15

Elastodynamic reciprocity theorems and their applications

In this chapter we discuss the basic reciprocity theorems for elastic wave fields in time-invariant configurations, together with a variety of their applications. The theorems will be presented both in the time domain and in the complex frequency domain. In view of the time invariance of the configurations to be considered, there exist two versions of the theorems as far as their operations on the time coordinate are concerned, viz. a version known as the *time convolution type* and a version known as the *time correlation type*. The two versions are related via a time inversion operation. Each of the two versions has its counterpart in the complex frequency domain.

The application of the theorems to the reciprocity in *transmitting/receiving properties of elastodynamic sources and receivers*, and to the formulations of the *direct (forward) source* and *inverse source* and the *direct (forward) scattering* and *inverse scattering problems* will be discussed. Furthermore, it is indicated how the theorems lead, in a natural way, to the *integral equation formulation* of elastic wave-field problems for numerical implementation. Finally, it is shown how the reciprocity theorems lead to a mathematical formulation of *Huygens' principle* and of the *Ewald–Oseen extinction theorem.*

15.1 The nature of the reciprocity theorems and the scope of their consequences

A reciprocity theorem interrelates, in a specific manner, the field or wave quantities that characterise two admissible states that could occur in one and the same time-invariant domain $\mathcal{D}\subset\mathcal{R}^3$ in space. Each of the two states can be associated with its own set of time-invariant medium parameters and its own set of source distributions. It is assumed that the media in the two states are linear in their elastodynamic behaviour, i.e. the medium parameters are independent of the values of the field or wave quantities. The domain $\mathcal{D}$ to which the reciprocity theorems apply may be bounded or unbounded. The application to unbounded domains will always be handled as a limiting case where the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ recedes (partially or entirely) to infinity.

From the pertaining elastic wave equations, first the *local form* of a reciprocity theorem will be derived, which form applies to each point of any subdomain of $\mathcal{D}$ where the elastic wave-field quantities are continuously differentiable. By integrating the local form over such subdomains and adding the results, the *global form* of the reciprocity theorem is arrived at. In it, a boundary integral over $\partial\mathcal{D}$ occurs, the integrand of which always contains the unit vector ν_m along the normal to $\partial\mathcal{D}$, oriented away from $\mathcal{D}$ (Figure 15.1-1). The two states will be denoted by the superscripts A and B.

The construction of the time-domain reciprocity theorems will be based on the elastic wave equations (see Equations (10.7-22), (10.7-23) and (10.7-26), (10.7-27))

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau^{A}_{p,q} + \partial_t C_t(\mu^{A}_{k,r}, v^{A}_r; \boldsymbol{x},t) = f^{A}_k, \tag{15.1-1}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{A}_r - \partial_t C_t(\chi^{A}_{i,j,p,q}, \tau^{A}_{p,q}; \boldsymbol{x},t) = h^{A}_{i,j}, \tag{15.1-2}$$

for state A, and

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau^{B}_{p,q} + \partial_t C_t(\mu^{B}_{k,r}, v^{B}_r; \boldsymbol{x},t) = f^{B}_k, \tag{15.1-3}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{B}_r - \partial_t C_t(\chi^{B}_{i,j,p,q}, \tau^{B}_{p,q}; \boldsymbol{x},t) = h^{B}_{i,j}, \tag{15.1-4}$$

for state B, where C_t denotes the time convolution operator (see Equation (B.1-11)) (Figure 15.1-2).

If, in $\mathcal{D}$, either surfaces of discontinuity in elastodynamic properties or elastodynamically impenetrable objects are present, Equations (15.1-1)–(15.1-4) are supplemented by boundary conditions of the type discussed in Section 10.6, both for state A and state B. These are either (see Equations (10.6-2) and (10.6-6))

$$\Delta^{+}_{k,m,p,q}\nu_m\tau^{A,B}_{p,q} \quad \text{is continuous across any interface,} \tag{15.1-5}$$

and

$$v^{A,B}_r \quad \text{is continuous across any interface,} \tag{15.1-6}$$

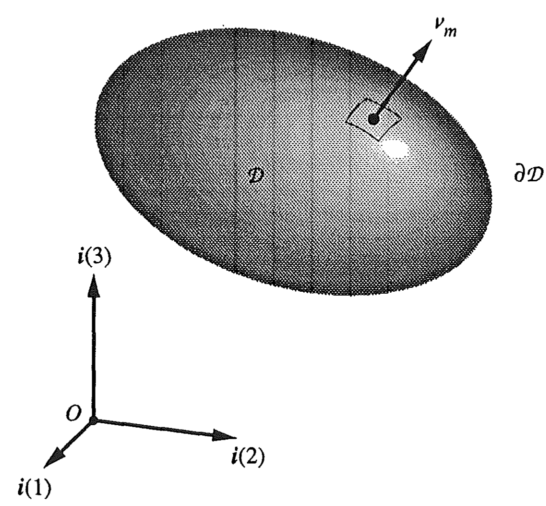


Figure 15.1-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ and unit vector ν_m along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$, to which the reciprocity theorems apply.

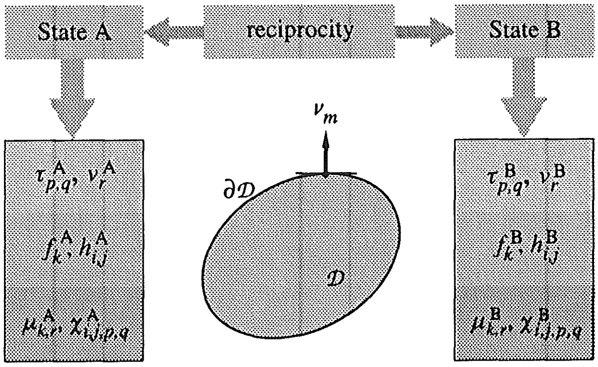


Figure 15.1-2 Bounded domain $\mathcal{D}$ and states A and B to which the time-domain reciprocity theorems apply.

where ν_m is the unit vector along the normal to the interface, or (see Equation (10.6-7))

$$\lim_{h\downarrow 0}\Delta^{+}_{k,m,p,q}\nu_m\tau^{A,B}_{p,q}(x+h\nu,t)=0 \quad \text{on the boundary of a void,} \tag{15.1-7}$$

where ν is the unit vector along the normal to the boundary of the void, pointing away from the void, or (see Equation (10.6-11))

$$\lim_{h\downarrow 0}v^{A,B}_{r}(x+h\nu,t)=0$$

$$\text{on the boundary of an immovable, perfectly rigid object,} \tag{15.1-8}$$

where ν is the unit vector along the normal to the boundary of the immovable, perfectly rigid object, pointing away from the object.

If the domain where the elastic wave fields are defined is a bounded domain (the *support* of the wave fields), the boundary surface of this domain is assumed to be elastodynamically impenetrable (Figure 15.1-3).

To handle unbounded domains, we assume that outside some sphere $\mathcal{S}(O,\Delta_0)$ with centre at the origin of the chosen reference frame and radius Δ_0, the solid is homogeneous, isotropic and lossless, with the volume density of mass ρ_0 and the compressional and shear wave speeds c_P and c_S, respectively, as constitutive parameters, as well as source-free (Figure 15.1-4). In the domain outside that sphere, the so-called *embedding*, the asymptotic causal far-field representations are (see Equations (13.8-7) and (13.8-10))

$$\{v^{A,B}_{r},\tau^{A,B}_{p,q}\}=\left[\frac{\{v^{P,\infty;A,B}_{r},\tau^{P,\infty;A,B}_{p,q}\}(\xi,t-|x|/c_P)}{4\pi c_P^2|x|}+\frac{\{v^{S,\infty;A,B}_{r},\tau^{S,\infty;A,B}_{p,q}\}(\xi,t-|x|/c_S)}{4\pi c_S^2|x|}\right]$$

$$\times[1+O(|x|^{-1})] \quad \text{as} \quad |x|\to\infty, \tag{15.1-9}$$

where $\{v^{P,\infty;A,B}_{r},\tau^{P,\infty;A,B}_{p,q}\}(\xi,t)$ denote the far-field P-wave amplitudes, which are interrelated through (see Equations (13.8-19) and (13.8-21))

$$\Delta^{+}_{k,m,p,q}(\xi_m/c_P)\tau^{P,\infty;A,B}_{p,q}+\rho_0 v^{P,\infty;A,B}_{k}=0\,, \tag{15.1-10}$$

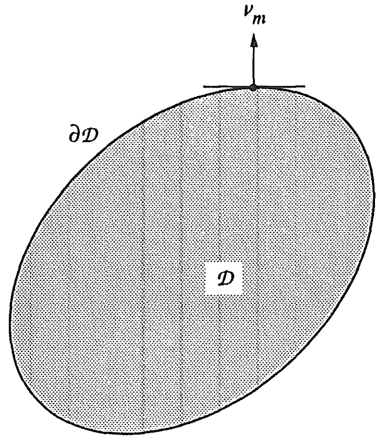


Figure 15.1-3 Bounded domain $\mathcal{D}$ for the application of a reciprocity theorem. The boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is assumed to be impenetrable. In $\mathcal{D}$, interfaces between different solids, voids and immovable perfectly rigid objects may be present.

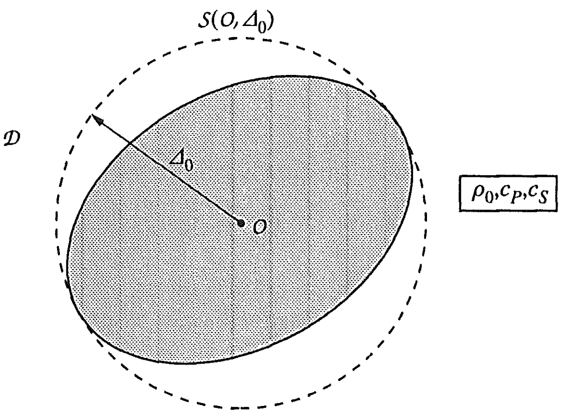


Figure 15.1-4 Unbounded domain $\mathcal{D}$ for the application of a reciprocity theorem. Outside the sphere $\mathcal{S}(O,\Delta_0)$, the solid is homogeneous, isotropic and lossless with constitutive parameters $\{\rho_0, c_P, c_S\}$. Inside $\mathcal{S}(O,\Delta_0)$, interfaces between different solids, voids and immovable perfectly rigid objects may be present.

with

$$v_k^{P,\infty;\mathrm{A,B}} = (\xi_r v_r^{P,\infty;\mathrm{A,B}})\xi_k \,, \tag{15.1-11}$$

and $\{v_r^{S,\infty;\mathrm{A,B}}, \tau_{p,q}^{S,\infty;\mathrm{A,B}}\}(\boldsymbol{\xi},t)$ denote the far-field S-wave amplitudes, which are interrelated through (see Equations (13.8-20) and (13.8-22))

$$\Delta^+_{k,m,p,q}(\xi_m/c_S)\tau_{p,q}^{S,\infty;\mathrm{A,B}} + \rho_0 v_k^{S,\infty;\mathrm{A,B}} = 0 \,, \tag{15.1-12}$$

with

$$\xi_r v_r^{S,\infty;\mathrm{A,B}} = 0 \,. \tag{15.1-13}$$

The construction of the *complex frequency-domain* reciprocity theorems will be based on the complex frequency-domain elastic wave equations (see Equations (12.5-1) and (12.5-2))

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{\mathrm{A}}_{p,q} + \hat{\zeta}^{\mathrm{A}}_{k,r}\hat{v}^{\mathrm{A}}_r = \hat{f}^{A}_k\,, \tag{15.1-14}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{\mathrm{A}}_r - \hat{\eta}^{\mathrm{A}}_{i,j,p,q}\hat{\tau}^{\mathrm{A}}_{p,q} = \hat{h}^{\mathrm{A}}_{i,j}\,, \tag{15.1-15}$$

for state A, and

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{\mathrm{B}}_{p,q} + \hat{\zeta}^{\mathrm{B}}_{k,r}\hat{v}^{\mathrm{B}}_r = \hat{f}^{B}_k\,, \tag{15.1-16}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{\mathrm{B}}_r - \hat{\eta}^{\mathrm{B}}_{i,j,p,q}\hat{\tau}^{\mathrm{B}}_{p,q} = \hat{h}^{\mathrm{B}}_{i,j}\,, \tag{15.1-17}$$

for state B. If, in $\mathcal{D}$, either surfaces of discontinuity in elastodynamic properties or elastodynamically impenetrable objects are present, Equations (15.1-14)–(15.1-17) are supplemented by boundary conditions of the type discussed in Section 12.3, both for state A and state B. These are either (see Equations (12.3-1) and (12.3-2))

$$\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}^{\mathrm{A,B}}_{p,q} \quad \text{is continuous across any interface,} \tag{15.1-18}$$

and

$$\hat{v}^{\mathrm{A,B}}_r \quad \text{is continuous across any interface,} \tag{15.1-19}$$

where ν_m is the unit vector along the normal to the interface, or (see Equation (12.3-3))

$$\lim_{h\downarrow 0}\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}^{\mathrm{A,B}}_{p,q}(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0 \quad \text{on the boundary of a void,} \tag{15.1-20}$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to the boundary of the void, pointing away from the void, or (see Equation (12.3-7))

$$\lim_{h\downarrow 0}\hat{v}^{\mathrm{A,B}}_r(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0$$

$$\text{on the boundary of an immovable, perfectly rigid object}\,, \tag{15.1-21}$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to the boundary of the immovable, perfectly rigid object, pointing away from the object.

If the domain where the elastic wave fields are defined is a bounded domain (the *support* of the wave fields), the boundary surface of this domain is assumed to be elastodynamically impenetrable.

In the complex frequency domain, too, we assume, to handle unbounded domains, that outside some sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ with centre at the origin of the chosen reference frame and radius Δ_0, the solid is homogeneous, isotropic and lossless, with the volume density of mass ρ_0 and the compressional and shear wave speeds c_P and c_S, respectively, as constitutive parameters, as well as source-free. In the domain outside that sphere, the *embedding*, the asymptotic causal far-field representations are (see Equations (13.7-17) and (13.7-20))

$$\{\hat{v}^{\mathrm{A,B}}_r,\hat{\tau}^{\mathrm{A,B}}_{p,q}\} = \left[\{\hat{v}^{P,\infty;\mathrm{A,B}}_r,\hat{\tau}^{P,\infty;\mathrm{A,B}}_{p,q}\}(\boldsymbol{\xi},s)\frac{\exp(-s|\boldsymbol{x}|/c_P)}{4\pi c_P^2|\boldsymbol{x}|}\right.$$
$$\left. + \{\hat{v}^{S,\infty;\mathrm{A,B}}_r,\hat{\tau}^{S,\infty;\mathrm{A,B}}_{p,q}\}(\boldsymbol{\xi},s)\frac{\exp(-s|\boldsymbol{x}|/c_S)}{4\pi c_S^2|\boldsymbol{x}|}\right][1+O(|\boldsymbol{x}|^{-1})] \quad \text{as} \quad |\boldsymbol{x}|\to\infty\,, \tag{15.1-22}$$

where $\{\hat{v}_r^{P,\infty;\mathrm{A,B}},\hat{\tau}_{p,q}^{P,\infty;\mathrm{A,B}}\}(\boldsymbol{\xi},s)$ denote the far-field P-wave amplitudes, which are interrelated through (see Equations (13.7-31) and (13.7-33))

$$\Delta^{+}_{k,m,p,q}(\xi_m/c_P)\hat{\tau}_{p,q}^{P,\infty;\mathrm{A,B}} + \rho_0\hat{v}_k^{P,\infty;\mathrm{A,B}} = 0\,, \tag{15.1-23}$$

with

$$\hat{v}_k^{P,\infty;\mathrm{A,B}} = (\xi_r\hat{v}_r^{P,\infty;\mathrm{A,B}})\xi_k\,, \tag{15.1-24}$$

and $\{\hat{v}_r^{S,\infty;\mathrm{A,B}},\hat{\tau}_{p,q}^{S,\infty;\mathrm{A,B}}\}(\boldsymbol{\xi},s)$ denote the far-field S-wave amplitudes, which are interrelated through (see Equations (13.7-32) and (13.7-34))

$$\Delta^{+}_{k,m,p,q}(\xi_m/c_S)\hat{\tau}_{p,q}^{S,\infty;\mathrm{A,B}} + \rho_0\hat{v}_k^{S,\infty;\mathrm{A,B}} = 0\,, \tag{15.1-25}$$

with

$$\xi_r\hat{v}_r^{S,\infty;\mathrm{A,B}} = 0\,. \tag{15.1-26}$$

As a rule, state A will be chosen to correspond to the actual elastic wave field in the configuration, or one of its constituents. This wave field will therefore satisfy the condition of causality, or, for short, will be a *causal wave field.* If state B is another physical state, for example a state that corresponds to source distributions and/or elastodynamic medium parameters that differ from those in state A, state B will also be a causal wave field. If, however, state B is a computational state, i.e. a state that is representative for the manner in which the wave-field quantities in state A are computed, or a state that is representative for the manner in which the elastic wave-field data pertaining to state A are processed, there is no necessity to take state B to be a causal wave field as well, and it may, for example, be taken to be an anti-causal wave field (i.e. a wave field that is time reversed with respect to a causal wave field) or no wave field at all (which happens, for example, if one of the corresponding constitutive parameters is taken to be zero). No matter how the source distributions and the constitutive parameters are chosen, the wave-field quantities will always be assumed to satisfy the pertaining elastic wave equations and the pertaining boundary conditions.

To accommodate causal, anti-causal as well as non-causal states in the complex frequency-domain analysis of reciprocity, the Laplace transform with respect to time of any *transient*, not necessarily causal or anti-causal, wave function $f = f(\boldsymbol{x},t)$ will always be taken as (see Equation (B.1-5))

$$\hat{f}(\boldsymbol{x},s) = \int_{t\in\mathcal{R}} \exp(-st)\,f(\boldsymbol{x},t)\,\mathrm{d}t \qquad \text{for } \mathrm{Re}(s) = s_0\,, \tag{15.1-27}$$

i.e. the support of the wave function is, in principle, taken to be the entire interval of real values of time. Whenever appropriate, the support of the wave function will be indicated explicitly. For wave fields that are neither causal nor anti-causal (but of a transient nature), the right-hand side of Equation (15.1-27) should exist for some value $\mathrm{Re}(s) = s_0$ on a line parallel to the imaginary axis of the complex s plane (Figure 15.1-5) for the transformation to make any sense at all.

