14

Plane elastic waves in homogeneous solids

In this chapter the notion of plane elastic wave, which has turned up in the local description of the elastic wave field in the far-field region of the elastodynamic radiation by extended sources, is generalised to cases where the wave amplitudes are arbitrary functions of the angular wave vector, not specifically those of the type that occurs in the far-field approximation. The concepts of dispersion equation and wave slowness are introduced for homogeneous, arbitrarily anisotropic solids, with the homogeneous, isotropic solid as a special case. For the real frequency domain the attenuation and the phase propagation of plane waves are discussed.

14.1 Plane waves in the complex frequency domain

In the complex frequency domain *plane waves* are solutions of the complex frequency-domain source-free elastic wave equations (cf. Equations (12.4-1) and (12.4-2))

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}_{p,q} + \hat{\zeta}_{k,r}\hat{v}_r = 0\,, \tag{14.1-1}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}_r - \hat{\eta}_{i,j,p,q}\hat{\tau}_{p,q} = 0\,, \tag{14.1-2}$$

of the form

$$\{\hat{v}_r,\hat{\tau}_{p,q}\} = \{\hat{V}_r,\hat{T}_{p,q}\}\exp(-\hat{\gamma}_s x_s)\,, \tag{14.1-3}$$

in which the *amplitudes* $\{\hat{V}_r,\hat{T}_{p,q}\}$ and the *propagation vector* $\hat{\gamma}_s$ are independent of $\boldsymbol{x}$. The factor $\exp(-\hat{\gamma}_s x_s)$ is called the *propagation factor.* Substitution of Equation (14.1-3) in Equations (14.1-1) and (14.1-2) leads, in view of the relation

$$\partial_m[\exp(-\hat{\gamma}_s x_s)] = -\hat{\gamma}_m\exp(-\hat{\gamma}_s x_s)\,, \tag{14.1-4}$$

to

$$\Delta^{+}_{k,m,p,q}\hat{\gamma}_m\hat{T}_{p,q} + \hat{\zeta}_{k,r}\hat{V}_r = 0\,, \tag{14.1-5}$$

$$-\Delta^{+}_{i,j,n,r}\hat{\gamma}_n\hat{V}_r - \hat{\eta}_{i,j,p,q}\hat{T}_{p,q} = 0\,. \tag{14.1-6}$$

Equations (14.1-5) and (14.1-6) constitute a homogeneous system of linear algebraic equations in $\hat{V}_r$ and $\hat{T}_{p,q}$, which for arbitrary values of $\hat{\gamma}_s$ has only the trivial solution $\hat{V}_r = 0$ and $\hat{T}_{p,q} = 0$.

For a non-trivial solution to exist, $\hat{\gamma}_s$ must be chosen appropriately. The condition to be put on $\hat{\gamma}_s$ in this respect could be found by setting equal to zero the determinant of the system of linear algebraic equations (Equations (14.1-5) and (14.1-6)), but a less complicated way to find the relevant relation is to solve $\hat{T}_{p,q}$ from Equation (14.1-5) and substitute the result in Equation (14.1-6). This yields

$$\hat{T}_{p,q} = -\hat{\eta}^{-1}_{p,q,i,j}\Delta^{+}_{i,j,n,r}\hat{\gamma}_n\hat{V}_r \tag{14.1-7}$$

and

$$(\hat{\gamma}_m\hat{\eta}^{-1}_{k,m,n,r}\hat{\gamma}_n - \hat{\zeta}_{k,r})\hat{V}_r = 0 \,, \tag{14.1-8}$$

respectively, where the property $\Delta^{+}_{k,m,p,q}\hat{\eta}^{-1}_{p,q,i,j}\Delta^{+}_{i,j,n,r} = \hat{\eta}^{-1}_{k,m,n,r}$ has been used. For a non-zero solution of $\hat{V}_r$ to the system of Equations (14.1-8) to exist, $\hat{\gamma}_s$ must satisfy the determinantal equation

$$\det(\hat{\gamma}_m\hat{\eta}^{-1}_{k,m,n,r}\hat{\gamma}_n - \hat{\zeta}_{k,r}) = 0 \,. \tag{14.1-9}$$

Equation (14.1-9) is known as the complex frequency-domain plane wave *dispersion equation* for the propagation vector. Once a value for $\hat{\gamma}_s$ satisfying Equation (14.1-9) has been chosen, one is free to choose one relation between the three components of $\hat{V}_r$ (normalisation condition), while the corresponding value of $\hat{T}_{p,q}$ subsequently follows from Equation (14.1-7). For arbitrary values of $\hat{\gamma}_s$, satisfying Equation (14.1-9), the resulting expressions of the type of Equation (14.1-3) are known as *non-uniform plane waves.*

Equation (14.1-9) shows that only the symmetric part in the first and second pair of subscripts of $\hat{\eta}^{-1}_{k,m,n,r}$ contributes to the admissible values of $\hat{\gamma}_s$ and hence to the propagation properties of the plane wave. Equation (14.1-9) further shows that with any value of $\hat{\gamma}_s$ satisfying the equation, also $-\hat{\gamma}_s$ is a solution. This property reduces the solution space in which admissible values of $\hat{\gamma}_s$ are to be sought.

