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The elastic wave equations, constitutive relations, and boundary conditions in the time Laplace-transform domain (complex frequency domain)

In a large number of cases met in practice, one is interested in the behaviour of causal elastic wave fields in linear, time-invariant configurations. Mathematically, one can take advantage of this situation by carrying out a Laplace transformation with respect to time and considering the equations governing the elastic wave field in the corresponding time Laplace-transform domain or *complex frequency domain.* In the complex frequency-domain relations, the time coordinate has been eliminated, and a field problem in space remains in which the Laplace-transform parameter s occurs as a parameter. Causality of the field is taken into account by taking $\mathrm{Re}(s) > 0$, and requiring that all causal field quantities are, in the case where the wave field is excited by sources of finite amplitude and bounded extent, analytic functions of s in the right half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane. The complex frequency-domain solution to a wave problem itself exhibits a number of features that are characteristic for the configuration in which the wave motion is present. If one is, in addition, interested in the actual pulse shapes of waves, one has to carry out the inverse Laplace transformation, either by analytical or by numerical methods.

In a number of wave propagation problems, the transform parameter s is profitably chosen to be real and positive. On the other hand, by taking $s = \mathrm{j}\omega$, where j is the imaginary unit and ω is real and positive, the complex steady-state representation of wave fields oscillating sinusoidally in time with *angular frequency* ω follows, the complex representation having the complex time factor $\exp(\mathrm{j}\omega t)$. For arbitrary complex values of s in the domain of analyticity, all complex frequency-domain wave-field quantities and constitutive relaxation functions are the Laplace transforms of real-valued functions of the time coordinate t. As a consequence, the complex frequency-domain wave-field quantities and constitutive relaxation functions are real-valued for real and positive values of s. On account of Schwarz's reflection principle of complex function theory, the relevant functions then take on complex conjugate values in conjugate complex points of the s plane.

In the present chapter the elastic wave equations, constitutive relations, and boundary conditions in the complex frequency domain are given and the complex frequency-domain elastodynamic wave potentials and Green's functions (point-source solutions) are introduced. The notations of Appendix B are used.

12.1 The complex frequency-domain elastic wave equations

We subject the elastic wave equations (10.7-22) and (10.7-23) to a Laplace transformation over the interval $\mathcal{T} = \{t \in \mathcal{R}; t > t_0\}$. For completeness, we allow a non-vanishing elastic wave field to be present at $t = t_0$, although in the majority of cases we are interested in the causal wave field generated by sources that are switched on at the instant $t = t_0$, in which case the initial values of the elastic wave field are taken to be zero. Since, with the use of the notations of Appendix B and the properties of the Laplace transformation,

$$\int_{t=t_0}^{\infty} \exp(-st)\partial_t \Phi_k(\boldsymbol{x},t)\,\mathrm{d}t = -\Phi_k(\boldsymbol{x},t_0)\exp(-st_0) + s\hat{\Phi}_k(\boldsymbol{x},s) \tag{12.1-1}$$

and

$$\int_{t=t_0}^{\infty} \exp(-st)\partial_t e_{i,j}(\boldsymbol{x},t)\,\mathrm{d}t = -e_{i,j}(\boldsymbol{x},t_0)\exp(-st_0) + s\hat{e}_{i,j}(\boldsymbol{x},s)\,, \tag{12.1-2}$$

we arrive at

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}_{p,q} + s\hat{\Phi}_k = \hat{f}_k + \exp(-st_0)\Phi_k(\boldsymbol{x},t_0)\,, \tag{12.1-3}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}_r - s\hat{e}_{i,j} = \hat{h}_{i,j} - \exp(-st_0)e_{i,j}(\boldsymbol{x},t_0)\,. \tag{12.1-4}$$

From Equations (12.1-3) and (12.1-4) it follows that, in the complex frequency domain, one can take into account the influence of a non-vanishing initial elastic wave field by properly incorporating its values in the complex frequency-domain volume densities of external force and external deformation rate. In the remainder of our analysis, it will be tacitly understood that non-zero initial elastic wave-field values have been accounted for in this manner.

After transforming back to the time domain, the reconstructed elastic wave-field values are zero in the interval $t \in \mathcal{T}'$, where $\mathcal{T}' = \{t \in \mathcal{R}; t < t_0\}$, and equal to the actual field values when $t \in \mathcal{T}$. In addition, many of the Laplace inversion algorithms, in particular the complex Bromwich inversion integral Equation (B.1-19) (of which the Fourier inversion integral is a limiting case), yield half the field values at the instant $t \in \partial\mathcal{T}$, where $\partial\mathcal{T} = \{t \in \mathcal{R}; t = t_0\}$. Notationally, this can be expressed by employing the characteristic function $\chi_{\mathcal{T}} = \chi_{\mathcal{T}}(t)$ of the set $\mathcal{T}$, which is defined as

$$\chi_{\mathcal{T}} = \{1, ½, 0\} \qquad \text{for } t \in \{\mathcal{T}, \partial\mathcal{T}, \mathcal{T}'\}\,. \tag{12.1-5}$$

With this notation, we have for the standard inversion applied to the Laplace transform $\hat{f}(\mathbf{x},s)$ of any space–time function $f = f(\mathbf{x},t)$ the result

Inverse Laplace transform of $\hat{f}(x,s) = \chi_{\mathcal{T}}(t)\, f(x,t)$. (12.1-6)

Exercises

Exercise 12.1-1

(a) What volume density of external volume force corresponds in the complex frequency domain to the initial field $\Phi_k(x,t_0)$?
(b) What would be the corresponding volume density of external volume force in the space–time domain?

