs as follows:

$$\hat{\Upsilon}(\boldsymbol{u},s) = a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} \mathrm{d}y_1\ \mathrm{d}y_2 \int_{y_3=-1}^{1} \exp\,[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\ \mathrm{d}y_3$$

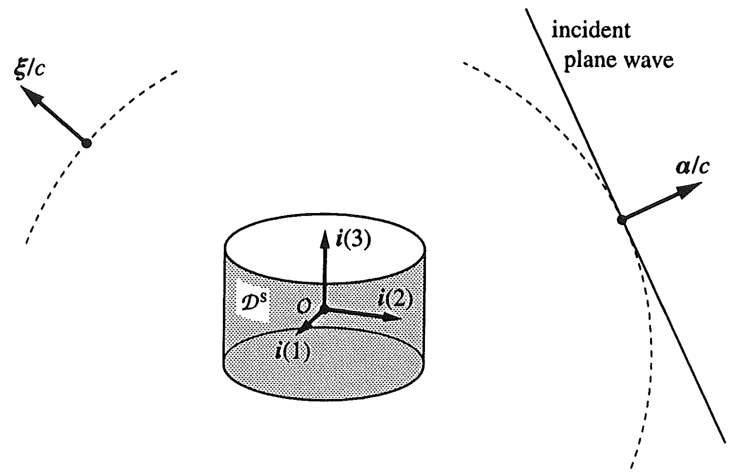

Figure 8.6-10 Scatterer in the shape of an elliptical cylinder of finite height.

$$= a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} (sU_3)^{-1} \left\{\exp\,[s(U_1 y_1 + U_2 y_2 + U_3)] - \exp\,[s(U_1 y_1 + U_2 y_2 - U_3)]\right\} \mathrm{d}y_1\,\mathrm{d}y_2 \,. \qquad (8.6\text{-}82)$$

Next, we observe that

$$\exp\,[s(U_1 y_1 + U_2 y_2 \pm U_3)] = (s^2 U_1^2 + s^2 U_2^2)^{-1}(\partial_{y_1}^2 + \partial_{y_2}^2)\exp\,[s(U_1 y_1 + U_2 y_2 \pm U_3)] \quad \text{for } U_1^2 + U_2^2 \neq 0 \,. \qquad (8.6\text{-}83)$$

Now, applying Gauss' divergence theorem to the integration over Δ^2, we obtain

$$\int_{(y_1,y_2)\in\Delta^2} (\partial_{y_1}^2 + \partial_{y_2}^2)\exp[s(U_1 y_1 + U_2 y_2 \pm U_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2$$

$$= \int_{(y_1,y_2)\in\mathcal{C}^2} (y_1\partial_{y_1} + y_2\partial_{y_2})\exp[s(U_1 y_1 + U_2 y_2 \pm U_3)]\,\mathrm{d}\sigma$$

$$= \int_{(y_1,y_2)\in\mathcal{C}^2} s(U_1 y_1 + U_2 y_2)\exp[s(U_1 y_1 + U_2 y_2 \pm U_3)]\,\mathrm{d}\sigma, \qquad (8.6\text{-}84)$$

where $\mathrm{d}\sigma$ is the elementary arc length along the unit circle $\mathcal{C}^2$ that forms the boundary of the unit disk Δ^2 and where we have used the property that the unit vector along the normal to $\mathcal{C}^2$ pointing away from Δ^2 is given by $\nu = y_1 i(1) + y_2 i(2)$. In the integral on the right-hand side of Equation (8.6-84) we introduce the polar coordinates $\{r,\phi\}$, with $r = 1$ and $0 \leqslant \phi < 2\pi$, about the vector $U_1 i(1) + U_2 i(2)$ as polar axis, as the variables of integration. This yields

$$\int_{(y_1,y_2)\in\mathcal{C}^2} (U_1 y_1 + U_2 y_2)\exp[s(U_1 y_1 + U_2 y_2 \pm U_3)]\,\mathrm{d}\sigma$$

$$= \int_{\phi=0}^{2\pi} U\cos(\phi)\exp[sU\cos(\phi) \pm sU_3]\,\mathrm{d}\phi = 2\pi U\exp(\pm sU_3)I_1(sU)\,, \tag{8.6-85}$$

where I_1 is the modified Bessel function of the first kind and of order 1 (Abramowitz and Stegun, 1964) and