Uniform plane waves

Uniform plane waves are a subset of the general class of non-uniform plane waves. For a *uniform plane wave* the propagation vector is of the special shape

$$\hat{\gamma}_s = \hat{\gamma}\xi_s \,, \tag{14.1-10}$$

where ξ_s is a real unit vector that specifies the *direction of propagation* of the wave. (Since ξ_s is a unit vector, we have $\xi_s\xi_s = 1$.) Now, $\hat{\gamma}$ is the (scalar) *propagation coefficient* of the uniform plane wave. Substitution of Equation (14.1-10) in Equation (14.1-9) yields

$$\det(\xi_m\hat{\eta}^{-1}_{k,m,n,r}\xi_n\hat{\gamma}^2 - \hat{\zeta}_{k,r}) = 0 \tag{14.1-11}$$

as the *dispersion equation for uniform plane waves.* Causality of the wave motion entails the condition that $\exp(-\hat{\gamma}\xi_s x_s)$ should remain bounded as $|x|\to\infty$ in the half-space where $\xi_s x_s > 0$. This yields the condition $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$. Equation (14.1-11) is a cubic equation in γ^2. Hence, for each given direction of propagation, three plane waves exist (together with the three waves propagating in the reverse direction). Equation (14.1-11) further clearly shows that the value of the propagation coefficients changes with the direction of propagation of the uniform plane wave, a property that is indicative of the *presence of anisotropy.*

Isotropic solids

For an isotropic solid we have

$$\hat{\zeta}_{k,r} = s\hat{\mu}(s)\delta_{k,r} \tag{14.1-12}$$

and

$$\hat{\eta}_{i,j,p,q} = s\left[\hat{\chi}^{\Lambda}(s)\delta_{i,j}\delta_{p,q} + \hat{\chi}^{M}(s)(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p})\right]. \tag{14.1-13}$$

The inverse of Equation (14.1-13) we write as

$$\hat{\eta}^{-1}_{p,q,i,j} = s^{-1}\left[\hat{\chi}^{\lambda}(s)\delta_{p,q}\delta_{i,j} + \hat{\chi}^{\mu}(s)(\delta_{p,i}\delta_{q,j} + \delta_{p,j}\delta_{q,i})\right], \tag{14.1-14}$$

where $\hat{\chi}^{\lambda} = \hat{\chi}^{\lambda}(s)$ and $\hat{\chi}^{\mu} = \hat{\chi}^{\mu}(s)$ are the Lamé coefficient relaxation functions. Hence, Equation (14.1-7) reduces to

$$\hat{T}_{p,q} = -s^{-1}\left[\hat{\chi}^{\lambda}(\hat{\gamma}_i\hat{V}_i)\delta_{p,q} + \hat{\chi}^{\mu}(\hat{\gamma}_p\hat{V}_q + \hat{\gamma}_q\hat{V}_p)\right]. \tag{14.1-15}$$

Substitution of Equations (14.1-15) and (14.1-12) in Equation (14.1-5) yields

$$(\hat{\chi}^{\lambda} + \hat{\chi}^{\mu})\hat{\gamma}_k(\hat{\gamma}_i\hat{V}_i) + \hat{\chi}^{\mu}(\hat{\gamma}_i\hat{\gamma}_i)\hat{V}_k - s^2\hat{\mu}\hat{V}_k = 0\,. \tag{14.1-16}$$

Upon contracting Equation (14.1-16) with $\hat{\gamma}_k$ it follows that

$$\left[(\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu})(\hat{\gamma}_i\hat{\gamma}_i) - s\hat{\mu}^2\right](\hat{\gamma}_k\hat{V}_k) = 0\,. \tag{14.1-17}$$

The first possibility to satisfy Equation (14.1-17) is

$$\hat{\gamma}_i\hat{\gamma}_i = s^2\frac{\hat{\mu}}{\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu}}, \tag{14.1-18}$$

combined with $\hat{\gamma}_k\hat{V}_k \neq 0$. Substitution of Equation (14.1-18) in Equation (14.1-16) then yields

$$(\hat{\gamma}_i\hat{\gamma}_i)\hat{V}_k = \hat{\gamma}_k(\hat{\gamma}_i\hat{V}_i)\,. \tag{14.1-19}$$

Since for this case $\hat{V}_k$ is "oriented" along $\hat{\gamma}_k$, this wave is obviously a (non-uniform) plane *P*-wave.

The second possibility to satisfy Equation (14.1-17) is

$$\hat{\gamma}_k\hat{V}_k = 0\,, \tag{14.1-20}$$

combined with $\hat{\gamma}_i\hat{\gamma}_i \neq s^2\left[\hat{\mu}/(\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu})\right]$. Substitution of Equation (14.1-20) in Equation (14.1-16) leads to

$$\hat{\gamma}_i\hat{\gamma}_i = s^2\frac{\hat{\mu}}{\hat{\chi}^{\mu}}\,. \tag{14.1-21}$$

Since for this case $\hat{\gamma}_k\hat{V}_k = 0$, this wave is obviously a (non-uniform) plane *S*-wave. (Note that if Equation (14.1-21) holds, we certainly have $\hat{\gamma}_i\hat{\gamma}_i \neq s^2\left[\hat{\mu}/(\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu})\right]$.)

Exercises

Exercise 14.1-1

Construct the one-dimensional P-wave solutions of the source-free complex frequency-domain elastic wave-field equations in a homogeneous, isotropic solid by taking a Euclidean reference frame such that the propagation takes place along the x_3-direction. What is the propagation factor for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3? Express $\hat{T}_{p,q}$ in terms of the non-vanishing component of $\hat{V}_r$.

Answer:

(a) propagation factor = $\exp(-\hat{\gamma}_P x_3)$,

$$\hat{T}_{1,1} = \hat{T}_{2,2} = -s^{-1}\hat{\chi}^{\lambda}(\hat{\gamma}_P\hat{V}_3),\ \hat{T}_{3,3} = -s^{-1}(\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu})(\hat{\gamma}_P\hat{V}_3),\ \hat{T}_{1,2} = \hat{T}_{2,3} = \hat{T}_{3,1} = 0\,;$$

(b) propagation factor = $\exp(\hat{\gamma}_P x_3)$,

$$\hat{T}_{1,1} = \hat{T}_{2,2}, = s^{-1}\hat{\chi}^{\lambda}(\hat{\gamma}_P\hat{V}_3),\ \hat{T}_{3,3} = s^{-1}(\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu})(\hat{\gamma}_P\hat{V}_3),\ \hat{T}_{1,2} = \hat{T}_{2,3} = \hat{T}_{3,1} = 0\,.$$

Here, $\hat{\gamma}_P = s\left[\hat{\mu}/(\hat{\chi}^{\lambda} + 2\hat{\chi}^{\mu})\right]^{1/2}$, and $\mathrm{Re}(\hat{\gamma}_P) > 0$ for $\mathrm{Re}(s) > 0$.

