13

Elastodynamic radiation from sources in an unbounded, homogeneous, isotropic solid

In this chapter we calculate the elastic wave field that is causally related to the action of sources of bounded extent in an unbounded homogeneous and isotropic solid that is linear, time invariant, and instantaneously and locally reacting in its elastodynamic behaviour. The wave-field quantities (particle velocity and dynamic stress) are determined with the aid of a spatial Fourier transformation method, which is applied to the complex frequency-domain coupled elastic wave equations discussed in Chapter 12. The elastodynamic vector and tensor potentials are employed. Several applications are given. In particular, the radiation from force sources and deformation sources is discussed, both in the complex frequency domain and in the time domain, and the solution to the initial-value problem (Cauchy problem) is given.

13.1 The coupled elastic wave equations in the angular wave-vector domain

The complex frequency-domain particle velocity $\hat{v}_r$ and dynamic stress $\hat{\tau}_{p,q}$ in a homogeneous, isotropic, lossless solid satisfy the complex frequency-domain coupled elastic wave equations (see Equations (12.5-1) and (12.5-2))

$$-\overset{+}{\Delta}_{k,m,p,q}\partial_m\hat{\tau}_{p,q} + s\rho\hat{v}_k = \hat{f}_k\,, \tag{13.1-1}$$

$$\overset{+}{\Delta}_{i,j,n,r}\partial_n\hat{v}_r - sS_{i,j,p,q}\hat{\tau}_{p,q} = \hat{h}_{i,j}\,, \tag{13.1-2}$$

in which

$$S_{i,j,p,q} = \Lambda\delta_{i,j}\delta_{p,q} + M(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p}) = 3\Lambda\overset{\delta}{\Delta}_{i,j,p,q} + 2M\overset{+}{\Delta}_{i,j,p,q}\,. \tag{13.1-3}$$

Since the solid is homogeneous, the volume density of mass ρ and the compliance $S_{i,j,p,q}$ are constants. We assume that the volume source density of force $\hat{f}_k$ and the volume source density of deformation rate $\hat{h}_{i,j}$ only differ from zero in some bounded subdomain $\mathcal{D}^T$ of $\mathcal{R}^3$; $\mathcal{D}^T$ is the spatial support of the source distributions and is denoted as the source domain of the transmitted (or radiated) elastic wave field (Figure 13.1-1).

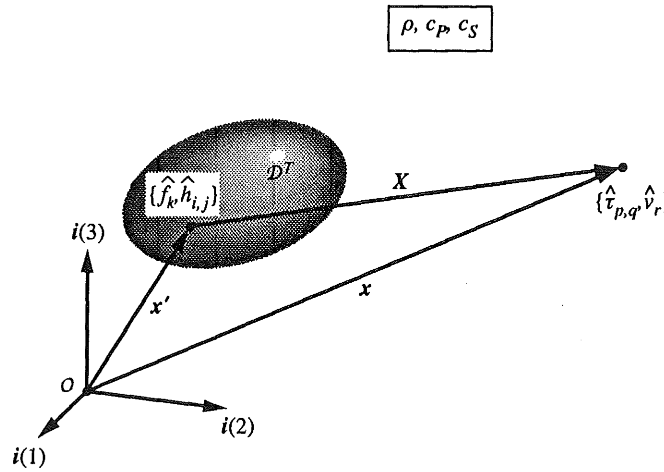

Figure 13.1-1 Sources $\{\hat{f}_k,\hat{h}_{i,j}\}$ at position $x'\in\mathcal{D}^T$ (source domain) generate elastodynamic radiation in a homogeneous, isotropic, lossless solid with constitutive coefficients $\{\rho,c_P,c_S\}$. The wave field $\{\hat{\tau}_{p,q},\hat{v}_r\}$ is observed at $x\in\mathcal{R}^3$.

The influence of non-zero initial field values is assumed to have been incorporated into the volume source densities. To solve Equations (13.1-1) and (13.1-2), we suject these equations to a three-dimensional Fourier transformation over the entire configuration space $\mathcal{R}^3$ (see Section B.2). The usefulness of this operation is associated with the property of shift invariance of the medium in all Cartesian directions. In accordance with Appendix B the spatial Fourier transformation is written as

$$\{\tilde{\tau}_{p,q},\tilde{v}_r\}(\mathrm{j}k,s) = \int_{x\in\mathcal{R}^3} \exp(\mathrm{j}k_s x_s)\{\hat{\tau}_{p,q},\hat{v}_r\}(x,s)\,\mathrm{d}V, \tag{13.1-4}$$

where j denotes the imaginary unit and

$$k = k_1 i(1) + k_2 i(2) + k_3 i(3) \qquad \text{with } k\in\mathcal{R}^3 \tag{13.1-5}$$

is the *angular wave vector* in three-dimensional Fourier-transform or k-space. According to Fourier's theorem we inversely have

$$\{\hat{\tau}_{p,q},\hat{v}_r\}(x,s) = (2\pi)^{-3}\int_{k\in\mathcal{R}^3} \exp(-\mathrm{j}k_s x_s)\{\tilde{\tau}_{p,q},\tilde{v}_r\}(\mathrm{j}k,s)\,\mathrm{d}V. \tag{13.1-6}$$

For the spatial derivatives we employ the relation

$$\int_{x\in\mathcal{R}^3} \exp(\mathrm{j}k_s x_s)\partial_m\{\hat{\tau}_{p,q},\hat{v}_r\}(x,s)\,\mathrm{d}V = -\mathrm{j}k_m\{\tilde{\tau}_{p,q},\tilde{v}_r\}(\mathrm{j}k,s), \tag{13.1-7}$$

where it has been taken into account that $\hat{\tau}_{p,q}$ and $\hat{v}_r$ will, due to causality, show, for $\mathrm{Re}(s) > 0$, an exponential decay as $|x|\to\infty$. (In Section 13.3 this will be shown indeed to be true.) With this, Equations (13.1-1) and (13.1-2) transform into

$$\Delta^+_{k,m,p,q}\,\mathrm{j}k_m\tilde{\tau}_{p,q} + s\rho\tilde{v}_k = \tilde{f}_k, \tag{13.1-8}$$

$$-\Delta^{+}_{i,j,n,r}\,\mathrm{j}k_n\tilde{v}_r - sS_{i,j,p,q}\tilde{\tau}_{p,q} = \tilde{h}_{i,j}\,, \tag{13.1-9}$$

where

$$\{\tilde{f}_k,\tilde{h}_{i,j}\}(\mathrm{j}\boldsymbol{k},s) = \int_{\boldsymbol{x}\in\mathcal{D}^T} \exp(\mathrm{j}k_s x_s)\{\hat{f}_k,\hat{h}_{i,j}\}(\boldsymbol{x},s)\,\mathrm{d}V \tag{13.1-10}$$

is the spatial Fourier transform of the source distributions. (Note that the integration on the right-hand side is extended only over the source domain $\mathcal{D}^T$.) To solve $\{\tilde{\tau}_{p,q},\tilde{v}_r\}$ from the linear algebraic equations (13.1-8) and (13.1-9), we eliminate $\tilde{\tau}_{p,q}$ from these equations. To this end, we need the stiffness $C_{p,q,i,j}$ of the solid, which is the inverse of the compliance. It is given by (see Equation (10.5-15))

$$C_{p,q,i,j} = 3\lambda\Delta^{\delta}_{p,q,i,j} + 2\mu\Delta^{+}_{p,q,i,j} = \lambda\delta_{p,q}\delta_{i,j} + \mu(\delta_{p,i}\delta_{q,j} + \delta_{p,j}\delta_{q,i})\,, \tag{13.1-11}$$

in which λ and μ are the Lamé coefficients. The coefficients $\{\Lambda,M\}$ and $\{\lambda,\mu\}$ are interrelated via Equations (10.5-23)–(10.5-26). Replacing the subscripts p and q in the second term on the left-hand side of Equation (13.1-9) by p' and q', respectively, and multiplying through in Equation (13.1-9) by $C_{p,q,i,j}$, we obtain

$$C_{p,q,i,j}\Delta^{+}_{i,j,n,r}(-\mathrm{j}k_n)\tilde{v}_r - \Delta^{+}_{p,q,p',q'}s\tilde{\tau}_{p',q'} = C_{p,q,i,j}\tilde{h}_{i,j}\,. \tag{13.1-12}$$

Using the expressions (10.3-33) and (13.1-11) for $\Delta^{+}_{p,q,p',q'}$ and $C_{p,q,i,j}$, respectively, Equation (13.1-12) is rewritten as

$$\lambda(-\mathrm{j}k_i\tilde{v}_i)\delta_{p,q} + \mu(-\mathrm{j}k_p\tilde{v}_q - \mathrm{j}k_q\tilde{v}_p) - s\left[(\tilde{\tau}_{p,q} + \tilde{\tau}_{q,p})/2\right] = C_{p,q,i,j}\tilde{h}_{i,j}\,. \tag{13.1-13}$$

Using the expression (10.3-33) for $\Delta^{+}_{k,m,p,q}$ and multiplying through in Equation (13.1-8) by s, we obtain

$$(s\mathrm{j}k_m)\left[(\tilde{\tau}_{k,m} + \tilde{\tau}_{m,k})/2\right] + \rho s^2\tilde{v}_k = s\tilde{f}_k\,. \tag{13.1-14}$$

Now, substituting the expression for $s\left[(\tilde{\tau}_{k,m} + \tilde{\tau}_{m,k})/2\right]$ that results from Equation (13.1-13) into Equation (13.1-14), we end up with

$$(\lambda k_k k_i\tilde{v}_i + \mu k_m k_m\tilde{v}_k + \mu k_m k_k\tilde{v}_m) + \rho s^2\tilde{v}_k = s\tilde{f}_k + C_{k,m,i,j}\,\mathrm{j}k_m\tilde{h}_{i,j}\,, \tag{13.1-15}$$

or

$$(c_P^2 - c_S^2)k_k k_i\tilde{v}_i + c_S^2 k_m k_m\tilde{v}_k + s^2\tilde{v}_k = \tilde{Q}_k\,, \tag{13.1-16}$$

in which

$$c_P = [(\lambda + 2\mu)/\rho]^{1/2} > 0\,, \tag{13.1-17}$$

$$c_S = (\mu/\rho)^{1/2} > 0\,, \tag{13.1-18}$$

and

$$\tilde{Q}_k = s\rho^{-1}\tilde{f}_k + \rho^{-1}C_{k,m,i,j}\,\mathrm{j}k_m\tilde{h}_{i,j} = s\rho^{-1}\tilde{f}_k + \rho^{-1}\left[\lambda\mathrm{j}k_k\tilde{h}_{i,i} + 2\mu\,\mathrm{j}k_m\tilde{h}_{k,m}\right]. \tag{13.1-19}$$

Equation (13.1-16) could also have been obtained directly by subjecting the *elastodynamic wave equation* (10.12-13) for the particle velocity of the elastic wave motion to the relevant integral transformations. For reasons to be discussed later, c_P is denoted as the *wave speed of P-waves* ("primary" waves, compressional waves, dilatational waves) and c_S as the *wave speed* *of S-waves* ("secondary" waves, shear waves, equivoluminal waves). Since $\lambda + 2\mu/3 > 0$ and $\mu > 0$ (see Equation (10.8-23)), we have $c_P > 2c_S/3^{1/2}$.

13.2 The elastodynamic wave equation for the particle velocity and its solution in the angular wave-vector domain

In this section the elastodynamic wave equation in the angular wave-vector domain Equation (13.1-16) will be solved. For convenience it is repeated here:

$$(c_P^2 - c_S^2)k_k k_i \tilde{v}_i + c_S^2 k_m k_m \tilde{v}_k + s^2 \tilde{v}_k = \tilde{Q}_k \,, \tag{13.2-1}$$

Solving $\tilde{v}_k$ from this equation would be simple if we had an expression for $k_i \tilde{v}_i$. To obtain this, we employ the auxiliary relation that follows by applying the operation k_k to Equation (13.2-1). The latter results in

$$(c_P^2 k_m k_m + s^2)k_i \tilde{v}_i = k_m \tilde{Q}_m \,, \tag{13.2-2}$$

from which we obtain

$$k_i \tilde{v}_i = \frac{k_m \tilde{Q}_m}{c_P^2 k_m k_m + s^2} \,. \tag{13.2-3}$$

Substitution of Equation (13.2-3) into Equation (13.2-1) yields

$$(c_S^2 k_m k_m + s^2)\tilde{v}_k = \tilde{Q}_k - \frac{(c_P^2 - c_S^2)k_k k_m \tilde{Q}_m}{c_P^2 k_m k_m + s^2} \,, \tag{13.2-4}$$

from which it follows that

$$\tilde{v}_k = \frac{\tilde{Q}_k}{c_S^2 k_m k_m + s^2} - \frac{(c_P^2 - c_S^2)k_k k_m \tilde{Q}_m}{(c_P^2 k_m k_m + s^2)(c_S^2 k_m k_m + s^2)} \,. \tag{13.2-5}$$

However,

$$\frac{(c_P^2 - c_S^2)}{(c_P^2 k_m k_m + s^2)(c_S^2 k_m k_m + s^2)} = s^{-2}\left[\frac{1}{k_m k_m + s^2/c_P^2} - \frac{1}{k_m k_m + s^2/c_S^2}\right]. \tag{13.2-6}$$

Consequently, the expression for $\tilde{v}_r$ can be written as

$$\tilde{v}_r = \tilde{G}_{r,k} \tilde{Q}_k \,, \tag{13.2-7}$$

in which

$$\tilde{G}_{r,k} = c_S^{-2} \tilde{G}_S \delta_{r,k} - s^{-2} k_r k_k (\tilde{G}_P - \tilde{G}_S) \,, \tag{13.2-8}$$

with

$$\tilde{G}_P = \frac{1}{k_m k_m + s^2/c_P^2} \tag{13.2-9}$$

and

$$\tilde{G}_S = \frac{1}{k_m k_m + s^2/c_S^2} \, . \tag{13.2-10}$$

Each term on the right-hand side of Equation (13.2-7) is the product of two transform-domain functions and some additional factors s^{-2}, s, $-jk_m$ and/or $-k_m k_m = (-jk_m)(-jk_m)$. In the transformation back to the space–time domain the product of two transform-domain quantities corresponds to the convolution of these quantities in space–time, while, further, we have the correspondences $s \to \partial_t$ and $-jk_m \to \partial_m$. (Note that the latter relationship only holds if applied to functions that change continuously with position.) Hence, the right-hand side of Equation (13.2-7) is easily transformed back to the space–time domain once the functions $G_P = G_P(\boldsymbol{x},t)$ and $G_S = G_S(\boldsymbol{x},t)$ that correspond to $\tilde{G}_P = \tilde{G}_P(j\boldsymbol{k},s)$ and $\tilde{G}_S = \tilde{G}_S(j\boldsymbol{k},s)$, respectively, have been determined. This is done in Section 13.3.