Answers: (a) $\exp(-st_0)\Phi_k(x,t_0)$; (b) $\Phi_k(x,t_0)\delta(t-t_0)$.

Exercise 12.1-2

(a) What volume density of external deformation rate corresponds in the complex frequency domain to the initial field $e_{i,j}(x,t_0)$?
(b) What would be the corresponding volume density of external deformation rate in the space–time domain?

Answers: (a) $-\exp(-st_0)e_{i,j}(x,t_0)$; (b) $-e_{i,j}(x,t_0)\delta(t-t_0)$.

12.2 The complex frequency-domain constitutive relations; the Kramers–Kronig causality relations for a solid with relaxation

In the time Laplace transformation of the constitutive relations we separately discuss: solids with relaxation, instantaneously reacting solids, and solids whose elastic behaviour is described by the frictional-force/viscosity elastodynamic loss mechanism given in Section 10.9.

Solid with relaxation

The constitutive relations for a linear, time-invariant, locally reacting solid with relaxation are, in their low-velocity linearised approximation, given by (see Equations (10.7-26) and (10.7-27))

$$\Phi_k(x,t) = \int_{t'=0}^{\infty} \mu_{k,r}(x,t')v_r(x,t-t')\,\mathrm{d}t' \tag{12.2-1}$$

and

$$e_{i,j}(x,t) = \int_{t'=0}^{\infty} \chi_{i,j,p,q}(x,t')\tau_{p,q}(x,t-t')\,\mathrm{d}t' . \tag{12.2-2}$$

Mathematically, the right-hand sides of Equations (12.2-1) and (12.2-2) are *convolutions in time*. (The notion that convolutions in time can serve as the mathematical description of mechanical relaxation goes back to Boltzmann (see Boltzmann 1876).) From this it can be expected that the Laplace transformation possibly reveals additional properties of the relaxation functions. Carrying out the Laplace transformation of Equations (12.2-1) and (12.2-2) over the interval $t \in \mathcal{R}$, we obtain, assuming the elastic wave field to be of a transient nature,

$$\hat{\Phi}_k(\boldsymbol{x},s) = \hat{\mu}_{k,r}(\boldsymbol{x},s)\hat{v}_r(\boldsymbol{x},s) \,, \tag{12.2-3}$$

and

$$\hat{e}_{i,j}(\boldsymbol{x},s) = \hat{\chi}_{i,j,p,q}(\boldsymbol{x},s)\hat{\tau}_{p,q}(\boldsymbol{x},s) \,, \tag{12.2-4}$$

respectively, where, in view of Equations (B.1-12) and (10.5-32),

$$\hat{\mu}_{k,r}(\boldsymbol{x},s) = \int_{t'=0}^{\infty} \exp(-st')\mu_{k,r}(\boldsymbol{x},t')\,\mathrm{d}t' \,, \tag{12.2-5}$$

and

$$\hat{\chi}_{i,j,p,q}(\boldsymbol{x},s) = \int_{t'=0}^{\infty} \exp(-st')\chi_{i,j,p,q}(\boldsymbol{x},t')\,\mathrm{d}t' \,. \tag{12.2-6}$$

Evidently, the quantities $\{\hat{\mu}_{k,r},\hat{\chi}_{i,j,p,q}\}$ are the Laplace transforms of causal functions of time. Therefore, as has been shown in Section B.3, their real and imaginary parts for imaginary values of $s = \mathrm{j}\omega$, with $\omega \in \mathcal{R}$, introduced according to

$$\{\hat{\mu}_{k,r},\hat{\chi}_{i,j,p,q}\}(\boldsymbol{x},\mathrm{j}\omega) = \{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega) - \mathrm{j}\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega) \,, \tag{12.2-7}$$

satisfy the Kramers–Kronig relations (see Equations (B.3-18) and (B.3-19))

$$\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega) = -\frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega')}{\omega' - \omega}\,\mathrm{d}\omega' \,, \tag{12.2-8}$$

and

$$\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega) = \frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega')}{\omega' - \omega}\,\mathrm{d}\omega' \,. \tag{12.2-9}$$

Equations (12.2-8) and (12.2-9) imply that $\{\mu'_{k,r},\chi'_{i,j,p,q}\}$ and $\{\mu''_{k,r},\chi''_{i,j,p,q}\}$ form pairs of Hilbert transforms. Another property of $\{\hat{\mu}_{k,r},\hat{\chi}_{i,j,p,q}\}(\boldsymbol{x},\mathrm{j}\omega)$ is that (see Equations (B.3-6) and (B.3-7))

$$\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},-\omega) = \{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega) \qquad \text{for all } \omega \in \mathcal{R}, \tag{12.2-10}$$

and

$$\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},-\omega) = -\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega) \qquad \text{for all } \omega \in \mathcal{R}, \tag{12.2-11}$$

i.e. $\mu'_{k,r}$ and $\chi'_{i,j,p,q}$ are even functions of ω and $\mu''_{k,r}$ and $\chi''_{i,j,p,q}$ are odd functions of ω for $\omega \in \mathcal{R}$. Using these properties in the right-hand sides of Equations (12.2-8) and (12.2-9), these relations can be rewritten as (see Equations (B.3-26) and (B.3-27))

$$\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega) = -\frac{2}{\pi}\int_{\omega'=0}^{\infty} \frac{\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega')\omega}{(\omega')^2-\omega^2}\,\mathrm{d}\omega' \quad \text{for } \omega\in\mathcal{R}, \tag{12.2-12}$$

and

$$\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega) = \frac{2}{\pi}\int_{\omega'=0}^{\infty} \frac{\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega')\omega'}{(\omega')^2-\omega^2}\,\mathrm{d}\omega' \quad \text{for } \omega\in\mathcal{R}. \tag{12.2-13}$$