Exercise 14.1-2

Construct the one-dimensional S-wave solutions of the source-free complex frequency-domain elastic wave-field equations in a homogeneous, isotropic solid by taking a Euclidean reference frame such that the propagation takes place along the x_3-direction. What is the propagation factor for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3? Express $\hat{T}_{p,q}$ in terms of the non-vanishing components of $\hat{V}_r$. Consider the cases $\hat{V}_1 \neq 0$ and $\hat{V}_2 \neq 0$ separately.

Answer:

(a) propagation factor = $\exp(-\hat{\gamma}_S x_3)$,

$$\hat{T}_{1,1} = \hat{T}_{2,2} = \hat{T}_{3,3} = 0,\ \hat{T}_{1,2} = \hat{T}_{2,3} = 0,\ \hat{T}_{3,1} = -s^{-1}\hat{\chi}^{\mu}(\hat{\gamma}_S\hat{V}_1),$$

or $\hat{T}_{1,1} = \hat{T}_{2,2} = \hat{T}_{3,3} = 0,\ \hat{T}_{1,2} = 0,\ \hat{T}_{2,3} = -s^{-1}\hat{\chi}^{\mu}(\hat{\gamma}_S\hat{V}_2),\ \hat{T}_{3,1} = 0\,;$

(b) propagation factor = $\exp(\hat{\gamma}_S x_3)$,

$$\hat{T}_{1,1} = \hat{T}_{2,2} = \hat{T}_{3,3} = 0,\ \hat{T}_{1,2} = \hat{T}_{2,3} = 0,\ \hat{T}_{3,1} = s^{-1}\hat{\chi}^{\mu}(\hat{\gamma}_S\hat{V}_1),$$

or $\hat{T}_{1,1} = \hat{T}_{2,2} = \hat{T}_{3,3} = 0,\ \hat{T}_{1,2} = 0,\ \hat{T}_{2,3} = s^{-1}\hat{\chi}^{\mu}(\hat{\gamma}_S\hat{V}_2),\ \hat{T}_{3,1} = 0\,.$

Here, $\hat{\gamma}_S = s(\hat{\mu}/(\hat{\chi}^{\mu})^{1/2}$, and $\mathrm{Re}(\hat{\gamma}_S) > 0$ for $\mathrm{Re}(s) > 0$.

14.2 Plane waves in lossless solids; the slowness surface

In a lossless solid the complex frequency-domain longitudinal elastodynamic impedance per length $\hat{\zeta}_{k,r}$ and the transverse elastodynamic admittance per length $\hat{\eta}_{i,j,p,q}$ reduce to

$$\hat{\zeta}_{k,r} = s\rho_{k,r} \qquad (14.2\text{-}1)$$

and

$$\hat{\eta}_{i,j,p,q} = sS_{i,j,p,q} , \qquad (14.2\text{-}2)$$

respectively, in which $\rho_{k,r}$ and $S_{i,j,p,q}$ are independent of s. Under these circumstances the complex propagation vector $\hat{\gamma}_s$ is written as

$$\hat{\gamma}_s = s\Lambda_s , \qquad (14.2\text{-}3)$$

in which Λ_2 is the *slowness vector*. Substitution of Equations (14.2-1)-(14.2-3) in the dispersion equation (14.1-9) leads to

$$\det(\Lambda_m C_{k,m,n,r}\Lambda_n - \rho_{r,k}) = 0 , \qquad (14.2\text{-}4)$$

which is the equation to be satisfied by the slowness vector. In this equation $C_{k,m,n,r}$ is the stiffness. Note that, although Equation (14.2-4) is independent of s and $C_{k,m,n,r}$ and ρ are real-valued, Λ_s can still be complex-valued.

Uniform plane waves

For a uniform plane wave in a lossless solid the propagation vector is written as

$$\hat{\gamma}_s = s\Lambda\xi_s , \qquad (14.2\text{-}5)$$

in which ξ_s is the (real) unit vector in the direction of propagation of the plane wave (note that $\xi_s\xi_s = 1$) and Λ is the *scalar slowness.*

Substitution of the corresponding

$$\Lambda_s = \Lambda\xi_s \qquad (14.2\text{-}6)$$

in Equation (14.2-4) yields

$$\det(\xi_m C_{k,m,n,r}\xi_n\Lambda^2 - \rho_{k,r}) = 0 , \qquad (14.2\text{-}7)$$

which is a cubic algebraic equation in Λ^2, which, in general, has three distinct roots.

In the three-dimensional Euclidean *slowness space* where $\Lambda_s = \Lambda\xi_s$ is the position vector, Equation (14.2-7) defines a surface known as the *slowness surface*. For the class of lossless solids, the slowness surface characterises geometrically the propagation properties of uniform plane waves. In particular, the shape of the slowness surface is indicative of the presence of anisotropy in the elastic and/or inertia properties of the solid. As Equation (14.2-7) shows, the slowness surface for an anisotropic, lossless solid consists, in general, of three sheets.