13.3 Determination of G_P and G_S

The functions $\tilde{G}_P$ and $\tilde{G}_S$ introduced in Equations (13.2-9) and (13.2-10) are both of the shape

$$\tilde{G} = \tilde{G}(j\boldsymbol{k},s) = \frac{1}{k_m k_m + s^2/c^2} \, . \tag{13.3-1}$$

The starting point for the evaluation of $\hat{G} = \hat{G}(\boldsymbol{x},s)$ is the inverse Fourier transformation

$$\hat{G}(\boldsymbol{x},s) = (2\pi)^{-3} \int_{\boldsymbol{k} \in \mathcal{R}^3} \exp(-jk_s x_s) \tilde{G}(j\boldsymbol{k},s) \, dV \, . \tag{13.3-2}$$

By applying the standard rules of the spatial Fourier transformation (see Appendix B), it is easily verified that $\tilde{G} = \tilde{G}(j\boldsymbol{k},s)$ is the three-dimensional Fourier transform, over the entire configuration space $\mathcal{R}^3$, of the function $\hat{G} = \hat{G}(\boldsymbol{x},s)$ that satisfies the three-dimensional scalar Helmholtz equation with point-source excitation

$$(\partial_m \partial_m - s^2/c^2)\hat{G} = -\delta(\boldsymbol{x}) \, , \tag{13.3-3}$$

where $\delta(\boldsymbol{x})$ is the three-dimensional Dirac distribution (impulse function) operative at $\boldsymbol{x} = \boldsymbol{0}$.

The simplest way to evaluate the right-hand side of Equation (13.3-2) is to introduce spherical coordinates in $\boldsymbol{k}$-space with centre at $\boldsymbol{k} = \boldsymbol{0}$ and the direction of $\boldsymbol{x}$ as polar axis. Let θ be the angle between $\boldsymbol{k}$ and $\boldsymbol{x}$, and ϕ the angle between the projection of $\boldsymbol{k}$ on the plane perpendicular to $\boldsymbol{x}$ and some fixed direction in this plane (Figure 13.3-1), then the range of integration is $0 \leqslant |\boldsymbol{k}| < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, while

$$k_s x_s = |\boldsymbol{k}||\boldsymbol{x}| \cos(\theta) \, , \tag{13.3-4}$$

$$k_m k_m = |\boldsymbol{k}|^2 \, , \tag{13.3-5}$$

and

$$dV = |\boldsymbol{k}|^2 \sin(\theta) \, d|\boldsymbol{k}| \, d\theta \, d\phi \, . \tag{13.3-6}$$

In the resulting right-hand side of Equation (13.3-2) we first carry out the integration with respect to ϕ. Since the integrand is independent of ϕ, this merely amounts to a multiplication

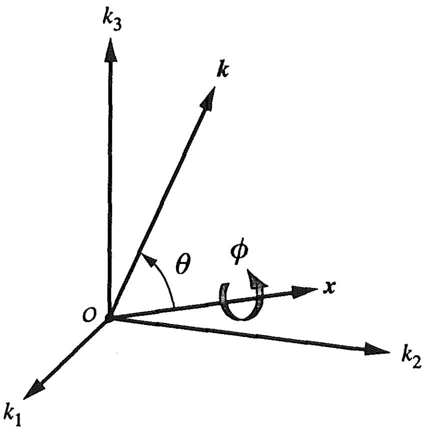

Figure 13.3-1 Integration in $\boldsymbol{k}$ space to evaluate $\hat{G}(\boldsymbol{x},s)$; $\{|\boldsymbol{k}|,\theta,\phi\}$ are the spherical polar coordinates, with $\boldsymbol{x}$ as polar axis and the ranges of integration $0 \leqslant |\boldsymbol{k}| < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$.

by a factor of 2π. Next, we carry out the integration with respect to θ, which is elementary. After this we have, for $|\boldsymbol{x}| \neq 0$,

$$\hat{G}(\boldsymbol{x},s) = \frac{1}{4\pi^2}\int_{|\boldsymbol{k}|=0}^{\infty} \frac{|\boldsymbol{k}|^2}{|\boldsymbol{k}|^2 + s^2/c^2}\left[\frac{\exp[-\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|\cos(\theta)]}{\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|}\right]_{\theta=0}^{\pi} \mathrm{d}|\boldsymbol{k}|$$

$$= \frac{1}{4\pi^2\mathrm{j}|\boldsymbol{x}|}\int_{|\boldsymbol{k}|=0}^{\infty} \frac{\exp(\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|) - \exp(-\mathrm{j}|\boldsymbol{k}||\boldsymbol{x}|)}{|\boldsymbol{k}|^2 + s^2/c^2}|\boldsymbol{k}|\,\mathrm{d}|\boldsymbol{k}|\,. \tag{13.3-7}$$

Considering $|\boldsymbol{k}|$ as a variable that can take on arbitrary complex values and denoting this variable by k, Equation (13.3-7) can be rewritten as

$$\hat{G}(\boldsymbol{x},s) = -\frac{1}{4\pi^2\mathrm{j}|\boldsymbol{x}|}\int_{k=-\infty}^{\infty} \frac{\exp(-\mathrm{j}k|\boldsymbol{x}|)}{k^2 + s^2/c^2}k\,\mathrm{d}k\,. \tag{13.3-8}$$

The integral on the right-hand side of Equation (13.3-8) is evaluated by continuing the integrand analytically into the complex k plane, supplementing the path of integration by a semicircle situated in the lower half-plane $-\infty < \mathrm{Im}(k) \leqslant 0$ and of infinitely large radius, and applying the theorem of residues (Figure 13.3-2).

On account of Jordan's lemma, the contribution from the semicircle at infinity vanishes. Furthermore, the only singularity of the integrand in the lower half of the complex k plane is the simple pole at $k = -\mathrm{j}s/c$ (note that $\mathrm{Re}(s) > 0$). Taking into account the residue of this pole and the fact that the contour integration is carried out clockwise in stead of counter-clockwise, we arrive at

$$\hat{G}(\boldsymbol{x},s) = \exp(-s|\boldsymbol{x}|/c)/4\pi|\boldsymbol{x}| \qquad \text{for} \quad |\boldsymbol{x}| \neq 0\,. \tag{13.3-9}$$

This expression will be used in the process of inversely Fourier transforming the angular wave-vector domain wave quantities obtained in Section 13.2.

The time-domain counterpart of Equation (13.3-9) follows as

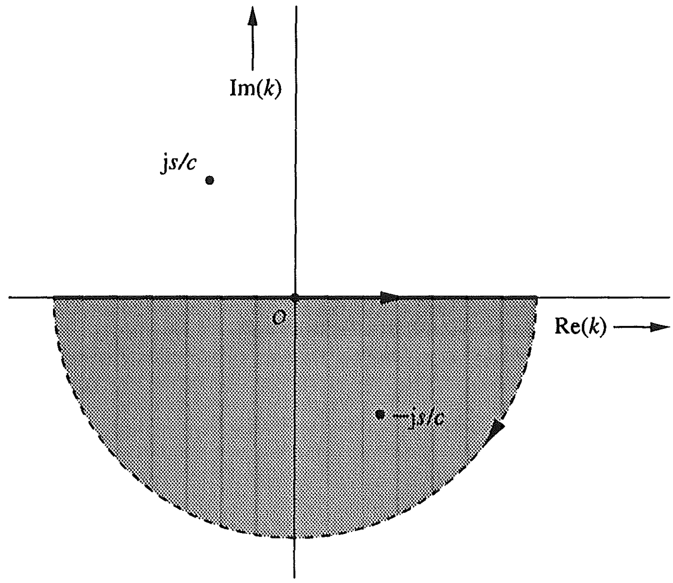

Figure 13.3-2 Integration in the complex k plane to evaluate $\hat{G}(\boldsymbol{x},s)$. Jordan's lemma and the theorem of residues are applied to the closed contour in the lower half-plane, where Im(k) < 0. The simple pole in the lower half-plane is located at $k = -\mathrm{j}s/c$ (note that Re(s) > 0).

$$G(\boldsymbol{x},t) = \delta(t - |\boldsymbol{x}|/c)/4\pi|\boldsymbol{x}| \quad \text{for} \quad |\boldsymbol{x}| \neq 0\,, \tag{13.3-10}$$

where $\delta(t - |\boldsymbol{x}|/c)$ is the one-dimensional unit impulse (Dirac distribution) operative at $t = |\boldsymbol{x}|/c$. Equation (13.3-10) is the expression for the space–time Green's function of the three-dimensional, scalar wave equation that is the time-domain counterpart of Equation (13.3-3), viz.

$$(\partial_m\partial_m - c^{-2}\partial_t^2)G = -\delta(\boldsymbol{x},t)\,, \tag{13.3-11}$$

where $\delta(\boldsymbol{x},t)$ denotes the four-dimensional unit impulse (Dirac distribution) operative at $\{\boldsymbol{x} = \boldsymbol{0}, t = 0\}$. Obviously, c is the speed with which the wave propagates away from the source that generates it. (Note that for the disturbance to occur, $|\boldsymbol{x}|$ must increase as t increases.) Furthermore, it is clear that we have indeed constructed the wave quantity that is causally related to the source (the source acts at $t = 0$ only, and $G = 0$ when $t < 0$, due to the fact that $|\boldsymbol{x}| > 0$ and $c > 0$).

With Equation (13.3-9) we have

$$\hat{G}_{P,S}(\boldsymbol{x},s) = \exp(-s|\boldsymbol{x}|/c_{P,S})/4\pi|\boldsymbol{x}| \quad \text{for} \quad |\boldsymbol{x}| \neq 0 \tag{13.3-12}$$

and with Equation (13.3-10)

$$G_{P,S}(\boldsymbol{x},t) = \delta(t - |\boldsymbol{x}|/c_{P,S})/4\pi|\boldsymbol{x}| \quad \text{for} \quad |\boldsymbol{x}| \neq 0\,, \tag{13.3-13}$$

which results will further be used.

In the course of our further analysis we also need the expressions for the spatial derivatives $\partial_r\hat{G}$, $\partial_k\partial_r\hat{G}$, $\partial_m\partial_k\partial_r\hat{G}$ and $\partial_n\partial_m\partial_k\partial_r\hat{G}$, in which $\hat{G} = \hat{G}(\boldsymbol{x},s)$ is given by Equation (13.3-9). Noting that

$$\partial_{|x|}\hat{G} = \left(-\frac{s}{c}\frac{1}{|x|} - \frac{1}{|x|^2}\right)\frac{\exp(-s|x|/c)}{4\pi} \qquad \text{for} \quad |x| \neq 0 \tag{13.3-14}$$

and

$$\partial_m|x| = \frac{x_m}{|x|} \qquad \text{for} \quad |x| \neq 0\,, \tag{13.3-15}$$

we obtain,

$$\partial_r\hat{G}(x,s) = \left(-\frac{1}{|x|^2}\frac{x_r}{|x|} - \frac{s}{c}\frac{1}{|x|}\frac{x_r}{|x|}\right)\frac{\exp(-s|x|/c)}{4\pi} \qquad \text{for} \quad |x| \neq 0\,, \tag{13.3-16}$$

$$\begin{aligned}\partial_k\partial_r\hat{G}(x,s) = \Bigg[&\frac{1}{|x|^3}\left(\frac{3x_kx_r}{|x|^2} - \delta_{k,r}\right) + \frac{s}{c}\frac{1}{|x|^2}\left(\frac{3x_kx_r}{|x|^2} - \delta_{k,r}\right)\\ &+ \frac{s^2}{c^2}\frac{1}{|x|}\frac{x_kx_r}{|x|^2}\Bigg]\frac{\exp(-s|x|/c)}{4\pi} \qquad \text{for} \quad |x| \neq 0\,,\end{aligned} \tag{13.3-17}$$

$$\begin{aligned}\partial_m\partial_k\partial_r\hat{G}(x,s) = \Bigg[&\frac{1}{|x|^4}\left(\frac{3\delta_{m,k}x_r + 3\delta_{m,r}x_k + 3\delta_{k,r}x_m}{|x|} - \frac{15x_mx_kx_r}{|x|^3}\right)\\ &+ \frac{s}{c}\frac{1}{|x|^3}\left(\frac{3\delta_{m,k}x_r + 3\delta_{m,r}x_k + 3\delta_{k,r}x_m}{|x|} - \frac{15x_mx_kx_r}{|x|^3}\right)\\ &+ \frac{s^2}{c^2}\frac{1}{|x|^2}\left(\frac{\delta_{m,k}x_r + \delta_{m,r}x_k + \delta_{k,r}x_m}{|x|} - \frac{6x_mx_kx_r}{|x|^3}\right)\\ &- \frac{s^3}{c^3}\frac{1}{|x|}\frac{x_mx_kx_r}{|x|^3}\Bigg]\frac{\exp(-s|x|/c)}{4\pi} \qquad \text{for} \quad |x| \neq 0\,,\end{aligned} \tag{13.3-18}$$