In case the right-hand sides of either Equations (12.2-8) and (12.2-9) or Equations (12.2-12) and (12.2-13) have to be evaluated numerically, the Cauchy principal values of the integrals may present a difficulty. To circumvent this difficulty, we can rewrite Equations (12.2-8) and (12.2-9) as (see Equations (B.3-24) and (B.3-25))

$$\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega) = -\frac{1}{\pi}\int_{\omega'=-\infty}^{\infty} \frac{\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega') - \{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega)}{\omega'-\omega}\,\mathrm{d}\omega'$$

$$\text{for } \omega\in\mathcal{R}, \tag{12.2-14}$$

and

$$\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega) = \frac{1}{\pi}\int_{\omega'=-\infty}^{\infty} \frac{\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega') - \{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega)}{\omega'-\omega}\,\mathrm{d}\omega'$$

$$\text{for } \omega\in\mathcal{R}, \tag{12.2-15}$$

and Equations (12.2-12) and (12.2-13) as (see Equations (B.3-29) and (B.3-30))

$$\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega) = -\frac{2}{\pi}\int_{\omega'=0}^{\infty} \frac{[\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega') - \{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega)]\,\omega}{(\omega')^2-\omega^2}\,\mathrm{d}\omega'$$

$$\text{for } \omega\in\mathcal{R}, \tag{12.2-16}$$

and

$$\{\mu'_{k,r},\chi'_{i,j,p,q}\}(\boldsymbol{x},\omega) = \frac{2}{\pi}\int_{\omega'=0}^{\infty} \frac{[\{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega')\omega' - \{\mu''_{k,r},\chi''_{i,j,p,q}\}(\boldsymbol{x},\omega)\omega]}{(\omega')^2-\omega^2}\,\mathrm{d}\omega'$$

$$\text{for } \omega\in\mathcal{R}. \tag{12.2-17}$$

Equations (12.2-14)–(12.2-17) are the Bode relations for the relaxation functions (see Bode 1959). In their right-hand sides only proper integrals occur.

Instantaneously reacting solid

For an instantaneously reacting solid, Equations (10.7-24) and (10.7-25) lead, after time Laplace transformation, to

$$\hat{\Phi}_k(\boldsymbol{x},s) = \rho_{k,r}(\boldsymbol{x})\hat{v}_r(\boldsymbol{x},s) \tag{12.2-18}$$

and

$$\hat{e}_{i,j}(\boldsymbol{x},s) = S_{i,j,p,q}(\boldsymbol{x})\hat{\tau}_{p,q}(\boldsymbol{x},s) , \tag{12.2-19}$$

respectively, in which the constitutive coefficients $\rho_{k,r}$ and $S_{i,j,p,q}$ are independent of s.

Solid with frictional-force/viscosity elastodynamic loss mechanism

For a solid whose loss behaviour can be described by the frictional-force/viscosity elastodynamic loss mechanism (see Equations (10.9-4) and (10.9-5)), the complex frequency-domain constitutive equations become

$$\hat{\Phi}_k(\boldsymbol{x},s) = [s^{-1}K_{k,r}(\boldsymbol{x}) + \rho_{k,r}(\boldsymbol{x})]\,\hat{v}_r(\boldsymbol{x},s) \tag{12.2-20}$$

and

$$\hat{e}_{i,j}(\boldsymbol{x},s) = [s^{-1}\Gamma_{i,j,p,q}(\boldsymbol{x}) + S_{i,j,p,q}(\boldsymbol{x})]\,\hat{\tau}_{p,q}(\boldsymbol{x},s) , \tag{12.2-21}$$

in which the constitutive coefficients $K_{k,r}$, $\rho_{k,r}$, $\Gamma_{i,j,p,q}$ and $S_{i,j,p,q}$ are independent of s. Note that, formally, Equations (12.2-20) and (12.2-21) have the same structure as Equations (12.2-3) and (12.2-4), and that the coefficients $s^{-1}K_{k,r} + \rho_{k,r}$ and $s^{-1}\Gamma_{i,j,p,q} + S_{i,j,p,q}$ are analytic functions of s in the right half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane (although they have a simple pole at $s = 0$).

12.3 The complex frequency-domain boundary conditions

The boundary conditions that have been discussed in Section 10.6 apply, in the linearised low-velocity approximation, to time-invariant boundaries. As a consequence of this, these boundary conditions can directly be transferred to the complex frequency domain. At the interface $\mathcal{S}$ of two different solids in rigid contact, therefore, the following conditions hold: on account of Equation (10.6-2))

$$\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}_{p,q} = \nu_m(\hat{\tau}_{m,k} + \hat{\tau}_{k,m})/2 = \hat{t}_k \quad \text{is continuous across } \mathcal{S}; \tag{12.3-1}$$

and on account of Equation (10.6-6)

$$\hat{v}_r \quad \text{is continuous across } \mathcal{S}. \tag{12.3-2}$$

Furthermore, from Equation (10.6-7) the condition

$$\lim_{h\downarrow 0}\hat{t}_k(\boldsymbol{x}, +h\boldsymbol{\nu},s) = 0 \quad \text{on the boundary of a void,} \tag{12.3-3}$$