Isotropic solids

For an isotropic, lossless solid we have

$$\rho_{k,r} = \rho\delta_{k,r} \qquad (14.2\text{-}8)$$

and

$$C_{i,j,p,q} = \lambda\delta_{i,j}\delta_{p,q} + \mu(\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p}) , \qquad (14.2\text{-}9)$$

where λ and μ are the Lamé coefficients. With this, Equation (14.1-15) reduces to

$$\hat{T}_{p,q} = -\left[\lambda(\Lambda_i\hat{V}_i)\delta_{p,q} + \mu(\Lambda_p\hat{V}_q + \Lambda_q\hat{V}_p)\right]. \tag{14.2-10}$$

Substitution of Equations (14.2-10) and (14.2-8) in Equation (14.1-5) yields

$$(\lambda+\mu)\Lambda_k(\Lambda_i\hat{V}_i) + \mu(\Lambda_i\Lambda_i)\hat{V}_k - \rho\hat{V}_k = 0\,. \tag{14.2-11}$$

Upon contracting Equation (14.2-11) with Λ_k it follows that

$$[(\lambda+2\mu)(\Lambda_i\Lambda_i) - \rho]\,\Lambda_k\hat{V}_k = 0\,. \tag{14.2-12}$$

The first possibility to satisfy Equation (14.2-12) is

$$\Lambda_i\Lambda_i = \rho/(\lambda+2\mu) = c_P^{-2}\,, \tag{14.2-13}$$

combined with $\Lambda_k\hat{V}_k \neq 0$. Substitution of Equation (14.2-13) in Equation (14.2-11) yields

$$\hat{V}_k = c_P^{-2}\Lambda_k(\Lambda_i\hat{V}_i)\,. \tag{14.2-14}$$

This wave is a (non-uniform) plane *P*-wave.

The second possibility to satisfy Equation (14.2-12) is

$$\Lambda_k\hat{V}_k = 0\,, \tag{14.2-15}$$

combined with $\Lambda_i\Lambda_i \neq \rho/(\lambda+2\mu)$. Substitution of Equation (14.2-15) in Equation (14.2-11) leads to

$$\Lambda_i\Lambda_i = \rho/\mu = c_S^{-2}\,. \tag{14.2-16}$$

This wave is a (non-uniform) plane *S*-wave. (Note that if Equation (14.2-16) holds, we certainly have $\Lambda_i\Lambda_i \neq \rho/(\lambda+2\mu)$.)

Uniform plane waves

For a uniform plane wave in an isotropic, lossles solid we have

$$\Lambda_s = \Lambda\xi_s\,, \tag{14.2-17}$$

where ξ_s is a real unit vector. In this case, Equation (14.2-13) reduces, since $\xi_i\xi_i = 1$, to

$$\Lambda^2 = \rho/(\lambda+2\mu) = c_P^{-2}\,, \tag{14.2-18}$$

or

$$\Lambda = [\rho/(\lambda+2\mu)]^{1/2} = c_P^{-1}\,, \tag{14.2-19}$$

and Equation (14.2-14) to

$$\hat{V}_k = \xi_k(\xi_i\hat{V}_i)\,. \tag{14.2-20}$$

Equation (14.2-20) shows that a uniform plane *P*-wave is *longitudinal* (with respect to its direction of propagation).

Furthermore, Equation (14.2-16) reduces, since $\xi_i\xi_i = 1$, to

$$\Lambda^2 = \rho/\mu = c_S^{-2}\,, \tag{14.2-21}$$

or

$$\Lambda = (\rho/\mu)^{1/2} = c_S^{-1}\,, \tag{14.2-22}$$

and Equation (14.2-15) to

$$\xi_k \hat{V}_k = 0\,. \tag{14.2-23}$$

Equation (14.2-23) shows that a uniform plane S-wave is *transverse* (with respect to its direction of propagation).

As Equations (14.2-18) and (14.2-21) show, the slowness surface for uniform plane waves in an isotropic, lossless solid consists of two concentric spheres with radii c_P^{-1} and c_S^{-1}, respectively. The relevant values could also have been obtained by solving the determinantal equation (14.2-7) after substituting Equations (14.2-8) and (14.2-9). After some algebraic manipulations we then arrive at

$$\left[(\lambda + 2\mu)\Lambda^2 - \rho\right](\mu\Lambda^2 - \rho)^2 = 0 \tag{14.2-24}$$

as the equation for the slowness surface. This equation shows that the sphere with radius c_S^{-1} is, in fact, a double sheet of the slowness surface (Figure 14.2-1).

Exercises

Exercise 14.2-1

Let $\{O; \Lambda_1, \Lambda_2, \Lambda_3\}$ be an orthogonal Cartesian reference frame in three-dimensional slowness space. Use Equations (14.2-7), (14.2-8) and (14.2-9) to construct the equation for the (three-sheeted) slowness surface of a homogeneous, isotropic, lossless elastic solid.

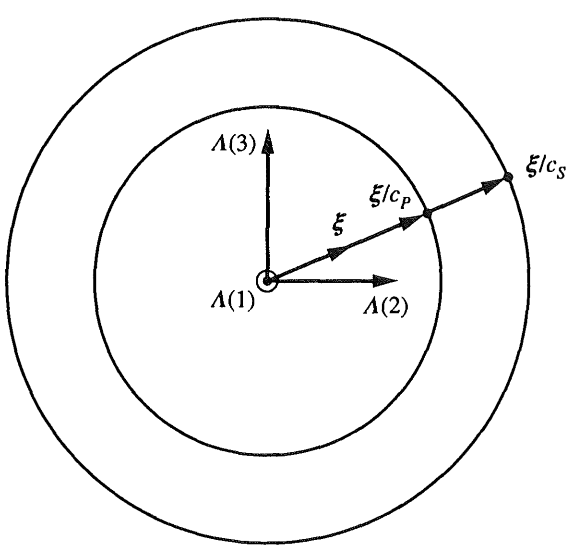

Figure 14.2-1 Slowness surface (two concentric spheres) for uniform plane waves in an isotropic, lossless solid.