and

$$\begin{aligned}&\partial_n\partial_m\partial_k\partial_r\hat{G}(x,s) =\\ &\Bigg[\frac{1}{|x|^5}\Bigg(3\delta_{m,k}\delta_{n,r} + 3\delta_{m,r}\delta_{n,k} + 3\delta_{k,r}\delta_{n,m} - \frac{15\delta_{m,k}x_nx_r + 15\delta_{m,r}x_nx_k + 15\delta_{k,r}x_nx_m}{|x|^2}\\ &- \frac{15\delta_{n,m}x_kx_r + 15\delta_{n,k}x_mx_r + 15\delta_{n,r}x_mx_k}{|x|^2} + \frac{105x_nx_mx_kx_r}{|x|^4}\Bigg)\\ &+ \frac{s}{c}\frac{1}{|x|^4}\Bigg(3\delta_{m,k}\delta_{n,r} + 3\delta_{m,r}\delta_{n,k} + 3\delta_{k,r}\delta_{n,m} - \frac{15\delta_{m,k}x_nx_r + 15\delta_{n,r}x_mx_k + 15\delta_{m,r}x_nx_k}{|x|^2}\\ &- \frac{15\delta_{n,k}x_mx_r + 15\delta_{k,r}x_nx_m + 15\delta_{n,m}x_kx_r}{|x|^2} + \frac{105x_nx_mx_kx_r}{|x|^4}\Bigg)\\ &+ \frac{s^2}{c^2}\frac{1}{|x|^3}\Bigg(\delta_{m,k}\delta_{n,r} + \delta_{m,r}\delta_{n,k} + \delta_{k,r}\delta_{n,m} - \frac{6\delta_{m,k}x_nx_r + 6\delta_{m,r}x_nx_k + 6\delta_{k,r}x_nx_m}{|x|^2}\end{aligned}$$

$$-\frac{6\delta_{n,m}x_kx_r + 6\delta_{n,k}x_mx_r + 6\delta_{n,r}x_mx_k}{|x|^2} + \frac{45x_nx_mx_kx_r}{|x|^4}\Bigg)$$

$$+\frac{s^3}{c^3}\frac{1}{|x|^2}\Bigg(-\frac{\delta_{n,m}x_kx_r + \delta_{n,k}x_mx_r + \delta_{n,r}x_mx_k}{|x|^2} - \frac{\delta_{m,k}x_nx_r + \delta_{m,r}x_nx_k + \delta_{k,r}x_nx_m}{|x|^2}$$

$$+\frac{10x_nx_mx_kx_r}{|x|^4}\Bigg) + \frac{s^4}{c^4}\frac{1}{|x|}\frac{x_nx_mx_kx_r}{|x|^4}\Bigg]\frac{\exp(-s|x|/c)}{4\pi} \quad \text{for} \quad |x| \neq 0 . \qquad (13.3\text{-}19)$$

Exercises

Exercise 13.3-1

Prove, by using Equation (13.3-2), and carrying out in the relevant Fourier integral a contour integration in the complex k_3 plane, that

$$\hat{G}(x,s) = \exp(-s|x|/c)/4\pi|x|$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{\{k_1,k_2\}\in\mathcal{R}^2} \frac{\exp\left[-\mathrm{j}(k_1x_1 + k_2x_2) - (k_1^2 + k_2^2 + s^2/c^2)^{1/2}|x_3|\right]}{2(k_1^2 + k_2^2 + s^2/c^2)^{1/2}}\, \mathrm{d}k_1\, \mathrm{d}k_2 . \qquad (13.3\text{-}20)$$

(*Hint*: observe that $k_3 = \pm\mathrm{j}(k_1^2 + k_2^2 + s^2/c^2)^{1/2}$ are simple poles of the analytically continued integrand (away from the real k_3 axis) in the upper and lower halves of the complex k_3 plane and that Jordan's lemma applies to a semicircle in the lower half of the k_3 plane for $x_3 > 0$ and to a semicircle in the upper half of the k_3 plane for $x_3 < 0$.) The representation of Equation (13.3-20) plays a major role in the analysis of elastodynamic radiation problems in subdomains of $\mathcal{R}^3$ with parallel, planar, boundaries.

13.4 The complex frequency-domain source-type integral representations for the particle velocity and the dynamic stress

The complex frequency-domain source-type integral representations for the particle velocity and the dynamic stress of the elastic wave field radiated by the sources located in $\mathcal{D}^T$ are obtained by carrying out the inverse spatial Fourier transformation of the angular wave-vector domain expressions in Equations (13.2-7) and (13.1-13). To this end, we first rewrite Equation (13.2-7) as (see also Equation (13.1-19))

$$\tilde{v}_r = s\rho^{-1}\tilde{\Phi}_r^f + \rho^{-1}C_{k,m,i,j}\,\mathrm{j}k_m\tilde{\Phi}_{r,k,i,j}^h , \qquad (13.4\text{-}1)$$

in which

$$\tilde{\Phi}_r^f = \tilde{G}_{r,k}\tilde{f}_k \qquad (13.4\text{-}2)$$

is the angular wave-vector domain elastodynamic force source vector potential and

$$\tilde{\Phi}_{r,k,i,j}^h = \tilde{G}_{r,k}\tilde{h}_{i,j} \qquad (13.4\text{-}3)$$

is the angular wave-vector domain elastodynamic deformation source tensor potential. (The interrelation between these source potentials in a homogeneous, isotropic solid and the general elastodynamic wave potentials introduced in Section 10.10 will be discussed later on.)

Furthermore, the corresponding angular wave-vector domain dynamic stress is expressed in terms of the angular wave-vector particle velocity through (see Equation (13.1-13))

$$\begin{aligned}(\tilde{\tau}_{p,q}+\tilde{\tau}_{q,p})/2 &= -s^{-1}C_{p,q,i,j}\tilde{h}_{i,j}+s^{-1}C_{p,q,n,r}(-\mathrm{j}k_n\tilde{v}_r)\\ &= -s^{-1}C_{p,q,i,j}\tilde{h}_{i,j}+\rho^{-1}C_{p,q,n,r}(-\mathrm{j}k_n\tilde{\Phi}^f_r)\\ &\quad-(s\rho)^{-1}C_{p,q,n,r}C_{k,m,i,j}(-\mathrm{j}k_n)(-\mathrm{j}k_m)\tilde{\Phi}^h_{r,k,i,j}\,, \end{aligned} \tag{13.4-4}$$

where the relation $C_{p,q,i,j}\Delta^+_{i,j,n,r}=C_{p,q,n,r}$ has been used.

The expressions for the complex frequency-domain elastodynamic force source vector potential $\hat{\Phi}^f_r$ and the complex frequency-domain elastodynamic deformation source tensor potential $\hat{\Phi}^f_{r,k,i,j}$ are obtained by carrying out the inverse spatial Fourier transformation of Equations (13.4-2) and (13.4-3), respectively. Since the product of two functions in angular wave-vector space corresponds to the convolution of these functions in configuration space (see Equations (B.2-11) and (B.2-12)), we obtain

$$\hat{\Phi}^f_r(\boldsymbol{x},s)=\int_{\boldsymbol{x}'\in\mathcal{D}^T}\hat{G}_{r,k}(\boldsymbol{x}-\boldsymbol{x}',s)\hat{f}_k(\boldsymbol{x}',s)\,\mathrm{d}V \tag{13.4-5}$$

and

$$\hat{\Phi}^h_{r,k,i,j}(\boldsymbol{x},s)=\int_{\boldsymbol{x}'\in\mathcal{D}^T}\hat{G}_{r,k}(\boldsymbol{x}-\boldsymbol{x}',s)\hat{h}_{i,j}(\boldsymbol{x}',s)\,\mathrm{d}V\,, \tag{13.4-6}$$

in which (see Equation (13.2-8) and using the rule that $-\mathrm{j}k_k$ corresponds to ∂_k (see Equation (13.1-7)))

$$\hat{G}_{r,k}=c_S^{-2}\hat{G}_S\delta_{r,k}+s^{-2}\partial_r\partial_k(\hat{G}_P-\hat{G}_S)\,, \tag{13.4-7}$$

with (see Equation (13.3-12))

$$\hat{G}_{P,S}(\boldsymbol{x},s)=\exp(-s|\boldsymbol{x}|/c_{P,S})/4\pi|\boldsymbol{x}| \qquad \text{for} \quad |\boldsymbol{x}|\neq 0\,. \tag{13.4-8}$$

Again using the rule that $-\mathrm{j}k_m$ corresponds to ∂_m, we obtain from Equation (13.4-1)

$$\hat{v}_r=s\rho^{-1}\hat{\Phi}^f_r-\rho^{-1}C_{k,m,i,j}\partial_m\hat{\Phi}^h_{r,k,i,j} \tag{13.4-9}$$

and from Equation (13.4-4)

$$\begin{aligned}(\hat{\tau}_{p,q}+\hat{\tau}_{q,p})/2 &= -s^{-1}C_{p,q,i,j}\hat{h}_{i,j}+\rho^{-1}C_{p,q,n,r}\partial_n\hat{\Phi}^f_r\\ &\quad-(s\rho)^{-1}C_{p,q,n,r}C_{k,m,i,j}\partial_n\partial_m\hat{\Phi}^h_{r,k,i,j}\,. \end{aligned} \tag{13.4-10}$$

Note that in the right-hand sides of Equations (13.4-5) and (13.4-6) we have taken care to distribute the arguments over the functions such that the spatial integration is carried out over the fixed domain $\mathcal{D}^T$ (Figure 13.4-1). (If we had provided the source distributions with the argument $\boldsymbol{x}-\boldsymbol{x}'$ and the Green's functions with the argument $\boldsymbol{x}'$, we would have to integrate over a domain that varies with $\boldsymbol{x}$.)

Equations (13.4-5)–(13.4-10) constitute the solution to the complex frequency-domain elastodynamic radiation problem in an unbounded homogeneous, isotropic, perfectly elastic

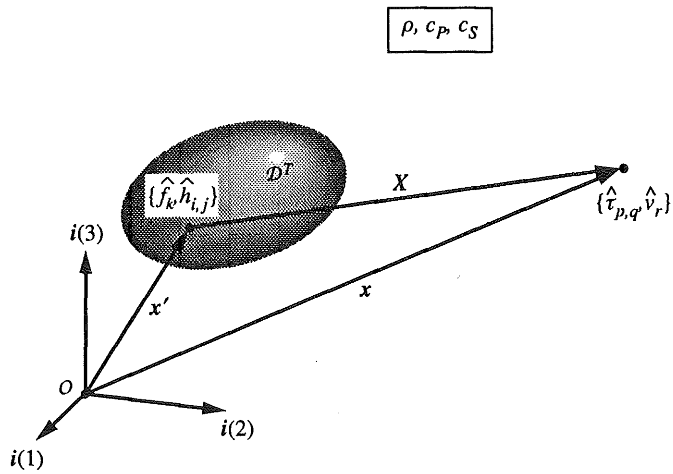

Figure 13.4-1 Complex frequency-domain source-type integral representations for the elastic wave field $\{\hat{\tau}_{p,q}, \hat{v}_r\}$ observed at position $x \in \mathcal{R}^3$, radiated by sources $\{\hat{f}_k, \hat{h}_{i,j}\}$ at position $x' \in \mathcal{D}^T$ (bounded source domain) in an unbounded homogeneous, isotropic lossless solid with elastodynamic parameters $\{\rho, c_P, c_S\}$.

solid. The expressions are used in the calculation and computation of multitudinous elastodynamic radiation problems. In a number of simple cases, the integrals in Equations (13.4-5) and (13.4-6) can be calculated analytically, and the differentiations in Equations (13.4-7), (13.4-9) and (13.4-10) can be carried out analytically as well. In more complicated cases, the integrals in Equations (13.4-5) and (13.4-6) must be computed with the aid of numerical methods. Since numerical integration can be carried out with any desired degree of accuracy, such an evaluation presents no difficulties (apart from the singularity in $\hat{G}_{k,r}$ at $x' = x$). Numerical differentiation, however, is much more difficult and inherently of restricted accuracy. (This also applies to the differentiations on the right-hand side of Equation (13.4-7).) Therefore, it is in general advantageous to carry out all the differentiations in Equations (13.4-7), (13.4-9) and (13.4-10) analytically, which can be done since they act on the position vector x that occurs in the argument of $\hat{G}_P$ and $\hat{G}_S$ in Equations (13.4-5) and (13.4-6) only. Once this has been done, only the integrals remain to be evaluated numerically.

The carrying out of the differentiations in Equations (13.4-7), (13.4-9) and (13.4-10) has also the advantage of making explicit the behaviour of the different terms on the right-hand side as far as their dependence on the distance from the source point to the point of observation is concerned.

In addition, each term in the expression for the particle velocity and the dynamic stress has its own directional characteristic in which only the unit vector in the direction of observation $x_m/|x|$ as viewed from the source point occurs. In Equations (13.3-14)–(13.3-19), care has been taken to make the two types of dependence explicit.

Finally, observe that $\hat{v}_r$ and $\hat{\tau}_{p,q}$ indeed show, since $\mathrm{Re}(s) > 0$, an exponential decay as $|x| \to \infty$, as has been assumed in Section 13.1.