follows and from Equations (10.6-8)–(10.6-10) the conditions

$$M^{\mathrm{R}}_{k,r}s\hat{v}^{\mathrm{R}}_r = \hat{F}^{\mathrm{R}}_k + M^{\mathrm{R}}_{k,r}\hat{v}^{\mathrm{R}}_r(t_0)\exp(-st_0) , \tag{12.3-4}$$

$$\hat{F}^{\mathrm{R}}_k = \int_{\boldsymbol{x}\in\mathcal{D}} \hat{f}^{\mathrm{R}}_k\,\mathrm{d}V + \int_{\boldsymbol{x}\in\partial\mathcal{D}} \hat{t}_k\,\mathrm{d}A , \tag{12.3-5}$$

$$\lim_{h\downarrow 0}\hat{v}_r(\boldsymbol{x}, +h\boldsymbol{\nu},s) = \hat{v}^{\mathrm{R}}_r(s) \quad \text{at the boundary of a perfectly rigid object} , \tag{12.3-6}$$

follow, where $\mathcal{D}$ is the domain occupied by the perfectly rigid object, $\partial\mathcal{D}$ is its boundary surface and ν_r is the unit vector along the normal to $\partial\mathcal{D}$ pointing away from $\mathcal{D}$. In the limiting case $M^{\rm R}_{k,r}\to\infty$, we have $\hat{v}_r^{\rm R}\to 0$, and Equations (12.3-4)–(12.3-6) reduce to

$$\lim_{h\downarrow 0}\hat{v}_r(\boldsymbol{x},+h\boldsymbol{\nu},s)=0$$

$$\text{at the boundary of an immovable, perfectly rigid object}\,. \qquad (12.3\text{-}7)$$

It is clear that these boundary conditions would also follow if the procedure of Section 10.6 had been applied to the complex frequency-domain elastic wave equations (12.1-3)–(12.1-4).

Exercises

Exercise 12.3-1

Apply the procedure of Section 10.6 to the complex frequency-domain elastic wave equations (12.1-3) and (12.1-4) to arrive at the boundary conditions given in Equations (12.3-1)–(12.3-7).

12.4 The complex frequency-domain coupled elastic wave equations

In the majority of our calculations we shall substitute the constitutive relations Equations (12.2-3) and (12.2-4), or Equations (12.2-18) and (12.2-19), or Equations (12.2-20) and (12.2-21), in Equations (12.1-3) and (12.1-4) and thus obtain a system of differential equations in space in which the number of unknowns is equal to the number of equations and in which s occurs as a parameter. The relevant equations are written as

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}_{p,q}+\hat{\zeta}_{k,r}\hat{v}_r=\hat{f}_k+\exp(-st_0)\Phi_k(\boldsymbol{x},t_0)\,, \qquad (12.4\text{-}1)$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}_r-\hat{\eta}_{i,j,p,q}\hat{\tau}_{p,q}=\hat{h}_{i,j}-\exp(-st_0)e_{i,j}(\boldsymbol{x},t_0)\,, \qquad (12.4\text{-}2)$$

in which

$\hat{\zeta}_{k,r}$ = Complex frequency-domain *longitudinal elastodynamic impedance per length* of the solid, (12.4-3)

and

$\hat{\eta}_{i,j,p,q}$ = Complex frequency-domain *transverse elastodynamic admittance per length* of the solid. (12.4-4)

(This terminology is borrowed from the conventional usage in electrical one-dimensional transmission-line theory.)

Equations (12.4-1) and (12.4-2) will be referred to as the complex frequency-domain coupled elastic wave equations. They serve as the point of departure in a number of subsequent analyses.

Solid with relaxation

For a solid with relaxation, Equations (12.2-3) and (12.2-4) lead to

$$\hat{\zeta}_{k,r} = s\hat{\mu}_{k,r} \tag{12.4-5}$$

and

$$\hat{\eta}_{i,j,p,q} = s\hat{\chi}_{i,j,p,q} \,. \tag{12.4-6}$$

Instantaneously reacting solid

For an instantaneously reacting solid, Equations (12.2-18) and (12.2-19) lead to

$$\hat{\zeta}_{k,r} = s\rho_{k,r} \tag{12.4-7}$$

and

$$\hat{\eta}_{i,j,p,q} = sS_{i,j,p,q} \,. \tag{12.4-8}$$

Solid with frictional-force/viscosity elastodynamic loss mechanism

For a solid with the frictional-force/viscosity elastodynamic loss mechanism, Equations (12.2-20) and (12.2-21) lead to

$$\hat{\zeta}_{k,r} = K_{k,r} + s\rho_{k,r} \tag{12.4-9}$$

and

$$\hat{\eta}_{i,j,p,q} = \Gamma_{i,j,p,q} + sS_{i,j,p,q} \,. \tag{12.4-10}$$

12.5 Complex frequency-domain elastodynamic vector and tensor potentials

Along the same lines as in Section 10.10 the complex frequency-domain elastodynamic vector and tensor potentials in the theory of radiation from sources are introduced. Starting points are the Equations (12.4-1) and (12.4-2), in which we incorporate the initial-value contributions in the volume source densities, i.e.