Answer:

$$A_1^2 + A_2^2 + A_3^2 = \rho/(\lambda + 2\mu) = c_P^{-2} , \tag{14.2-25}$$

together with

$$A_1^2 + A_2^2 + A_3^2 = \rho/\mu = c_S^{-2} \text{ (double).} \tag{14.2-26}$$

14.3 Plane waves in the real frequency domain; attenuation vector and phase vector

In the signal processing of elastic wave phenomena extensive use is made of the highly efficient Fast-Fourier-Transform (FFT) algorithms that apply to the imaginary values $s = \mathrm{j}\omega$ (j = imaginary unit, ω = (real) angular frequency) of the complex frequency s to transform wave-field quantities from the time domain to the complex frequency domain, and vice versa. As a consequence, the corresponding imaginary values of s are of particular interest. Now, for imaginary values of s, the condition of causality can no longer be easily invoked on the frequency-domain wave quantities. To control the causality one must always consider the imaginary values of s as the limiting ones upon approaching, in the complex s plane, the imaginary axis via the right half $\mathrm{Re}(s) > 0$ of the complex s plane. For $s = \mathrm{j}\omega$ it is customary to decompose the complex propagation vector $\hat{\gamma}_s = \hat{\gamma}_s(\mathrm{j}\omega)$ into its real and imaginary parts according to

$$\hat{\gamma}_s(\mathrm{j}\omega) = \alpha_s(\omega) + \mathrm{j}\beta_s(\omega) , \tag{14.3-1}$$

where α_s = attenuation vector (SI-unit: neper/meter = Np/m), β_s = phase vector (SI-unit: radian/meter = rad/m). In view of the property

$$|\exp(-\hat{\gamma}_s x_s)| = \exp(-\alpha_s x_s) , \tag{14.3-2}$$

which holds since $|\exp(-\mathrm{j}\beta_s x_s)| = 1$, the family of planes $\{\alpha\in\mathcal{R}^3, x\in\mathcal{R}^3; \alpha_s x_s = \text{constant}\}$ defines a set of *planes of equal amplitude*, while in view of the property

$$\arg[\exp(-\hat{\gamma}_s x_s)] = -\beta_s x_s , \tag{14.3-3}$$

which holds since $\arg[\exp(-\alpha_s x_s)] = 0$, the family of planes $\{\beta\in\mathcal{R}^3, x\in\mathcal{R}^3; \beta_s x_s = \text{constant}\}$ defines a set of *planes of equal phase*. These two properties elucidate the term "plane wave" for complex frequency-domain solutions of the elastodynamic wave equations of the type given in Equation (14.1-3).

Uniform plane waves

For a uniform plane wave propagating in the direction of the unit vector ξ_s we have (see Equations (14.1-10) and (14.3-1))

$$\alpha_s = \alpha\xi_s \tag{14.3-4}$$

and

$$\beta_s = \beta\xi_s , \tag{14.3-5}$$

where

α is the (scalar) attenuation coefficient (Np/m),

β is the (scalar) phase coefficient (rad/m)

and

$$\hat{\gamma}(j\omega) = \alpha(\omega) + j\beta(\omega) . \tag{14.3-6}$$

For uniform plane waves, the set of planes of equal amplitude coincides with the set of planes of equal phase (Figure 14.3-1).

Propagation in a lossless, isotropic solid

As an illustrative example we shall discuss the propagation of plane waves in a lossless, isotropic solid. For such a solid, it follows from Equations (14.2-3), (14.2-13) and (14.2-16) that

$$(\alpha_s + j\beta_s)(\alpha_s + j\beta_s) = -\omega^2/c_P^2 \quad \text{for plane } P\text{-waves} \tag{14.3-7}$$

and

$$(\alpha_s + j\beta_s)(\alpha_s + j\beta_s) = -\omega^2/c_S^2 \quad \text{for plane } S\text{-waves.} \tag{14.3-8}$$

Since the right-hand sides of Equations (14.3-7) and (14.3-8) are real-valued, a separation of these equations into real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2/c_P^2 \quad \text{and} \quad \alpha_s\beta_s = 0 \quad \text{for plane } P\text{-waves} \tag{14.3-9}$$

and

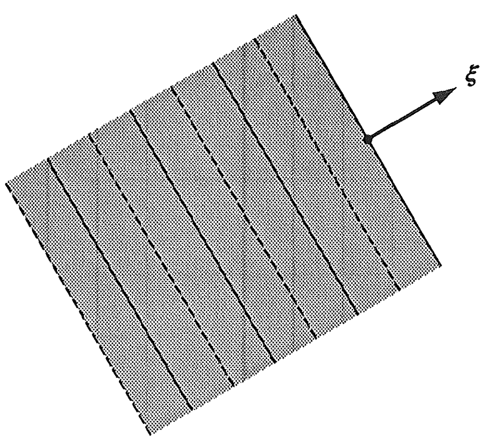

Figure 14.3-1 Planes of equal amplitude (——) and planes of equal phase (– – –) of a uniform plane *P*- or *S*-wave in the real frequency domain.

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2/c_S^2 \quad \text{and} \quad \alpha_s\beta_s = 0 \qquad \text{for plane } S\text{-waves.} \tag{14.3-10}$$

Equation (14.3-9) admits solutions for which $\alpha_s \neq 0$ and $\beta_s \neq 0$, subject to the condition $\alpha_s\beta_s = 0$. These solutions correspond to *non-uniform plane P-waves*, for which the propagation vector and the attenuation vector are mutually perpendicular, while the magnitude of the propagation vector is greater than the magnitude of the attenuation vector. The planes of equal amplitudes are in this case perpendicular to the planes of equal phase (Figure 14.3-2).

Equation (14.3-9) also admits solutions for which $\alpha_s = 0$ and $\beta_s\beta_s = \omega^2/c_P^2$. In this case we can write $\beta_s = \beta\xi_s$, where ξ_s is an arbitrary unit vector and $\beta = \omega/c_P$. This type of solution corresponds to a *uniform plane P-wave.* (Note that $\alpha_s \neq 0$ and $\beta_s = 0$ does not lead to a solution of Equation (14.3-9).)

Similarly, Equation (14.3-10) admits solutions for which $\alpha_s \neq 0$ and $\beta_s \neq 0$, subject to the condition $\alpha_s\beta_s = 0$. These solutions correspond to *non-uniform plane S-waves*, for which the propagation vector and the attenuation vector are mutually perpendicular, while the magnitude of the propagation vector is greater than the magnitude of the attenuation vector. The planes of equal amplitudes are in this case perpendicular to the planes of equal phase (Figure 14.3-2).