As the structure of $\hat{G}_{r,k}$ as given by Equation (13.4-7) shows, the total elastic wave motion decomposes, outside the sources, into a part that has the complex frequency-domain spherical wave propagation factor $\hat{G}_P$, in which the delay factor $\exp(-s|x|/c_P)$ containing c_P occurs, and a part that has the complex frequency-domain spherical wave propagation factor $\hat{G}_S$, in which the delay factor $\exp(-s|x|/c_S)$ containing c_S occurs. In view of the conditions $\lambda + 2\mu/3 > 0$ and $\mu > 0$ (see Equation (10.8-23)), we have $c_P > 2c_S/3^{1/2}$, which in the realm of earthquake seismology historically has led to the designation of the waves propagating with the (larger) wave speed c_P as "primary waves" or *P-waves* and of the waves propagating with the (smaller) wave speed c_S as "secondary waves" or *S-waves*. Furthermore, in connection with Equation (13.4-7) we observe that

$$\varepsilon_{n,m,r}\partial_m\partial_r\partial_k\hat{G}_P = 0 \quad \text{for} \quad |x| \neq 0 \tag{13.4-11}$$

and (see Equation (13.3-3))

$$\partial_r\left[c_S^{-2}\hat{G}_S\delta_{r,k} - s^{-2}\partial_r\partial_k\hat{G}_S\right] = \partial_k\left[c_S^{-2}\hat{G}_S - s^{-2}\partial_r\partial_r\hat{G}_S\right] = 0 \quad \text{for} \quad |x| \neq 0 . \tag{13.4-12}$$

As a consequence, the particle velocity of the P-waves is, outside the source domain, curl-free or rotation-free, while the particle velocity of the S-waves is, outside the source domain, divergence-free. For this reason, P-waves are also denoted as *irrotational, dilatational*, or *compressional waves*, and S-waves as *rotational, equivoluminal*, or *shear waves*. Obviously, in the homogeneous, isotropic, perfectly elastic solid, the two types of waves travel independently.

As far as the singularity of the elastic wave-field quantities at the location of the source point is concerned, this turns out to be less severe than it would seem at first sight. The nature of this singularity follows upon using in the neighbourhood of $x' = x$ the Taylor expansion of the exponential functions occurring in Equation (13.4-7). With

$$X = x - x' , \tag{13.4-13}$$

we have

$$\hat{G}_S = \frac{1}{4\pi|X|} + O(1) \quad \text{as} \quad |X| \to 0 , \tag{13.4-14}$$

and

$$\begin{aligned}\hat{G}_P - \hat{G}_S &= \frac{1 - s|X|/c_P + s^2|X|^2/2c_P^2}{4\pi|X|} - \frac{1 - s|X|/c_S + s^2|X|^2/2c_S^2}{4\pi|X|} \\ &\quad + O(|X|^2) \quad \text{as} \quad |X| \to 0 \\ &= \frac{s}{4\pi}\left(\frac{1}{c_S} - \frac{1}{c_P}\right) + \frac{s^2}{4\pi}\left(\frac{1}{2c_P^2} - \frac{1}{2c_S^2}\right)|X| + O(|X|^2) \quad \text{as } |X| \to 0 .\end{aligned} \tag{13.4-15}$$

From the latter equation it follows that

$$s^{-2}\partial_r\partial_k(\hat{G}_P - \hat{G}_S) = \frac{1}{4\pi}\left(\frac{1}{2c_P^2} - \frac{1}{2c_S^2}\right)\left(\delta_{r,k} - \frac{X_rX_k}{|X|^2}\right)\frac{1}{|X|} + O(1) \quad \text{as } |X| \to 0 . \tag{13.4-16}$$

Consequently,

$$\begin{aligned}\hat{G}_{r,k} &= \frac{1}{4\pi|X|}\left[\frac{1}{c_S^2}\delta_{r,k} + \left(\frac{1}{2c_P^2} - \frac{1}{2c_S^2}\right)\left(\delta_{r,k} - \frac{X_rX_k}{|X|^2}\right)\right] + O(1) \quad \text{as } |X| \to 0 \\ &= \frac{1}{4\pi|X|}\left[\left(\frac{1}{2c_P^2} + \frac{1}{2c_S^2}\right)\delta_{r,k} - \left(\frac{1}{2c_P^2} - \frac{1}{2c_S^2}\right)\frac{X_rX_k}{|X|^2}\right] + O(1) \quad \text{as } |X| \to 0 .\end{aligned} \tag{13.4-17}$$

Equation (13.4-17) shows that the singularity in $\hat{G}_{r,k}$ at $X = \mathbf{0}$ is of Order ($|X|^{-1}$) rather than of Order ($|X|^{-3}$) as would at first sight be expected from Equations (13.4-7) and (13.3-16)–(13.3-17). In view of Equations (13.4-9) and (13.4-10) the singularity in the particle velocity due to a point source of force is then of Order ($|X|^{-1}$), the dynamic stress due to a point source of deformation rate is of Order ($|X|^{-3}$), and the singularity in the particle velocity due to a point force of deformation rate and the dynamic stress due to a point source of force is of Order ($|X|^{-2}$). In addition to this, a direct source term occurs in the expression for the dynamic stress due to a point source of deformation rate.

Exercises

Exercise 13.4-1

Employ three-dimensional spatial Fourier-transform methods to construct the solution of the "grad-div" vector Helmholtz equation

$$\partial_k\partial_r\hat{F}_r - (s^2/c^2)\,\hat{F}_k = -\hat{Q}_k \qquad (13.4\text{-}18)$$

that shows an exponential decay as $|x|\to\infty$ for $\mathrm{Re}(s) > 0$. In Equation (13.4-18), $\hat{Q}_k$ differs from zero in a bounded subdomain $\mathcal{D}^T$ of $\mathcal{R}^3$ only. (*Hint*: Use the Fourier transformation:

$$\tilde{F}_k(\mathrm{j}k,s) = \int_{x\in\mathcal{R}^3} \exp(\mathrm{j}k_s x_s)\hat{F}_k(x,s)\,\mathrm{d}V\,.)$$

(a) Determine the equation that results upon Fourier transforming Equation (13.4-18). (b) Derive an expression for $k_r\tilde{F}_r$ from the resulting equation. (c) Use the expression obtained under (b) to solve for $\tilde{F}_k$. (d) Transform the expression for $\tilde{F}_k$ back to the complex frequency-domain configuration space; write the result as

$$\tilde{F}_k(x,s) = \int_{x'\in\mathcal{D}^T} \hat{\Gamma}_{k,r}(x - x',s)\hat{Q}_r(x',s)\,\mathrm{d}V \qquad (13.4\text{-}19)$$

and determine $\hat{\Gamma}_{k,k'}$.

Answers:

(a) $k_k k_r\tilde{F}_r + (s^2/c^2)\tilde{F}_k = \tilde{Q}_k$;

(b) $k_r\tilde{F}_r = k_r\tilde{Q}_r\tilde{G}$, with $\tilde{G} = (k_m k_m + s^2/c^2)^{-1}$;

(c) $\tilde{F}_k = (c^2/s^2)\left[\tilde{Q}_k - k_k k_r\tilde{Q}_r\tilde{G}\right]$;

(d) Equation (13.4-19), with

$$\hat{\Gamma}_{k,r} = (c^2/s^2)\left[\delta_{k,r}\delta(x) + \partial_k\partial_r\hat{G}\right], \quad \text{with} \quad \hat{G}(x,s) = \frac{\exp(-s|x|/c)}{4\pi|x|} \quad \text{for} \quad |x| \neq 0\,. \qquad (13.4\text{-}20)$$

13.5 The time-domain source-type integral representations for the particle velocity and the dynamic stress

The time-domain source-type integral representations for the particle velocity and the dynamic stress that are causally related to the action of a volume source density of force $f_k = f_k(\boldsymbol{x},t)$ and a volume source density of deformation rate $h_{i,j} = h_{i,j}(\boldsymbol{x},t)$, both with support $\mathcal{D}^T$, follow from the results of Section 13.4 by employing some elementary rules of the Laplace transformation of causal functions of time. These are: (1) the product of two Laplace transforms corresponds to the time convolution of the corresponding time functions; (2) the factor $\exp(-s|X|/c)$ corresponds to a time delay by the amount of $|X|/c$; (3) the factor s corresponds to a time differentiation; (4) the factor s^{-1} corresponds to a time integration; (5) the factor s^{-2} in the expression of $\hat{G}_{r,k}$ corresponds to $tH(t)$, where $H(t)$ is the Heaviside unit step function: $H(t) = \{0,½,1\}$ for $\{t<0, t=0, t>0\}$.

Application of these rules to the complex frequency-domain expressions given in Equations (13.4-9) and (13.4-10) yields (Figure 13.5-1)

$$v_r(\boldsymbol{x},t) = \rho^{-1}\partial_t \Phi^f_r(\boldsymbol{x},t) - \rho^{-1} C_{k,m,i,j}\partial_m \Phi^h_{r,k,i,j}(\boldsymbol{x},t)\,, \tag{13.5-1}$$

and

$$\begin{aligned}&[\tau_{p,q}(\boldsymbol{x},t) + \tau_{q,p}(\boldsymbol{x},t)]/2\\ &= -C_{p,q,i,j}\int_{t'=t_0}^{t} h_{i,j}(\boldsymbol{x},t')\,\mathrm{d}t' + \rho^{-1}C_{p,q,n,r}\partial_n \Phi^f_r(\boldsymbol{x},t)\\ &\quad -\rho^{-1}C_{p,q,n,r}C_{k,m,i,j}\partial_n\partial_m \int_{t'=t_0}^{t} \Phi^h_{r,k,i,j}(\boldsymbol{x},t')\,\mathrm{d}t'\,,\end{aligned} \tag{13.5-2}$$

in which the elastodynamic force source vector potential follows from Equation (13.4-5) as

$$\Phi^f_r(\boldsymbol{x},t) = \int_{t'=t_0}^{\infty}\mathrm{d}t'\int_{\boldsymbol{x}'\in\mathcal{D}^T} G_{r,k}(\boldsymbol{x}-\boldsymbol{x}',t-t')\,f_k(\boldsymbol{x}',t')\,\mathrm{d}V\,, \tag{13.5-3}$$

the elastodynamic deformation source tensor potential from Equation (13.4-6) as

$$\Phi^h_{r,k,i,j}(\boldsymbol{x},t) = \int_{t'=t_0}^{\infty}\mathrm{d}t'\int_{\boldsymbol{x}'\in\mathcal{D}^T} G_{r,k}(\boldsymbol{x}-\boldsymbol{x}',t-t')\,h_{i,j}(\boldsymbol{x}',t')\,\mathrm{d}V\,, \tag{13.5-4}$$

and the elastodynamic Green's tensor follows from Equations (13.4-7), (13.4-8) and (13.3-10) as

$$\begin{aligned}G_{r,k}(\boldsymbol{x},t) &= \frac{\delta(t-|\boldsymbol{x}|/c_S)}{4\pi c_S^2|\boldsymbol{x}|}\,\delta_{r,k}\\ &\quad + \partial_r\partial_k\left[\frac{(t-|\boldsymbol{x}|/c_P)H(t-|\boldsymbol{x}|/c_P)}{4\pi|\boldsymbol{x}|} - \frac{(t-|\boldsymbol{x}|/c_S)H(t-|\boldsymbol{x}|/c_S)}{4\pi|\boldsymbol{x}|}\right].\end{aligned} \tag{13.5-5}$$

The instant $t = t_0$ marks the instant at which the sources are switched on. Substituting Equation (13.5-5) in Equations (13.5-3) and (13.5-4), and rearranging the terms in the time convolutions, we arrive at the alternative expressions for the elastodynamic source potentials

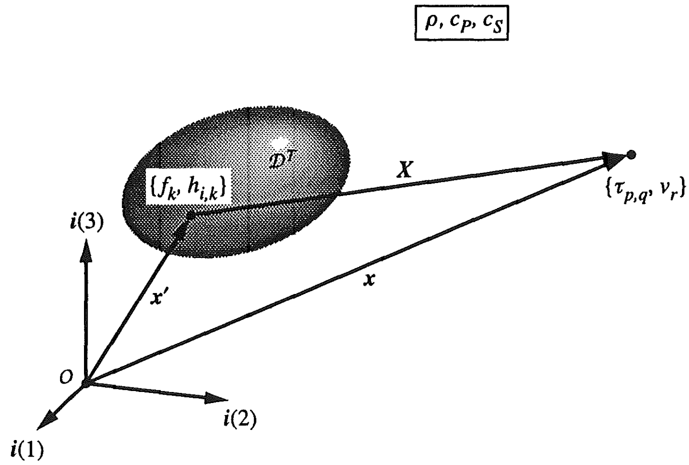

Figure 13.5-1 Time-domain source-type integral representations for the elastic wave field $\{\tau_{p,q},v_r\}$ observed at position $x\in\mathcal{R}^3$, radiated by sources $\{f_k,h_{i,j}\}$ at position $x'\in\mathcal{D}^T$ (bounded source domain) in an unbounded homogeneous, isotropic lossless solid with elastodynamic parameters $\{\rho,c_P,c_S\}$.

$$\Phi_r^f(x,t) = \int_{x'\in\mathcal{D}^T} \frac{f_r(x',t-|x-x'|/c_S)}{4\pi c_S^2|x-x'|}\,\mathrm{d}V$$
$$+\partial_r\partial_k\left[\int_{x'\in\mathcal{D}^T}\left(\int_{t''=0}^{t-t_0-|x-x'|/c_P} t''\frac{f_k(x',t-t''-|x-x'|/c_P)}{4\pi|x-x'|}\,\mathrm{d}t''\right.\right.$$
$$\left.\left.-\int_{t''=0}^{t-t_0-|x-x'|/c_S} t''\frac{f_k(x',t-t''-|x-x'|/c_S)}{4\pi|x-x'|}\,\mathrm{d}t''\right)\mathrm{d}V\right] \qquad (13.5\text{-}6)$$

and

$$\Phi_{r,k,i,j}^h(x,t) = \delta_{r,k}\int_{x'\in\mathcal{D}^T} \frac{h_{i,j}(x',t-|x-x'|/c_S)}{4\pi c_S^2|x-x'|}\,\mathrm{d}V$$
$$+\partial_r\partial_k\left[\int_{x'\in\mathcal{D}^T}\left(\int_{t''=0}^{t-t_0-|x-x'|/c_P} t''\frac{h_{i,j}(x',t-t''-|x-x'|/c_P)}{4\pi|x-x'|}\,\mathrm{d}t''\right.\right.$$
$$\left.\left.-\int_{t''=0}^{t-t_0-|x-x'|/c_S} t''\frac{h_{i,j}(x',t-t''-|x-x'|/c_S)}{4\pi|x-x'|}\,\mathrm{d}t''\right)\mathrm{d}V\right]. \qquad (13.5\text{-}7)$$

The expressions Equations (13.5-3)–(13.5-7) clearly exhibit the decomposition into P-and S-waves, while Equations (13.5-6) and (13.5-7) show explicitly the retarded potential nature of the elastodynamic source potentials. The terms containing c_P yield only a non-vanishing contribution if $t > t_0 + |x-x'|/c_P$, i.e. if the elapse time after switching on the source exceeds the travel time of a disturbance to traverse the distance $|x-x'|$ from a particular source point

x' to the point of observation x with the *P-wave speed* c_P. The relevant terms are denoted as the *P-wave contribution.* Similarly, the terms containing c_S yield only a non-vanishing contribution if $t > t_0 + |x - x'|/c_S$, i.e. if the elapse time after switching on the source exceeds the travel time of a disturbance to traverse the distance $|x - x'|$ from a particular source point x' to the point of observation x with the *S-wave speed* c_S. The relevant terms are denoted as the *S-wave contribution.* As to the evaluation of the expressions occurring in Equations (13.5-1)–(13.5-7) the same remarks as in Section 13.4 apply. Here, too, it is, in case numerical evaluations are necessary, advantageous to carry out the differentiations with respect to the spatial coordinates analytically. The relevant results directly follow from Equations (13.3-14)–(13.3-19) by applying the rules that the factors s, s^2, s^3 and s^4 correspond to ∂_t, ∂_t^2, ∂_t^3 and ∂_t^4, respectively, while the factor $\exp(-s|x|/c)$ corresponds to a time delay to the amount of $|x|/c$. The resulting expressions are not reproduced here.