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0\,. \tag{8.6-86}$$

Collecting the results, we end up with

$$\begin{aligned}\hat{\Gamma}(\boldsymbol{u},s) &= 2\pi a_1 a_2 h s^{-2} U^{-1} U_3^{-1} I_1(sU)\,[\exp(sU_3) - \exp(-sU_3)] \\ &= 2V^{\mathrm{s}} s^{-2} U^{-1} U_3^{-1} I_1(sU)\sinh(sU_3)\,.\end{aligned} \tag{8.6-87}$$

Special cases occur for $U{\downarrow}0$ and/or $U_3{\to}0$. The corresponding limits easily follow from Equation (8.6-87) by using the pertaining Taylor expansions on the right-hand side. In particular, it can be verified that for $U{\downarrow}0$ and $U_3{\to}0$ the result is in accordance with Equation (8.6-61).

Elliptical cone of finite height

Let the elliptical cone of finite height be defined by (Figure 8.6-11)

$$\mathcal{D}^{\mathrm{s}} = \left\{x\in\mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < x_3^2/h^2,\ 0 < x_3 < h\right\}. \tag{8.6-88}$$

Its volume is

$$V^{\mathrm{s}} = \pi a_1 a_2 h/3\,. \tag{8.6-89}$$

In the integral on the right-hand side of Equation (8.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{8.6-90}$$

as the variables of integration. In y space, the domain of integration is then $\{y\in\mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2,\ 0 < y_3 < 1\}$. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \tag{8.6-91}$$

we have

$$\begin{aligned}u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3\,,\end{aligned} \tag{8.6-92}$$

while

$$\mathrm{d}V = a_1 a_2 h\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3\,. \tag{8.6-93}$$

The integration then runs as follows:

$$\hat{\Gamma}(\boldsymbol{u},s) = a_1 a_2 h \int_{y_3=0}^{1} \mathrm{d}y_3 \int_{(y_1,y_2)\in\Delta^2(y_3)} \exp[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2\,, \tag{8.6-94}$$

where $\Delta^2((y_3) = \{(y_1,y_2)\in\mathcal{R}^2;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2\}$ is the circular disk of radius y_3. With a reasoning similar to the one as used in Equations (8.6-83)–(8.6-85), we obtain

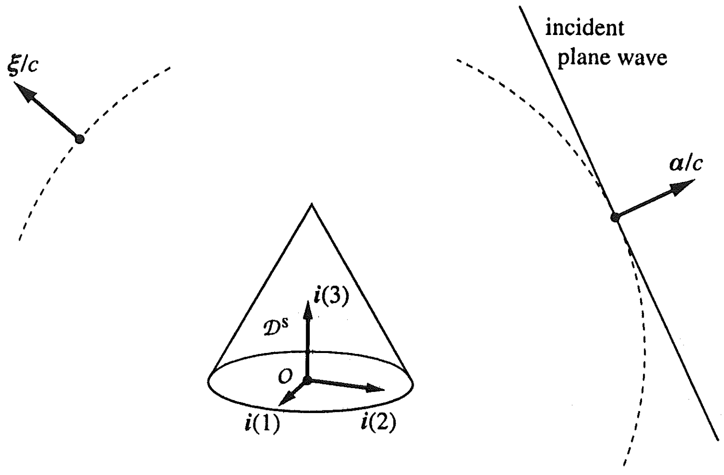

Figure 8.6-11 Scatterer in the shape of an elliptical cone of finite height.