Equation (14.3-10) also admits solutions for which $\alpha_s = 0$ and $\beta_s\beta_s = \omega^2/c_S^2$. In this case we can write $\beta_s = \beta\xi_s$, where ξ_s is an arbitrary unit vector and $\beta = \omega/c_S$. This type of solution corresponds to a *uniform plane S-wave.* (Note that $\alpha_s \neq 0$ and $\beta_s = 0$ does not lead to a solution of Equation (14.3-10).)

Exercises

Exercise 14.3-1

Carry out the steps that lead from Equation (14.3-7) to Equation (14.3-9) and verify the statements about the admissible solutions of the latter equation.

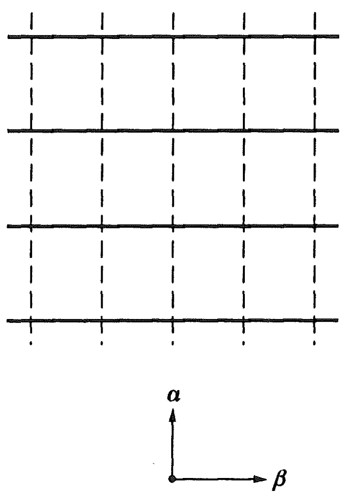

Figure 14.3-2 Planes of equal amplitude (——) and planes of equal phase (– – –) of a non-uniform plane P- or S-wave in a homogenous, isotropic, perfectly elastic solid.

Exercise 14.3-2

Carry out the steps that lead from Equation (14.3-8) to Equation (14.3-10) and verify the statements about the admissible solutions of the latter equation.

14.4 Time-domain uniform plane waves in an isotropic, lossless solid

Time-domain *uniform plane waves* in an isotropic, lossless solid are solutions of the source-free elastic wave equations (see Equations (10.7-22)–(10.7-25) and (10.5-4), (10.5-5))

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau_{p,q} + \rho\partial_t v_k = 0 , \tag{14.4-1}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v_r - S_{i,j,p,q}\partial_t\tau_{p,q} = 0 , \tag{14.4-2}$$

of the form

$$\{\tau_{p,q}, v_r\} = \{T_{p,q}, V_r\}(t - \Lambda\xi_s x_s) , \tag{14.4-3}$$

in which ξ_s is the *unit vector in the direction of propagation* of the wave, Λ is its *slowness*, and $T_{p,q}(t)$ and $V_r(t)$ are the *pulse shapes* ("signatures") of the dynamic stress $\tau_{p,q}$ and the particle velocity v_r, respectively. Substitution of Equation (14.4-3) in Equations (14.4-1), (14.4-2) leads, in view of the relation

$$\partial_m\{T_{p,q}, V_r\}(t - \Lambda\xi_s x_s) = -\Lambda\xi_m\partial_t\{T_{p,q}, V_r\}(t - \Lambda\xi_s x_s) , \tag{14.4-4}$$

to

$$\Lambda\Delta^{+}_{k,m,p,q}\xi_m\partial_t T_{p,q} + \rho\partial_t V_k = 0 , \tag{14.4-5}$$

$$-\Lambda\Delta^{+}_{i,j,n,r}\xi_n\partial_t V_r - S_{i,j,p,q}\partial_t T_{p,q} = 0 . \tag{14.4-6}$$

Since Equations (14.4-5) and (14.4-6) have to be satisfied for all values of t at each position $\boldsymbol{x}$ of the source-free domain under consideration, all components of $\partial_t T_{p,q}$ and all components of $\partial_t V_r$ must have a common pulse shape. In view of this, and of the condition of causality by which $T_{p,q}$ and V_r have the value zero prior to some instant in the finite past, all components of $T_{p,q}$ and all components of V_r must have a common pulse shape (i.e. $T_{p,q}(t) = T_{p,q}a(t)$ and $V_r(t) = V_r a(t)$, where $\{T_{p,q}, V_r\}$ are now the (constant) amplitudes of $\{\tau_{p,q}, v_r\}$ and $a(t)$ is the somehow normalised pulse shape of the wave motion). In accordance with this, Equation (14.4-3) is replaced by

$$\{\tau_{p,q}, v_r\} = \{T_{p,q}, V_r\}a(t - \Lambda\xi_s x_s) . \tag{14.4-7}$$

With Equation (14.4-7), Equations (14.4-5), (14.4-6) reduce to

$$\Lambda\Delta^{+}_{k,m,p,q}\xi_m T_{p,q} + \rho V_k = 0 , \tag{14.4-8}$$

$$-\Lambda\Delta^{+}_{i,j,n,r}\xi_n V_r - S_{i,j,p,q}T_{p,q} = 0 . \tag{14.4-9}$$

Substitution of the expression for $T_{p,q}$ resulting from Equation (14.4-9), viz.

$$(T_{p,q} + T_{q,p})/2 = -\Lambda C_{p,q,i,j}\xi_i V_j = -\Lambda\left[\lambda\delta_{p,q}\xi_i V_i + \mu(\xi_p V_q + \xi_q V_p)\right] , \tag{14.4-10}$$

where λ and μ are the Lamé coefficients, in Equation (14.4-8) yields

$$\Lambda^2(\lambda+\mu)\xi_k(\xi_i V_i) + \Lambda^2 \mu V_k - \rho V_k = 0\,. \tag{14.4-11}$$

Upon contracting Equation (14.4-11) with ξ_k it follows that

$$\left[\Lambda^2(\lambda+2\mu) - \rho\right]\xi_k V_k = 0\,. \tag{14.4-12}$$

The first possibility to satisfy Equation (14.4-12) is

$$\Lambda = [\rho/(\lambda+2\mu)]^{1/2} = c_P^{-1}\,, \tag{14.4-13}$$

combined with $\xi_k V_k \neq 0$. Substitution of Equation (14.4-13) in Equation (14.4-11) then yields

$$V_k = (\xi_i V_i)\xi_k\,, \tag{14.4-14}$$

i.e. the particle velocity of this wave is longitudinal with respect to its direction of propagation. This wave is obviously a uniform plane P-wave.