13.6 Point-source solutions

From the general results pertaining to a distributed volume source the corresponding point-source solutions are obtained in the limiting case where the maximum diameter of the source domain $\mathcal{D}^T$ becomes vanishingly small with respect to the distance from the source to the point of observation. For this case, let $\mathcal{D}^T$ be centred around the point $x' = b$ (for example, its barycentre defined by

$$b = (V^T)^{-1} \int_{x' \in \mathcal{D}^T} x' \,\mathrm{d}V , \tag{13.6-1}$$

where

$$V^T = \int_{x' \in \mathcal{D}^T} \mathrm{d}V , \tag{13.6-2}$$

is the volume of $\mathcal{D}^T$).

Complex frequency-domain analysis

Under these conditions, for points of observation x not too close to the source domain, Equation (13.4-5) can be replaced by

$$\Phi_r^f(x,s) = \hat{G}_{r,k}(X,s)\hat{F}_k(s) , \tag{13.6-3}$$

where

$$\hat{F}_k(s) = \int_{x' \in \mathcal{D}^T} \hat{f}_k(x',s) \,\mathrm{d}V \tag{13.6-4}$$

is the complex frequency-domain total *point force* operative on the solid, and Equation (13.4-6) by

$$\hat{\Phi}_{r,k,i,j}(\boldsymbol{x},s) = \hat{G}_{r,k}(\boldsymbol{X},s)\hat{H}_{i,j}(s)\,, \tag{13.6-5}$$

where

$$\hat{H}_{i,j}(s) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} \hat{h}_{i,j}(\boldsymbol{x}',s)\,\mathrm{d}V \tag{13.6-6}$$

is the complex frequency-domain *point volume deformation rate* inflicted on the solid. Furthermore,

$$\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{b} \tag{13.6-7}$$

is the position vector from the point $\boldsymbol{b}$ where the source is located to the point of observation $\boldsymbol{x}$. The relevant expressions for the particle velocity and the dynamic stress easily follow from Equations (13.4-9) and (13.4-10) and (13.3-14)–(13.3-19), by using Equations (13.6-3)–(13.6-7), and replacing $\boldsymbol{x}$ by $\boldsymbol{X}$ in Equations (13.3-14)–(13.3-19); all terms in these expressions, except the direct source term, have the complex frequency-domain propagation factors $\exp(-s|\boldsymbol{X}|/c_P)$ or $\exp(-s|\boldsymbol{X}|/c_S)$ in common, thus showing their decomposition into P-and S-waves. Furthermore, each term has its dependence on $|\boldsymbol{X}|$, i.e. the distance from the location of the point source to the point of observation, and its directional characteristic that contains the unit vector $X_m/|\boldsymbol{X}|$ from the location of the point source to the point of observation. Furthermore, in the direct source term we now have $\hat{h}_{i,j}(\boldsymbol{x},s) = \hat{H}_{i,j}(s)\delta(\boldsymbol{X})$. The relevant expressions will not be reproduced here.

Time-domain analysis

For points of observation $\boldsymbol{x}$ not too close to the source domain, Equation (13.5-6) can be replaced by

$$\Phi_r^f(\boldsymbol{x},t) = \frac{F_r(t-|\boldsymbol{X}|/c_S)}{4\pi c_S^2|\boldsymbol{X}|} + \partial_r\partial_k\left[\int_{t''=0}^{t-t_0-|\boldsymbol{X}|/c_P} \frac{F_k(t-t''-|\boldsymbol{X}|/c_P)}{4\pi|\boldsymbol{X}|}\,t''\,\mathrm{d}t'' - \int_{t''=0}^{t-t_0-|\boldsymbol{X}|/c_S} \frac{F_k(t-t''-|\boldsymbol{X}|/c_S)}{4\pi|\boldsymbol{X}|}\,t''\,\mathrm{d}t''\right], \tag{13.6-8}$$

where

$$F_k(t) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} f_k(\boldsymbol{x}',t)\,\mathrm{d}V \tag{13.6-9}$$

is the *point force* operative on the solid, and Equation (13.5-7) by

$$\Phi_{r,k,i,j}^h(\boldsymbol{x},t) = \frac{H_{i,j}(t-|\boldsymbol{X}|/c_S)}{4\pi c_S^2|\boldsymbol{X}|}\delta_{r,k} + \partial_r\partial_k\left[\int_{t''=0}^{t-t_0-|\boldsymbol{X}|/c_P} \frac{H_{i,j}(t-t''-|\boldsymbol{X}|/c_P)}{4\pi|\boldsymbol{X}|}\,t''\,\mathrm{d}t'' - \int_{t''=0}^{t-t_0-|\boldsymbol{X}|/c_S} \frac{H_{i,j}(t-t''-|\boldsymbol{X}|/c_S)}{4\pi|\boldsymbol{X}|}\,t''\,\mathrm{d}t''\right], \tag{13.6-10}$$

where

$$H_{i,j}(t) = \int_{\boldsymbol{x}'\in\mathcal{D}^T} h_{i,j}(\boldsymbol{x}',t)\,\mathrm{d}V \tag{13.6-11}$$

is the *point volume deformation rate* inflicted on the solid. Furthermore,

$$X = x - b \, , \tag{13.6-12}$$

is the position vector from the point b where the source is located to the point of observation x. The relevant expressions for the particle velocity and the dynamic stress easily follow from Equations (13.5-1) and (13.5-2) and using the time-domain counterparts of Equations (13.3-14)–(13.3-19). All terms in these expressions, except the direct source term, show their decomposition into P- and S-waves. Furthermore, each term has its dependence on $|X|$, i.e. the distance from the location of the point source to the point of observation, and its directional characteristic that contains the unit vector

$$\Xi_m = X_m/|X| \quad \text{for} \quad |X| \neq 0 \tag{13.6-13}$$

from the location of the point source to the point of observation. Furthermore, in the direct source term we have $h_{i,j}(x,t) = H_{i,j}(t)\delta(X)$. The relevant expressions are not reproduced here.

As far as the singularity of the elastic wave-field quantities at the location of the point source is concerned, it follows from Equation (13.4-17) that

$$G_{r,k} = \frac{1}{4\pi|X|}\left[\frac{1}{c_S^2}\delta_{r,k} + \left(\frac{1}{2c_P^2} - \frac{1}{2c_S^2}\right)(\delta_{r,k} - \Xi_r\Xi_k)\right]\delta(t) + O(1) \quad \text{as} \quad |X| \to 0$$

$$= \frac{1}{4\pi|X|}\left[\left(\frac{1}{2c_P^2} + \frac{1}{2c_S^2}\right)\delta_{r,k} - \left(\frac{1}{2c_P^2} - \frac{1}{2c_S^2}\right)\Xi_r\Xi_k\right]\delta(t) + O(1) \quad \text{as} \quad |X| \to 0 \, . \tag{13.6-14}$$

Equation (13.6-14) shows that the singularity in $G_{r,k}$ at $X = \mathbf{0}$ is of Order ($|X|^{-1}$) rather than of Order ($|X|^{-3}$) as would at first sight be expected from Equations (13.4-7), (13.3-16) and (13.3-17). In view of Equations (13.5-1) and (13.5-2) the singularity in the particle velocity due to a point source of force is then of Order ($|X|^{-1}$), the dynamic stress due to a point source of deformation rate is of Order ($|X|^{-3}$) and the singularity in the particle velocity due to a point force of deformation rate and the dynamic stress due to a point source of force is of Order ($|X|^{-2}$). In addition to this, a direct source term occurs in the expression for the dynamic stress due to a point source of deformation rate. Furthermore, the $\delta(t)$ time behaviour in Equation (13.6-14) entails a reproduction of the time behaviour of the point source strengths (source signatures).

The point source of force is a useful model for the mechanical vibrator used in the seismic exploration for fossil energy resources; the point source of deformation rate is a useful model in the description of earthquake mechanisms and for the acoustic emission from cracks under formation, while the special case $h_{1,1} = h_{2,2} = h_{3,3}$ and $h_{i,j} = 0$ for $i \neq j$ models an explosion source.

13.7 Far-field radiation characteristics of extended sources (complex frequency-domain analysis)

In many applications of elastodynamic radiation one is often particularly interested in the behaviour of the radiated field at large distances from the radiating structures. To investigate this behaviour, we consider the leading term in the expansions of the right-hand sides of

Equations (13.4-5)–(13.4-10) as $|x|\to\infty$; this term is known as the *far-field approximation* of the relevant elastic wave field. The region in space where the far-field approximation represents the wave-field values with sufficient accuracy is known as the *far-field region.* Since in the far-field region the mutual relationships between the dynamic stress and the particle velocity (though not their radiation characteristics) prove to be the same for the wave-field constituents generated by force sources and the wave-field constituents generated by deformation rate sources, it is advantageous to investigate those relationships for the total wave field, which will be done below.

To construct the far-field approximation we first observe that

$$|x-x'| = [(x_s-x_s')(x_s-x_s')]^{1/2} = |x|\left[1-2x_sx_s'/|x|^2+|x'|^2/|x|^2\right]^{1/2}, \tag{13.7-1}$$

from which, by a Taylor expansion of the square-root expression about $|x|=\infty$, it follows that (Figure 13.7-1)

$$|x-x'| = |x| - \xi_s x_s' + O(|x|^{-1}) \quad \text{as} \quad |x|\to\infty, \tag{13.7-2}$$

where

$$\xi_s = x_s/|x| \tag{13.7-3}$$

is the unit vector in the direction of observation (note that in the far-field approximation certainly $|x|\neq 0$). For the derivatives of $|x\text{-}x'|$ furthermore, we have

$$\partial_m|x-x'| = (x_m-x_m')/|x-x'|. \tag{13.7-4}$$

This leads to

$$\partial_m|x-x'| = \xi_m + O(|x|^{-1}) \quad \text{as} \quad |x|\to\infty, \tag{13.7-5}$$

where the order term follows from a Taylor expansion of $|(x_m-x_m')/|x-x'|-\xi_m|$ about $|x|=\infty$. Using these results, the Green's function of the scalar Helmholtz equation, Equation (13.3-9), can, in the far-field region, be approximated by

$$\hat{G}(x-x',s) = \frac{\exp(-s|x|/c)}{4\pi|x|}\exp(s\xi_s x_s'/c)\,[1+O(|x|^{-1})] \quad \text{as} \quad |x|\to\infty, \tag{13.7-6}$$

and its spatial derivatives by

$$\partial_r\hat{G}(x-x',s) \sim -(s\xi_r/c)\frac{\exp(-s|x|/c)}{4\pi|x|}\exp(s\xi_s x_s'/c)\,[1+O(|x|^{-1})] \quad \text{as} \quad |x|\to\infty, \tag{13.7-7}$$

Using Equations (13.7-6) and (13.7-7) in the expression (13.4-7) for the complex frequency-domain elastodynamic Green's function $\hat{G}_{r,k}$, we obtain the latter's far-field approximation as

$$\hat{G}_{r,k}(x-x',s) = \left[\hat{G}^{P,\infty}_{r,k}(\xi,s)\frac{\exp(-s|x|/c_P)}{4\pi c_P^2|x|} + \hat{G}^{S,\infty}_{r,k}(\xi,s)\frac{\exp(-s|x|/c_S)}{4\pi c_S^2|x|}\right] \times [1+O(|x|^{-1})] \quad \text{as} \quad |x|\to\infty, \tag{13.7-8}$$

in which

$$\hat{G}^{P,\infty}_{r,k}(\xi,s) = \xi_r\xi_k\exp(s\xi_s x_s'/c_P), \tag{13.7-9}$$

$$\hat{G}^{S,\infty}_{r,k}(\xi,s) = (\delta_{r,k}-\xi_r\xi_k)\exp(s\xi_s x_s'/c_S). \tag{13.7-10}$$

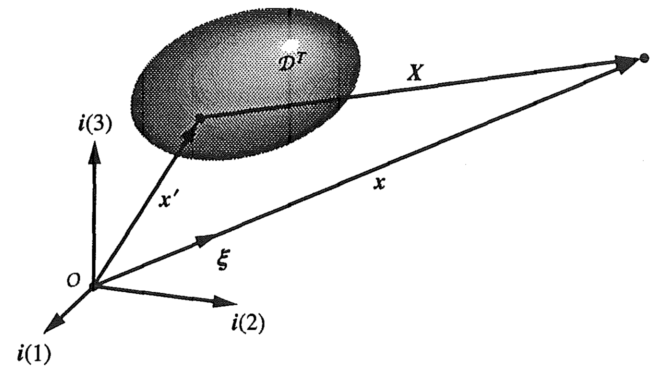

Figure 13.7-1 Far-field approximation to the distance function from source point $x' \in \mathcal{D}^T$ to observation point $x \in \mathcal{R}^3$: $|X| = |x - x'| = |x| - \xi_s x'_s + O(|x|^{-1})$ as $|x| \to \infty$.