$$\int_{(y_1,y_2)\in\Delta^2(y_3)} \exp[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2$$

$$= (sU)^{-1} y_3 \int_{\phi=0}^{2\pi} \cos(\phi)\exp[s(U y_3\cos(\phi) + U_3 y_3)]\,\mathrm{d}\phi\,, \tag{8.6-95}$$

in which

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0\,. \tag{8.6-96}$$

Furthermore,

$$\int_{y_3=0}^{1} y_3 \exp[s(U y_3\cos(\phi) + U_3 y_3)]\,\mathrm{d}y_3$$

$$= [s(U\cos(\phi) + U_3)]^{-1}\left\{\exp[s(U\cos(\phi) + U_3)] - \int_{y_3=0}^{1} \exp[s(U y_3\cos(\phi) + U_3 y_3)]\,\mathrm{d}y_3\right\}$$

$$= [s(U\cos(\phi) + U_3)]^{-1}\exp[s(U\cos(\phi) + U_3)]$$

$$- [s(U\cos(\phi) + U_3)]^{-2}\left\{\exp[s(U y_3\cos(\phi) + U_3)] - 1\right\}. \tag{8.6-97}$$

Collecting the results, we end up with

$$\hat{\Gamma}(\boldsymbol{u},s) = 6V^{\mathrm{s}}(sU)^{-1}\int_{\phi=0}^{2\pi} \cos(\phi)$$

$$\times\frac{1}{2\pi}\left\{\frac{\exp[s(U\cos(\phi) + U_3)]}{s(U\cos(\phi) + U_3)} - \frac{\exp[s(U\cos(\phi) + U_3)] - 1}{s^2(U\cos(\phi) + U_3)^2}\right\}\mathrm{d}\phi\,. \tag{8.6-98}$$

Special cases occur for $U \downarrow 0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (8.6-98) by using the pertaining Taylor expansions on the right-hand side. In particular, it can be verified that for $U \downarrow 0$ and $U_3 \to 0$ the result is in accordance with Equation (8.6-61).

Tetrahedron

Let the tetrahedron be defined by (see Equation (A.9-17) and Figure 8.6-12)

$$\mathcal{D}^{\mathrm{s}} = \left\{ \boldsymbol{x} \in \mathcal{R}^3;\ \boldsymbol{x} = \sum_{I=0}^{3} \lambda(I)\boldsymbol{x}(I),\ 0 < \lambda(I) < 1,\ \sum_{I=0}^{3} \lambda(I) = 1 \right\}, \tag{8.6-99}$$

in which $\{\boldsymbol{x}(0),\boldsymbol{x}(1),\boldsymbol{x}(2),\boldsymbol{x}(3)\}$ are the position vectors of the vertices and $\{\lambda(0),\ \lambda(1),\ \lambda(2),\lambda(3)\}$ are the barycentric coordinates. Its volume is given by (see Equations (A.10-29) and (A.10-33))

$$V^{\mathrm{s}} = \det[\boldsymbol{x}(1) - \boldsymbol{x}(0), \boldsymbol{x}(2) - \boldsymbol{x}(1), \boldsymbol{x}(3) - \boldsymbol{x}(2)]\,/6\,. \tag{8.6-100}$$

In the integral on the right-hand side of Equation (8.6-60) we replace $\lambda(0)$ by $1 - \lambda(1) - \lambda(2) - \lambda(3)$ and introduce $\{\lambda(1),\lambda(2),\lambda(3)\}$ as the (dimensionless) variables of integration. In $\{\lambda(1),\lambda(2),\lambda(3)\}$ space the domain of integration is then $\{0 < \lambda(1) < 1,\ 0 < \lambda(2) < 1 - \lambda(1),\ 0 < \lambda(3) < 1 - \lambda(1) - \lambda(2)\}$. Then, with

$$U(I) = u_s x_s(I) \qquad \text{for } I = 0, 1, 2, 3\,, \tag{8.6-101}$$

we have

$$\begin{aligned} u_s x_s &= \lambda(0)U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3) \\ &= [1 - \lambda(1) - \lambda(2) - \lambda(3)]\ U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3) \\ &= U(0) + [U(1) - U(0)]\,\lambda(1) + [U(2) - U(0)]\,\lambda(2) + [U(3) - U(0)]\,\lambda(3)\,, \end{aligned} \tag{8.6-102}$$

while, with the Jacobian (see Equation (A.10-31))

$$\frac{\partial(x_1 x_2 x_3)}{\partial\,[\lambda(1),\lambda(2),\lambda(3)]} = 6V^{\mathrm{s}}, \tag{8.6-103}$$

the elementary volume is expressed as

$$\mathrm{d}V = 6V^{\mathrm{s}}\,\mathrm{d}\lambda(1)\,\mathrm{d}\lambda(2)\,\mathrm{d}\lambda(3)\,. \tag{8.6-104}$$