The second possibility to satisfy Equation (14.4-12) is

$$\xi_k V_k = 0\,, \tag{14.4-15}$$

combined with $\Lambda \neq c_P^{-1}$. Substitution of Equation (14.4-15) in Equation (14.4-11) leads to

$$\Lambda = (\rho/\mu)^{1/2} = c_S^{-1}\,. \tag{14.4-16}$$

On account of Equation (14.4-15) the particle velocity of this wave is transverse with respect to its direction of propagation. This wave is obviously a uniform plane S-wave. (Note that if Equation (14.4-15) holds, we must have $\Lambda = c_S^{-1}$ and hence $\Lambda \neq c_P^{-1}$.)

Next, Equation (14.4-10) is contracted with $\Delta^+_{k,m,p,q}\xi_m$. Since $\Delta^+_{k,m,p,q}\xi_m T_{p,q}(t-\Lambda\xi_s x_s) = t_k(t-\Lambda\xi_s x_s)$ where $t_k = t_k(t-\Lambda\xi_s x_s)$ is the dynamic traction of the uniform plane wave along its direction of propagation, this leads to

$$t_k = -\Lambda\left[\lambda\xi_k\xi_i V_i + \mu(\xi_k\xi_m V_m + \xi_m\xi_m V_k)\right] = -\Lambda\left[\lambda\xi_k\xi_i V_i + \mu(\xi_k\xi_m V_m + V_k)\right], \tag{14.4-17}$$

since $\xi_m\xi_m = 1$. For a *uniform plane P-wave*, we have $\Lambda = c_P^{-1}$ and $V_k = \xi_k\xi_m V_m$, and hence

$$t_k^P = -\rho c_P V_k^P\,, \tag{14.4-18}$$

where we have used the relation $\lambda + 2\mu = \rho c_P^2$. For a *uniform plane S-wave*, we have $\Lambda = c_S^{-1}$ and $\xi_i V_i = 0$, and hence

$$t_k^S = -\rho c_S V_k^S\,, \tag{14.4-19}$$

where we have used the relation $\mu = \rho c_S^2$. Equations (14.4-18) and (14.4-19) show that for a uniform plane P- or S-wave the traction along the direction of propagation and the particle velocity have the same direction (in fact, opposite directions) and that they are proportional with proportionality factors

$$Z_P = \rho c_P = [\rho(\lambda+2\mu)]^{1/2}\,, \tag{14.4-20}$$

which is denoted as the *plane elastodynamic P-wave impedance*, and

$$Z_S = \rho c_S = (\rho\mu)^{1/2}\,, \tag{14.4-21}$$

which is denoted as the *plane elastodynamic S-wave impedance*, respectively.

For the *elastodynamic power flow density* in the plane P- or S-wave we have to evaluate the elastodynamic Poynting vector

$$S_m^{\mathrm{a}} = -\Delta^+_{m,r,p,q}\tau_{p,q}v_r\,. \tag{14.4-22}$$

Using Equations (14.4-3) and (14.4-10) in the right-hand side, we obtain the following expression in terms of the particle velocity:

$$S_m^{\mathrm{a}} = \Lambda\{\lambda(\xi_r V_r)V_m + \mu[(V_q V_q)\xi_m + (\xi_r V_r)V_m]\} \,. \tag{14.4-23}$$

However, the particle velocity V_r^P of a uniform plane P-wave is longitudinal, and hence

$$(\xi_r V_r^P)V_m^P = (V_r^P V_r^P)\xi_m \,. \tag{14.4-24}$$

With this, the expression for the elastodynamic Poynting vector $S_m^{\mathrm{a};P}$ associated with a plane P-wave becomes

$$S_m^{\mathrm{a};P} = c_P^{-1}(\lambda + 2\mu)(V_r^P V_r^P)\xi_m = c_P\rho(V_r^P V_r^P)\xi_m \,. \tag{14.4-25}$$

Furthermore, the particle velocity V_r^S of a uniform plane S-wave is transverse, and hence

$$\xi_r V_r^S = 0 \,. \tag{14.4-26}$$

With this, the expression for the elastodynamic Poynting vector $S_m^{\mathrm{a};S}$ associated with a plane S-wave becomes

$$S_m^{\mathrm{a};S} = c_S^{-1}\mu(V_q^S V_q^S)\xi_m = c_S\rho(V_q^S V_q^S)\xi_m \,. \tag{14.4-27}$$

Equations (14.4-25) and (14.4-27) express the elastodynamic Poynting vector in terms of the particle velocity of the plane waves. They show that the elastodynamic Poynting vector has only a component along the direction of propagation of the wave. In view of this property we have, upon changing some subscripts and using the symmetry property $\Delta^+_{k,r,p,q} = \Delta^+_{p,q,k,r}$,

$$\begin{aligned} S_m^{\mathrm{a}} &= -\Delta^+_{m,r,p,q}\tau_{p,q}v_r = -\xi_m(\xi_k\Delta^+_{k,r,p,q}\tau_{p,q}v_r) \\ &= -\xi_m(\tau_{p,q}\Delta^+_{p,q,k,r}\xi_k v_r) = -\xi_m(\tau_{i,j}\Delta^+_{i,j,k,r}\xi_k v_r) \,. \end{aligned} \tag{14.4-28}$$

Using Equation (14.4-9) in the right-hand side, we obtain the following expression for the elastodynamic Poynting vector in terms of the dynamic stress:

$$S_m^{\mathrm{a}} = c\xi_m(T_{i,j}S_{i,j,p,q}T_{p,q}) \,. \tag{14.4-29}$$

For a uniform plane P-wave this results in

$$S_m^{\mathrm{a};P} = c_P\xi_m(T_{i,j}^P S_{i,j,p,q}T_{p,q}^P) \tag{14.4-30}$$

and for a uniform plane S-wave in

$$S_m^{\mathrm{a};S} = c_S\xi_m(T_{i,j}^S S_{i,j,p,q}T_{p,q}^S) \,. \tag{14.4-31}$$