For *causal wave functions* with support $\mathcal{T}^+ = \{t\in\mathcal{R};t > t_0\}$ Equation (15.1-27) yields

$$\hat{f}(\boldsymbol{x},s) = \int_{t=t_0}^{\infty} \exp(-st)\,f(\boldsymbol{x},t)\,\mathrm{d}t \qquad \text{for } \mathrm{Re}(s) > s_0^{+}\,. \tag{15.1-28}$$

Here, the right-hand side is regular in some *right half* $\mathrm{Re}(s) > s_0^{+}$ of the complex s plane (Figure 15.1-6).

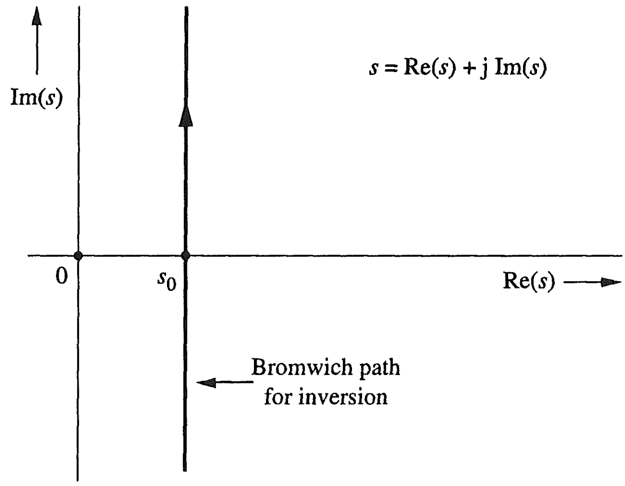


Figure 15.1-5 Line $\mathrm{Re}(s) = s_0$, parallel to the imaginary axis of the complex s plane, at which the time Laplace transform of a wave field that is neither causal, nor anti-causal, but of a *transient* nature, exists.

For *anti-causal wave functions* with support $\mathcal{T}^- = \{t \in \mathcal{R}; t < t_0\}$ Equation (15.1-27) yields

$$\hat{f}(\boldsymbol{x},s) = \int_{t=-\infty}^{t_0} \exp(-st)\, f(\boldsymbol{x},t)\, \mathrm{d}t \qquad \text{for } \mathrm{Re}(s) < s_0^- . \tag{15.1-29}$$

Here, the right-hand side is regular in some *left half* $\mathrm{Re}(s) < s_0^-$ of the complex s plane (Figure 15.1-7).

A consequence of Equation (15.1-29) is that $\hat{f}(\boldsymbol{x},-s)$ is regular in the right half $\mathrm{Re}(s) > -s_0^-$ of the complex s plane if $\hat{f}(\boldsymbol{x},s)$ is regular in the left half $\mathrm{Re}(s) < s_0^-$. This result will be needed in reciprocity theorems of the time correlation type.

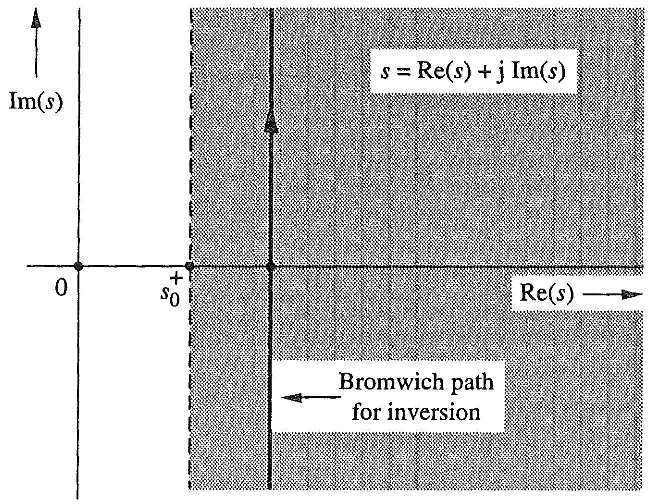
For the *time convolution* $\mathrm{C}_t\,(f_1, f_2; \boldsymbol{x},t)$ of any two transient wave functions we have (see Equations (B.1-11) and (B.1-12))

$$\hat{\mathrm{C}}_t(f_1, f_2; \boldsymbol{x},s) = \hat{f}_1(\boldsymbol{x},s)\hat{f}_2(\boldsymbol{x},s) . \tag{15.1-30}$$

This relation only holds in the common domain of regularity of $\hat{f}_1(\boldsymbol{x},s)$ and $\hat{f}_2(\boldsymbol{x},s)$. If, in particular, both $f_1 = f_1(\boldsymbol{x},t)$ and $f_2 = f_2(\boldsymbol{x},t)$ are causal wave functions, they have a certain right half of the complex s plane as the domain of regularity in common. (Note that in this case $\hat{f}_1(\boldsymbol{x},s)$ is regular in some right half of the complex s plane, while $\hat{f}_2(\boldsymbol{x},s)$ is also regular in some right half of the complex s plane.) If, on the other hand, $f_1 = f_1(\boldsymbol{x},t)$ is a causal wave function and $f_2 = f_2(\boldsymbol{x},t)$ is an anti-causal wave function, the common domain of regularity where Equation (15.1-30) holds is at most a strip of finite width parallel to the imaginary axis of the complex s plane. (Note that in this case $\hat{f}_1(\boldsymbol{x},s)$ is regular in some right half of the complex s plane, while $\hat{f}_2(\boldsymbol{x},s)$ is regular in some left half of the complex s plane.)

For the *time correlation* $\mathrm{R}_t(f_1, f_2; \boldsymbol{x},t)$ of any two transient wave functions we have (see Equations (B.1-14) and (B.1-15))

$$\hat{\mathrm{R}}_t(f_1, f_2; \boldsymbol{x},s) = \hat{f}_1(\boldsymbol{x},s)\hat{f}_2(\boldsymbol{x},-s) . \tag{15.1-31}$$



15.1-6 Right half $\mathrm{Re}(s) > s_0^+$ of the complex s plane, in which the time Laplace transform of a *causal* wave function exists.

This relation only holds in the common domain of regularity of $\hat{f}_1(x,s)$ and $\hat{f}_2(x,-s)$. If, in particular, both $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are causal wave functions, the common domain of regularity where Equation (15.1-31) holds is at most a strip of finite width parallel to the imaginary axis of the complex s plane. (Note in this case that $\hat{f}_1(x,s)$ is regular in some right half of the complex s plane, and that $\hat{f}_2(x,-s)$ is regular in some left half of the complex s plane.) If, on the other hand, $f_1 = f_1(x,t)$ is a causal wave function and $f_2 = f_2(x,t)$ is an anti-causal wave function, the common domain of regularity where Equation (15.1-31) holds is some right half of the complex s plane. (Note that in this case $\hat{f}_1(x,s)$ is regular in some right half of the complex s plane, while also $\hat{f}_2(x,-s)$ is regular in some right half of the complex s plane.)

In subsequent calculations the time correlation will, whenever appropriate, be replaced by (see Equation (B.1-18))

$$\mathrm{R}_t(f_1,f_2;x,t) = \mathrm{C}_t(f_1,\mathrm{J}_t(f_2);x,t)\,, \qquad (15.1\text{-}32)$$

where J_t is the *time reversal* operator. The latter operator changes causal wave functions into anti-causal ones, and vice versa.

Exercises

Exercise 15.1-1

Of what type is the domain of regularity of the Laplace transform of the time convolution $\mathrm{C}_t(f_1,f_2;x,t)$ of two wave functions $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ that are both anti-causal?

Answer: Some left half of the complex s plane.

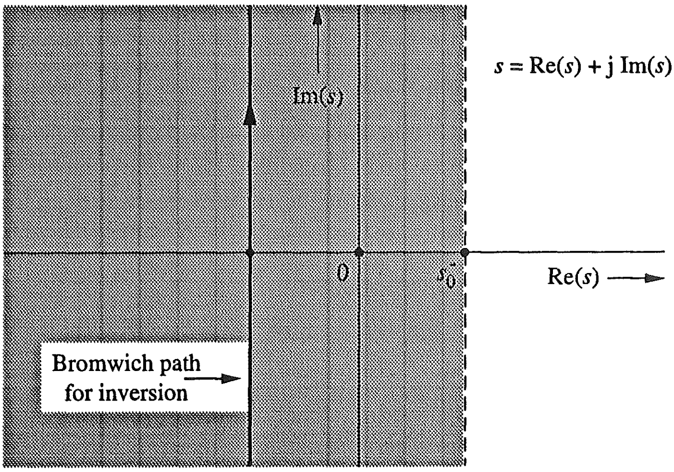


Figure 15.1-7 Left half $\mathrm{Re}(s) < s_0^-$ of the complex s plane, in which the time Laplace transform of an *anti-causal* wave function exists.

Exercise 15.1-2

Of what type is the domain of regularity of the Laplace transform of the time correlation $R_t(f_1, f_2;x,t)$ of the anti-causal wave function $f_1 = f_1(x,t)$ and the causal wave function $f_2 = f_2(x,t)$?

Answer: Some left half of the complex s plane.

15.2 The time-domain reciprocity theorem of the time convolution type

The time-domain reciprocity theorem of the time convolution type follows upon considering the *local interaction quantity* $\Delta^+_{m,r,p,q}\partial_m\,[C_t(-\tau^A_{p,q},v^B_r;x,t) - C_t(\tau^B_{p,q},v^A_r;x,t)]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$
\begin{aligned}
&\Delta^+_{m,r,p,q}\partial_m\left[C_t(-\tau^A_{p,q},v^B_r;x,t) - C_t(-\tau^B_{p,q},v^A_r;x,t)\right]\\
&\quad= \Delta^+_{k,m,p,q}\partial_m C_t(-\tau^A_{p,q},v^B_k;x,t) - \Delta^+_{i,j,n,r}\partial_n C_t(-\tau^B_{i,j},v^A_r;x,t)\\
&\quad= -C_t(\Delta^+_{k,m,p,q}\partial_m\tau^A_{p,q},v^B_k;x,t) - C_t(\tau^A_{p,q},\Delta^+_{k,m,p,q}\partial_m v^B_k;x,t)\\
&\qquad+ C_t(\Delta^+_{i,j,n,r}\partial_n\tau^B_{i,j},v^A_r;x,t) + C_t(\tau^B_{i,j},\Delta^+_{i,j,n,r}\partial_n v^A_r;x,t)\,. \qquad (15.2\text{-}1)
\end{aligned}
$$

With the aid of Equations (15.1-1)–(15.1-4), the different terms on the right-hand side become

$$-C_t(\Delta^+_{k,m,p,q}\partial_m\tau^A_{p,q},v^B_k;x,t) = -\partial_t C_t(\mu^A_{k,r},v^A_r,v^B_k;x,t) + C_t(f^A_k,v^B_k;x,t)\,, \qquad (15.2\text{-}2)$$

$$-C_t(\tau^A_{p,q},\Delta^+_{k,m,p,q}\partial_m v^B_k;x,t) = -\partial_t C_t(\tau^A_{p,q},\chi^B_{p,q,i,j},\tau^B_{i,j};x,t) - C_t(\tau^A_{p,q},h^B_{p,q};x,t) \qquad (15.2\text{-}3)$$

and

$$-C_t(\Delta^+_{i,j,n,r}\partial_n\tau^{B}_{i,j},v^{A}_r;x,t) = -\partial_t C_t(\mu^{B}_{r,k},v^{B}_k,v^{A}_r;x,t) + C_t(f^{B}_r,v^{A}_r;x,t)\,, \tag{15.2-4}$$

$$-C_t(\tau^{B}_{i,j},\Delta^+_{i,j,n,r}\partial_n v^{A}_r;x,t) = -\partial_t C_t(\tau^{B}_{i,j},\chi^{A}_{i,j,p,q},\tau^{A}_{p,q};x,t) - C_t(\tau^{B}_{i,j},h^{A}_{i,j};x,t)\,, \tag{15.2-5}$$

in which the convolution of three functions is a shorthand notation for the convolution of a function with the convolution of two other functions. (Note that in the convolution operation the order of the operators is immaterial.) Combining Equations (15.2-2)–(15.2-5) with Equation (15.2-1), it is found that

$$\begin{aligned}&\Delta^+_{m,r,p,q}\partial_m\left[C_t(-\tau^{A}_{p,q},v^{B}_r;x,t) - C_t(-\tau^{B}_{p,q},v^{A}_r;x,t)\right]\\
&= \partial_t C_t(\mu^{B}_{r,k} - \mu^{A}_{k,r},v^{A}_r,v^{B}_k;x,t) - \partial_t C_t(\chi^{B}_{p,q,i,j} - \chi^{A}_{i,j,p,q},-\tau^{A}_{p,q},-\tau^{B}_{i,j};x,t)\\
&\quad + C_t(f^{A}_k,v^{B}_k;x,t) - C_t(\tau^{A}_{p,q},h^{B}_{p,q};x,t) - C_t(f^{B}_r,v^{A}_r;x,t) + C_t(\tau^{B}_{i,j},h^{A}_{i,j};x,t)\,.\end{aligned} \tag{15.2-6}$$

Equation (15.2-6) is the *local form of elastodynamic reciprocity theorem of the time convolution type*. The first two terms on the right-hand side are representative for the differences (contrasts) in the elastodynamic properties of the solids present in the two states; these terms vanish at those locations where $\mu^{B}_{r,k}(x,t) = \mu^{A}_{k,r}(x,t)$ and $\chi^{B}_{p,q,i,j}(x,t) = \chi^{A}_{i,j,p,q}(x,t)$ for all $t\in\mathcal{R}$. In case the latter conditions hold, the two media are denoted as each other's *adjoint*. Note in this respect that the adjoint of a causal (anti-causal) medium is causal (anti-causal) as well. The last four terms on the right-hand side of Equation (15.2-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (15.2-6) are continuous. Upon integrating Equation (15.2-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 15.2-1)

$$\begin{aligned}&\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[C_t(-\tau^{A}_{p,q},v^{B}_r;x,t) - C_t(-\tau^{B}_{p,q},v^{A}_r;x,t)\right]\mathrm{d}A\\
&= \int_{x\in\mathcal{D}}\left[\partial_t C_t(\mu^{B}_{r,k} - \mu^{A}_{k,r},v^{A}_r,v^{B}_k;x,t) - \partial_t C_t(\chi^{B}_{p,q,i,j} - \chi^{A}_{i,j,p,q},-\tau^{A}_{p,q},-\tau^{B}_{i,j};x,t)\right]\mathrm{d}V\\
&\quad + \int_{x\in\mathcal{D}}\left[C_t(f^{A}_k,v^{B}_k;x,t) - C_t(\tau^{A}_{p,q},h^{B}_{p,q};x,t) - C_t(f^{B}_r,v^{A}_r;x,t) + C_t(\tau^{B}_{i,j},h^{A}_{i,j};x,t)\right]\mathrm{d}V\,.\end{aligned} \tag{15.2-7}$$

Equation (15.2-7) is the *global form, for the bounded domain $\mathcal{D}$, of the elastodynamic reciprocity theorem of the time convolution type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (15.1-5) and (15.1-6)), and that the contributions from boundary surfaces of elastodynamically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (15.1-7) or Equation (15.1-8)). In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an elastodynamically impenetrable object. In the right-hand side, the first integral is representative for the differences (contrasts) in the elastodynamic properties of the solids present in the two states; this term vanishes if the media

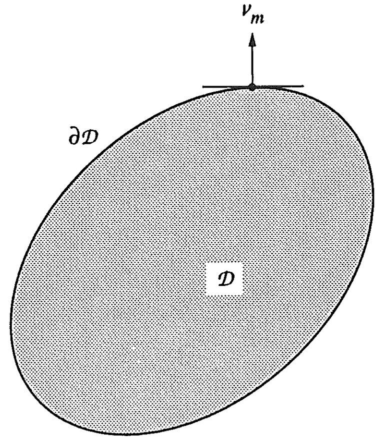


Figure 15.2-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

in the two states are, throughout $\mathcal{D}$, each other's adjoint. The second integral on the right-hand side is representative of the action of the sources present in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem Equation (15.2-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 15.1 are made and Equation (15.2-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with centre at the origin and radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 15.2-2).

Whether or not the surface integral contribution over $\mathcal{S}(\mathcal{O},\Delta)$ does vanish as $\Delta\to\infty$ depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (15.1-9)-(15.1-13) apply for sufficiently large values of Δ. Then, the time convolutions occurring in the integrands of Equation (15.2-7) are also causal in time, and at any finite value of t, Δ can be chosen so large that on $\mathcal{S}(\mathcal{O},\Delta)$ the integrand vanishes. In this case, the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes. If, however, at least one of the two states is chosen to be non-causal (which, for example, can apply to the case where one of the two states is a computational one), the time convolutions occurring in Equation (15.2-7) are non-causal as well and the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ does not vanish, no matter how large Δ is chosen. Since outside the sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ that is used to define the embedding (see Section 15.1) the media are each other's adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(\mathcal{O},\Delta)$ is, however, independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 15.2-2).

The time-domain reciprocity theorem of the time convolution type is mainly used for investigating the transmission/reception reciprocity properties of elastodynamic sources and receivers (see Sections 15.6 and 15.7) and for the modelling of direct (forward) source problems

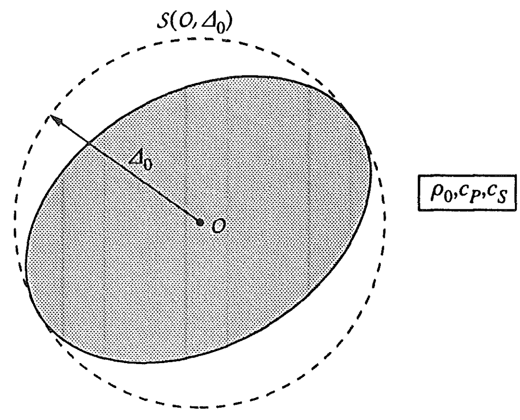


Figure 15.2-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the solid is homogeneous, isotropic and lossless.