After some lengthy, but elementary, calculations it is found that

$$\begin{aligned} \hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{s}}s^{-3} \Bigg\{ & \frac{1}{U(0) - U(1)}\,\frac{1}{U(0) - U(2)}\,\frac{1}{U(0) - U(3)} \exp[sU(0)] \\ & + \frac{1}{U(1) - U(0)}\,\frac{1}{U(1) - U(2)}\,\frac{1}{U(1) - U(3)} \exp[sU(1)] \\ & + \frac{1}{U(2) - U(0)}\,\frac{1}{U(2) - U(1)}\,\frac{1}{U(2) - U(3)} \exp[sU(2)] \\ & + \frac{1}{U(3) - U(0)}\,\frac{1}{U(3) - U(1)}\,\frac{1}{U(3) - U(2)} \exp[sU(3)] \Bigg\}\,. \end{aligned} \tag{8.6-105}$$

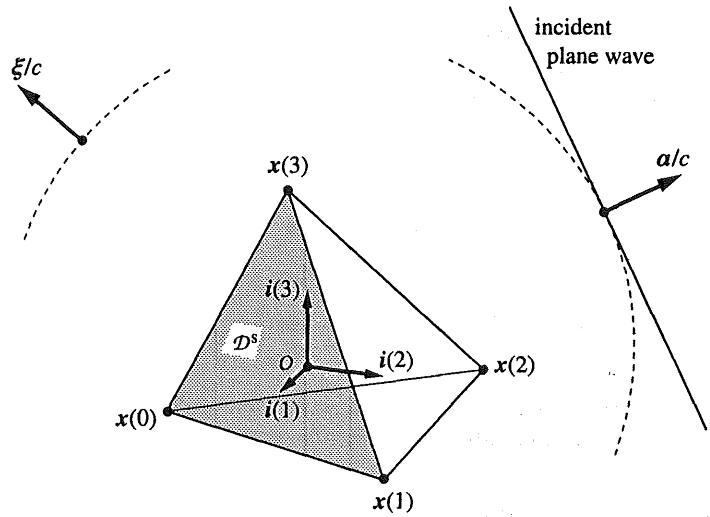

Figure 8.6-12 Scatterer in the shape of a tetrahedron (3-simplex).

In a symmetrical fashion, this result can be written as

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{s}s^{-3}\sum_{I=0}^{3}\frac{1}{U(I)-U(J)}\,\frac{1}{U(I)-U(K)}\,\frac{1}{U(I)-U(L)}\exp[sU(I)]\,, \qquad (8.6\text{-}106)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$.

Special cases occur for $U(I) = U(J)$ and/or $U(I) = U(K)$ and/or $U(I) = U(L)$. The easiest way to arrive at the expressions for the relevant cases is to redo the integrations that need modification.

(*Note*: Since the first-order Rayleigh–Gans–Born approximation is additive in the domains occupied by the scatterers, the scattering by an arbitrary union of canonical scatterers follows by superposition. In particular, the result for the tetrahedron is the building block for scatterers in the shape of an arbitrary polyhedron.)

The first-order Rayleigh–Gans–Born scattering finds numerous applications both in the forward (direct) and the inverse scattering theory. References to the earlier literature can be found in Quak *et al.* (1986).

Exercises

Exercise 8.6-1

Show that Equation (8.6-60) follows from the time Laplace transform of Equation (8.6-7).

Exercise 8.6-2

Show that Equation (8.6-61) follows from the time Laplace transform of Equation (8.6-8).

Exercise 8.6-3

Show that Equation (8.6-68) follows from the time Laplace transform of Equation (8.6-15).

Exercise 8.6-4

Show that Equation (8.6-75) follows from the time Laplace transform of Equation (8.6-22).

Exercise 8.6-5

Show that Equation (8.6-87) follows from the time Laplace transform of Equation (8.6-34).

Exercise 8.6-6

Show that Equation (8.6-98) follows from the time Laplace transform of Equation (8.6-45).