Equations (14.4-30) and (14.4-31) express the elastodynamic Poynting vector in terms of the dynamic stress of the plane wave. Comparing Equation (14.4-25) with Equation (14.4-30), we conclude that

$$\tfrac{1}{2}\rho V_r^P V_r^P = \tfrac{1}{2} T_{i,j}^P S_{i,j,p,q}T_{p,q}^P \,, \tag{14.4-32}$$

i.e. the volume density of kinetic energy in a uniform plane P-wave is equal to the volume density of deformation energy in it. In view of this, we can write

$$S_m^{\mathrm{a};P} = c_P\left[\tfrac{1}{2}\rho V_r^P V_r^P + \tfrac{1}{2} T_{i,j}^P S_{i,j,p,q}T_{p,q}^P\right]\xi_m \,. \tag{14.4-33}$$

Similarly, comparing Equation (14.4-27) with Equation (14.4-31), we conclude that

$$\tfrac{1}{2}\rho V_r^S V_r^S = \tfrac{1}{2} T_{i,j}^S S_{i,j,p,q}T_{p,q}^S \,, \tag{14.4-34}$$

i.e. the volume density of kinetic energy in a uniform plane S-wave is equal to the volume density of deformation energy in it. In view of this, we can write

$$S_m^{a;S} = c_S\left[\tfrac{1}{2}\rho V_r^S V_r^S + \tfrac{1}{2} T_{i,j}^S S_{i,j,p,q} T_{p,q}^S\right] \xi_m \, . \qquad (14.4\text{-}35)$$

Equations (14.4-33) and (14.4-35) lead to the picture that for a uniform plane P-wave and a uniform plane S-wave the elastodynamic Poynting vector is the quantity that carries the sum of the volume densities of kinetic and deformation energy with the corresponding elastic P- or S-wave speed in the direction of propagation of the wave.

Exercises

Exercise 14.4-1

Construct the one-dimensional P-wave solutions of the source-free time-domain elastic wave-field equations in a homogeneous, isotropic, lossless solid by taking a Euclidean reference frame such that the propagation takes place along the x_3-direction, (a) for propagation in the direction of increasing x_3, (b) for propagation in the direction of decreasing x_3. Express $T_{p,q}$ in terms of the non-vanishing component of V_r.

Answer:

(a) $V_3 = V_3(t - x_3/c_P)$, $T_{1,1} = T_{2,2} = -(\lambda/c_P)V_3$, $T_{3,3} = -[(\lambda + 2\mu)/c_P]V_3$, $T_{1,2} = T_{2,3} = T_{3,1} = 0$;

(b) $V_3 = V_3(t - x_3/c_P)$, $T_{1,1} = T_{2,2} = (\lambda/c_P)V_3$, $T_{3,3} = [(\lambda + 2\mu)/c_P]V_3$, $T_{1,2} = T_{2,3} = T_{3,1} = 0$.

Here, $c_P = [(\lambda + 2\mu)/\rho]^{1/2}$.

Exercise 14.4-2

Construct the one-dimensional S-wave solutions of the source-free time-domain elastic wave-field equations in a homogeneous, isotropic, lossless solid by taking a Euclidean reference frame such that the propagation takes place along the x_3-direction, for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3. Express $T_{p,q}$ in terms of the non-vanishing components of V_r. Consider the cases $V_1 \neq 0$ and $V_2 \neq 0$ separately.

Answer:

(a) $V_1 = V_1(t - x_3/c_S)$, $T_{1,1} = T_{2,2} = T_{3,3} = 0$, $T_{1,2} = T_{2,3} = 0$, $T_{3,1} = -(\mu/c_S)V_1$,

or

$V_2 = V_2(t - x_3/c_S)$, $T_{1,1} = T_{2,2} = T_{3,3} = 0$, $T_{1,2} = 0$, $T_{2,3} = -(\mu/c_S)V_2$, $T_{3,1} = 0$;

(b) $V_1 = V_1(t + x_3/c_S)$, $T_{1,1} = T_{2,2} = T_{3,3} = 0$, $T_{1,2} = T_{2,3} = 0$, $T_{3,1} = (\mu/c_S)V_1$,

or

$V_2 = V_2(t + x_3/c_S)$, $T_{1,1} = T_{2,2} = T_{3,3} = 0$, $T_{1,2} = 0$, $T_{2,3} = (\mu/c_S)V_2$, $T_{3,1} = 0$.

Here, $c_S = (\mu/\rho)^{1/2}$.

Exercise 14.4-3

Determine the value of (a) the elastodynamic plane P-wave impedance, (b) the elastodynamic plane S-wave impedance for a solid with $\rho = 8\times10^3$ kg/m^3, $c_P = 5000$ m/s, $c_S = 3000$ m/s.

Answer: (a) $Z_P = 40\times10^6$ kg/m^2•s; (b) $Z_S = 24\times10^6$ kg/m^2•s.

Exercise 14.4-4

For a uniform plane wave, periodicity in time entails periodicity in space. Let T denote the time period of the wave and $f = 1/T$ its frequency (Hz). Show, with the aid of Equation (14.4-7), that the spatial period λ in the direction of propagation of the wave is related to T or f via

$$\lambda = cT = c/f . \qquad (14.4\text{-}36)$$

(The quantity λ is known as the *wavelength* of the time-periodic plane wave.)

Exercise 14.4-5

Determine the wavelength of a time-periodic plane elastic wave for which $c = 2000$ m/s if this wave has a frequency of (a) $f = 10$ Hz, (b) $f = 100$ Hz, (c) $f = 1000$ Hz, (d) $f = 10$ kHz, (e) $f = 1$ MHz.

Answer: (a) $\lambda = 200$ m, (b) $\lambda = 20$ m, (c) $\lambda = 2.0$ m, (d) $\lambda = 0.2$ m, (e) $\lambda = 2.0$ mm.