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}_{p,q} + \hat{\zeta}_{k,r}\hat{v}_r = \hat{f}_k \,, \tag{12.5-1}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}_r - \hat{\eta}_{i,j,p,q}\hat{\tau}_{p,q} = \hat{h}_{i,j} \,, \tag{12.5-2}$$

and in which inertia relaxation effects, if present, have been incorporated in $\hat{\zeta}_{k,r} = s\hat{\mu}_{k,r}$ and compliance relaxation effects, if present, have been incorporated in $\hat{\eta}_{i,j,p,q} = s\hat{\chi}_{i,j,p,q}$. For an instantaneously reacting solid, we have $\hat{\zeta}_{k,r} = s\rho_{k,r}$ and $\hat{\eta}_{i,j,p,q} = sS_{i,j,p,q}$; for a solid with the frictional-force/viscosity elastodynamic loss mechanism we have $\hat{\zeta}_{k,r} = K_{k,r} + s\rho_{r,k}$ and $\hat{\eta}_{i,j,p,q} = \Gamma_{i,j,p,q} + sS_{i,j,p,q}$.

Now let $\{\hat{v}_r, \hat{\tau}_{p,q}\} = \{\hat{v}_r^f, \hat{\tau}_{p,q}^f\}$ denote the elastic wave motion that is generated by the force source distribution $\hat{f}_k = \hat{f}_k(\boldsymbol{x},s)$, in the absence of a deformation source distribution, i.e. $\hat{h}_{i,j} = 0$. Then,

$$-\Delta^+_{k,m,p,q}\partial_m\hat{\tau}^f_{p,q} + \hat{\zeta}_{k,r}\hat{v}^f_r = \hat{f}_k\,, \tag{12.5-3}$$

$$\Delta^+_{i,j,n,r}\partial_n\hat{v}^f_r - \hat{\eta}_{i,j,p,q}\hat{\tau}^f_{p,q} = 0\,. \tag{12.5-4}$$

Taking advantage of the fact that the right-hand side of Equation (12.5-4) is zero, this equation is rewritten as

$$\hat{\tau}^f_{p,q} = \hat{\eta}^{-1}_{p,q,i,j}\Delta^+_{i,j,n,r}\partial_n\hat{v}^f_r = \hat{\eta}^{-1}_{p,q,n,r}\partial_n\hat{v}^f_r, \tag{12.5-5}$$

where $\hat{\eta}^{-1}_{p,q,i,j}$ is the tensor of rank four that is inverse to $\hat{\eta}_{p,q,i,j}$, i.e.

$$\hat{\eta}_{i,j,p,q}\hat{\eta}^{-1}_{p,q,i',j'} = \Delta^+_{i,j,i',j'}\,. \tag{12.5-6}$$

Substitution of Equation (12.5-5) in Equation (12.5-3) yields, with the use of $\Delta^+_{k,m,p,q}\hat{\eta}^{-1}_{p,q,n,r} = \hat{\eta}^{-1}_{k,m,n,r}$, the second-order vector differential equation

$$\partial_m(\hat{\eta}^{-1}_{k,m,n,r}\partial_n\hat{v}^f_r) - \hat{\zeta}_{k,r}\hat{v}^f_r = -\hat{f}_k\,. \tag{12.5-7}$$

By analogy with Equation (10.10-11) we now introduce the *complex frequency-domain elastodynamic vector potential* $\hat{A}_r = \hat{A}_r(\boldsymbol{x},s)$ as the solution to the second-order differential equation (*elastodynamic vector Helmholtz equation*)

$$\partial_m(s\hat{\eta}^{-1}_{k,m,n,r}\partial_n\hat{A}_r) - s\hat{\zeta}_{k,r}\hat{A}_r = -\hat{f}_k\,, \tag{12.5-8}$$

with the volume source density of force as the forcing term on the right-hand side. Comparison of Equations (12.5-7) and (12.5-8) leads to

$$\hat{v}^f_r = s\hat{A}_r\,, \tag{12.5-9}$$

while Equation (12.5-5) leads to

$$\hat{\tau}^f_{p,q} = s\hat{\eta}^{-1}_{p,q,i,j}\Delta^+_{i,j,n,r}\partial_n\hat{A}_r = s\hat{\eta}^{-1}_{p,q,n,r}\partial_n\hat{A}_r\,. \tag{12.5-10}$$

Next let $\{\hat{v}_r, \hat{\tau}_{p,q}\} = \{\hat{v}^h_r, \hat{\tau}^h_{p,q}\}$ denote the wave motion that is generated by the deformation source distribution $\hat{h}_{i,j} = \hat{h}_{i,j}(\boldsymbol{x},s)$, in the absence of a force source distribution, i.e. $\hat{f}_k = 0$. Then,

$$-\Delta^+_{k,m,p,q}\partial_m\hat{\tau}^h_{p,q} + \hat{\zeta}_{k,r}\hat{v}^h_r = 0\,, \tag{12.5-11}$$

$$\Delta^+_{i,j,n,r}\partial_n\hat{v}^h_r - \hat{\eta}_{i,j,p,q}\hat{\tau}^h_{p,q} = \hat{h}_{i,j}\,. \tag{12.5-12}$$

Taking advantage of the fact that the right-hand side of Equation (12.5-11) is zero, this equation is rewritten as

$$\hat{v}^h = \hat{\zeta}^{-1}_{r,k}\Delta^+_{k,m,p,q}\partial_m\hat{\tau}^h_{p,q}, \tag{12.5-13}$$

where $\hat{\zeta}^{-1}_{r,k}$ is the tensor of rank two that is inverse to $\hat{\zeta}_{k,r}$, i.e.

$$\hat{\zeta}_{k,r}\hat{\zeta}^{-1}_{r,k'} = \delta_{k,k'}\,. \tag{12.5-14}$$

Substitution of Equation (12.5-13) in Equation (12.5-12) yields the second-order tensor differential equation

$$\overset{+}{\Delta}_{i,j,n,r}\overset{+}{\Delta}_{k,m,p,q}\partial_n(\hat{\xi}^{-1}_{r,k}\partial_m\hat{\tau}^h_{p,q}) - \hat{\eta}_{i,j,p,q}\hat{\tau}^h_{p,q} = \hat{h}_{i,j} . \tag{12.5-15}$$