(see Section 15.8) and direct (forward) scattering problems (see Section 15.9). References to the earlier literature on the subject can be found in a paper by De Hoop (1988).

Exercises

Exercise 15.2-1

To what form do the contrast-in-media terms in the reciprocity theorems Equations (15.2-6) and (15.2-7) reduce if the solids in states A and B are both instantaneously reacting?

Answer:

$$\mathrm{C}_t(\mu^{\mathrm{B}}_{r,k} - \mu^{\mathrm{A}}_{k,r}, v^{\mathrm{A}}_r, v^{\mathrm{B}}_k; \boldsymbol{x},t) = \left[\rho^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \rho^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right] \mathrm{C}_t(v^{\mathrm{A}}_r, v^{\mathrm{B}}_k; \boldsymbol{x},t)$$

and

$$\mathrm{C}_t(\chi^{\mathrm{B}}_{p,q,i,j} - \chi^{\mathrm{A}}_{i,j,p,q}, -\tau^{\mathrm{A}}_{p,q}, -\tau^{\mathrm{B}}_{i,j}; \boldsymbol{x},t) = \left[s^{\mathrm{B}}_{p,q,i,j}(\boldsymbol{x}) - s^{\mathrm{A}}_{i,j,p,q}(\boldsymbol{x})\right] \mathrm{C}_t(-\tau^{\mathrm{A}}_{p,q}, -\tau^{\mathrm{B}}_{i,j}; \boldsymbol{x},t) .$$

Exercise 15.2-2

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 15.2-3.

The reciprocity theorem Equation (15.2-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, either in state A or in state B, and the solid in $\mathcal{D}$ in state B is in its elastodynamic properties adjoint to the one in state A. Prove that

$$\Delta^{+}_{m,r,p,q} \int_{\boldsymbol{x} \in \mathcal{S}_1} \nu_m \left[\mathrm{C}_t(-\tau^{\mathrm{A}}_{p,q}, v^{\mathrm{B}}_r; \boldsymbol{x},t) - \mathrm{C}_t(-\tau^{\mathrm{B}}_{p,q}, v^{\mathrm{A}}_r; \boldsymbol{x},t)\right] \mathrm{d}A$$

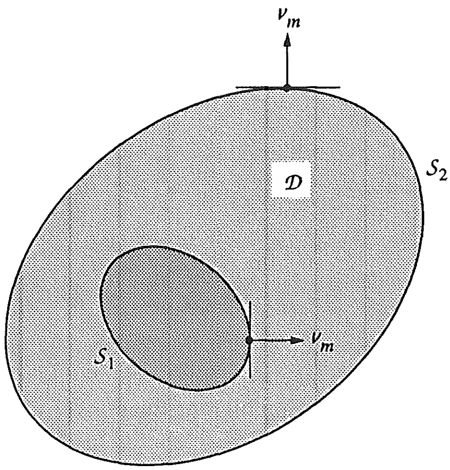


Figure 15.2-3 Domain $\mathcal{D}$ bounded internally by the closed surface S_1 and externally by the closed surface S_2.

$$= \Delta^{+}_{m,r,p,q} \int_{x\in S_2} \nu_m \left[C_t(-\tau^{A}_{p,q}, v^{B}_r; x,t) - C_t(-\tau^{B}_{p,q}, v^{A}_r; x,t) \right] dA , \qquad (15.2\text{-}8)$$

i.e. that the surface integral is an invariant.

15.3 The time-domain reciprocity theorem of the time correlation type

The time-domain reciprocity theorem of the time correlation type follows upon considering the *local interaction quantity* $\Delta^{+}_{m,r,p,q}\partial_m[R_t(-\tau^{A}_{p,q},v^{B}_r;x,t) + R_t(-\tau^{B}_{p,q},v^{A}_r;x,-t)]$. On account of Equations (B.1-14) and (B.1-18) and the symmetry of the convolution operator, this quantity can be rewritten as $\Delta^{+}_{m,r,p,q}\partial_m[C_t(-\tau^{A}_{p,q},J_t(v^{B}_r);x,t) + C_t(J_t(\tau^{B}_{p,q}),v^{A}_r;x,t)]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\begin{aligned}
&\Delta^{+}_{m,r,p,q}\partial_m\left[C_t(-\tau^{A}_{p,q},J_t(v^{B}_r);x,t) + C_t(J_t(-\tau^{B}_{p,q}),v^{A}_r;x,t)\right] \\
&\quad = \Delta^{+}_{k,m,p,q}\,\partial_m C_t(-\tau^{A}_{p,q},J_t(v^{B}_k);x,t) + \Delta^{+}_{i,j,n,r}\partial_n C_t(J_t(-\tau^{B}_{i,j}),v^{A}_r;x,t) \\
&\quad = -C_t(\Delta^{+}_{k,m,p,q}\partial_m\tau^{A}_{p,q},J_t(v^{B}_k);x,t) - C_t(\tau^{A}_{p,q},J_t(\Delta^{+}_{k,m,p,q}\partial_m v^{B}_k);x,t) \\
&\qquad - C_t(J_t(\Delta^{+}_{i,j,n,r}\partial_n\tau^{B}_{i,j}),v^{A}_r;x,t) - C_t(J_t(\tau^{B}_{i,j}),\Delta^{+}_{i,j,n,r}\partial_n v^{A}_r;x,t) \,. \qquad (15.3\text{-}1)
\end{aligned}$$

With the aid of Equations (15.1-1)–(15.1-4) and the rule $J_t(\partial_t f) = -\partial_t(J_t(f))$, the different terms on the right-hand side become

$$-C_t(\Delta^{+}_{k,m,p,q}\partial_m\tau^{A}_{p,q},J_t(v^{B}_k);x,t) = -\partial_t C_t(\mu^{A}_{k,r},v^{A}_r,J_t(v^{B}_k);x,t) + C_t(f^{A}_k,J_t(v^{B}_k);x,t) , \qquad (15.3\text{-}2)$$

$$-C_t(\tau^{A}_{p,q},J_t(\Delta^{+}_{k,m,p,q}\partial_m v^{B}_k);x,t) = \partial_t C_t(\tau^{A}_{p,q},J_t(\chi^{B}_{p,q,i,j}),J_t(\tau^{B}_{i,j});x,t)$$
$$- C_t(\tau^{A}_{p,q},J_t(h^{B}_{p,q});x,t)\,, \quad (15.3\text{-}3)$$

and

$$-C_t(J_t(\Delta^{+}_{i,j,n,r}\partial_n\tau^{B}_{i,j}),v^{A}_r;x,t) = \partial_t C_t(J_t(\mu^{B}_{r,k}),J_t(v^{B}_k),v^{A}_r;x,t) + C_t(J_t(f^{B}_r),v^{A}_r;x,t)\,, \quad (15.3\text{-}4)$$

$$-C_t(J_t(\tau^{B}_{i,j}),\Delta^{+}_{i,j,n,r}\partial_n v^{A}_r;x,t) = -\partial_t C_t(J_t(\tau^{B}_{i,j}),\chi^{A}_{i,j,p,q},\tau^{A}_{p,q};x,t)$$
$$- C_t(J_t(\tau^{B}_{i,j}),h^{A}_{i,j};x,t)\,, \quad (15.3\text{-}5)$$

in which the convolution of three functions is a shorthand notation for the convolution of a function with the convolution of two other functions. (Note that in the convolution operation the order of the operators is immaterial.) Combining Equations (15.3-2)–(15.3-5) with Equation (15.3-1), it is found that

$$\Delta^{+}_{m,r,p,q}\partial_m\left[R_t(-\tau^{A}_{p,q},v^{B}_r;x,t) + R_t(-\tau^{B}_{p,q},v^{A}_r;x,-t)\right]$$
$$= \Delta^{+}_{m,r,p,q}\partial_m\left[C_t(-\tau^{A}_{p,q},J_t(v^{B}_r);x,t) + C_t(J_t(-\tau^{B}_{p,q}),v^{A}_r;x,t)\right]$$
$$= \partial_t C_t(J_t(\mu^{B}_{r,k}) - \mu^{A}_{k,r},v^{A}_r,J_t(v^{B}_k);x,t)$$
$$+ \partial_t C_t(J_t(\chi^{B}_{p,q,i,j}) - \chi^{A}_{i,j,p,q},\tau^{A}_{p,q},J_t(\tau^{B}_{i,j});x,t) + C_t(f^{A}_k,J_t(v^{B}_k);x,t)$$
$$- C_t(\tau^{A}_{p,q},J_t(h^{B}_{p,q});x,t) + C_t(J_t(f^{B}_r),v^{A}_r;x,t) - C_t(J_t(\tau^{B}_{i,j}),h^{A}_{i,j};x,t)\,. \quad (15.3\text{-}6)$$

Equation (15.3-6) is the *local form of elastodynamic reciprocity theorem of the time correlation type*. The first two terms on the right-hand side are representative for the differences (contrasts) in the elastodynamic properties of the solids present in the two states; these terms vanish at those locations where $J_t\mu^{B}_{r,k}(x,t) = \mu^{A}_{k,r}(x,t)$ and $J_t(\chi^{B}_{p,q,i,j})(x,t) = \chi^{A}_{i,j,p,q}(x,t)$ for all $t\in\mathcal{R}$. In case the latter conditions hold, the two media are known as each other's *time-reverse adjoint*. Note is this respect that the time-reverse adjoint of a causal (anti-causal) medium is an anti-causal (causal) medium. The last four terms on the right-hand side of Equation (15.3-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (15.3-6) are continuous. Upon integrating Equation (15.3-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 15.3-1)

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[R_t(-\tau^{A}_{p,q},v^{B}_r;x,t) + R_t(-\tau^{B}_{p,q},v^{A}_r;x,-t)\right]\mathrm{d}A$$
$$= \Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[-C_t(\tau^{A}_{p,q},J_t(v^{B}_r);x,t) - C_t(J_t(\tau^{B}_{p,q}),v^{A}_r;x,t)\right]\mathrm{d}A$$
$$= \int_{x\in\mathcal{D}}\left[\partial_t C_t(J_t(\mu^{B}_{r,k}) - \mu^{A}_{k,r},v^{A}_r,J_t(v^{B}_k);x,t) + \partial_t C_t(J_t(\chi^{B}_{p,q,i,j}) - \chi^{A}_{i,j,p,q},\tau^{A}_{p,q},J_t(\tau^{B}_{i,j});x,t)\right]\mathrm{d}V$$

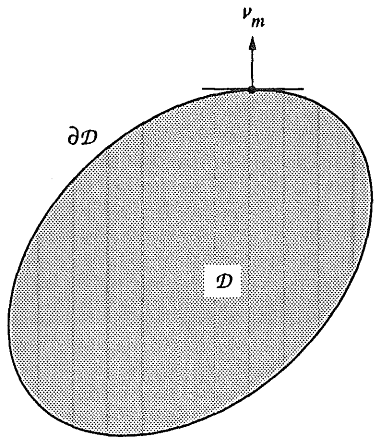


Figure 15.3-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

$$+\int_{x\in\mathcal{D}}\Big[\mathrm{C}_t(f_k^{\mathrm{A}},\mathrm{J}_t(v_k^{\mathrm{B}});x,t)-\mathrm{C}_t(\tau_{p,q}^{\mathrm{A}},\mathrm{J}_t(h_{p,q}^{\mathrm{B}});x,t)$$

$$+\mathrm{C}_t(\mathrm{J}_t(f_r^{\mathrm{B}}),v_r^{\mathrm{A}};x,t)-\mathrm{C}_t(\mathrm{J}_t(\tau_{i,j}^{\mathrm{B}}),h_{i,j}^{\mathrm{A}};x,t)\Big]\,\mathrm{d}V\,. \tag{15.3-7}$$

Equation (15.3-7) is the *global form, for the bounded domain $\mathcal{D}$, of the elastodynamic reciprocity theorem of the time correlation type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (15.1-5) and (15.1-6)), and that the contributions from boundary surfaces of elastodynamically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (15.1-7) or Equation (15.1-8)). In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an elastodynamically impenetrable object. In the right-hand side, the first integral is representative for the differences (contrasts) in the elastodynamic properties of the solids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's time-reverse adjoint. The second integral on the right-hand side is representative for the action of the sources present in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In a number of cases the reciprocity theorem Equation (15.3-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 15.1 are made and Equation (15.3-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with centre at the origin and radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 15.3-2).

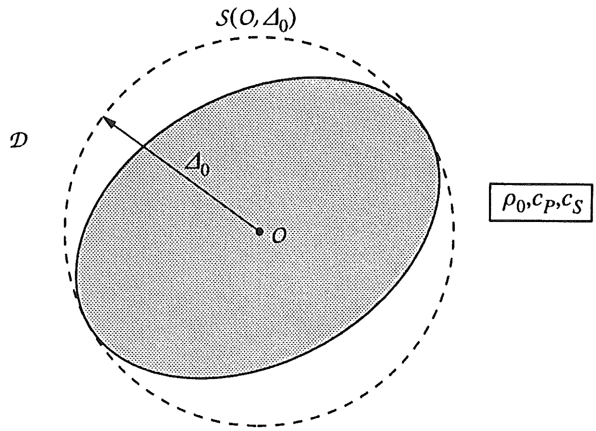


Figure 15.3-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the solid is homogeneous, isotropic and lossless.

Since outside the sphere $\mathcal{S}(O,\Delta_0)$ that is used to define the embedding (see Section 15.1) the media are each other's time-reverse adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(O,\Delta)$ is, in any case, independent of the value of Δ for $\Delta > \Delta_0$ (see Exercise 15.3-2). Whether or not this contribution vanishes as $\Delta \to \infty$ depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the time correlations occurring in the integrands of Equation (15.3-7) are neither causal nor anti-causal, and the contribution of $\mathcal{S}(O,\Delta)$ is a non-vanishing function that is independent of the value of Δ. If, however, state A is chosen to be causal and state B is chosen to be anti-causal (which, for example, can apply to the case where state B is a computational one), the correlations occurring in Equation (15.3-7) are causal as well and the contribution from $\mathcal{S}(O,\Delta)$ vanishes for sufficiently large values of Δ.

The time-domain reciprocity theorem of the time correlation type is mainly used in the modelling of inverse source problems (see Section 15.10) and inverse scatttering problems (see Section 15.11). References to the earlier literature on the subject can be found in a paper by De Hoop (1988).

Exercises

Exercise 15.3-1

To what form do the contrast-in-media terms in the reciprocity theorems Equations (15.3-6) and (15.3-7) reduce if the solids in states A and B are both instantaneously reacting?

Answer:

$$\partial_t \mathrm{C}_t(\mathrm{J}_t(\mu^{\mathrm{B}}_{r,k}) - \mu^{\mathrm{A}}_{k,r}, v^{\mathrm{A}}_r, \mathrm{J}_t(v^{\mathrm{B}}_k); x, t) = \left[\rho^{\mathrm{B}}_{r,k}(x) - \rho^{\mathrm{A}}_{k,r}(x)\right] \partial_t \mathrm{C}_t(v^{\mathrm{A}}_r, \mathrm{J}_t(v^{\mathrm{B}}_k); x, t)$$

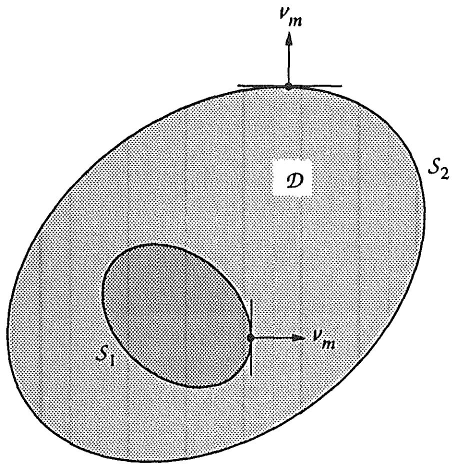


Figure 15.3-3 Domain $\mathcal{D}$ bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

and

$$\partial_t \mathrm{C}_t(\mathrm{J}_t(\chi^{\mathrm{B}}_{p,q,i,j}) - \chi^{\mathrm{A}}_{i,j,p,q}, \tau^{\mathrm{A}}_{p,q}, \mathrm{J}_t(\tau^{\mathrm{B}}_{i,j}); x,t) = \left[S^{\mathrm{B}}_{p,q,i,j}(x) - S^{\mathrm{A}}_{i,j,p,q}(x)\right] \partial_t \mathrm{C}_t(\tau^{\mathrm{A}}_{p,q}, \mathrm{J}_t(\tau^{\mathrm{B}}_{i,j}); x,t) \, .$$

Exercise 15.3-2

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 15.3-3. The reciprocity theorem Equation (15.3-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, in either state A or state B, and the solid in $\mathcal{D}$ in state B is in its elastodynamic properties the time-reverse adjoint of the one in state A. Prove that

$$\Delta^{+}_{m,r,p,q} \int_{x \in \mathcal{S}_1} \nu_m \left[\mathrm{C}_t(-\tau^{\mathrm{A}}_{p,q}, \mathrm{J}_t(v^{\mathrm{B}}_r); x,t) + \mathrm{C}_t(\mathrm{J}_t(-\tau^{\mathrm{B}}_{p,q}), v^{\mathrm{A}}_r; x,t)\right] \mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q} \int_{x \in \mathcal{S}_2} \nu_m \left[\mathrm{C}_t(-\tau^{\mathrm{A}}_{p,q}, \mathrm{J}_t(v^{\mathrm{B}}_r); x,t) + \mathrm{C}_t(\mathrm{J}_t(-\tau^{\mathrm{B}}_{p,q}), v^{\mathrm{A}}_r; x,t)\right] \mathrm{d}A \, , \qquad (15.3\text{-}8)$$

i.e. that the surface integral is an invariant.