Exercise 8.6-7

Show that Equation (8.6-106) follows from the time Laplace transform of Equation (8.6-53).

Exercise 8.6-8

Show that for $U \downarrow 0$, Equation (8.6-15) becomes Equation (8.6-8). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 8.6-9

Show that for $U_3 \to 0$, Equation (8.6-22) becomes

$$\Upsilon(\boldsymbol{u},t) = 2a_1a_2a_3(U_1U_2)^{-1}\,[a(t+U_1+U_2) - a(t+U_1-U_2) - a(t-U_1+U_2) + a(t-U_1-U_2)]\,. \quad (8.6\text{-}107)$$

(In this case, $\boldsymbol{u}$ is parallel to the x_1,x_2 plane.)

Exercise 8.6-10

Show that for $U_2 \to 0$ and $U_3 \to 0$, Equation (8.6-22) becomes

$$\Upsilon(\boldsymbol{u},t) = 4a_1a_2a_3U_1^{-1}\,[\partial_t a(t+U_1) - \partial_t a(t-U_1)]\,. \quad (8.6\text{-}108)$$

(In this case, $\boldsymbol{u}$ is parallel to the x_1 axis.)

Exercise 8.6-11

Show that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$, Equation (8.6-22) becomes Equation (8.6-8). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 8.6-12

Show that for $U \downarrow 0$, Equation (8.6-34) becomes

$$\Upsilon(\boldsymbol{u},t) = \pi a_1 a_2 h U_3^{-1}\,[\partial_t a(t + U_3) - \partial_t a(t - U_3)]\,. \tag{8.6-109}$$

(In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 8.6-13

Show that for $U_3 \to 0$, Equation (8.6-34) becomes

$$\Upsilon(\boldsymbol{u},t) = 2a_1 a_2 h U^{-1} \int_{\phi=0}^{2\pi} \cos(\phi)\partial_t a\,[t + U\cos(\phi)]\,\mathrm{d}\phi\,. \tag{8.6-110}$$

(In this case $\boldsymbol{u}$ is perpendicular to the axis of the cylinder.)

Exercise 8.6-14

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (8.6-34) becomes Equation (8.6-8). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 8.6-15

Show that for $U \downarrow 0$, Equation (8.6-45) becomes

$$\Upsilon(\boldsymbol{u},t) = \pi a_1 a_2 h \left\{U_3^{-1}\,[\partial_t a(t + U_3) - 2U_3^{-2} a(t + U_3)] + 2U_3^{-3}\,[\mathrm{I}_t a(t + U_3) - \mathrm{I}_t a(t)]\right\}. \tag{8.6-111}$$

(In this case, $\boldsymbol{u}$ is parallel to the axis of the cone.)

Exercise 8.6-16

Show that for $U_3 \to 0$, Equation (8.6-45) becomes

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} \int_{\phi=0}^{2\pi} \left\{[U\cos(\phi)\,]^{-1}\, a(t + U\cos(\phi)) - [U\cos(\phi)]^{-2}[\mathrm{I}_t a(t + U\cos(\phi)) - \mathrm{I}_t a(t)]\right\} \cos(\phi)\,\mathrm{d}\phi\,. \tag{8.6-112}$$

(In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cone.)

Exercise 8.6-17

Show that for $U{\downarrow}0$ and $U_3 \to 0$, Equation (8.6-45) becomes Equation (8.6-8). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 8.6-18

Show that for $U(J) \to U(I)$, Equation (8.6-53) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left[\left\{\frac{1}{U(I)-U(K)}\,\frac{1}{U(I)-U(L)}\right\} a[t+U(I)]\right.$$
$$-\left\{\frac{1}{[U(I)-U(K)]^2}\,\frac{1}{U(I)-U(L)}+\frac{1}{U(I)-U(K)}\,\frac{1}{[U(I)-U(L)]^2}\right\}\mathrm{I}_t a[t+U(I)]$$
$$+\left\{\frac{1}{[U(K)-U(I)]^2}\,\frac{1}{U(K)-U(L)}\,\mathrm{I}_t a[t+U(K)]\right\}$$
$$\left.+\left\{\frac{1}{[U(L)-U(I)]^2}\,\frac{1}{U(L)-U(K)}\,\mathrm{I}_t a[t+U(L)]\right\}\right], \tag{8.6-113}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$.)