By analogy with Equation (10.10-20) we now introduce the *complex frequency-domain elastodynamic tensor potential* $\hat{W}_{p,q} = \hat{W}_{p,q}(x,s)$ as the solution to the second-order differential equation (*elastodynamic tensor Helmholtz equation*)

$$\overset{+}{\Delta}_{i,j,n,r}\overset{+}{\Delta}_{k,m,p,q}\partial_n(s\hat{\xi}^{-1}_{r,k}\partial_m\hat{W}_{p,q}) - s\hat{\eta}_{i,j,p,q}\hat{W}_{p,q} = -\hat{h}_{i,j} , \tag{12.5-16}$$

with the volume source density of deformation rate as the forcing term on the right-hand side. Comparison of Equations (12.5-16) and (12.5-15) leads to

$$\hat{\tau}^h_{p,q} = -s\hat{W}_{p,q} , \tag{12.5-17}$$

while Equation (12.5-13) leads to

$$\hat{v}^h_r = -s\hat{\xi}^{-1}_{r,k}\overset{+}{\Delta}_{k,m,p,q}\partial_m\hat{W}_{p,q} . \tag{12.5-18}$$

Since the total wave field is the superposition of the two constituents, i.e.

$$\{\hat{v}_r,\hat{\tau}_{p,q}\} = \{\hat{v}^f_r + \hat{v}^h_r , \hat{\tau}^f_{p,q} + \hat{\tau}^h_{p,q}\} , \tag{12.5-19}$$

we end up with

$$\hat{\tau}_{p,q} = -s\hat{W}_{p,q} + s\hat{\eta}^{-1}_{p,q,i,j}\overset{+}{\Delta}_{i,j,n,r}\partial_n\hat{A}_r , \tag{12.5-20}$$

$$\hat{v}_r = s\hat{A}_r - s\hat{\xi}^{-1}_{r,k}\overset{+}{\Delta}_{k,m,p,q}\partial_m\hat{W}_{p,q} , \tag{12.5-21}$$

which are the complex frequency-domain counterparts of Equations (10.10-24) and (10.10-25).

Exercises

Exercise 12.5-1

Verify that Equations (12.5-20) and (12.5-21) satisfy Equations (12.5-1) and (12.5-2), provided that Equations (12.5-8) and (12.5-16) are satisfied.

12.6 Complex frequency-domain point-source solutions; complex frequency-domain Green's functions

Along the same lines as in Section 10.11 the complex frequency-domain point-source solutions and the corresponding complex frequency-domain Green's functions are introduced. To this end the volume source density of force $\hat{f} = \hat{f}(x,s)$ and the volume source density of deformation rate $\hat{h}_{i,j} = \hat{h}_{i,j}(x,s)$ are written as a continuous superposition of point sources through the representations (see Equations (10.11-1) and (10.11-2))

$$\hat{f}_k(x,s) = \int_{x' \in \mathcal{D}^T} \delta_{k,k'}\delta(x - x')\hat{f}_{k'}(x',s)\,\mathrm{d}V , \tag{12.6-1}$$

and

$$\hat{h}_{i,j}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \Delta^+_{i,j,i',j'}\delta(\boldsymbol{x}-\boldsymbol{x}')\hat{h}_{i',j'}(\boldsymbol{x}',s)\,\mathrm{d}V\,, \tag{12.6-2}$$

where $\mathcal{D}^T$ is the spatial support of the distributed sources and the sifting property of the Dirac delta distribution $\delta(\boldsymbol{x}-\boldsymbol{x}')$ operative at $\boldsymbol{x}'=\boldsymbol{x}$ has been used. Now let the tensor function of rank two $\hat{G}^A_{r,k'} = \hat{G}^A_{r,k'}(\boldsymbol{x},\boldsymbol{x}',s)$ satisfy the second-order tensor differential equation (see Equation (12.5-8))

$$\partial_m(s\hat{\eta}^{-1}_{k,m,n,r}\partial_n\hat{G}^A_{r,k'}) - s\hat{\zeta}_{k,r}\hat{G}^A_{r,k'} = -\hat{\delta}_{k,k'}\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{12.6-3}$$

and let the tensor function of rank four $\hat{G}^W_{p,q,i',j'} = \hat{G}^W_{p,q,i',j'}(\boldsymbol{x},\boldsymbol{x}',s)$ satisfy the second-order tensor differential equation (see Equation (12.5-16))

$$\Delta^+_{i,j,n,r}\Delta^+_{k,m,p,q}\partial_n(s\hat{\zeta}^{-1}_{r,k}\partial_m\hat{G}^W_{p,q,i',j'}) - s\hat{\eta}_{i,j,p,q}\hat{G}^W_{p,q,i',j'} = -\Delta^+_{i,j,i',j'}\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{12.6-4}$$

then Equations (12.5-8) and (12.5-16) are satisfied by

$$\hat{A}_r(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \hat{G}^A_{r,k'}(\boldsymbol{x},\boldsymbol{x}',s)\hat{f}_{k'}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{12.6-5}$$

and

$$\hat{W}_{p,q}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \hat{G}^W_{p,q,i',j'}(\boldsymbol{x},\boldsymbol{x}',s)\hat{h}_{i',j'}(\boldsymbol{x}',s)\,\mathrm{d}V\,, \tag{12.6-6}$$