15.4 The complex frequency-domain reciprocity theorem of the time convolution type

The complex frequency-domain reciprocity theorem of the time convolution type follows upon considering the *local interaction quantity* $\Delta^{+}_{m,r,p,q}\partial_m[-\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{v}^{\mathrm{B}}_r(x,s) + \hat{\tau}^{\mathrm{B}}_{p,q}(x,s)\hat{v}^{\mathrm{A}}_r(x,s)]$.

Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\Delta^{+}_{m,r,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{v}^{B}_{r}(\boldsymbol{x},s)+\hat{\tau}^{B}_{p,q}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)\right]$$

$$=\Delta^{+}_{k,m,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{v}^{B}_{k}(\boldsymbol{x},s)\right]+\Delta^{+}_{i,j,n,r}\partial_n\left[\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)\right]$$

$$=\left[-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\right]\hat{v}^{B}_{k}(\boldsymbol{x},s)-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\left[\Delta^{+}_{k,m,p,q}\partial_m\hat{v}^{B}_{k}(\boldsymbol{x},s)\right]$$

$$+\left[\Delta^{+}_{i,j,n,r}\partial_n\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\right]\hat{v}^{A}_{r}(\boldsymbol{x},s)+\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\left[\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{A}_{r}(\boldsymbol{x},s)\right]. \tag{15.4-1}$$

With the aid of Equations (15.1-14)–(15.1-17), the different terms on the right-hand side become

$$\left[-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\right]\hat{v}^{B}_{k}(\boldsymbol{x},s)=-\hat{\zeta}^{A}_{k,r}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)\hat{v}^{B}_{k}(\boldsymbol{x},s)+\hat{f}^{A}_{k}(\boldsymbol{x},s)\hat{v}^{B}_{k}(\boldsymbol{x},s)\,, \tag{15.4-2}$$

$$-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\left[\Delta^{+}_{k,m,p,q}\partial_m\hat{v}^{B}_{k}(\boldsymbol{x},s)\right]=-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{\eta}^{B}_{p,q,i,j}(\boldsymbol{x},s)\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{h}^{B}_{p,q}(\boldsymbol{x},s)\,, \tag{15.4-3}$$

and

$$\left[\Delta^{+}_{i,j,n,r}\partial_n\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\right]\hat{v}^{A}_{r}(\boldsymbol{x},s)=+\hat{\zeta}^{B}_{r,k}(\boldsymbol{x},s)\hat{v}^{B}_{k}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)-\hat{f}^{B}_{r}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)\,, \tag{15.4-4}$$

$$\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\left[\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{A}_{r}(\boldsymbol{x},s)\right]=\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\hat{\eta}^{A}_{i,j,p,q}(\boldsymbol{x},s)\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)+\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\hat{h}^{A}_{i,j}(\boldsymbol{x},s)\,. \tag{15.4-5}$$

Combining Equations (15.4-2)–(15.4-5) with Equation (15.4-1), it is found that

$$\Delta^{+}_{m,r,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{v}^{B}_{r}(\boldsymbol{x},s)+\hat{\tau}^{B}_{p,q}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)\right]$$

$$=\left[\hat{\zeta}^{B}_{r,k}(\boldsymbol{x},s)-\hat{\zeta}^{A}_{k,r}(\boldsymbol{x},s)\right]\hat{v}^{A}_{r}(\boldsymbol{x},s)\hat{v}^{B}_{k}(\boldsymbol{x},s)$$

$$-\left[\hat{\eta}^{B}_{p,q,i,j}(\boldsymbol{x},s)-\hat{\eta}^{A}_{i,j,p,q}(\boldsymbol{x},s)\right]\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)$$

$$+\hat{f}^{A}_{k}(\boldsymbol{x},s)\hat{v}^{B}_{k}(\boldsymbol{x},s)-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{h}^{B}_{p,q}(\boldsymbol{x},s)-\hat{f}^{B}_{r}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)+\hat{\tau}^{B}_{i,j}(\boldsymbol{x},s)\hat{h}^{A}_{i,j}(\boldsymbol{x},s)\,. \tag{15.4-6}$$

Equation (15.4-6) is the *local form of the complex frequency-domain counterpart of the elastodynamic reciprocity theorem of the time convolution type.* The first two terms on the right-hand side are representative for the differences (contrasts) in the elastodynamic properties of the solids present in the two states; these terms vanish at those locations where $\hat{\zeta}^{B}_{r,k}(\boldsymbol{x},s)=\hat{\zeta}^{A}_{k,r}(\boldsymbol{x},s)$ and $\hat{\eta}^{B}_{p,q,i,j}(\boldsymbol{x},s)=\hat{\eta}^{A}_{i,j,p,q}(\boldsymbol{x},s)$ for all s in the domain in the complex s plane where Equation (15.4-6) holds. In case the latter conditions hold, the two media are denoted as each other's *adjoint*. The last four terms on the right-hand side of Equation (15.4-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (15.4-6) are continuous. Upon integrating Equation (15.4-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 15.4-1)

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[-\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{v}^{\mathrm{B}}_r(x,s)+\hat{\tau}^{\mathrm{B}}_{p,q}(x,s)\hat{v}^{\mathrm{A}}_r(x,s)\right]\mathrm{d}A$$

$$=\int_{x\in\mathcal{D}}\left\{\left[\hat{\zeta}^{\mathrm{B}}_{r,k}(x,s)-\hat{\zeta}^{\mathrm{A}}_{k,r}(x,s)\right]\hat{v}^{\mathrm{A}}_r(x,s)\hat{v}^{\mathrm{B}}_k(x,s)\right.$$

$$\left.-\left[\hat{\eta}^{\mathrm{B}}_{p,q,i,j}(x,s)-\hat{\eta}^{\mathrm{A}}_{i,j,p,q}(x,s)\right]\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{\tau}^{\mathrm{B}}_{i,j}(x,s)\right\}\mathrm{d}V$$

$$+\int_{x\in\mathcal{D}}\left[\hat{f}^{\mathrm{A}}_k(x,s)\hat{v}^{\mathrm{B}}_k(x,s)-\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{h}^{\mathrm{B}}_{p,q}(x,s)\right.$$

$$\left.-\hat{f}^{\mathrm{B}}_r(x,s)\hat{v}^{\mathrm{A}}_r(x,s)+\hat{\tau}^{\mathrm{B}}_{i,j}(x,s)\hat{h}^{\mathrm{A}}_{i,j}(x,s)\right]\mathrm{d}V\,. \tag{15.4-7}$$

Equation (15.4-7) is the *global form*, for the bounded domain $\mathcal{D}$, of the *complex frequency-domain counterpart of the elastodynamic reciprocity theorem of the time convolution type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (15.1-18) and (15.1-19)), and that the contributions from boundary surfaces of elastodynamically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (15.1-20) or Equation (15.1-21)). In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains

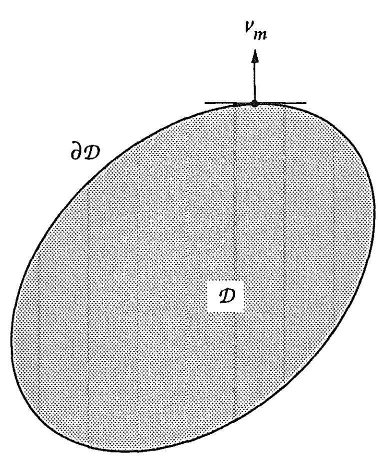


Figure 15.4-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

insofar as parts of this boundary do not coincide with the boundary surface of an elastodynamically impenetrable object. In the right-hand side, the first integral is representative of the differences (contrasts) in the elastodynamic properties of the solids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's adjoint. The second integral on the right-hand side is representative for the action of the sources in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem Equation (15.4-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 15.1 are made and Equation (15.4-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with centre at the origin and radius Δ, after which the limit $\Delta \to \infty$ is taken (Figure 15.4-2).

Whether or not the surface integral contribution over $\mathcal{S}(\mathcal{O},\Delta)$ does vanish as $\Delta \to \infty$ depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (15.1-22)–(15.1-26) apply for sufficiently large values of Δ. Then, the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes in the limit $\Delta \to \infty$. If, however, at least one of the two states is chosen to be non-causal (which, for example, can apply to the case where one of the two states is a computational one), the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ does not vanish, no matter how large Δ is chosen. Since outside the sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ that is used to define the embedding (see Section 15.1) the media are each other's adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(\mathcal{O},\Delta)$ is, however, independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 15.4-4).

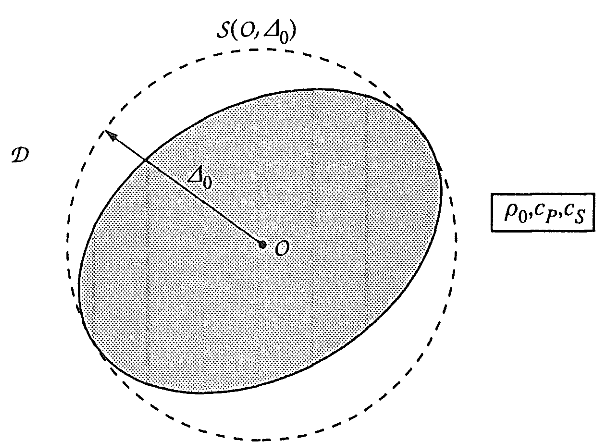


Figure 15.4-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(\mathcal{O},\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(\mathcal{O},\Delta_0)$ is the sphere outside which the solid is homogeneous, isotropic and lossless.

The complex frequency-domain reciprocity theorem of the time convolution type is mainly used for investigating the transmission/reception reciprocity properties of elastodynamic sources and receivers (see Sections 15.6 and 15.7) and for modelling direct (forward) source problems (see Section 15.8) and direct (forward) scattering problems (see Section 15.9).

Exercises

Exercise 15.4-1

Show, by taking the Laplace transform with respect to time, that Equation (15.4-6) follows from Equation (15.2-6).

Exercise 15.4-2

Show, by taking the Laplace transform with respect to time, that Equation (15.4-7) follows from Equation (15.2-7).

Exercise 15.4-3

To what form do the contrast-in-media terms in the reciprocity theorems Equations (15.4-6) and (15.4-7) reduce if the solids in states A and B are both instantaneously reacting?

Answers:

$$\left[\hat{\zeta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},s) - \hat{\zeta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)\right]\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_k(\boldsymbol{x},s) = s\left[\rho^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \rho^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right]\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_k(\boldsymbol{x},s)$$

and

$$\left[\hat{\eta}^{\mathrm{B}}_{p,q,i,j}(\boldsymbol{x},s) - \hat{\eta}^{\mathrm{A}}_{i,j,p,q}(\boldsymbol{x},s)\right]\hat{\tau}^{\mathrm{A}}_{p,q}(\boldsymbol{x},s)\hat{\tau}^{\mathrm{B}}_{i,j}(\boldsymbol{x},s) = s\left[S^{\mathrm{B}}_{p,q,i,j}(\boldsymbol{x}) - S^{\mathrm{A}}_{i,j,p,q}(\boldsymbol{x})\right]\hat{\tau}^{\mathrm{A}}_{p,q}(\boldsymbol{x},s)\hat{\tau}^{\mathrm{B}}_{i,j}(\boldsymbol{x},s)\,.$$

Exercise 15.4-4

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 15.4-3. The reciprocity theorem Equation (15.4-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, in either state A or state B, and the solid in $\mathcal{D}$ in state B is in its elastodynamic properties adjoint to the one in state A. Prove that

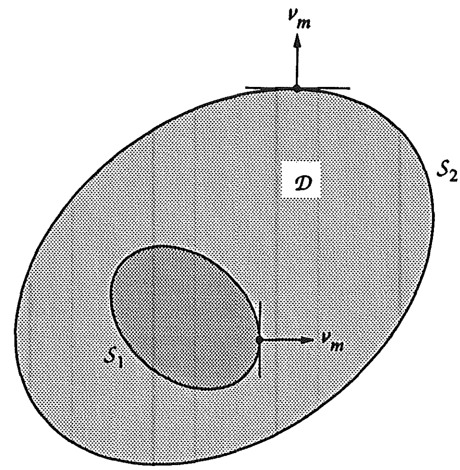


Figure 15.4-3 Domain $\mathcal{D}$ bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

$$\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}_1}\nu_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,s)+\hat{\tau}^{B}_{p,q}(x,s)\hat{v}^{A}_{r}(x,s)\right]\mathrm{d}A$$

$$=\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}_2}\nu_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,s)+\hat{\tau}^{B}_{p,q}(x,s)\hat{v}^{A}_{r}(x,s)\right]\mathrm{d}A\,, \tag{15.4-8}$$

i.e. the surface integral is an invariant.

15.5 The complex frequency-domain reciprocity theorem of the time correlation type

The complex frequency-domain reciprocity theorem of the time correlation type follows upon considering the *local interaction quantity* $\Delta^{+}_{m,r,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,-s)-\hat{\tau}^{B}_{p,q}(x,-s)\hat{v}^{A}_{r}(x,s)\right]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\Delta^{+}_{m,r,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,-s)-\hat{\tau}^{B}_{p,q}(x,-s)\hat{v}^{A}_{r}(x,s)\right]$$

$$=\Delta^{+}_{k,m,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{k}(x,-s)\right]+\Delta^{+}_{i,j,n,r}\partial_n\left[-\hat{\tau}^{B}_{i,j}(x,-s)\hat{v}^{A}_{r}(x,s)\right]$$

$$=\left[-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{A}_{p,q}(x,s)\right]\hat{v}^{B}_{k}(x,-s)-\hat{\tau}^{A}_{p,q}(x,s)\left[\Delta^{+}_{k,m,p,q}\partial_m\hat{v}^{B}_{k}(x,-s)\right]$$

$$+\left[-\Delta^{+}_{i,j,n,r}\partial_n\hat{\tau}^{B}_{i,j}(x,-s)\right]\hat{v}^{A}_{r}(x,s)-\hat{\tau}^{B}_{i,j}(x,-s)\left[\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{A}_{r}(x,s)\right]. \tag{15.5-1}$$

With the aid of Equations (15.1-14)–(15.1-17), the different terms on the right-hand side become

$$\left[-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{A}_{p,q}(x,s)\right]\hat{v}^{B}_{k}(x,-s)$$

$$= -\hat{\xi}^{A}_{k,r}(x,s)\hat{v}^{A}_{r}(x,s)\hat{v}^{B}_{k}(x,-s) + \hat{f}^{A}_{k}(x,s)\hat{v}^{B}_{k}(x,-s)\,, \tag{15.5-2}$$

$$-\hat{\tau}^{A}_{p,q}(x,s)\left[\Delta^{+}_{k,m,p,q}\partial_m\hat{v}^{B}_{k}(x,-s)\right]$$

$$= -\hat{\tau}^{A}_{p,q}(x,s)\hat{\eta}^{B}_{p,q,i,j}(x,-s)\hat{\tau}^{B}_{i,j}(x,-s) - \hat{\tau}^{A}_{p,q}(x,s)\hat{h}^{B}_{p,q}(x,-s)\,, \tag{15.5-3}$$

and

$$\left[-\Delta^{+}_{i,j,n,r}\partial_n\hat{\tau}^{B}_{i,j}(x,-s)\right]\hat{v}^{A}_{r}(x,s)$$

$$= -\hat{\xi}^{B}_{r,k}(x,-s)\hat{v}^{B}_{k}(x,-s)\hat{v}^{A}_{r}(x,s) + \hat{f}^{B}_{r}(x,-s)\hat{v}^{A}_{r}(x,s)\,, \tag{15.5-4}$$

$$-\hat{\tau}^{B}_{i,j}(x,-s)\left[\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{A}_{r}(x,s)\right]$$

$$= -\hat{\tau}^{B}_{i,j}(x,-s)\hat{\eta}^{A}_{i,j,p,q}(x,s)\hat{\tau}^{A}_{p,q}(x,s) - \hat{\tau}^{B}_{i,j}(x,-s)\hat{h}^{A}_{i,j}(x,s)\,. \tag{15.5-5}$$

Combining Equations (15.5-2)–(15.5-5) with Equation (15.5-1), it is found that

$$\begin{aligned}
&\Delta^{+}_{m,r,p,q}\partial_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,-s) - \hat{\tau}^{B}_{p,q}(x,-s)\hat{v}^{A}_{r}(x,s)\right]\\
&= \left[-\hat{\xi}^{B}_{r,k}(x,-s) - \hat{\xi}^{A}_{k,r}(x,s)\right]\hat{v}^{A}_{r}(x,s)\hat{v}^{B}_{k}(x,-s)\\
&\quad + \left[-\hat{\eta}^{B}_{p,q,i,j}(x,-s) - \hat{\eta}^{A}_{i,j,p,q}(x,s)\right]\hat{\tau}^{A}_{p,q}(x,s)\hat{\tau}^{B}_{i,j}(x,-s)\\
&\quad + \hat{f}^{A}_{k}(x,s)\hat{v}^{B}_{k}(x,-s) - \hat{\tau}^{A}_{p,q}(x,s)\hat{h}^{B}_{p,q}(x,-s)\\
&\quad + \hat{f}^{B}_{r}(x,-s)\hat{v}^{A}_{r}(x,s) - \hat{\tau}^{B}_{i,j}(x,-s)\hat{h}^{A}_{i,j}(x,s)\,.
\end{aligned} \tag{15.5-6}$$

Equation (15.5-6) is the *local form of the complex frequency-domain counterpart of the elastodynamic reciprocity theorem of the time correlation type.* The first two terms on the right-hand side are representative of the differences (contrasts) in the elastodynamic properties of the solids present in the two states; these terms vanish at those locations where $\hat{\xi}^{B}_{r,k}(x,-s) = -\hat{\xi}^{A}_{k,r}(x,s)$ and $\hat{\eta}^{B}_{p,q,i,j}(x,-s) = -\hat{\eta}^{A}_{i,j,p,q}(x,s)$ for all s in the domain in the complex s plane where Equation (15.5-6) holds. In case the latter conditions hold, the two media are denoted as each other's *time-reverse adjoint*. The last four terms on the right-hand side of Equation (15.5-6) are representative of the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (15.5-6) are continuous. Upon integrating Equation (15.5-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 15.5-1)