With this, the far-field approximations to the elastodynamic force source vector potential $\hat{\Phi}_r^f$ and the elastodynamic deformation source tensor potential $\hat{\Phi}_{r,k,i,j}^h$ follow from Equations (13.4-5) and (13.4-6) as

$$\hat{\Phi}_r^f(x,s) = \left[\hat{\Phi}_r^{f;P,\infty}(\xi,s) \frac{\exp(-s|x|/c_P)}{4\pi c_P^2 |x|} + \hat{\Phi}_r^{f;S,\infty}(\xi,s) \frac{\exp(-s|x|/c_S)}{4\pi c_S^2 |x|} \right]$$

$$\times [1 + O(|x|^{-1})] \quad \text{as } |x| \to \infty , \tag{13.7-11}$$

with

$$\hat{\Phi}_r^{f;P,\infty}(\xi,s) = \xi_r \xi_k \int_{x' \in \mathcal{D}^T} \exp(s\xi_s x'_s / c_P) \hat{f}_k(x',s) \, \mathrm{d}V , \tag{13.7-12}$$

$$\hat{\Phi}_r^{f;S,\infty}(\xi,s) = (\delta_{r,k} - \xi_r \xi_k) \int_{x' \in \mathcal{D}^T} \exp(s\xi_s x'_s / c_S) \hat{f}_k(x',s) \, \mathrm{d}V , \tag{13.7-13}$$

and

$$\hat{\Phi}_{r,k,i,j}^h(x,s) = \left[\hat{\Phi}_{r,k,i,j}^{h;P,\infty}(\xi,s) \frac{\exp(-s|x|/c_P)}{4\pi c_P^2 |x|} + \hat{\Phi}_{r,k,i,j}^{h;S,\infty}(\xi,s) \frac{\exp(-s|x|/c_S)}{4\pi c_S^2 |x|} \right]$$

$$\times [1 + O(|x|^{-1})] \quad \text{as } |x| \to \infty , \tag{13.7-14}$$

with

$$\hat{\Phi}_{r,k,i,j}^{h;P,\infty}(\xi,s) = \xi_r \xi_k \int_{x' \in \mathcal{D}^T} \exp(s\xi_s x'_s / c_P) \hat{h}_{i,j}(x',s) \, \mathrm{d}V , \tag{13.7-15}$$

$$\hat{\Phi}_{r,k,i,j}^{h;S,\infty}(\xi,s) = (\delta_{r,k} - \xi_r \xi_k) \int_{x' \in \mathcal{D}^T} \exp(s\xi_s x'_s / c_S) \hat{h}_{i,j}(x',s) \, \mathrm{d}V , \tag{13.7-16}$$

respectively. Using Equations (13.7-7) and (13.7-11)–(13.7-16) in the expressions (13.4-9) and (13.4-10) for the particle velocity and the dynamic stress, we arrive at the far-field approximations for these quantities as

$$\hat{v}_r(\boldsymbol{x},s) = \left[\hat{v}_r^{P,\infty}(\boldsymbol{\xi},s) \frac{\exp(-s|\boldsymbol{x}|/c_P)}{4\pi c_P^2|\boldsymbol{x}|} + \hat{v}_r^{S,\infty}(\boldsymbol{\xi},s) \frac{\exp(-s|\boldsymbol{x}|/c_S)}{4\pi c_S^2|\boldsymbol{x}|} \right]$$

$$\times [1 + O(|\boldsymbol{x}|^{-1})] \quad \text{as} \quad |\boldsymbol{x}| \to \infty , \tag{13.7-17}$$

with

$$\hat{v}_r^{P,\infty} = s\rho^{-1} \hat{\Phi}_r^{f;P,\infty} + s(\rho c_P)^{-1} C_{k,m,i,j} \xi_m \hat{\Phi}_{r,k,i,j}^{h;P,\infty} , \tag{13.7-18}$$

$$\hat{v}_r^{S,\infty} = s\rho^{-1} \hat{\Phi}_r^{f;S,\infty} + s(\rho c_S)^{-1} C_{k,m,i,j} \xi_m \hat{\Phi}_{r,k,i,j}^{h;S,\infty} , \tag{13.7-19}$$

and

$$[\hat{\tau}_{p,q}(\boldsymbol{x},s) + \hat{\tau}_{q,p}(\boldsymbol{x},s)]/2 = \left(\left\{ \left[\hat{\tau}_{p,q}^{P,\infty}(\boldsymbol{\xi},s) + \hat{\tau}_{q,p}^{P,\infty}(\boldsymbol{\xi},s) \right]/2 \right\} \frac{\exp(-s|\boldsymbol{x}|/c_P)}{4\pi c_P^2|\boldsymbol{x}|} \right.$$

$$\left. + \left\{ \left[\hat{\tau}_{p,q}^{S,\infty}(\boldsymbol{\xi},s) + \hat{\tau}_{q,p}^{S,\infty}(\boldsymbol{\xi},s) \right]/2 \right\} \frac{\exp(-s|\boldsymbol{x}|/c_S)}{4\pi c_S^2|\boldsymbol{x}|} \right) [1 + O(|\boldsymbol{x}|^{-1})] \quad \text{as} \quad |\boldsymbol{x}| \to \infty , \tag{13.7-20}$$

with

$$(\hat{\tau}_{p,q}^{P,\infty} + \hat{\tau}_{q,p}^{P,\infty})/2 = -s(\rho c_P)^{-1} C_{p,q,n,r} \xi_n \hat{\Phi}_r^{f;P,\infty}$$
$$- s(\rho c_P^2)^{-1} C_{p,q,n,r} C_{k,m,i,j} \xi_n \xi_m \hat{\Phi}_{r,k,i,j}^{h;P,\infty} , \tag{13.7-21}$$

$$(\hat{\tau}_{p,q}^{S,\infty} + \hat{\tau}_{q,p}^{S,\infty})/2 = -s(\rho c_S)^{-1} C_{p,q,n,r} \xi_n \hat{\Phi}_r^{f;S,\infty}$$
$$- s(\rho c_S^2)^{-1} C_{p,q,n,r} C_{k,m,i,j} \xi_n \xi_m \hat{\Phi}_{r,k,i,j}^{h;S,\infty} . \tag{13.7-22}$$

In obtaining Equation (13.7-20) the fact has been used that the far-field region is outside the support of the direct source term in the right-hand side of Equation (13.4-10), in view of which the direct source term yields no contribution to the far-field approximation.

As Equations (13.7-17)–(13.7-22) show, the particle velocity and the dynamic stress have, in the far-field region, the structure of the superposition of a spherical P-wave that expands radially with the P-wave speed and a spherical S-wave that expands radially with the S-wave speed, both away from the origin of the chosen reference frame (which is also denoted as the *phase centre* of the far-field approximation), the latter being chosen in the neighbourhood of the source domain, with amplitudes that depend on the direction of observation and that decrease inversely proportionally to the distance from the source domain. The corresponding amplitude radiation characteristics $\{\hat{v}_r^{P,\infty}, \hat{\tau}_{p,q}^{P,\infty}\}$ and $\{\hat{v}_r^{S,\infty}, \hat{\tau}_{p,q}^{S,\infty}\}$ only depend on the direction of observation $\boldsymbol{\xi}$, and on s. Their dependence on $\boldsymbol{\xi}$ is the resultant of the dependence on $\boldsymbol{\xi}$ of the integrals

$$\int_{\boldsymbol{x}' \in \mathcal{D}^T} \exp(s\xi_s x_s'/c_{P,S}) \hat{f}_k(\boldsymbol{x}',s) \, \mathrm{d}V = \tilde{f}_k(s\boldsymbol{\xi}/c_{P,S},s) \tag{13.7-23}$$

and

$$\int_{x' \in \mathcal{D}^T} \exp(s\xi_s x'_s / c_{P,S}) \hat{h}_{i,j}(x', s)\, \mathrm{d}V = \tilde{h}_{i,j}(s\boldsymbol{\xi}/c_{P,S}, s)\,, \tag{13.7-24}$$

which are the spatial Fourier transforms of the source distributions at the values $\mathrm{j}\boldsymbol{k} = s\boldsymbol{\xi}/c_{P,S}$, and the purely directional characteristics, which for the P-wave particle velocity is $\xi_r \xi_k$ and for the S-wave particle velocity is $(\delta_{r,k} - \xi_r \xi_k)$. From Equations (13.7-12), (13.7-15) and (13.7-18), it follows that $\hat{v}_r^{P,\infty} = \xi_r(\xi_k \hat{v}_k^{P,\infty})$, i.e. the P-wave particle velocity is, in the far-field region, *longitudinal* with respect to the radial direction of propagation, while from Equations (13.7-13), (13.7-16) and (13.7-19) it follows that $\xi_r \hat{v}_r^{S,\infty} = 0$, since $\xi_r(\delta_{r,k} - \xi_r \xi_k) = \xi_k - \xi_r \xi_r \xi_k = \xi_k - \xi_k = 0$, i.e. the S-wave particle velocity is, in the far-field region, *transverse* with respect to the radial direction of propagation.

The far-field radiation characteristics of the particle velocity and the dynamic stress are not independent of each other. First, from Equations (13.7-18), (13.7-19) and Equations (13.7-21), (13.7-22), it is observed that

$$(\hat{\tau}_{p,q}^{P,\infty} + \hat{\tau}_{q,p}^{P,\infty})/2 = -c_P^{-1} C_{p,q,n,r} \xi_n \hat{v}_r^{P,\infty} \tag{13.7-25}$$

and

$$(\hat{\tau}_{p,q}^{S,\infty} + \hat{\tau}_{q,p}^{S,\infty})/2 = -c_S^{-1} C_{p,q,n,r} \xi_n \hat{v}_r^{S,\infty}\,, \tag{13.7-26}$$

in which

$$C_{p,q,n,r} = \lambda \delta_{p,q} \delta_{n,r} + \mu(\delta_{p,n}\delta_{q,r} + \delta_{p,r}\delta_{q,n}) \tag{13.7-27}$$

is the stiffness. By contracting the left-and right-hand sides of Equations (13.7-25) and (13.7-26) with the compliance $S_{i,j,p,q}$ it follows that

$$S_{i,j,p,q} \hat{\tau}_{p,q}^{P,\infty} = -c_P^{-1} \Delta_{i,j,n,r}^{+} \xi_n \hat{v}_r^{P,\infty} \tag{13.7-28}$$

and

$$S_{i,j,p,q} \hat{\tau}_{p,q}^{S,\infty} = -c_S^{-1} \Delta_{i,j,n,r}^{+} \xi_n \hat{v}_r^{S,\infty}\,, \tag{13.7-29}$$

respectively. Now, Equations (13.7-28) and (13.7-29) would have resulted if, in the source-free deformation rate equation pertaining to the homogeneous, isotropic, perfectly elastic solid under consideration, expressions of the type

$$\{\hat{\tau}_{p,q}, \hat{v}_r\} = \{\hat{\tau}_{p,q}^{\infty}, \hat{v}_r^{\infty}\} \exp(-s\xi_s x_s / c) \tag{13.7-30}$$

had been substituted, for $c = c_P$ and $c = c_S$, respectively, where $\hat{\tau}_{p,q}^{\infty}$ and $\hat{v}_r^{\infty}$ only depend on the real unit vector $\boldsymbol{\xi}$ and the Laplace transform variable s, and not on $\boldsymbol{x}$. Wave fields of the type of Equation (13.7-30) are denoted as complex frequency-domain *uniform plane waves* that propagate in the direction of the unit vector ξ_s.

Next, Equations (13.7-25) and (13.7-26) are contracted with $\Delta_{k,m,p,q}^{+} \xi_m$ and use is made of Equation (13.7-27). The result is

$$\hat{t}_k^{P,\infty} = -\rho c_P \hat{v}_k^{P,\infty}\,, \tag{13.7-31}$$

where the relations $\hat{v}_r^{P,\infty} = \xi_r(\xi_k \hat{v}_k^{P,\infty})$ and $\lambda + 2\mu = \rho c_P^2$ have been used, and

$$\hat{t}_k^{S,\infty} = -\rho c_S \hat{v}_k^{S,\infty}\,, \tag{13.7-32}$$

where the relations $\xi_r\hat{v}_r^{S,\infty} = 0$ and $\mu = \rho c_S^2$ have been used, respectively. In Equations (13.7-31) and (13.7-32),

$$\hat{t}_k^{P,\infty} = \Delta^+_{k,m,p,q}\xi_m\hat{\tau}_{p,q}^{P,\infty} \tag{13.7-33}$$

is the *radial P-wave traction* and

$$\hat{t}_k^{S,\infty} = \Delta^+_{k,m,p,q}\xi_m\hat{\tau}_{p,q}^{S,\infty} \tag{13.7-34}$$

is the *radial S-wave traction.* Equations of the type of Equations (13.7-31)–(13.7-34) would have resulted if in the source-free equation of motion pertaining to the homogeneous, isotropic, perfectly elastic solid under consideration, expressions of the type of Equation (13.7-30) had been substituted for $c = c_P$ and $c = c_S$, respectively.

Observing that $|\boldsymbol{x}| = \xi_s x_s$, we can therefore say that, after compensating for the (distance)$^{-1}$ decay, the spherical P-and S-wave amplitudes in the far-field region radiation pattern locally behave as if the waves were uniform plane waves travelling radially away from the source.