respectively. The proof follows by observing that the differentiations in the left-hand sides of Equations (12.5-8) and (12.5-16) are with respect to $\boldsymbol{x}$, whereas the integrations in the right-hand sides of Equations (12.6-1) and (12.6-2), and (12.6-5) and (12.6-6) are with respect to $\boldsymbol{x}'$. The function $\hat{G}^A_{r,k'} = \hat{G}^A_{r,k'}(\boldsymbol{x},\boldsymbol{x}',s)$ is the complex frequency-domain (tensor) Green's function associated with the elastodynamic vector potential $\hat{A}_r = \hat{A}_r(\boldsymbol{x},s)$; the function $\hat{G}^W_{p,q,i',j'} = \hat{G}^W_{p,q,i',j'}(\boldsymbol{x},\boldsymbol{x}',s)$ is the complex frequency-domain (tensor) Green's function associated with the elastodynamic tensor potential $\hat{W}_{p,q} = \hat{W}_{p,q}(\boldsymbol{x},s)$. The role that these Green's functions play in the solution of elastic wave problems will be more extensively discussed in Chapter 13.

Exercises

Exercise 12.6-1

Let $\hat{u} = \hat{u}(\boldsymbol{x},s)$ be the solution to the scalar Helmholtz equation

$$\partial_m\partial_m\hat{u} - (s^2/c^2)\hat{u} = -\hat{\rho}\,. \tag{12.6-7}$$

The spatial support of the sources with volume density $\hat{\rho} = \hat{\rho}(\boldsymbol{x},s)$ is $\mathcal{D}^T$. (a) Give the differential equation for the Green's function $\hat{G} = \hat{G}(\boldsymbol{x},\boldsymbol{x}',s)$. (b) Express $\hat{u} = \hat{u}(\boldsymbol{x},s)$ as a superposition of point-source solutions.

Answers:

(a) $$\partial_m\partial_m\hat{G} - (s^2/c^2)\hat{G} = -\delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{12.6-8}$$

and

$$\text{(b)} \quad \hat{u}(\boldsymbol{x},s) = \int_{\boldsymbol{x}' \in \mathcal{D}^T} \hat{G}(\boldsymbol{x},\boldsymbol{x}',s)\hat{\rho}(\boldsymbol{x}',s)\,\mathrm{d}V. \tag{12.6-9}$$

12.7 The complex frequency-domain elastic wave equations for dilatational waves (equivalent fluid model)

The s-domain elastic wave equations for dilatational waves in a solid (equivalent fluid model) follow from Equations (10.13-7) and (10.13-10) as

$$\partial_r \hat{v}_r - s\kappa\hat{\sigma} = \hat{h}_{i,i} \tag{12.7-1}$$

and

$$-\partial_k \hat{\sigma} + s\rho_{k,r}\hat{v}_r = \hat{f}_k, \tag{12.7-2}$$

respectively.

Boundary conditions at a source-free interface

An analysis of the type developed in Section 10.6, applied to Equations (12.7-1) and (12.7-2) leads to the following boundary conditions to be satisfied at a source-free interface between two media with different constitutive coefficients:

$$\hat{\sigma} \quad \text{is continuous across interface,} \tag{12.7-3}$$

and

$$\nu_r \hat{v}_r \quad \text{is continuous across interface,} \tag{12.7-4}$$

where ν_m is the unit vector along the normal to the interface. These boundary conditions also follow directly from Equations (10.13-18) and (10.13-19), respectively, since in the linearised low-velocity approximation the latter apply to time-invariant boundaries.

Boundary conditions at the boundary surface of the solid

If the solid under consideration occupies the bounded domain $\mathcal{D}$ in space, explicit boundary conditions may be prescribed at the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$. Admissible boundary conditions for the equivalent fluid model are: a prescribed isotropic dynamic tension on some part $\mathcal{S}_1$ of $\partial\mathcal{D}$ and a prescribed normal component of the particle velocity at the remaining part $\mathcal{S}_2$ of $\partial\mathcal{D}$. In the present approximation we are not allowed to prescribe the tangential part of the particle velocity on $\partial\mathcal{D}$.

Boundary conditions at a void

A subdomain $\mathcal{D}$ of the solid is denoted as a void if in it the isotropic dynamic tension is negligibly small, while the continuity of the isotropic dynamic tension across the boundary surface $\partial\mathcal{D}$ of the void is maintained. Consequently, the boundary condition upon approaching the boundary surface $\partial\mathcal{D}$ of a void via its exterior is in the present approximation given by

$$\lim_{h\downarrow 0}\hat{\sigma}(\boldsymbol{x}+h\boldsymbol{\nu},s)=0 \qquad \text{for any } \boldsymbol{x}\in\partial\mathcal{D}, \qquad (12.7\text{-}5)$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to $\partial\mathcal{D}$ pointing away from $\mathcal{D}$. We are not free to prescribe the normal component of particle velocity in this case. In fact, the particle velocity will, in general, have a non-zero value at $\partial\mathcal{D}$, while it is not defined in $\mathcal{D}$.