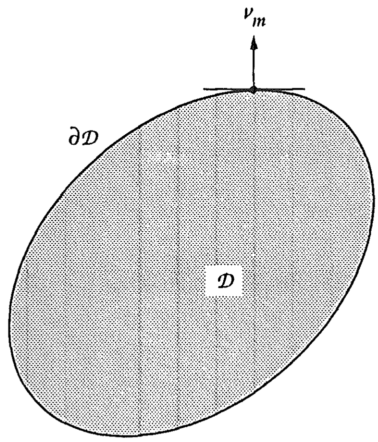


Figure 15.5-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

$$\begin{aligned}
&\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}}\nu_m\left[-\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{v}^{\mathrm{B}}_r(x,-s)-\hat{\tau}^{\mathrm{B}}_{p,q}(x,-s)\hat{v}^{\mathrm{A}}_r(x,s)\right]\mathrm{d}A\\
&=\int_{x\in\mathcal{D}}\Big\{\left[-\hat{\xi}^{\mathrm{B}}_{r,k}(x,-s)-\hat{\xi}^{\mathrm{A}}_{k,r}(x,s)\right]\hat{v}^{\mathrm{A}}_r(x,s)\hat{v}^{\mathrm{B}}_k(x,-s)\\
&+\left[-\hat{\eta}^{\mathrm{B}}_{p,q,i,j}(x,-s)-\hat{\eta}^{\mathrm{A}}_{i,j,p,q}(x,s)\right]\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{\tau}^{\mathrm{B}}_{i,j}(x,-s)\Big\}\,\mathrm{d}V\\
&+\int_{x\in\mathcal{D}}\Big[\hat{f}^{\mathrm{A}}_k(x,s)\hat{v}^{\mathrm{B}}_k(x,-s)-\hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{h}^{\mathrm{B}}_{p,q}(x,-s)\\
&+\hat{f}^{\mathrm{B}}_r(x,-s)\hat{v}^{\mathrm{A}}_r(x,s)-\hat{\tau}^{\mathrm{B}}_{i,j}(x,-s)\hat{h}^{\mathrm{A}}_{i,j}(x,s)\Big]\,\mathrm{d}V\,. \qquad (15.5\text{-}7)
\end{aligned}$$

Equation (15.5-7) is the *global form*, for the bounded domain $\mathcal{D}$, of the *complex frequency-domain counterpart of the elastodynamic reciprocity theorem of the time correlation type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (15.1-18) and (15.1-19)), and that the contributions from boundary surfaces of elastodynamically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (15.1-20) or Equation (15.1-21)). On the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an elastodynamically impenetrable object. On the right-hand side, the first integral is representative of the differences (contrasts) in the elastodynamic properties of the solids present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's time-reverse adjoint. The second integral on the right-hand side is representative of the action of the sources in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

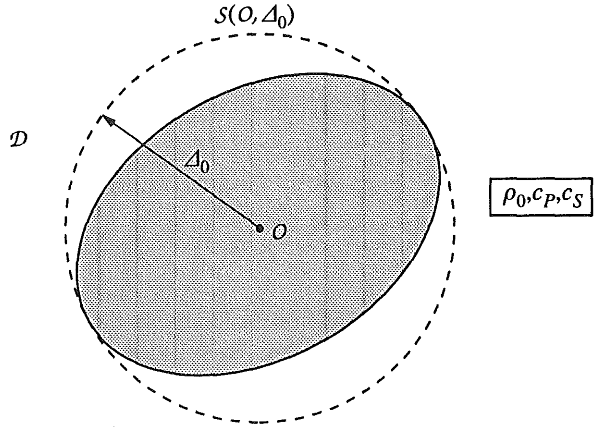


Figure 15.5-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the solid is homogeneous, isotropic and lossless.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem Equation (15.5-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 15.1 are made and Equation (15.5-7) is first applied to the domain interior to the sphere $\mathcal{S}(O,\Delta)$ with centre at the origin and radius Δ, after which the limit $\Delta \rightarrow \infty$ is taken (Figure 15.5-2).

Whether or not the surface integral contribution over $\mathcal{S}(O,\Delta)$ does vanish as $\Delta \rightarrow \infty$, depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (15.1-22)–(15.1-26) apply for sufficiently large values of Δ. Then, since outside the sphere $\mathcal{S}(O,\Delta_0)$ that is used to define the embedding (see Section 15.1) the media are each other's time-reverse adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(O,\Delta)$ is, in any case, independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 15.5-4). If, however, state A is chosen to be causal and state B is chosen to be anti-causal (which, for example, can apply to the case where state B is a computational one), the contribution from $\mathcal{S}(O,\Delta)$ vanishes for sufficiently large values of Δ.

The complex frequency-domain reciprocity theorem of the time correlation type is mainly used in modelling inverse source problems (see Section 15.10) and inverse scattering problems (see Section 15.11).

Exercises

Exercise 15.5-1

Show, by taking the Laplace transform with respect to time, that Equation (15.5-6) follows from Equation (15.3-6).

Exercise 15.5-2

Show, by taking the Laplace transform with respect to time, that Equation (15.5-7) follows from Equation (15.3-7).

Exercise 15.5-3

To what form do the contrast-in-media terms in the reciprocity theorems Equations (15.5-6) and (15.5-7) reduce if the solids in states A and B are both instantaneously reacting?

Answer:

$$\left[-\hat{\xi}^{\,B}_{r,k}(x,-s) - \hat{\xi}^{\,A}_{k,r}(x,s)\right]\hat{v}^{A}_{r}(x,s)\hat{v}^{B}_{k}(x,-s) = s\left[\hat{\rho}^{B}_{r,k}(x) - \hat{\rho}^{A}_{k,r}(x)\right]\hat{v}^{A}_{r}(x,s)\hat{v}^{B}_{k}(x,-s)$$

and

$$\left[-\hat{\eta}^{B}_{p,q,i,j}(x,-s) - \hat{\eta}^{A}_{i,j,p,q}(x,s)\right]\hat{\tau}^{A}_{p,q}(x,s)\hat{\tau}^{B}_{i,j}(x,-s) = s\left[S^{B}_{p,q,i,j}(x) - S^{A}_{i,j,p,q}(x)\right]\hat{\tau}^{A}_{p,q}(x,s)\hat{\tau}^{B}_{i,j}(x,-s)\,.$$

Exercise 15.5-4

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 15.5-3. The reciprocity theorem (Equation (15.5-7)) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present either in state A or in state B, and the solid in $\mathcal{D}$ in state B is in its elastodynamic properties the time-reverse adjoint of the one in state A. Prove that

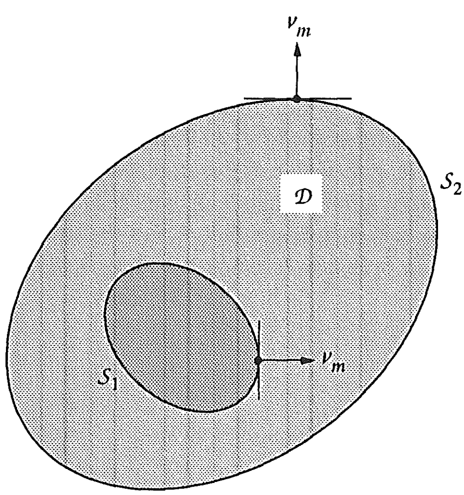


Figure 15.5-3 Domain $\mathcal{D}$ bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

$$\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}_1}\nu_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,-s)-\hat{\tau}^{B}_{p,q}(x,-s)\hat{v}^{A}_{r}(x,s)\right]\mathrm{d}A$$

$$=\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}_2}\nu_m\left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,-s)-\hat{\tau}^{B}_{p,q}(x,-s)\hat{v}^{A}_{r}(x,s)\right]\mathrm{d}A\,, \tag{15.5-8}$$

i.e. the surface integral is an invariant.

15.6 Transmission/reception reciprocity properties of a pair of elastodynamic transducers

The transmission and the reception of elastic waves take place through the action of elastodynamic sources and receivers (both also known as *transducers*). The sources are divided into two types, viz., those whose action can be computationally modelled by prescribed values of (volume or surface) *forces*, and those whose action can be computationally modelled by prescribed values of the (volume or surface) *deformation rates.* Examples of force sources are: the hydraulically driven, truck-mounted mechanical vibrator and piezoelectric, magnetoelastic or electrodynamic devices. Examples of deformation rate sources are: instantaneous stress release through rupture as an earthquake mechanism, crack formation in solid materials, and rigid mechanical stamps. Also the receivers are divided into two types, viz., those whose action can computationally be modelled by their sensitivity to the *particle displacement* (throughout their volume or at their surface), and those whose action can computationally be modelled by their sensitivity to the *dynamic stress.* Examples of particle velocity sensitive receivers are: geophones in seismic prospecting, earthquake seismographs, and piezoelectric, magnetoelastic or electrodynamic transducers, all of them if their inertia effects can be neglected. Examples of dynamic stress sensitive receivers (usually at their surface) are all devices listed under particle velocity sensitive receivers in case their inertia effects must be taken into account.

To analyse the reciprocity properties of the different transducers in their transmitting and receiving situations, we consider the fundamental configuration of two transducers that are surrounded by a solid. The entire configuration occupies a bounded or unbounded domain $\mathcal{D}$. Transducer A occupies the bounded domain Tr_A with boundary surface $\partial\mathrm{Tr}_A$, and unit vector along the normal ν_m oriented away from Tr_A. Transducer B occupies the bounded domain Tr_B with boundary surface $\partial\mathrm{Tr}_B$, and unit vector ν_m along the normal oriented away from Tr_B. The domain exterior to $\mathrm{Tr}_A\cup\partial\mathrm{Tr}_A$ is denoted by Tr_A'; the domain exterior to $\mathrm{Tr}_B\cup\partial\mathrm{Tr}_B$ is denoted by Tr_B'. The domains Tr_A and Tr_B are disjoint (Figure 15.6-1).

As to the boundary conditions across interfaces between solid parts with different elastodynamic properties and the boundary conditions at the boundary surfaces of elastodynamically impenetrable objects, the provisions necessary for the global reciprocity theorems to hold are made. If the embedding solid occupies a bounded domain, the exterior of this domain is assumed to be elastodynamically impenetrable. If the embedding solid occupies an unbounded domain, the standard limiting procedure of Section 15.1 for handling an unbounded domain is applied. Since transmission and reception are both causal phenomena, the transmission/reception reciprocity properties are based on the reciprocity theorems (15.2-7) and (15.4-7) of the time convolution type, in which theorem causality is preserved.

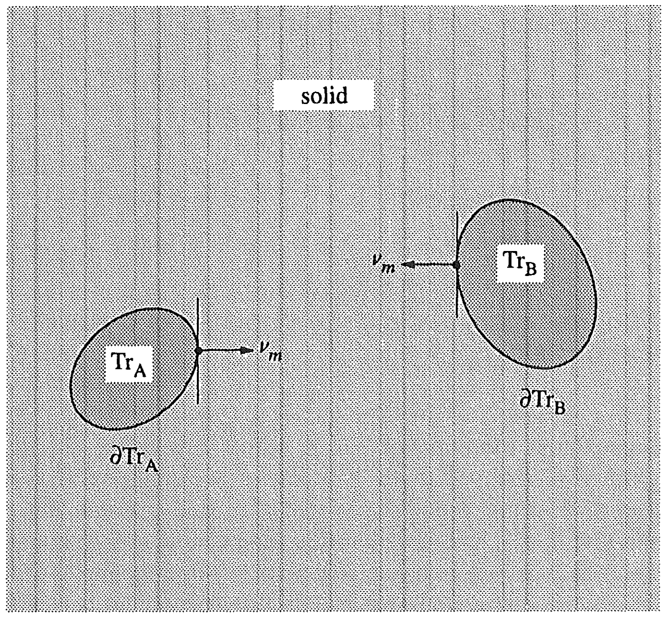


Figure 15.6-1 Configuration of the transmission/reception reciprocity properties of a pair of elastodynamic transducers ($\mathrm{Tr_A}$ and $\mathrm{Tr_B}$) surrounded by a solid.

Volume action transducers

A *volume action transducer* is characterised by the property that in the transmitting mode its action can be accounted for by prescribed values of the volume source densities of force and/or deformation rate, whose common support is the domain occupied by that transducer, while in the receiving mode it is sensitive to the particle velocity and/or the dynamic stress over the domain it occupies. To investigate the transmission/reception reciprocity properties of a pair of such transducers, we take state A to be the causal elastodynamic state for which the volume source densities have the support $\mathrm{Tr_A}$ (i.e. in state A, Transducer A is the transmitting transducer and Transducer B is the receiving transducer). Furthermore, we take state B to be the causal elastodynamic state for which the volume source densities have the support $\mathrm{Tr_B}$ (i.e. in state B, Transducer B is the transmitting transducer and Transducer A is the receiving transducer). Application of the global time-domain reciprocity theorem of the time convolution type, Equation (15.2-7), to the entire domain $\mathcal{D}$ occupied by the configuration yields, assuming the embedding solid to be self-adjoint,

$$\int_{x\in\mathrm{Tr_A}} \left[\mathrm{C}_t(f_k^{\mathrm{A}},v_k^{\mathrm{B}};x,t) - \mathrm{C}_t(-\tau_{i,j}^{\mathrm{B}},h_{i,j}^{\mathrm{A}};x,t)\right]\mathrm{d}V$$

$$= \int_{x\in\mathrm{Tr_B}} \left[\mathrm{C}_t(f_r^{\mathrm{B}},v_r^{\mathrm{A}};x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{A}},h_{p,q}^{\mathrm{B}};x,t)\right]\mathrm{d}V. \qquad (15.6\text{-}1)$$

The complex frequency-domain counterpart of Equation (15.6-1) follows from Equation (15.4-7) as

$$\int_{x\in \mathrm{Tr}_\mathrm{A}} \left[\hat{f}_k^{\mathrm{A}}(x,s)\hat{v}_k^{\mathrm{B}}(x,s) + \hat{\tau}_{i,j}^{\mathrm{B}}(x,s)\hat{h}_{i,j}^{\mathrm{A}}(x,s)\right] \mathrm{d}V$$

$$= \int_{x\in \mathrm{Tr}_\mathrm{B}} \left[\hat{f}_r^{\mathrm{B}}(x,s)\hat{v}_r^{\mathrm{A}}(x,s) + \hat{\tau}_{p,q}^{\mathrm{A}}(x,s)\hat{h}_{p,q}^{\mathrm{B}}(x,s)\right] \mathrm{d}V . \tag{15.6-2}$$

In Equations (15.6-1) and (15.6-2), the terms containing the volume source densities of force are representative of the action of the transducer as a (volume distributed) *force type transmitting transducer*, while the terms containing the volume source densities of deformation rate are representative of the action of the transducer as a (volume distributed) *deformation rate type transmitting transducer*. Furthermore, the terms containing the particle velocity quantify the action of the transducer as a (volume distributed) *particle velocity sensitive receiving transducer*, while the terms containing the dynamic stress quantify the action of the transducer as a (volume distributed) *dynamic stress sensitive receiving transducer*. From Equations (15.6-1) and (15.6-2), and hence from the principle of reciprocity, it is concluded that a spatially distributed force type transducer is sensitive only to the particle velocity (and insensitive to the dynamic stress), while a spatially distributed deformation rate type transducer is sensitive only to the dynamic stress (and insensitive to the particle velocity). The reciprocity relations imply that the different sensitivities are related (viz. through Equations (15.6-1) and (15.6-2)).

Surface action transducers

A *surface action transducer* is characterised by the property that in its transmitting mode its action can be accounted for by prescribed values of the particle velocity and the dynamic traction (i.e. the normal component of the dynamic stress) at its boundary surface, while in its receiving mode it is sensitive to the particle velocity and the dynamic traction at that surface. This description of the action of the transducer is employed when the description of its action by volume sources is either inapplicable or irrelevant. To investigate the reciprocity properties of a pair of such transducers, we take state A to be the causal elastodynamic state for which the prescribed surface source densities have the support $\partial\mathrm{Tr}_\mathrm{A}$ (i.e. in state A, Transducer A is the transmitting transducer and Transducer B is the receiving transducer). Furthermore, we take state B to be the causal elastodynamic state for which the prescribed surface source densities have the support $\partial\mathrm{Tr}_\mathrm{B}$ (i.e. in state B, Transducer B is the transmitting transducer and Transducer A is the receiving transducer). Application of the global time-domain reciprocity theorem of the time convolution type, Equation (15.2-7), to the entire domain $\mathcal{D}\cap\mathrm{Tr}_\mathrm{A}'\cap\mathrm{Tr}_\mathrm{B}'$ exterior to the transducers yields, assuming the embedding solid to be self-adjoint,

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}_\mathrm{A}} \nu_m\left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{A}},v_r^{\mathrm{B}};x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{B}},v_r^{\mathrm{A}};x,t)\right]\mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}_\mathrm{B}} \nu_m\left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{B}},v_r^{\mathrm{A}};x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{A}},v_r^{\mathrm{B}};x,t)\right]\mathrm{d}A . \tag{15.6-3}$$

The complex frequency-domain counterpart of Equation (15.6-3) follows from Equation (15.4-7) as

$$\Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathrm{Tr}_A} \nu_m \left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,s) + \hat{\tau}^{B}_{p,q}(x,s)\hat{v}^{A}_{r}(x,s)\right] \mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathrm{Tr}_B} \nu_m \left[-\hat{\tau}^{B}_{p,q}(x,s)\hat{v}^{A}_{r}(x,s) + \hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_{r}(x,s)\right] \mathrm{d}A \,. \qquad (15.6\text{-}4)$$

In Equations (15.6-3) and (15.6-4), the terms containing the source densities of dynamic surface traction (i.e. $-\Delta^{+}_{m,r,p,q}\nu_m\tau_{p,q}$) are representative of the action of the transducer as a (surface distributed) *force type transmitting transducer*, while the terms containing the surface densities of deformation rate (i.e. $\Delta^{+}_{m,r,p,q}\nu_m v_r$) are representative of the action of the transducer as a (surface distributed) *deformation rate type transmitting transducer*. Furthermore, the terms containing the dynamic traction quantify the sensitivity of the transducer as a (surface distributed) *dynamic traction type receiving transducer*, while the terms containing the particle velocity quantify the sensitivity of the transducer as a (surface distributed) *particle velocity type receiving transducer*. From Equations (15.6-3) and (15.6-4) it is concluded that a surface distributed force type transducer is only sensitive to the particle velocity (and insensitive to the dynamic traction), while a surface distributed deformation rate type transducer is only sensitive to the dynamic traction (and insensitive to the particle velocity). The reciprocity relations imply that the different sensitivities are related (viz. through Equations (15.6-3) and (15.6-4)).