Exercise 8.6-19

Show that for $U(J) \to U(I)$ and $U(L) \to U(K)$, Equation (8.6-53) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left(\frac{1}{[U(I)-U(K)]^2}\{a\,[t+U(I)]+a\,[t+U(K)]\}\right.$$
$$\left.-\frac{2}{[U(I)-U(K)]^3}\{\mathrm{I}_t a\,[t+U(I)]-\mathrm{I}_t a\,[t+U(K)]\}\right), \tag{8.6-114}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$, as well as perpendicular to the edge connecting the vertex $\boldsymbol{x}(K)$ with the vertex $\boldsymbol{x}(L)$.)

Exercise 8.6-20

Show that for $U(J) \to U(I)$ and $U(K) \to U(I)$, Equation (8.6-53) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{S}}\left(\frac{1}{U(I)-U(L)}\partial_t a\,[t+U(I)] - \frac{1}{[U(I)-U(L)]^2}a\,[t+U(I)] + \frac{1}{[U(I)-U(L)]^3}\left\{\mathrm{I}_t a\,[t+U(I)] - \mathrm{I}_t a\,[t+U(L)]\right\}\right), \tag{8.6-115}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the plane containing the triangle of which $\boldsymbol{x}(I)$, $\boldsymbol{x}(J)$ and $\boldsymbol{x}(K)$ are the vertices.)

Exercise 8.6-21

Show that for $\boldsymbol{u} = \boldsymbol{0}$, Equation (8.6-53) becomes Equation (8.6-8).

Exercise 8.6-22

Show that for $U \downarrow 0$, Equation (8.6-68) becomes Equation (8.6-61).

Exercise 8.6-23

Show that for $U_3 \to 0$, Equation (8.6-75) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = V^{\mathrm{S}}\frac{\sinh(sU_2)}{sU_2}\frac{\sinh(sU_1)}{sU_1} \tag{8.6-116}$$

and show that the result follows from the time Laplace transform of Equation (8.6-107). (In this case, $\boldsymbol{u}$ is parallel to the x_1,x_2 plane.)

Exercise 8.6-24

Show that for $U_2 \to 0$ and $U_3 \to 0$, Equation (8.6-75) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = V^{\mathrm{S}}\frac{\sinh(sU_1)}{sU_1} \tag{8.6-117}$$

and show that the result follows from the time Laplace transform of Equation (8.6-108). (In this case, $\boldsymbol{u}$ is parallel to the x_1 axis.)

Exercise 8.6-25

Show that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$, Equation (8.6-75) becomes Equation (8.6-61). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 8.6-26

Show that for $U \downarrow 0$, Equation (8.6-87) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 2\pi a_1 a_2 h s^{-1} U_3^{-1} \sinh(sU_3) \tag{8.6-118}$$

and show that the result follows from the time Laplace transform of Equation (8.6-109). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 8.6-27

Show that for $U_3 \to 0$, Equation (8.6-87) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 4\pi a_1 a_2 h s^{-1} U^{-1} I_1(sU) \tag{8.6-119}$$

and show that the result follows from the time Laplace transform of Equation (8.6-110). (In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cylinder.)

Exercise 8.6-28

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (8.6-87) becomes Equation (8.6-61). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 8.6-29

Show that for $U \downarrow 0$, Equation (8.6-98) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = \pi a_1 a_2 h s^{-2} \left\{ \left[sU_3^{-1} - 2U_3^{-2} + 2s^{-1} U_3^{-3} \right] \exp(sU_3) - 2s^{-1} U_3^{-3} \right\} \tag{8.6-120}$$

and show that the result follows from the time Laplace transform of Equation (8.6-111). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cone.)

Exercise 8.6-30

Show that for $U_3 \to 0$, Equation (8.6-98) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = a_1 a_2 h s^{-2} U^{-1} \int_{\phi=0}^{2\pi} \left\{ [U\cos(\phi)]^{-1} \exp\,[sU\cos(\phi)] - s^{-1}\,[U\cos(\phi)]^{-2}\,[\exp(sU\cos(\phi)) - 1] \right\} \cos(\phi)\,\mathrm{d}\phi \tag{8.6-121}$$

and show that this result is the time Laplace transform of Equation (8.6-112). (In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cone.)