Boundary condition at a perfectly rigid object

A material body occupying a domain $\mathcal{D}$ in the solid, is denoted as a perfectly rigid object if it cannot be deformed and, hence, can only be in rigid motion, and if its surface is impenetrable to the surrounding solid. Let $M^{\rm R}_{k,r}$ be the possibly anisotropic mass of the rigid object and let $\hat{v}^{\rm R}_r = \hat{v}^{\rm R}_r(s)$ be its velocity. Then, according to Newton's law of motion, we have as a first condition

$$M^{\rm R}_{k,r}s\hat{v}^{\rm R}_r=\hat{F}^{\rm R}_k+M^{\rm R}_{k,r}v^{\rm R}_r(t_0)\exp(-st_0), \qquad (12.7\text{-}6)$$

where $\hat{F}^{\rm R}_k$ is the total force acting on the object. Let $\hat{f}^{\rm R}_k$ be the volume density of body force acting on the object and $\hat{\sigma}$ be the isotropic dynamic tension at its surface, then $\hat{F}^{\rm R}_k$ is given by

$$\hat{F}^{\rm R}_k=\int_{\boldsymbol{x}\in\mathcal{D}}\hat{f}^{\rm R}_k\,{\rm d}V+\int_{\boldsymbol{x}\in\partial\mathcal{D}}\hat{\sigma}\nu_k\,{\rm d}A\,. \qquad (12.7\text{-}7)$$

As a second conditon, we have, at the surface of the object,

$$\lim_{h\downarrow 0}\nu_r\hat{v}_r(\boldsymbol{x}+h\boldsymbol{\nu},s)=\nu_r\hat{v}^{\rm R}_r \qquad \text{for any } \boldsymbol{x}\in\partial\mathcal{D}. \qquad (12.7\text{-}8)$$

The conditions given in Equations (12.7-6)–(12.7-8) must be satisfied simultaneously.

In the limiting case $M^{\rm R}_{k,r}\to\infty$, we have $\hat{v}^{\rm R}_r(s)\to 0$, and Equations (12.7-6)–(12.7-8) are replaced by

$$\lim_{h\downarrow 0}\nu_r\hat{v}_r(\boldsymbol{x}+h\boldsymbol{\nu},s)=0 \qquad \text{for any } \boldsymbol{x}\in\partial\mathcal{D}. \qquad (12.7\text{-}9)$$

This condition also holds if the object is held immovable by external means. If Equation (12.7-9) holds, we are not free to prescribe the isotropic dynamic tension on $\partial\mathcal{D}$. Furthermore, we are in the present approximation not allowed to put conditions on the tangential part of the particle velocity on $\partial\mathcal{D}$.

Boundary condition at an interface with surface sources

In the modelling of some elastodynamic problems (in particular in seismics), it is advantageous to allow for the presence of surface sources at the interface between two media with different constitutive coefficients. These surface sources must be compatible with Equations (12.7-1) and (12.7-2), i.e. they must at most give rise to jump discontinuities in the elastic wave-field quantities across the relevant interface. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote domains at either side of the interface $\mathcal{S}$ and let $\nu_r^{(1)}$ and $\nu_r^{(2)}$ denote the unit vectors along the normal to $\mathcal{S}$ pointing away from $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. From Equation (12.7-1) it then follows that at most the normal component of the particle velocity can jump by a finite amount, i.e.

$$\lim_{h\downarrow 0}\left[\nu_r^{(1)}(\boldsymbol{x})\hat{v}_r(\boldsymbol{x}-h\boldsymbol{\nu}^{(1)},s)+\nu_r^{(2)}(\boldsymbol{x})\hat{v}_r(\boldsymbol{x}-h\boldsymbol{\nu}^{(2)},s)\right]=\hat{h}^S(\boldsymbol{x},s) \text{ for any } \boldsymbol{x}\in\mathcal{S}, \quad (12.7\text{-}10)$$

where $\hat{h}^S$ is the area density of normal deformation rate, while from Equation (12.7-2) it follows that the isotropic dynamic tension can at most jump by a finite amount, i.e.

$$-\lim_{h\downarrow 0}\left[\nu_k^{(1)}(\boldsymbol{x})\hat{\sigma}(\boldsymbol{x}-h\boldsymbol{\nu}^{(1)},s)+\nu_k^{(2)}(\boldsymbol{x})\hat{\sigma}(\boldsymbol{x}-h\boldsymbol{\nu}^{(2)},s)\right]=\hat{f}_k^S(\boldsymbol{x},s) \text{ for any } \boldsymbol{x}\in\mathcal{S} \quad (12.7\text{-}11)$$

where $\hat{f}_k^S$ is the area density of normal surface force. (Note that $\hat{f}_k^S$ is oriented along the normal to the interface.)

The introduction of the scalar and vector wave potentials associated with Equations (12.7-1) and (12.7-2), as well as the introduction of the corresponding Green's functions is discussed in Part 1.

Exercises

Exercise 12.7-1

Show that, for a solid with relaxation, the complex frequency-domain equations for the equivalent fluid model for dilatational waves in a solid are given by

$$\partial_r\hat{v}_r-\hat{\eta}\hat{\sigma}=\hat{h}_{i,i}, \quad (12.7\text{-}12)$$

$$-\partial_k\hat{\sigma}+\hat{\zeta}_{k,r}\hat{v}_r=\hat{f}_k, \quad (12.7\text{-}13)$$

in which

$$\hat{\eta}=s\hat{\chi}_{i,i,p,p} \quad (12.7\text{-}14)$$

and

$$\hat{\zeta}_{k,r}=s\hat{\mu}_{k,r}. \quad (12.7\text{-}15)$$

References

Bode, H.W., 1959, *Network Analysis and Feedback Amplifier Design*, New York: Van Nostrand, 13th edn., 551 pp.

Boltzmann, L., 1876, *Zur Theorie der Elastischen Nachwirkung*, Annalen der Physik und Chemie, Ergaenzungsband 7, pp. 624–654.