Exercises

Exercise 15.6-1

Use Equation (15.2-7) to derive the time-domain transmission/reception reciprocity theorem for a pair of transducers A and B if transducer A is a volume action transducer and transducer B is a surface action transducer. (Note the orientation of the unit vector ν_m along the normal to $\partial\mathrm{Tr}_B$.)

Answer:

$$\int_{x\in\mathrm{Tr}_A} \left[\mathrm{C}_t(f^{A}_{k},v^{B}_{k};x,t) - \mathrm{C}_t(-\tau^{B}_{i,j},h^{A}_{i,j};x,t)\right] \mathrm{d}V$$

$$= \Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathrm{Tr}_B} \nu_m \left[\mathrm{C}_t(-\tau^{B}_{p,q},v^{A}_{r};x,t) - \mathrm{C}_t(-\tau^{A}_{p,q},v^{B}_{r};x,t)\right] \mathrm{d}A \,. \qquad (15.6\text{-}5)$$

Exercise 15.6-2

Use Equation (15.4-7) to derive the complex frequency-domain transmission/reception reciprocity theorem for a pair of transducers A and B if transducer A is a volume action transducer and transducer B is a surface action transducer. (Note the orientation of the unit vector ν_m along the normal to $\partial\mathrm{Tr}_\mathrm{B}$.)

Answer:

$$\int_{x\in\mathrm{Tr}_\mathrm{A}} \left[\hat{f}_k^{\mathrm{A}}(x,s)\hat{v}_k^{\mathrm{B}}(x,s) + \hat{\tau}_{i,j}^{\mathrm{B}}(x,s)\hat{h}_{i,j}^{\mathrm{A}}(x,s)\right] \mathrm{d}V$$

$$= \Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathrm{Tr}_\mathrm{B}} \nu_m \left[-\hat{\tau}_{p,q}^{\mathrm{B}}(x,s)\hat{v}_r^{\mathrm{A}}(x,s) + \hat{\tau}_{p,q}^{\mathrm{A}}(x,s)\hat{v}_r^{\mathrm{B}}(x,s)\right] \mathrm{d}A \,. \tag{15.6-6}$$

Exercise 15.6-3

If in the interior of Tr_A and Tr_B the elastic wave-field quantities were set equal to zero and these wave-field quantities on $\partial\mathrm{Tr}_\mathrm{A}$ and $\partial\mathrm{Tr}_\mathrm{B}$ jumped to their respective boundary values, the jumps would, on account of Equations (15.1-1)–(15.1-4), give rise to surface force sources with volume densities $f_k^{\mathrm{A,B}} = -\Delta^{+}_{k,m,p,q}\nu_m\tau_{p,q}^{\mathrm{A,B}}\delta_{\partial\mathrm{Tr}_{\mathrm{A,B}}}(x)$ and surface deformation rate sources with volume densities $h_{i,j}^{\mathrm{A,B}} = \Delta^{+}_{i,j,n,r}\nu_n v_r^{\mathrm{A,B}}\delta_{\partial\mathrm{Tr}_{\mathrm{A,B}}}(x)$, where $\delta_{\mathcal{S}}(x)$ is the surface Dirac delta distribution operative on the surface $\mathcal{S}$. Show, by taking the time convolution of the inner products of $f_k^{\mathrm{A,B}}$ with $v_k^{\mathrm{B,A}}$ and of $h_{i,j}^{\mathrm{A,B}}$ with $-\tau_{i,j}^{\mathrm{B,A}}$, that in this physical picture Equation (15.6-3) is compatible with Equation (15.6-1).

Exercise 15.6-4

Show, in a manner similar to Exercise 15.6-3, that Equation (15.6-4) is compatible with Equation (15.6-2).

15.7 Transmission/reception reciprocity properties of a single elastodynamic transducer

To analyse the transmission/reception reciprocity properties of a single transducer, we consider the fundamental configuration of a single transducer surrounded by a solid. The entire configuration occupies a bounded or unbounded domain $\mathcal{D}$. The transducer occupies the bounded domain Tr, with boundary surface $\partial\mathrm{Tr}$ and unit vector along the normal ν_m oriented away from Tr (Figure 15.7-1). The domain exterior to $\mathrm{Tr}\cup\partial\mathrm{Tr}$ is denoted by Tr'.

As to the boundary conditions across interfaces between solid parts with different elastodynamic properties and the boundary conditions at the boundary surfaces of elastodynamically impenetrable objects, the provisions necessary for the global reciprocity theorems to hold are made. If the embedding solid occupies a bounded domain, the exterior of this domain is assumed to be elastodynamically impenetrable. If the embedding solid occupies an unbounded domain, the standard limiting procedure of Section 15.1 for handling an unbounded domain is applied. Since transmission and reception are both causal phenomena, the transmission/reception reciprocity properties are based on the reciprocity theorems (15.2-7) and (15.4-7) of the time convolution type in which theorem causality is preserved.

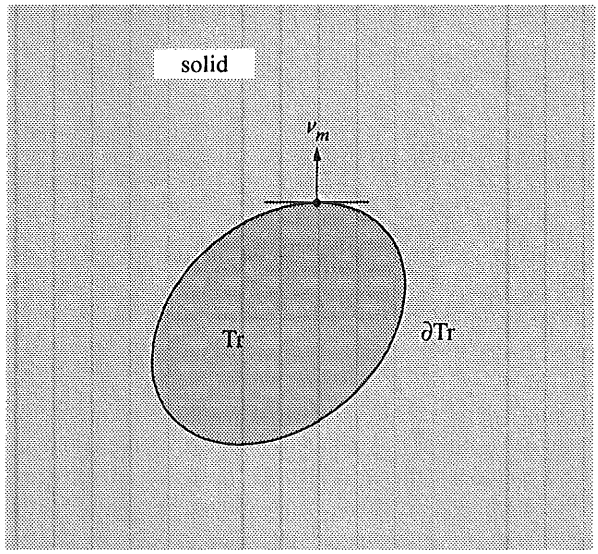


Figure 15.7-1 Configuration of the transmission/reception reciprocity properties of a single elastodynamic transducer Tr surrounded by a solid.

Volume action transducer

If Tr is a *volume action transducer*, its action in the transmitting mode is accounted for by prescribed values of the volume source densities of force and/or deformation rate, whose support is the domain occupied by the transducer, while in the receiving mode it is sensitive to the particle velocity and/or the dynamic stress over the domain it occupies. To investigate the transmission/reception reciprocity properties of a single transducer of this kind, state A is taken to be the causal state associated with the wave field $\{\tau^{\mathrm{T}}_{p,q}, v^{\mathrm{T}}_r\}$ generated by the prescribed volume source densities $\{h^{\mathrm{T}}_{i,j}, f^{\mathrm{T}}_k\}$ whose support is Tr. This state is denoted as the *transmitting state* and will be denoted by the superscript T. Next, state B is taken to be the causal state associated with the wave field that is generated by unspecified sources located in the domain Tr′ exterior to the transducer. In the surrounding solid these sources generate an *incident wave field* $\{\tau^{\mathrm{i}}_{p,q}, v^{\mathrm{i}}_r\}$ if the transducer is not activated. The total wave field $\{\tau^{\mathrm{R}}_{p,q}, v^{\mathrm{R}}_r\}$ in the presence of the transducer is then the superposition of the incident wave field and the *scattered wave field* $\{\tau^{\mathrm{s}}_{p,q}, v^{\mathrm{s}}_r\}$ i.e.

$$\{\tau^{\mathrm{R}}_{p,q}, v^{\mathrm{R}}_r\} = \{\tau^{\mathrm{i}}_{p,q} + \tau^{\mathrm{s}}_{p,q}, v^{\mathrm{i}}_r + v^{\mathrm{s}}_r\} \; . \tag{15.7-1}$$

The relevant state is denoted as the *receiving state* and will be denoted by the superscript R. Note that in the receiving state the domain Tr occupied by the transducer is source-free and that the scattered wave field in this state is source-free in the domain Tr′ exterior to the domain occupied by the transducer. Application of the time-domain reciprocity theorem of the time convolution type, Equation (15.2-7), to the transmitted and the scattered wave fields and to the domain Tr′ exterior to the transducer yields, assuming the solid to be self-adjoint,

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{s}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{s}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A=0\,. \tag{15.7-2}$$

Here, we have used the property that the total wave field in the transmitting state and the scattered wave field in the receiving state are both source-free in the domain Tr′ exterior to the transducer and causally related to the action of (primary or secondary) source distributions with the domain Tr occupied by the transducer as their supports, on account of which both the volume integral over the domain exterior to the transducer and the surface integral over the outer boundary of the domain of application of Equation (15.2-7) vanish. Using Equation (15.7-1), it follows from Equation (15.7-2) that

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{R}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{R}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A$$

$$=\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{i}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{i}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A\,. \tag{15.7-3}$$

Next, Equation (15.2-7) is applied to the total wave fields in the transmitting and the receiving states and to the domain Tr occupied by the transducer. This yields, again assuming the solid to be self-adjoint,

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{R}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{R}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A$$

$$=\int_{x\in\mathrm{Tr}}\left[\mathrm{C}_t(f^{\mathrm{T}}_k,v^{\mathrm{R}}_k;x,t)-\mathrm{C}_t(-\tau^{\mathrm{R}}_{i,j},h^{\mathrm{T}}_{i,j};x,t)\right]\mathrm{d}V\,. \tag{15.7-4}$$

Combining Equations (15.7-3) and (15.7-4), and using the continuity of the dynamic traction and the particle velocity across ∂Tr in both states, we arrive at

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{i}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{i}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A$$

$$=\int_{x\in\mathrm{Tr}}\left[\mathrm{C}_t(f^{\mathrm{T}}_k,v^{\mathrm{R}}_k;x,t)-\mathrm{C}_t(-\tau^{\mathrm{R}}_{i,j},h^{\mathrm{T}}_{i,j};x,t)\right]\mathrm{d}V\,. \tag{15.7-5}$$

The complex frequency-domain counterpart of Equation (15.7-5) follows, in a similar manner, from Equation (15.4-7) as

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}} \nu_m\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(x,s)\hat{v}^{\mathrm{i}}_r(x,s)+\hat{\tau}^{\mathrm{i}}_{p,q}(x,s)\hat{v}^{\mathrm{T}}_r(x,s)\right]\mathrm{d}A$$

$$=\int_{x\in\mathrm{Tr}}\left[\hat{f}^{\mathrm{T}}_k(x,s)\hat{v}^{\mathrm{R}}_k(x,s)+\hat{\tau}^{\mathrm{R}}_{i,j}(x,s)\hat{h}^{\mathrm{T}}_{i,j}(x,s)\right]\mathrm{d}V\,. \tag{15.7-6}$$

In view of what has been found in Section 15.6, the right-hand sides of Equations (15.7-5) and (15.7-6) are representative of the sensitivity of the transducer to a received elastic wave field generated elsewhere in the domain exterior to the transducer. The left-hand sides express that the transducer can, in the receiving state, be conceived as being excited, across its boundary surface, by the incident wave field. Equations (15.7-5) and (15.7-6) relate these two aspects quantitatively.

Surface action transducer

If Tr is a *surface action transducer*, its action in the transmitting mode is accounted for by prescribed values of the particle velocity and the dynamic traction at its boundary surface, while in the receiving mode it is sensitive to the dynamic traction and the particle velocity at that surface. This description of the action of the transducer is employed when the description of its action by volume sources is either inapplicable or irrelevant. To investigate the transmission/reception reciprocity properties of a single transducer of this kind, state A is, as above, taken to be the causal state associated with the wave field $\{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}$ generated by the prescribed surface source densities of force (i.e. $-\Delta^{+}_{m,r,p,q}\nu_m\tau^{\mathrm{T}}_{p,q}$) and of deformation rate (i.e. $\Delta^{+}_{m,r,p,q}\nu_m v^{\mathrm{T}}_r$), whose support is ∂Tr. This state is denoted as the *transmitting state* and will be denoted by the superscript T. Next, state B is taken to be the causal state associated with the wave field that is generated by unspecified sources located in the domain Tr′ exterior to the transducer. In the surrounding solid these sources would generate an *incident wave field* $\{\tau^{\mathrm{i}}_{p,q},v^{\mathrm{i}}_r\}$ if the transducer were not activated. The total wave field $\{\tau^{\mathrm{R}}_{p,q},v^{\mathrm{R}}_r\}$ in the presence of the transducer is again the superposition of the incident wave field and the *scattered wave field* $\{\tau^{\mathrm{s}}_{p,q},v^{\mathrm{s}}_r\}$ i.e.

$$\{\tau^{\mathrm{R}}_{p,q},v^{\mathrm{R}}_r\} = \{\tau^{\mathrm{i}}_{p,q}+\tau^{\mathrm{s}}_{p,q},v^{\mathrm{i}}_r+v^{\mathrm{s}}_r\} \ . \tag{15.7-7}$$

The relevant state is denoted as the *receiving state* and will be denoted by the superscript R. Note that in the receiving state the scattered wave field is source-free in the domain Tr′ exterior to the domain occupied by the transducer. Application of the time-domain reciprocity theorem of the time convolution type, Equation (15.2-7), to the transmitted and the scattered wave fields and to the domain Tr′ exterior to the transducer yields, assuming the solid to be self-adjoint,

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}}\nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{s}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{s}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A=0\ . \tag{15.7-8}$$

Next, using Equation (15.7-7), it follows that

$$\begin{aligned}&\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}}\nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{R}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{R}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A\\&=\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}}\nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{\mathrm{i}}_r;x,t)-\mathrm{C}_t(-\tau^{\mathrm{i}}_{p,q},v^{\mathrm{T}}_r;x,t)\right]\mathrm{d}A\ .\end{aligned} \tag{15.7-9}$$

The complex frequency-domain counterpart of Equation (15.7-9) follows, in a similar manner, from Equation (15.4-7) as

$$\begin{aligned}&\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}}\nu_m\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(x,s)\hat{v}^{\mathrm{R}}_r(x,s)+\hat{\tau}^{\mathrm{R}}_{p,q}(x,s)\hat{v}^{\mathrm{T}}_r(x,s)\right]\mathrm{d}A\\&=\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathrm{Tr}}\nu_m\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(x,s)\hat{v}^{\mathrm{i}}_r(x,s)+\hat{\tau}^{\mathrm{i}}_{p,q}(x,s)\hat{v}^{\mathrm{T}}_r(x,s)\right]\mathrm{d}A\ .\end{aligned} \tag{15.7-10}$$

In view of what has been found in Section 15.6, the left-hand sides of Equations (15.7-9) and (15.7-10) are representative of the sensitivity of the transducer to a received elastic wave field generated elsewhere in the domain exterior to the transducer. The right-hand sides express that the transducer can, in the receiving state, be conceived as to be excited, across its boundary surface, by the incident wave field. Equations (15.7-9) and (15.7-10) relate these two aspects quantitatively.

15.8 The direct (forward) source problem. Point-source solutions and Green's functions

In the direct (or forward) source problem we want to express the elastic wave-field quantities in a configuration with given elastodynamic properties in terms of the source distributions that generate the wave field. Let $\mathcal{D}$ be the domain in which expressions for the generated elastodynamic wave field $\{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\} = \{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}(\boldsymbol{x},t)$ are to be found. If $\mathcal{D}$ is a bounded domain, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable. If $\mathcal{D}$ is an unbounded domain, the standard provisions of Section 15.1 for handling an unbounded domain are made. Since $\{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}$ is a physical wave field, it satisfies the condition of causality. The source distributions $\{h^{\mathrm{T}}_{i,j},f^{\mathrm{T}}_k\} = \{h^{\mathrm{T}}_{i,j},f^{\mathrm{T}}_k\}(\boldsymbol{x},t)$ that generate the wave field, have the bounded support $\mathcal{D}^T$ that is a proper subdomain of $\mathcal{D}$ (Figure 15.8-1).

The elastodynamic properties of the solid present in $\mathcal{D}$ are characterised by the relaxation functions $\{\mu_{k,r},\chi_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t)$ which are causal functions of time. The case of an instantaneously reacting solid easily follows from the more general case of a solid with relaxation.

Time-domain analysis

For the time-domain analysis of the problem the global reciprocity theorem of the time convolution type, Equation (15.2-7), is taken as the point of departure. In it, state A is taken to be the generated elastodynamic wave field under consideration, i.e.

$$\{\tau^{\mathrm{A}}_{p,q},v^{\mathrm{A}}_r\} = \{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}(\boldsymbol{x},t) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.8-1}$$

$$\{h^{\mathrm{A}}_{i,j},f^{\mathrm{A}}_k\} = \{h^{\mathrm{T}}_{i,j},f^{\mathrm{T}}_k\}(\boldsymbol{x},t) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^T, \tag{15.8-2}$$

and

$$\{\mu^{\mathrm{A}}_{k,r},\chi^{\mathrm{A}}_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.8-3}$$

Next, state B is chosen such that the application of Equation (15.2-7) to the domain $\mathcal{D}$ leads to the values of $\{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}$ at some arbitrary point $\boldsymbol{x}'\in\mathcal{D}$. Inspection of the right-hand side of Equation (15.2-7) reveals that this is accomplished if we take for the source distributions of state B a point source of deformation rate at $\boldsymbol{x}'$, in case we want an expression for the dynamic stress at $\boldsymbol{x}'$, and a point source of force at $\boldsymbol{x}'$ in case we want an expression for the particle velocity at $\boldsymbol{x}'$, while the solid in state B must be taken to be the adjoint of the one in state A, i.e.