Equations (13.7-31) and (13.7-32) further show that in the far-field region the radial traction and the particle velocity of the P-and S-wave are proportional, with proportionality factors

$$Z_P = \rho c_P \tag{13.7-35}$$

and

$$Z_S = \rho c_S\,, \tag{13.7-36}$$

respectively. The quantity Z_P is known as the *elastodynamic plane P-wave impedance*; the quantity Z_S is known as the *elastodynamic plane S-wave impedance.*

Exercises

Exercise 13.7-1

Verify that Equations (13.7-18) and (13.7-19), and (13.7-21) and (13.7-22) indeed satisfy Equations (13.7-28) and (13.7-29), and (13.7-31)–(13.7-34).

13.8 Far-field radiation characteristics of extended sources (time-domain analysis)

In this section we investigate the time-domain far-field radiation characteristics of the elastodynamic radiation emitted by extended sources. To this end, we start from the results obtained in Section 13.7 and transform the relevant complex frequency-domain results to the time domain. Since the factor $\exp(-s|\boldsymbol{x}|/c)$ in the complex frequency domain corresponds in the time domain to a time delay by the amount of $|\boldsymbol{x}|/c$ and the factor $\exp(s\xi_s x_s'/c)$ to a time advance by the amount of $\xi_s x_s'/c$, the time-domain far-field approximations to the elastodynamic force source vector potential Φ_r^f and the elastodynamic deformation source tensor potential $\Phi_{r,k,i,j}^h$ follow from Equations (13.7-11)–(13.7-16) as (Figure 13.8-1)

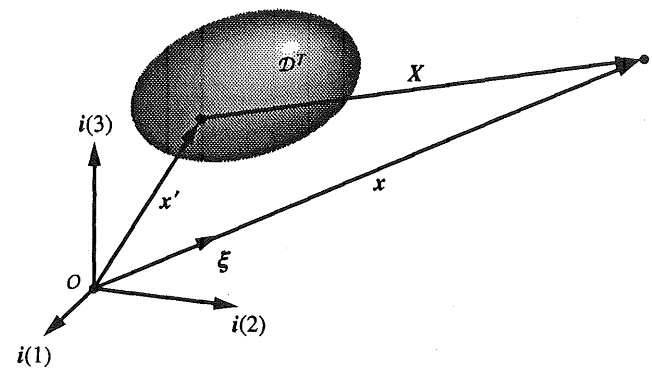

Figure 13.8-1 Far-field approximation to the distance function from source point $x' \in \mathcal{D}^T$ to observation point $x \in \mathcal{R}^3$: $|X| = |x - x'| = |x| - \xi_s x'_s + O(|x|^{-1})$ as $|x| \to \infty$.

$$\Phi_r^f(x,t) = \left[\frac{\Phi_r^{f;P,\infty}(\xi, t - |x|/c_P)}{4\pi c_P^2 |x|} + \frac{\Phi_r^{f;S,\infty}(\xi, t - |x|/c_S)}{4\pi c_S^2 |x|} \right]$$

$$\times [1 + O(|x|^{-1})] \quad \text{as } |x| \to \infty, \tag{13.8-1}$$

with

$$\Phi_r^{f;P,\infty}(\xi,t) = \xi_r \xi_k \int_{x' \in \mathcal{D}^T} f_k(x', t + \xi_s x'_s / c_P) \, \mathrm{d}V, \tag{13.8-2}$$

$$\Phi_r^{f;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r \xi_k) \int_{x' \in \mathcal{D}^T} f_k(x', t + \xi_s x'_s / c_S) \, \mathrm{d}V, \tag{13.8-3}$$

and

$$\Phi_{r,k,i,j}^h(x,t) = \left[\frac{\Phi_{r,k,i,j}^{h;P,\infty}(\xi, t - |x|/c_P)}{4\pi c_P^2 |x|} + \frac{\Phi_{r,k,i,j}^{h;S,\infty}(\xi, t - |x|/c_S)}{4\pi c_S^2 |x|} \right]$$

$$\times [1 + O(|x|^{-1})] \quad \text{as } |x| \to \infty, \tag{13.8-4}$$

with

$$\Phi_{r,k,i,j}^{h;P,\infty}(\xi,t) = \xi_r \xi_k \int_{x' \in \mathcal{D}^T} h_{i,j}(x', t + \xi_s x'_s / c_P) \, \mathrm{d}V, \tag{13.8-5}$$

$$\Phi_{r,k,i,j}^{h;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r \xi_k) \int_{x' \in \mathcal{D}^T} h_{i,j}(x', t + \xi_s x'_s / c_S) \, \mathrm{d}V. \tag{13.8-6}$$

Using Equations (13.8-1) and (13.8-4) in the expressions Equations (13.7-17)–(13.7-22) for the particle velocity and the dynamic stress, we arrive at the far-field representations of these quantities as

$$v_r(\boldsymbol{x},t) = \left[\frac{v_r^{P,\infty}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c_P)}{4\pi c_P^2|\boldsymbol{x}|} + \frac{v_r^{S,\infty}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c_S)}{4\pi c_S^2|\boldsymbol{x}|}\right]$$

$$\times [1+O(|\boldsymbol{x}|^{-1})] \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{13.8-7}$$

with

$$v_r^{P,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t\Phi_r^{f;P,\infty}(\boldsymbol{\xi},t) + (\rho c_P)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi_{r,k,i,j}^{h;P,\infty}(\boldsymbol{\xi},t), \tag{13.8-8}$$

$$v_r^{S,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t\Phi_r^{f;S,\infty}(\boldsymbol{\xi},t) + (\rho c_S)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi_{r,k,i,j}^{h;S,\infty}(\boldsymbol{\xi},t), \tag{13.8-9}$$

and

$$[\tau_{p,q}(\boldsymbol{x},t)+\tau_{q,p}(\boldsymbol{x},t)]/2 = \left\{\frac{\left[\tau_{p,q}^{P,\infty}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c_P)+\tau_{q,p}^{P,\infty}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c_P)\right]/2}{4\pi c_P^2|\boldsymbol{x}|}\right.$$

$$\left. + \frac{\left[\tau_{p,q}^{S,\infty}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c_S)+\tau_{q,p}^{S,\infty}(\boldsymbol{\xi},t-|\boldsymbol{x}|/c_S)\right]/2}{4\pi c_S^2|\boldsymbol{x}|}\right\}[1+O(|\boldsymbol{x}|^{-1})] \quad \text{as } |\boldsymbol{x}|\to\infty, \tag{13.8-10}$$

with

$$\left[\tau_{p,q}^{P,\infty}(\boldsymbol{\xi},t)+\tau_{q,p}^{P,\infty}(\boldsymbol{\xi},t)\right]/2 = -(\rho c_P)^{-1}C_{p,q,n,r}\xi_n\partial_t\Phi_r^{f;P,\infty}(\boldsymbol{\xi},t)$$
$$-(\rho c_P^2)^{-1}C_{p,q,n,r}C_{k,m,i,j}\xi_n\xi_m\partial_t\Phi_{r,k,i,j}^{h;P,\infty}(\boldsymbol{\xi},t), \tag{13.8-11}$$

$$\left[\tau_{p,q}^{S,\infty}(\boldsymbol{\xi},t)+\tau_{q,p}^{S,\infty}(\boldsymbol{\xi},t)\right]/2 = -(\rho c_S)^{-1}C_{p,q,n,r}\xi_n\partial_t\Phi_r^{f;S,\infty}(\boldsymbol{\xi},t)$$
$$-(\rho c_S^2)^{-1}C_{p,q,n,r}C_{k,m,i,j}\xi_n\xi_m\partial_t\Phi_{r,k,i,j}^{h;S,\infty}(\boldsymbol{\xi},t). \tag{13.8-12}$$

As Equations (13.8-7)–(13.8-12) show, the particle velocity and the dynamic stress have, in the far-field region, the shape of the superposition of a spherical P-wave that expands radially with the P-wave speed and a spherical S-wave that expands radially with the S-wave speed, both away from the origin of the chosen reference frame (which is also denoted as the *time reference centre* of the far-field approximation), the latter being chosen in the neighbourhood of the radiating sources. The amplitudes of the waves decrease inversely proportionally to the distance from the origin, and their amplitude radiation characteristics $\{v_r^{P,\infty},\tau_{p,q}^{P,\infty}\}$ and $\{\tau_{p,q}^{S,\infty},v_r^{S,\infty}\}$ only depend on the direction of observation $\boldsymbol{\xi}$ and on the pulse shapes of the source distributions. Their dependence on the unit vector $\boldsymbol{\xi}$ is the resultant of the dependence of the integrals over $\mathcal{D}^T$ in the right-hand sides of Equations (13.8-2), (13.8-3) and (13.8-5), (13.8-6) on $\boldsymbol{\xi}$ and the purely directional characteristics, which for the P-wave particle velocity is $\xi_r\xi_k$ and for the S-wave particle velocity is $(\delta_{r,k}-\xi_r\xi_k)$. From Equations (13.8-2), (13.8-5) and (13.8-8) it follows that $v_r^{P,\infty} = \xi_r(\xi_k v_k^{P,\infty})$, i.e. the *P-wave particle velocity* is, in the far-field region, *longitudinal* with respect to the radial direction of propagation, while from Equations (13.8-3), (13.8-6) and (13.8-9) it follows that $\xi_r v_r^{S,\infty} = 0$, since $\xi_r(\delta_{r,k}-\xi_r\xi_k) = \xi_k - \xi_r\xi_r\xi_k = \xi_k - \xi_k = 0$, i.e. the *S-wave particle velocity* is, in the far-field region, *transverse* with respect to its radial direction of propagation. Note that in the right-hand sides of Equations (13.8-8), (13.8-9) and (13.8-11), (13.8-12) only the time-differentiated forms of the volume source densities occur.

The far-field amplitude radiation characteristics of the particle velocity and the dynamic stress are not independent of each other. First, from Equations (13.8-8), (13.8-9) and (13.8-11), (13.8-12) it is observed that

$$(\tau_{p,q}^{P,\infty} + \tau_{q,p}^{P,\infty})/2 = -c_P^{-1} C_{p,q,n,r} \xi_n v_r^{P,\infty} \tag{13.8-13}$$

and

$$(\tau_{p,q}^{S,\infty} + \tau_{q,p}^{S,\infty})/2 = -c_S^{-1} C_{p,q,n,r} \xi_n v_r^{S,\infty}, \tag{13.8-14}$$

in which

$$C_{p,q,n,r} = \lambda \delta_{p,q} \delta_{n,r} + \mu(\delta_{p,n}\delta_{q,r} + \delta_{p,r}\delta_{q,n}) \tag{13.8-15}$$

is the stiffness. By contracting the left-and right-hand sides of Equations (13.8-13) and (13.8-14) with the compliance $S_{i,j,p,q}$ it follows further that

$$S_{i,j,p,q} \tau_{p,q}^{P,\infty} = -c_P^{-1} \Delta_{i,j,n,r}^{+} \xi_n v_r^{P,\infty} \tag{13.8-16}$$

and

$$S_{i,j,p,q} \tau_{p,q}^{S,\infty} = -c_S^{-1} \Delta_{i,j,n,r}^{+} \xi_n v_r^{S,\infty}, \tag{13.8-17}$$

respectively. Equations (13.8-16) and (13.8-17) would have resulted if in the source-free deformation rate equation pertaining to the homogeneous, isotropic, perfectly elastic solid under consideration, expressions of the type

$$\{\tau_{p,q}, v_r\} = \{\tau_{p,q}^{\infty}, v_r^{\infty}\}(t - \xi_s x_s / c), \tag{13.8-18}$$

had been substituted for $c = c_P$ and $c = c_S$, respectively and the causal relation between this wave field and its sources (which are located elsewhere in space), which entails zero initial values in time, had been used. Wave fields of the type of Equation (13.8-18) are denoted as *uniform plane waves* that propagate in the direction of the unit vector ξ_s.

Next, Equations (13.8-13) and (13.8-14) are contracted with $\Delta_{k,m,p,q}^{+}\xi_m$ and use is made of Equation (13.8-15). The result is

$$t_k^{P,\infty} = -\rho c_P v_k^{P,\infty}, \tag{13.8-19}$$

where the relations $v_r^{P,\infty} = \xi_r(\xi_k v_k^{P,\infty})$ and $\lambda + 2\mu = \rho c_P^2$ have been used, and

$$t_k^{S,\infty} = -\rho c_S v_k^{S,\infty}, \tag{13.8-20}$$

where the relations $\xi_r v_r^{S,\infty} = 0$ and $\mu = \rho c_S^2$ have been used, respectively. In Equations (13.8-19) and (13.8-20)

$$t_k^{P,\infty} = \Delta_{k,m,p,q}^{+} \xi_m \tau_{p,q}^{P,\infty} \tag{13.8-21}$$

is the *radial P-wave traction* and

$$t_k^{S,\infty} = \Delta_{k,m,p,q}^{+} \xi_m \tau_{p,q}^{S,\infty} \tag{13.8-22}$$

is the *radial S-wave traction*. Equations of the type of Equations (13.8-19)–(13.8-22) would have resulted if in the source-free equation of motion pertaining to the homogeneous, isotropic, perfectly elastic solid under consideration, expressions of the type of Equation (13.8-18) had been substituted for $c = c_P$ and $c = c_S$, respectively, and causality had been used.

Observing that $|x| = \xi_s x_s$, we can therefore say that, after compensating for the (distance)$^{-1}$ decay, the spherical P-and S-wave amplitudes in the far-field region locally behave as if the waves were uniform plane waves travelling radially away from the source.

Equations (13.8-19) and (13.8-20) further show that in the far-field region the traction and the particle velocity of the P-and S-wave are proportional, with proportionality factors

$$Z_P = \rho c_P \tag{13.8-23}$$

and

$$Z_S = \rho c_S \,, \tag{13.8-24}$$

respectively. The quantity Z_P is denoted as the *elastodynamic plane P-wave impedance*, the quantity Z_S is denoted as the *elastodynamic plane S-wave impedance*.