Exercise 8.6-31

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (8.6-98) becomes Equation (8.6-61). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 8.6-32

Show that for $U(J)\rightarrow U(I)$, Equation (8.6-105) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}} s^{-2} \Bigg[\left\{ \frac{1}{U(I)-U(K)} \frac{1}{U(I)-U(L)} \exp\,[sU(I)] \right\}$$

$$- \left\{ \frac{1}{[U(I)-U(K)]^2} \frac{1}{U(I)-U(L)} + \frac{1}{U(I)-U(K)} \frac{1}{[U(I)-U(L)]^2} \right\} s^{-1} \exp\,[sU(I)]$$

$$+ \left\{ \frac{1}{[U(K)-U(L)]^2} \frac{1}{U(K)-U(L)} s^{-1} \exp\,[sU(K)] \right\}$$

$$+ \left\{ \frac{1}{[U(L)-U(I)]^2} \frac{1}{U(L)-U(K)} s^{-1} \exp\,[sU(L)] \right\} \Bigg], \tag{8.6-122}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$ and show that this result follows from the time Laplace transform of Equation (8.6-113). (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$.)

Exercise 8.6-33

Show that for $U(J)\rightarrow U(I)$ and $U(L)\rightarrow U(K)$, Equation (8.6-105) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}} s^{-2} \left(\frac{1}{[U(I)-U(K)]^2} \{\exp\,[sU(I)] + \exp\,[sU(K)]\} \right.$$

$$\left. - \frac{2s^{-1}}{[U(I)-U(K)]^3} \{\exp\,[sU(I)] - \exp\,[sU(L)]\} \right), \tag{8.6-123}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$ and show that this result follows from the time Laplace transform of Equation (8.6-114). (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$ as well as perpendicular to the edge connecting the vertex $\boldsymbol{x}(K)$ with the vertex $\boldsymbol{x}(L)$.)

Exercise 8.6-34

Show that for $U(J)\rightarrow U(I)$ and $U(K)\rightarrow U(I)$, Equation (8.6-105) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}} s^{-2} \left(\frac{s}{U(I)-U(K)} \exp\,[sU(I)] - \frac{1}{[U(I)-U(K)]^2} \exp\,[sU(I)] \right.$$

$$\left. + \frac{s^{-1}}{[U(I)-U(K)]^3} \{\exp\,[sU(I)] - \exp\,[sU(L)]\} \right), \tag{8.6-124}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$ and show that this result follows from the time Laplace transform of Equation (8.6-115). (In this case, $\boldsymbol{u}$ is perpendicular to the plane containing the triangle of which $\boldsymbol{x}(I)$, $\boldsymbol{x}(J)$ and $\boldsymbol{x}(K)$ are the vertices.)

Exercise 8.6-35

Show that for $\boldsymbol{u} = \boldsymbol{0}$, Equation (8.6-105) becomes Equation (8.6-61).

References

Abromowitz, M., and Stegun, I.E. 1964, *Handbook of Mathematical Functions*, Washington DC: National Bureau of Standards, Formula 9.6.19, p. 376.

De Hoop, A.T., 1959, On the plane wave extinction cross-section of an obstacle, *Applied Scientific Research, Section B*, **7**, 463–469.

De Hoop, A.T., 1960, A reciprocity theorem for the electromagnetic field scattered by an obstacle, *Applied Scientific Research, Section B*, **8**, 135–140.

De Hoop, A.T., 1985, A time-domain energy theorem for scattering of plane acoustic waves in fluids, *Journal of the Acoustical Society of America*, **77** (1), 11–14.

De Hoop, A.T., 1991, Convergence criterion for the time-domain iterative Born approximation to scattering by an inhomogeneous, dispersive object, *Journal of the Optical Society of America A*, **8** (8), 1256–1260.

Quak, D., De Hoop, A.T., and Stam, H.J., 1986, Time-domain farfield scattering of plane acoustic waves by a penetrable object in the Born approximation, *Journal of the Acoustical Society of America*, **80** (4), 1228–1234.