15.8-1(a)

15.8-1(b)

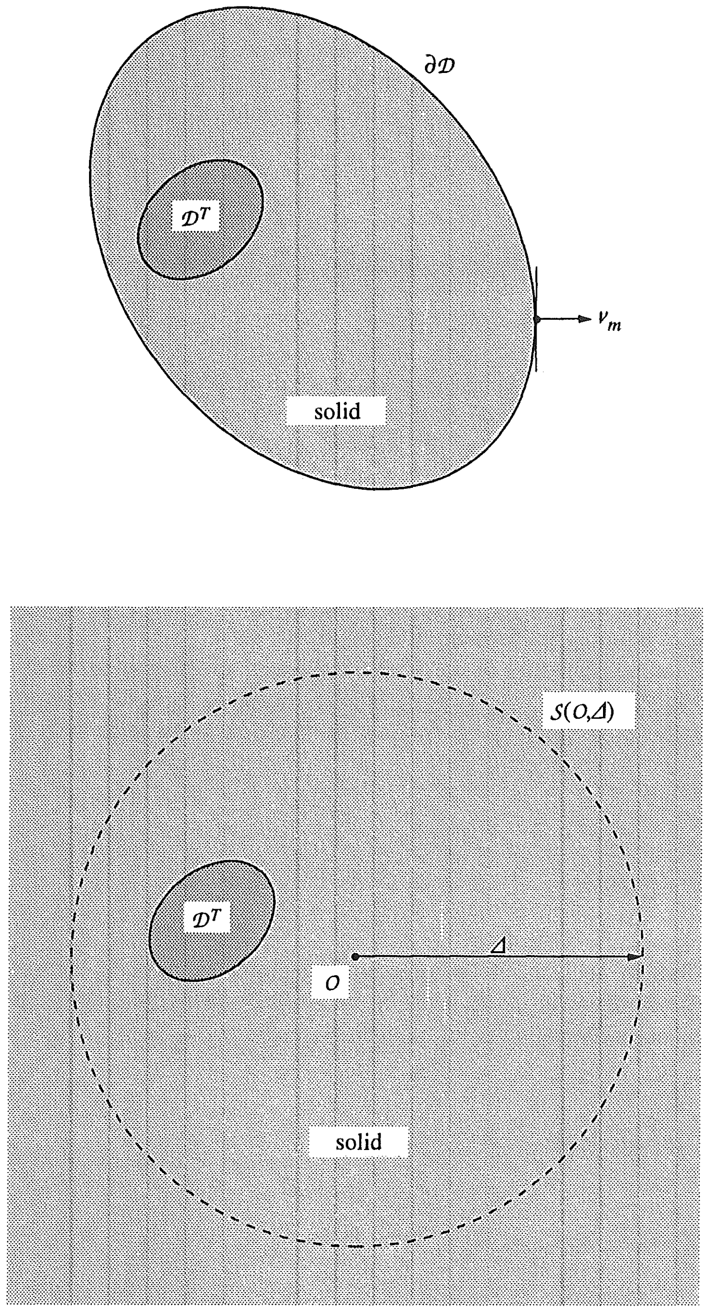


Figure 15.8-1 Configuration of the direct (forward) source problem. $\mathcal{D}^T$ is the bounded support of the source distributions. (a) The solid occupies the bounded domain $\mathcal{D}$ with elastodynamically impenetrable boundary $\partial\mathcal{D}$. (b) The solid occupies the unbounded domain $\mathcal{D}$; $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity.

$$\{\overset{\mathrm{B}}{\mu}_{r,k},\overset{\mathrm{B}}{\chi}_{p,q,i,j}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t) \qquad \text{for all } \boldsymbol{x}\in\mathcal{D}. \tag{15.8-4}$$

Furthermore, if $\mathcal{D}$ is bounded, the elastic wave field in state B must satisfy on $\partial\mathcal{D}$ the same boundary conditions for an elastodynamically impenetrable boundary as in state A, while if $\mathcal{D}$ is unbounded, the elastic wave field in state B must be causally related to the action of its (point) sources. The two choices for the source distributions will be discussed separately below.

First, we choose

$$\overset{\mathrm{B}}{h}_{p,q} = a_{p,q}\delta(\boldsymbol{x}-\boldsymbol{x}',t) \quad \text{and} \quad \overset{\mathrm{B}}{f}_r = 0\,, \tag{15.8-5}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x}',t)$ represents the four-dimensional unit impulse (Dirac distribution) operative at the point $\boldsymbol{x}=\boldsymbol{x}'$ and at the instant $t=0$, while $a_{p,q}$ is an arbitrary constant tensor of rank two. The elastic wave field causally radiated by this source is denoted as

$$\{\overset{\mathrm{B}}{\tau}_{i,j},\overset{\mathrm{B}}{v}_k\} = \{\tau_{i,j}^{h;\mathrm{B}},v_k^{h;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',t)\,, \tag{15.8-6}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (15.2-7) is applied to the domain $\mathcal{D}$. In case $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\partial\mathcal{D}}\nu_m\left[\mathrm{C}_t(-\overset{\mathrm{A}}{\tau}_{p,q},\overset{\mathrm{B}}{v}_r;\boldsymbol{x},t)-\mathrm{C}_t(-\overset{\mathrm{B}}{\tau}_{p,q},\overset{\mathrm{A}}{v}_r;\boldsymbol{x},t)\right]\mathrm{d}A$$

$$=\Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\partial\mathcal{D}}\nu_m\left[\mathrm{C}_t(-\overset{\mathrm{T}}{\tau}_{p,q},v_r^{h;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t)-\mathrm{C}_t(-\tau_{p,q}^{h;\mathrm{B}},\overset{\mathrm{T}}{v}_r;\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}A=0\,, \tag{15.8-7}$$

while if $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain yield

$$\Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)}\nu_m\left[\mathrm{C}_t(-\overset{\mathrm{A}}{\tau}_{p,q},\overset{\mathrm{B}}{v}_r;\boldsymbol{x},t)-\mathrm{C}_t(-\overset{\mathrm{B}}{\tau}_{p,q},\overset{\mathrm{A}}{v}_r;\boldsymbol{x},t)\right]\mathrm{d}A$$

$$=\Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}(O,\Delta)}\nu_m\left[\mathrm{C}_t(-\overset{\mathrm{T}}{\tau}_{p,q},v_r^{h;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t)-\mathrm{C}_t(-\tau_{p,q}^{h;\mathrm{B}},\overset{\mathrm{T}}{v}_r;\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}A\to 0$$

$$\text{as } \Delta\to\infty\,. \tag{15.8-8}$$

Furthermore, in view of Equation (15.8-5) and the properties of $\delta(\boldsymbol{x}-\boldsymbol{x}',t)$,

$$\int_{\boldsymbol{x}\in\mathcal{D}}\left[-\mathrm{C}_t(\overset{\mathrm{B}}{f}_r,\overset{\mathrm{A}}{v}_r;\boldsymbol{x},t)+\mathrm{C}_t(-\overset{\mathrm{A}}{\tau}_{p,q},\overset{\mathrm{B}}{h}_{p,q};\boldsymbol{x},t)\right]\mathrm{d}V$$

$$=\int_{\boldsymbol{x}\in\mathcal{D}}\left[\mathrm{C}_t(-\overset{\mathrm{T}}{\tau}_{p,q},a_{p,q}\delta(\boldsymbol{x}-\boldsymbol{x}',t);\boldsymbol{x},t)\right]\mathrm{d}V=-a_{p,q}\overset{\mathrm{T}}{\tau}_{p,q}(\boldsymbol{x}',t)\,. \tag{15.8-9}$$

Since, further, the sources have the support $\mathcal{D}^T$,

$$\int_{\boldsymbol{x}\in\mathcal{D}}\left[\mathrm{C}_t(\overset{\mathrm{A}}{f}_k,\overset{\mathrm{B}}{v}_k;\boldsymbol{x},t)-\mathrm{C}_t(-\overset{\mathrm{B}}{\tau}_{i,j},\overset{\mathrm{A}}{h}_{i,j};\boldsymbol{x},t)\right]\mathrm{d}V$$

$$=\int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\mathrm{C}_t(\overset{\mathrm{T}}{f}_k,v_k^{h;\mathrm{B}};\boldsymbol{x},\boldsymbol{x}',t)-\mathrm{C}_t(-\tau_{i,j}^{h;\mathrm{B}},\overset{\mathrm{T}}{h}_{i,j};\boldsymbol{x},\boldsymbol{x}',t)\right]\mathrm{d}V\,. \tag{15.8-10}$$

Collecting the results, we arrive at

$$-a_{p,q}\tau^{\mathrm{T}}_{p,q}(x',t) = \int_{x\in\mathcal{D}^{\mathrm{T}}} \left[\mathrm{C}_t(-\tau^{h;\mathrm{B}}_{i,j},h^{\mathrm{T}}_{i,j};x,x',t) - \mathrm{C}_t(v^{h;\mathrm{B}}_k,f^{\mathrm{T}}_k;x,x',t)\right]\mathrm{d}V \text{ for } x'\in\mathcal{D}, \tag{15.8-11}$$

where, in the second term on the right-hand side, we have used the symmetry of the convolution in its functional arguments. From Equation (15.8-11) a representation for $\tau^{\mathrm{T}}_{p,q}(x',t)$ is obtained by taking into account that $\tau^{h;\mathrm{B}}_{i,j}$ and $v^{h;\mathrm{B}}_k$ are linearly related to $a_{p,q}$. The latter relationship is expressed by

$$\{-\tau^{h;\mathrm{B}}_{i,j},v^{h;\mathrm{B}}_k\}(x,x',t) = \{G^{\tau h;\mathrm{B}}_{i,j,p,q},G^{vh;\mathrm{B}}_{k,p,q}\}(x,x',t)a_{p,q}\,. \tag{15.8-12}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 15.8-1 and 15.8-3)

$$\{G^{\tau h;\mathrm{B}}_{i,j,p,q},G^{vh;\mathrm{B}}_{k,p,q}\}(x,x',t) = \{G^{\tau h}_{p,q,i,j},-G^{\tau f}_{p,q,k}\}(x',x,t) \tag{15.8-13}$$

hold, Equation (15.8-11) leads, with Equations (15.8-12) and (15.8-13), and invoking the condition that the resulting equation has to hold for arbitrary values of $a_{p,q}$, to the final result

$$-\tau^{\mathrm{T}}_{p,q}(x',t) = \int_{x\in\mathcal{D}^{\mathrm{T}}} \left[\mathrm{C}_t(G^{\tau h}_{p,q,i,j},h^{\mathrm{T}}_{i,j};x',x,t) + \mathrm{C}_t(G^{\tau f}_{p,q,k},f^{\mathrm{T}}_k;x',x,t)\right]\mathrm{d}V \quad \text{for } x'\in\mathcal{D}\,. \tag{15.8-14}$$

Equation (15.8-14) expresses the dynamic stress $\tau^{\mathrm{T}}_{p,q}$ of the generated elastic wave field at x' as the superposition of the contributions from the elementary distributed sources $h^{\mathrm{T}}_{i,j}\,\mathrm{d}V$ and $f^{\mathrm{T}}_k\,\mathrm{d}V$ at x. The intervening kernel functions are the *dynamic stress/deformation rate source Green's function* $G^{\tau h}_{p,q,i,j} = G^{\tau h}_{p,q,i,j}(x',x,t)$ and the *dynamic stress/force source Green's function* $G^{\tau f}_{p,q,k} = G^{\tau f}_{p,q,k}(x',x,t)$. These Green's functions are the dynamic stress at x', radiated in the actual solid with constitutive parameters $\{\mu_{k,r},\chi_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t)$, by a point source of deformation rate at x and a point source of force at x, respectively.

Secondly, we choose

$$h^{\mathrm{B}}_{p,q} = 0 \quad \text{and } f^{\mathrm{B}}_r = b_r\delta(x-x',t)\,, \tag{15.8-15}$$

where b_r is an arbitrary constant vector. The elastic wave field causally radiated by this source is denoted as

$$\{\tau^{\mathrm{B}}_{i,j},v^{\mathrm{B}}_k\} = \{\tau^{f;\mathrm{B}}_{i,j},v^{f;\mathrm{B}}_k\}(x,x',t)\,, \tag{15.8-16}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (15.2-7) is applied to the domain $\mathcal{D}$. In case $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{A}}_{p,q},v^{\mathrm{B}}_r;x,t) - \mathrm{C}_t(-\tau^{\mathrm{B}}_{p,q},v^{\mathrm{A}}_r;x,t)\right]\mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{T}}_{p,q},v^{f;\mathrm{B}}_r;x,x',t) - \mathrm{C}_t(-\tau^{f;\mathrm{B}}_{p,q},v^{\mathrm{T}}_r;x,x',t)\right]\mathrm{d}A = 0\,, \tag{15.8-17}$$

while if $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain yield

$$\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[C_t(-\tau^{A}_{p,q},v^{B}_r;x,t)-C_t(-\tau^{B}_{p,q},v^{A}_r;x,t)\right]\mathrm{d}A$$
$$=\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[C_t(-\tau^{T}_{p,q},v^{f;B}_r;x,x',t)-C_t(-\tau^{f;B}_{p,q},v^{T}_r;x,x',t)\right]\mathrm{d}A\to 0$$
$$\text{as }\ \Delta\to\infty\,. \tag{15.8-18}$$

Furthermore, in view of Equation (15.8-15) and the properties of $\delta(x-x',t)$,

$$\int_{x\in\mathcal{D}}\left[-C_t(f^{B}_r,v^{A}_r;x,t)+C_t(-\tau^{A}_{p,q},h^{B}_{p,q};x,t)\right]\mathrm{d}V$$
$$=-\int_{x\in\mathcal{D}}\left[C_t(b_r\delta(x-x',t),v^{T}_r;x,t)\right]\mathrm{d}V=-b_r v^{T}_r(x',t)\,. \tag{15.8-19}$$

Since, further, the sources have the support $\mathcal{D}^T$,

$$\int_{x\in\mathcal{D}}\left[C_t(f^{A}_k,v^{B}_k;x,t)-C_t(-\tau^{B}_{i,j},h^{A}_{i,j};x,t)\right]\mathrm{d}V$$
$$=\int_{x\in\mathcal{D}^T}\left[C_t(f^{T}_k,v^{f;B}_k;x,x',t)-C_t(-\tau^{f;B}_{i,j},h^{T}_{i,j};x,x',t)\right]\mathrm{d}V\,. \tag{15.8-20}$$

Collecting the results, we arrive at

$$b_r v^{T}_r(x',t)=\int_{x\in\mathcal{D}^T}\left[-C_t(\tau^{f;B}_{i,j},h^{T}_{i,j};x,x',t)+C_t(v^{f;B}_k,f^{T}_k;x,x',t)\right]\mathrm{d}V\quad\text{for}\quad x'\in\mathcal{D}\,, \tag{15.8-21}$$

where, in the second term on the right-hand side, we have used the symmetry of the convolution in its functional arguments. From Equation (15.8-21) a representation for $v_r^{T}(x',t)$ is obtained by taking into account that $\tau^{f;B}_{i,j}$ and $v^{f;B}_k$ are linearly related to b_r. The latter relationship is expressed by

$$\{-\tau^{f;B}_{i,j},v^{f;B}_k\}(x,x',t)=\{G^{\tau f}_{i,j,r},G^{vf;B}_{k,r}\}(x,x',t)b_r\,. \tag{15.8-22}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 15.8-2 and 15.8-4)

$$\{G^{\tau f;B}_{i,j,r},G^{vf;B}_{k,r}\}(x,x',t)=\{-G^{vh}_{r,i,j},G^{vf}_{r,k}\}(x',x,t) \tag{15.8-23}$$

hold, Equation (15.8-21) leads, with Equations (15.8-22) and (15.8-23), and invoking the condition that the resulting equation has to hold for arbitrary values of b_r, to the final result

$$v^{T}_r(x',t)=\int_{x\in\mathcal{D}^T}\left[C_t(G^{vh}_{r,i,j},h^{T}_{i,j};x',x,t)+C_t(G^{vf}_{r,k},f^{T}_k;x',x,t)\right]\mathrm{d}V\qquad\text{for}\quad x'\in\mathcal{D}\,. \tag{15.8-24}$$

Equation (15.8-24) expresses the particle velocity v_r^{T} of the generated elastic wave field at x' as the superposition of the contributions from the elementary distributed sources $h^{T}_{i,j}\,\mathrm{d}V$ and $f^{T}_k\,\mathrm{d}V$ at x. The intervening kernel functions are the *particle velocity/deformation rate source Green's function* $G^{vh}_{r,i,j}=G^{vh}_{r,i,j}(x',x,t)$ and the *particle velocity/force source Green's function* $G^{vf}_{r,k}=G^{vf}_{r,k}(x',x,t)$. These Green's functions are the particle velocity at x', radiated in the actual solid with constitutive parameters $\{\mu_{k,r},\chi_{i,j,p,q}\}=\{\mu_{k,r},\chi_{i,j,p,q}\}(x,t)$, by a point source of deformation rate at x and a point source of force at x, respectively.