Exercises

Exercise 13.8-1

Let $F = F(x,t)$ be a tensor function of arbitrary rank, defined over some subdomain $\mathcal{D}$ of $\mathcal{R}^3$ and for all $t \in \mathcal{R}$. Furthermore, let $\partial_t F(x,t) = 0$ for all $x \in \mathcal{D}$ and all $t \in \mathcal{R}$, while $F(x,t_0) = 0$ for all $x \in \mathcal{D}$. Show that also $F(x,t) = 0$ for all $x \in \mathcal{D}$ and $t > t_0$. (*Hint*: note that:

$$0 = \int_{t'=t_0}^{t} \partial'_t F(x,t')\,\mathrm{d}t' = F(x,t) - F(x,t_0).)$$

Exercise 13.8-2

Verify that Equations (13.8-8) and (13.8-9), and (13.8-11) and (13.8-22) indeed satisfy Equations (13.8-16) and (13.8-17), and (13.8-19)–(13.8-22).

13.9 The time evolution of an elastic wave field. The initial-value problem (Cauchy problem) for a homogeneous, isotropic, perfectly elastic solid

In this section a solution is presented for the initial-value problem (Cauchy problem) for elastic waves in a homogeneous, isotropic, perfectly elastic solid with volume density of mass ρ, P-wave speed c_P and S-wave speed c_S. From the given initial values $v_r(x,t_0)$ of the particle velocity and $\tau_{p,q}(x,t_0)$ of the dynamic stress in all space at the instant t_0, the values of $v_r = v_r(x,t)$ and $\tau_{p,q} = \tau_{p,q}(x,t)$ at all succeeding instants $t > t_0$ are to be constructed in case nowhere in the solid sources are active for $t \geqslant t_0$. Thus, we are looking for the pure time evolution for $t > t_0$ of the elastic wave field, given its values of the particle velocity and the dynamic stress at $t = t_0$. From Equations (12.1-3) and (12.1-4) we learn that this problem can be solved by transforming Equations (13.4-5)–(13.4-10) back to the time domain for the particular case where

$$\hat{f}_k = \rho v_k(\boldsymbol{x},t_0)\exp(-st_0) \tag{13.9-1}$$

and

$$\hat{h}_{i,j} = -S_{i,j,p,q}\tau_{p,q}(\boldsymbol{x},t_0)\exp(-st_0)\,, \tag{13.9-2}$$

with

$$S_{i,j,p,q} = \Lambda\delta_{i,j}\delta_{p,q} + M(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p})\,, \tag{13.9-3}$$

in which

$$\Lambda = -\lambda/(3\lambda + 2\mu)2\mu\,, \tag{13.9-4}$$

$$M = 1/4\mu\,, \tag{13.9-5}$$

$$\lambda = \rho(c_P^2 - 2c_S^2)\,, \tag{13.9-6}$$

$$\mu = \rho c_S^2\,. \tag{13.9-7}$$

Substitution of Equations (13.9-1), (13.9-2) and (13.4-7), (13.4-8) in Equations (13.4-5), (13.4-6) leads to the separation into P-wave and S-wave constituents according to

$$\hat{\Phi}_r^f = \hat{\Phi}_r^{f;P} + \hat{\Phi}_r^{f;S}\,, \tag{13.9-8}$$

and

$$\hat{\Phi}_{r,k,i,j}^h = \hat{\Phi}_{r,k,i,j}^{h;P} + \hat{\Phi}_{r,k,i,j}^{h;S}\,, \tag{13.9-9}$$

in which

$$\hat{\Phi}_r^{f;P}(\boldsymbol{x},s) = \rho s^{-2}\partial_r\partial_k \int_{\boldsymbol{x}'\in\mathcal{R}^3} f_k(\boldsymbol{x}',t_0)\frac{\exp(-s|\boldsymbol{x}-\boldsymbol{x}'|/c_P - st_0)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V\,, \tag{13.9-10}$$

$$\hat{\Phi}_r^{f;S}(\boldsymbol{x},s) = \rho(c_S^{-2}\delta_{r,k} - s^{-2}\partial_r\partial_k) \int_{\boldsymbol{x}'\in\mathcal{R}^3} f_k(\boldsymbol{x}',t_0)\frac{\exp(-s|\boldsymbol{x}-\boldsymbol{x}'|/c_S - st_0)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V\,, \tag{13.9-11}$$

and

$$\hat{\Phi}_{r,k,i,j}^{h;P}(\boldsymbol{x},s) = -S_{i,j,p,q}s^{-2}\partial_r\partial_k \int_{\boldsymbol{x}'\in\mathcal{R}^3} \tau_{p,q}(\boldsymbol{x}',t_0)\frac{\exp(-s|\boldsymbol{x}-\boldsymbol{x}'|/c_P - st_0)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V\,, \tag{13.9-12}$$

$$\hat{\Phi}_{r,k,i,j}^{h;S}(\boldsymbol{x},s)$$
$$= -S_{i,j,p,q}(c_S^{-2}\delta_{r,k} - s^{-2}\partial_r\partial_k) \int_{\boldsymbol{x}'\in\mathcal{R}^3} \tau_{p,q}(\boldsymbol{x}',t_0)\frac{\exp(-s|\boldsymbol{x}-\boldsymbol{x}'|/c_S - st_0)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V\,. \tag{13.9-13}$$

The integrals on the right-hand sides of Equations (13.9-10)–(13.9-13) will be rewritten such that their time-domain counterparts can be obtained by inspection. This is accomplished by introducing spherical polar coordinates about the observation point $\boldsymbol{x}$ as the variables of integration. Consider, to this end, the generic expression

$$\hat{\Phi}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{R}^3} q(\boldsymbol{x}',t_0)\frac{\exp(-s|\boldsymbol{x}-\boldsymbol{x}'|/c - st_0)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V\,. \tag{13.9-14}$$

In the right-hand side, $c(\tau - t_0)$, with $\tau \geqslant t_0$, is now taken as the radial variable of integration and the unit vector $\boldsymbol{\theta}$, with $\boldsymbol{\theta} \in \Omega$, where Ω denotes the sphere of unit radius, as the angular variable of integration. Then,

$$\boldsymbol{x}' = \boldsymbol{x} + c(\tau - t_0)\boldsymbol{\theta} , \tag{13.9-15}$$

and, since $\boldsymbol{\theta} \cdot \boldsymbol{\theta} = 1$,

$$|\boldsymbol{x} - \boldsymbol{x}'| = c(\tau - t_0) , \tag{13.9-16}$$

and hence

$$|\boldsymbol{x} - \boldsymbol{x}'|/c + t_0 = \tau , \tag{13.9-17}$$

while

$$\mathrm{d}V = c^3(\tau - t_0)^2 \, \mathrm{d}\tau \, \mathrm{d}\Omega , \tag{13.9-18}$$

where $\mathrm{d}\Omega$ is the elementary area on Ω. With this, Equation (13.9-14) is rewritten as

$$\hat{\Phi}(\boldsymbol{x},s) = \int_{\tau=t_0}^{\infty} \exp(-s\tau)c^2(\tau - t_0)\langle q(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]} \, \mathrm{d}\tau , \tag{13.9-19}$$

in which

$$\langle q(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]} = \frac{1}{4\pi} \int_{\boldsymbol{\theta} \in \Omega} q[\boldsymbol{x} + c(\tau - t_0)\boldsymbol{\theta},t_0] \, \mathrm{d}\Omega \tag{13.9-20}$$

denotes the spherical mean over the sphere $\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]$ with centre at $\boldsymbol{x}$ and radius $c(\tau - t_0)$. Now, the right-hand side of Equation (13.9-19) has the form of the Laplace transformation of a causal function of time whose support is $\{t \in \mathcal{R}; t > t_0\}$. In view of the uniqueness of the Laplace transformation with real, positive transform parameter (see Section B.1), the time-domain counterpart $\Phi(\boldsymbol{x},t)$ of $\hat{\Phi}(\boldsymbol{x},s)$ is given by

$$\Phi(\boldsymbol{x},t) = c^2(t - t_0)\langle q(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c(t - t_0)]} \quad \text{for} \quad t \geqslant t_0 . \tag{13.9-21}$$

Using this generic result and some standard rules of the time Laplace transformation, the time-domain counterparts of Equations (13.9-8)–(13.9-13) are obtained as

$$\Phi_r^{f;P}(\boldsymbol{x},t) = \rho\partial_r\partial_k \mathrm{I}_t^2 \left\{c_P^2(t - t_0)\langle v_k(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_P(t - t_0)]}\right\} \quad \text{for} \quad t \geqslant t_0 , \tag{13.9-22}$$

$$\begin{aligned} \Phi_r^{f;S}(\boldsymbol{x},t) = {} & \rho(t - t_0)\langle v_r(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_S(t - t_0)]} \\ & - \rho\partial_r\partial_k \mathrm{I}_t^2 \left\{c_S^2(t - t_0)\langle v_k(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_S(t - t_0)]}\right\} \text{for} \quad t \geqslant t_0 , \end{aligned} \tag{13.9-23}$$

and

$$\Phi_{r,k,i,j}^{h;P}(\boldsymbol{x},t) = -S_{i,j,p,q}\partial_r\partial_k \mathrm{I}_t^2 \left\{c_P^2(t - t_0)\langle \tau_{p,q}(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_P(t - t_0)]}\right\} \quad \text{for} \quad t \geqslant t_0 , \tag{13.9-24}$$

$$\begin{aligned} \Phi_{r,k,i,j}^{h;S}(\boldsymbol{x},t) = {} & -S_{i,j,p,q}(t - t_0)\langle \tau_{p,q}(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_S(t - t_0)]} \\ & + S_{i,j,p,q} \mathrm{I}_t^2 \left\{c_S^2(t - t_0)\langle \tau_{p,q}(\boldsymbol{x}',t_0)\rangle_{\mathcal{S}[\boldsymbol{x},c_S(t - t_0)]}\right\} \quad \text{for} \quad t \geqslant t_0 . \end{aligned} \tag{13.9-25}$$

Here, I_t^2 denotes the double time integration operator or, alternatively, the convolution with $tH(t)$. In terms of these source potentials the expressions for the particle velocity and the dynamic stress follow from Equations (13.5-1) and (13.5-2) as

$$v_r(\boldsymbol{x},t) = \rho^{-1}\partial_t \Phi_r^f(\boldsymbol{x},t) - \rho^{-1} C_{k,m,i,j}\partial_m \Phi_{r,k,i,j}^h(\boldsymbol{x},t) , \tag{13.9-26}$$

and

$$[\tau_{p,q}(\boldsymbol{x},t) + \tau_{q,p}(\boldsymbol{x},t)]/2 = [\tau_{p,q}(\boldsymbol{x},t_0) + \tau_{q,p}(\boldsymbol{x},t_0)]/2 + \rho^{-1} C_{p,q,n,r}\partial_n \Phi_r^f(\boldsymbol{x},t)$$
$$- \rho^{-1} C_{p,q,n,r} C_{k,m,i,j}\partial_n\partial_m \int_{t'=t_0}^{t} \Phi_{r,k,i,j}^h(\boldsymbol{x},t')\,\mathrm{d}t' , \tag{13.9-27}$$

with

$$\Phi_r^f = \Phi_r^{f;P} + \Phi_r^{f;S} , \tag{13.9-28}$$

$$\Phi_{r,k,i,j}^h = \Phi_{r,k,i,j}^{h;P} + \Phi_{r,k,i,j}^{h;S} . \tag{13.9-29}$$

Exercises

Exercise 13.9-1

Construct the solution to the initial-value problem (Cauchy problem) of the three-dimensional scalar wave equation

$$\partial_m\partial_m u - c^{-2}\partial_t^2 u = 0 \tag{13.9-30}$$

for $t > t_0$ if $u(\boldsymbol{x},t_0) = u_0(\boldsymbol{x})$ and $\partial_t u(\boldsymbol{x},t_0) = v_0(\boldsymbol{x})$.
(a) Take the time Laplace transform of Equation (13.9-30) over the interval $t_0 < t < \infty$ and show that

$$\partial_m\partial_m \hat{u} - (s^2/c^2)\hat{u} = -c^{-2} v_0(\boldsymbol{x}) \exp(-st_0) - c^{-2} s u_0(\boldsymbol{x}) \exp(-st_0) . \tag{13.9-31}$$

(b) Show that the solution to Equation (13.9-31) is given by

$$\hat{u}(\boldsymbol{x},s) = \int_{\boldsymbol{x}'\in\mathcal{R}^3} \hat{q}(\boldsymbol{x}')\,\frac{\exp[-s|\boldsymbol{x}-\boldsymbol{x}'|/c - st_0]}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V , \tag{13.9-32}$$

in which

$$\hat{q}(\boldsymbol{x}') = c^{-2}[v_0(\boldsymbol{x}') + s u_0(\boldsymbol{x}')] . \tag{13.9-33}$$

(c) Introduce spherical polar coordinates about the observation point $\boldsymbol{x}$ as the variables of integration and show that

$$\hat{u}(\boldsymbol{x},s) = \int_{\tau=t_0}^{\infty} \exp(-s\tau)(\tau - t_0)\langle v_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]}\,\mathrm{d}\tau$$
$$+ s\int_{\tau=t_0}^{\infty} \exp(-s\tau)(\tau - t_0)\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]}\,\mathrm{d}\tau , \tag{13.9-34}$$

in which

$$\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]} = \frac{1}{4\pi}\int_{\boldsymbol{\theta}\in\Omega} u_0[\boldsymbol{x} + c(\tau - t_0)\boldsymbol{\theta}]\,\mathrm{d}\Omega \tag{13.9-35}$$

is the spherical mean over the sphere $\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]$ with centre at $\boldsymbol{x}$ and radius $c(\tau - t_0)$.

(d) Use the uniqueness of the time Laplace transformation to show that

$$u(\boldsymbol{x},t) = (t - t_0)\langle v_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(\tau - t_0)]} + \partial_t\left\{(t - t_0)\langle u_0(\boldsymbol{x}')\rangle_{\mathcal{S}[\boldsymbol{x},c(t - t_0)]}\right\} \quad \text{for} \quad t \geqslant t_0\,. \tag{13.9-36}$$

Equation (13.9-36) is Poisson's solution to the initial-value problem of the three-dimensional scalar wave equation.