Complex frequency-domain analysis

For the complex frequency-domain analysis of the problem the complex frequency-domain global reciprocity theorem of the time convolution type, Equation (15.4-7), is taken as the point of departure. In it, state A is taken to be the generated elastic wave field under consideration, i.e.

$$\{\hat{\tau}^{A}_{p,q},\hat{v}^{A}_{r}\} = \{\hat{\tau}^{T}_{p,q},\hat{v}^{T}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.8-25}$$

$$\{\hat{h}^{A}_{i,j},\hat{f}^{A}_{k}\} = \{\hat{h}^{T}_{i,j},\hat{f}^{T}_{k}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{T}, \tag{15.8-26}$$

and

$$\{\hat{\zeta}^{A}_{k,r},\hat{\eta}^{A}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.8-27}$$

Next, state B is chosen such that the application of Equation (15.4-7) to the domain $\mathcal{D}$ leads to the values of $\{\hat{\tau}^{T}_{p,q},\hat{v}^{T}_{r}\}$ at some arbitrary point $\boldsymbol{x}'\in\mathcal{D}$. Inspection of the right-hand side of Equation (15.4-7) reveals that this is accomplished if we take for the source distributions of state B a point source of deformation rate at $\boldsymbol{x}'$ in case we want an expression for the dynamic stress at $\boldsymbol{x}'$ and a point source of force at $\boldsymbol{x}'$ in case we want an expression for the particle velocity at $\boldsymbol{x}'$, while the solid in state B must be taken to be the adjoint of that in state A, i.e.

$$\{\hat{\zeta}^{B}_{r,k},\hat{\eta}^{B}_{p,q,i,j}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for all} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.8-28}$$

Furthermore, if $\mathcal{D}$ is bounded, the elastic wave field in state B must satisfy on $\partial\mathcal{D}$ the same boundary conditions for an elastodynamically impenetrable boundary as in state A, while if $\mathcal{D}$ is unbounded, the elastic wave field in state B must be causally related to the action of its (point) sources. The two choices for the source distributions will be discussed separately below.

First, we choose

$$\hat{h}^{B}_{p,q} = \hat{a}_{p,q}(s)\delta(\boldsymbol{x}-\boldsymbol{x}') \quad \text{and} \quad \hat{f}^{B}_{r} = 0\,, \tag{15.8-29}$$

where $\delta(\boldsymbol{x}-\boldsymbol{x}')$ represents the three-dimensional unit impulse (Dirac distribution) operative at the point $\boldsymbol{x}=\boldsymbol{x}'$, while $\hat{a}_{p,q}=\hat{a}_{p,q}(s)$ is an arbitrary tensor function of rank two of s. The elastic wave field causally radiated by this source is denoted as

$$\{\hat{\tau}^{B}_{i,j},\hat{v}^{B}_{k}\} = \{\hat{\tau}^{h;B}_{i,j},\hat{v}^{h;B}_{k}\}(\boldsymbol{x},\boldsymbol{x}',s)\,, \tag{15.8-30}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (15.4-7) is applied to the domain $\mathcal{D}$. In case $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[-\hat{\tau}^{A}_{p,q}(\boldsymbol{x},s)\hat{v}^{B}_{r}(\boldsymbol{x},s)+\hat{\tau}^{B}_{p,q}(\boldsymbol{x},s)\hat{v}^{A}_{r}(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[-\hat{\tau}^{T}_{p,q}(\boldsymbol{x},s)\hat{v}^{h;B}_{r}(\boldsymbol{x},\boldsymbol{x}',s)+\hat{\tau}^{h;B}_{p,q}(\boldsymbol{x},\boldsymbol{x}',s)\hat{v}^{T}_{r}(\boldsymbol{x},s)\right]\mathrm{d}A = 0\,, \tag{15.8-31}$$

while if $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain yield

$$\Delta^+_{m,r,p,q} \int_{x\in S(O,\Delta)} \nu_m \left[-\hat{\tau}^{A}_{p,q}(x,s)\hat{v}^{B}_r(x,s) + \hat{\tau}^{B}_{p,q}(x,s)\hat{v}^{A}_r(x,s)\right] \mathrm{d}A$$

$$= \Delta^+_{m,r,p,q} \int_{x\in S(O,\Delta)} \nu_m \left[-\hat{\tau}^{T}_{p,q}(x,s)\hat{v}^{h;B}_r(x,x',s) + \hat{\tau}^{h;B}_{p,q}(x,x',s)\hat{v}^{T}_r(x,s)\right] \mathrm{d}A \to 0$$

$$\text{as } \Delta\to\infty . \tag{15.8-32}$$

Furthermore, in view of Equation (15.8-29) and the properties of $\delta(x - x')$,

$$\int_{x\in\mathcal{D}} \left[-\hat{f}^{B}_r(x,s)\hat{v}^{A}_r(x,s) - \hat{\tau}^{A}_{p,q}(x,s)\hat{h}^{B}_{p,q}(x,s)\right] \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \hat{\tau}^{T}_{p,q}(x,s)\hat{a}_{p,q}(s)\delta(x-x')\,\mathrm{d}V = -\hat{a}_{p,q}(s)\hat{\tau}^{T}_{p,q}(x',s) . \tag{15.8-33}$$

In addition, as the sources have the support $\mathcal{D}^T$,

$$\int_{x\in\mathcal{D}} \left[\hat{f}^{A}_k(x,s)\hat{v}^{B}_k(x,s) + \hat{\tau}^{B}_{i,j}(x,s)\hat{h}^{A}_{i,j}(x,s)\right] \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^T} \left[\hat{f}^{T}_k(x,s)\hat{v}^{h;B}_k(x,x',s) + \hat{\tau}^{h;B}_{i,j}(x,x',s)\hat{h}^{T}_{i,j}(x,s)\right] \mathrm{d}V . \tag{15.8-34}$$

Collecting the results, we arrive at

$$-\hat{a}_{p,q}(s)\hat{\tau}^{T}_{p,q}(x',s) = \int_{x\in\mathcal{D}^T} \left[-\hat{\tau}^{h;B}_{i,j}(x,x',s)\hat{h}^{T}_{i,j}(x,s) - \hat{v}^{h;B}_k(x,x',s)\hat{f}^{T}_k(x,s)\right] \mathrm{d}V \quad \text{for } x'\in\mathcal{D} . \tag{15.8-35}$$

From Equation (15.8-35) a representation for $\hat{\tau}^{T}_{p,q}(x',s)$ is obtained by taking into account that $\hat{\tau}^{h;B}_{i,j}$ and $\hat{v}^{h;B}_k$ are linearly related to $\hat{a}_{p,q}(s)$. The latter relationship is expressed by

$$\{-\hat{\tau}^{h;B}_{i,j},\hat{v}^{h;B}_k\}(x,x',s) = \{\hat{G}^{\tau h;B}_{i,j,p,q},\hat{G}^{vh;B}_{k,p,q}\}(x,x',s)\hat{a}_{p,q}(s) . \tag{15.8-36}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 15.8-5 and 15.8-7)

$$\{\hat{G}^{\tau h;B}_{i,j,p,q},\hat{G}^{vh;B}_{k,p,q}\}(x,x',s) = \{\hat{G}^{\tau h}_{p,q,i,j},-\hat{G}^{\tau f}_{p,q,k}\}(x',x,s) \tag{15.8-37}$$

hold, Equation (15.8-35) leads, with Equations (15.8-36) and (15.8-37), and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{a}_{p,q}(s)$, to the final result

$$-\hat{\tau}^{T}_{p,q}(x',s) = \int_{x\in\mathcal{D}^T} \left[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\hat{h}^{T}_{i,j}(x,s) + \hat{G}^{\tau f}_{p,q,k}(x',x,s)\hat{f}^{T}_k(x,s)\right] \mathrm{d}V \text{ for } x'\in\mathcal{D} . \tag{15.8-38}$$

Equation (15.8-38) expresses the dynamic stress $\hat{\tau}^{T}_{p,q}$ of the generated elastic wave field at x' as the superposition of the contributions from the elementary distributed sources $\hat{h}^{T}_{i,j}\,\mathrm{d}V$ and $\hat{f}^{T}_k\,\mathrm{d}V$ at x. The intervening kernel functions are the *dynamic stress/deformation rate source Green's function* $\hat{G}^{\tau h}_{p,q,i,j} = \hat{G}^{\tau h}_{p,q,i,j}(x',x,s)$ and the *dynamic stress/force source Green's function* $\hat{G}^{\tau f}_{p,q,k} = \hat{G}^{\tau f}_{p,q,k}(x',x,s)$. These Green's functions are the dynamic stress at x', radiated in the actual solid with constitutive parameters $\{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s)$, by a point source of deformation rate at $\boldsymbol{x}$ and a point source of force at $\boldsymbol{x}$, respectively.

Secondly, we choose

$$\hat{h}^{\mathrm{B}}_{p,q} = 0 \quad \text{and} \quad \hat{f}^{\mathrm{B}}_r = \hat{b}_r(s)\delta(\boldsymbol{x}-\boldsymbol{x}') \, , \tag{15.8-39}$$

where $\hat{b}_r = \hat{b}_r(s)$ is an arbitrary vector function of s. The elastic wave field causally radiated by this source is denoted as

$$\{\hat{\tau}^{\mathrm{B}}_{i,j},\hat{f}^{\mathrm{B}}_k\} = \{\hat{\tau}^{f;\mathrm{B}}_{i,j},\hat{v}^{f;\mathrm{B}}_k\}(\boldsymbol{x},\boldsymbol{x}',s) \, , \tag{15.8-40}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (15.4-7) is applied to the domain $\mathcal{D}$. In case $\mathcal{D}$ is bounded, we have for the integral over its boundary surface

$$\Delta^+_{m,r,p,q}\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[-\hat{\tau}^{\mathrm{A}}_{p,q}(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_r(\boldsymbol{x},s) + \hat{\tau}^{\mathrm{B}}_{p,q}(\boldsymbol{x},s)\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \Delta^+_{m,r,p,q}\int_{\boldsymbol{x}\in\partial\mathcal{D}} \nu_m\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(\boldsymbol{x},s)\hat{v}^{f;\mathrm{B}}_r(\boldsymbol{x},\boldsymbol{x}',s) + \hat{\tau}^{f;\mathrm{B}}_{p,q}(\boldsymbol{x},\boldsymbol{x}',s)\hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\right]\mathrm{d}A = 0 \, , \tag{15.8-41}$$

while if $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain yield

$$\Delta^+_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}(\mathcal{O},\Delta)} \nu_m\left[-\hat{\tau}^{\mathrm{A}}_{p,q}(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_r(\boldsymbol{x},s) + \hat{\tau}^{\mathrm{B}}_{p,q}(\boldsymbol{x},s)\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \Delta^+_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}(\mathcal{O},\Delta)} \nu_m\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(\boldsymbol{x},s)\hat{v}^{f;\mathrm{B}}_r(\boldsymbol{x},\boldsymbol{x}',s) + \hat{\tau}^{f;\mathrm{B}}_{p,q}(\boldsymbol{x},\boldsymbol{x}',s)\hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\right]\mathrm{d}A \to 0$$

$$\text{as} \quad \Delta\to\infty \, . \tag{15.8-42}$$

Furthermore, in view of Equation (15.8-39) and the properties of $\delta(\boldsymbol{x}-\boldsymbol{x}')$,

$$\int_{\boldsymbol{x}\in\mathcal{D}}\left[-\hat{f}^{\mathrm{B}}_r(\boldsymbol{x},s)\hat{v}^{\mathrm{A}}_r(\boldsymbol{x},s) - \hat{\tau}^{\mathrm{A}}_{p,q}(\boldsymbol{x},s)\hat{h}^{\mathrm{B}}_{p,q}(\boldsymbol{x},s)\right]\mathrm{d}V$$

$$= -\int_{\boldsymbol{x}\in\mathcal{D}} \hat{b}_r(s)\delta(\boldsymbol{x}-\boldsymbol{x}')\hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\,\mathrm{d}V = -\hat{b}_r(s)\hat{v}^{\mathrm{T}}_r(\boldsymbol{x}',s) \, . \tag{15.8-43}$$

In addition, as the sources have the support $\mathcal{D}^T$,

$$\int_{\boldsymbol{x}\in\mathcal{D}}\left[\hat{f}^{\mathrm{A}}_k(\boldsymbol{x},s)\hat{v}^{\mathrm{B}}_k(\boldsymbol{x},s) + \hat{\tau}^{\mathrm{B}}_{i,j}(\boldsymbol{x},s)\hat{h}^{\mathrm{A}}_{i,j}(\boldsymbol{x},s)\right]\mathrm{d}V$$

$$= \int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\hat{f}^{\mathrm{T}}_k(\boldsymbol{x},s)\hat{v}^{f;\mathrm{B}}_k(\boldsymbol{x},\boldsymbol{x}',s) + \hat{\tau}^{f;\mathrm{B}}_{i,j}(\boldsymbol{x},\boldsymbol{x}',s)\hat{h}^{\mathrm{T}}_{i,j}(\boldsymbol{x},s)\right]\mathrm{d}V \, . \tag{15.8-44}$$

Collecting the results, we arrive at

$$\hat{b}_r(s)v^{\mathrm{T}}_r(\boldsymbol{x}',s) = \int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\hat{\tau}^{f;\mathrm{B}}_{i,j}(\boldsymbol{x},\boldsymbol{x}',s)\hat{h}^{\mathrm{T}}_{i,j}(\boldsymbol{x},s) + \hat{v}^{f;\mathrm{B}}_k(\boldsymbol{x},\boldsymbol{x}',s)\hat{f}^{\mathrm{T}}_k(\boldsymbol{x},s)\right]\mathrm{d}V \quad \text{for } \boldsymbol{x}'\in\mathcal{D} \, . \tag{15.8-45}$$

From Equation (15.8-45) a representation for $\hat{v}_r^{\mathrm{T}}(\boldsymbol{x}',s)$ is obtained by taking into account that $\hat{\tau}_{i,j}^{f;\mathrm{B}}$ and $\hat{v}_k^{f;\mathrm{B}}$ are linearly related to $\hat{b}_r(s)$. The latter relationship is expressed by

$$\{-\hat{\tau}_{i,j}^{f;\mathrm{B}},\hat{v}_k^{f;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s) = \{\hat{G}_{i,j,r}^{\tau f;\mathrm{B}},\hat{G}_{k,r}^{vf;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s)\hat{b}_r(s)\,. \tag{15.8-46}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 15.8-6 and 15.8-8)

$$\{\hat{G}_{i,j,r}^{\tau f;\mathrm{B}},\hat{G}_{k,r}^{vf;\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}',s) = \{-\hat{G}_{r,i,j}^{vh},\hat{G}_{r,k}^{vf}\}(\boldsymbol{x}',\boldsymbol{x},s) \tag{15.8-47}$$

hold, Equation (15.8-45) leads, with Equations (15.8-46) and (15.8-47), and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{b}_r(s)$, to the final result

$$\hat{v}_r^{\mathrm{T}}(\boldsymbol{x}',s) = \int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\hat{G}_{r,i,j}^{vh}(\boldsymbol{x}',\boldsymbol{x},s)\hat{h}_{i,j}^{\mathrm{T}}(\boldsymbol{x},s) + \hat{G}_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},s)\hat{f}_k^{\mathrm{T}}(\boldsymbol{x},s)\right]\mathrm{d}V \quad \text{for } \boldsymbol{x}'\in\mathcal{D}. \tag{15.8-48}$$

Equation (15.8-48) expresses the particle velocity $\hat{v}_r^{\mathrm{T}}$ of the generated elastic wave field at $\boldsymbol{x}'$ as the superposition of the contributions from the elementary distributed sources $\hat{h}_{i,j}^{\mathrm{T}}\,\mathrm{d}V$ and $\hat{f}_k^{\mathrm{T}}\,\mathrm{d}V$ at $\boldsymbol{x}$. The intervening kernel functions are the *particle velocity/deformation rate source Green's function* $\hat{G}_{r,i,j}^{vh} = \hat{G}_{r,i,j}^{vh}(\boldsymbol{x}',\boldsymbol{x},s)$ and the *particle velocity/force source Green's function* $\hat{G}_{r,k}^{vf} = \hat{G}_{r,k}^{vf}(\boldsymbol{x}',\boldsymbol{x},s)$. These Green's functions are the particle velocity at $\boldsymbol{x}'$, radiated in the actual solid with constitutive parameters $\{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s)$, by a point source of deformation rate at $\boldsymbol{x}$ and a point source of force at $\boldsymbol{x}$, respectively.

Exercises

Exercise 15.8-1

Let $\{\tau_{p,q}^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{\tau_{p,q}^{\mathrm{A}},v_r^{\mathrm{A}}\}(\boldsymbol{x},\boldsymbol{x}',t)$ be the elastic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}'$ with volume source density $\{h_{i,j}^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{a_{i,j}^{\mathrm{A}}\delta(\boldsymbol{x}-\boldsymbol{x}',t),0\}$ and let $\{\tau_{i,j}^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{\tau_{i,j}^{\mathrm{B}},v_k^{\mathrm{B}}\}(\boldsymbol{x},\boldsymbol{x}'',t)$ be the elastic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}''$ with volume source density $\{h_{p,q}^{\mathrm{B}},f_r^{\mathrm{B}}\} = \{a_{p,q}^{\mathrm{B}}\delta(\boldsymbol{x}-\boldsymbol{x}'',t),0\}$, with $\boldsymbol{x}'\neq\boldsymbol{x}''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.2-7)) to the domain $\mathcal{D}$. (b) Write

$$-\tau_{p,q}^{\mathrm{A}} = G_{p,q,i,j}^{\tau h;A}(\boldsymbol{x},\boldsymbol{x}',t)a_{i,j}^{\mathrm{A}}, \qquad v_r^{\mathrm{A}} = G_{r,i,j}^{vh;A}(\boldsymbol{x},\boldsymbol{x}',t)a_{i,j}^{\mathrm{A}},$$
$$-\tau_{i,j}^{\mathrm{B}} = G_{i,j,p,q}^{\tau h;\mathrm{B}}(\boldsymbol{x},\boldsymbol{x}'',t)a_{p,q}^{\mathrm{B}}, \qquad v_k^{\mathrm{B}} = G_{k,p,q}^{vh;\mathrm{B}}(\boldsymbol{x},\boldsymbol{x}'',t)a_{p,q}^{\mathrm{B}}\,,$$

invoke the condition that the result should hold for arbitrary $a_{i,j}^{\mathrm{A}}$ and $a_{p,q}^{\mathrm{B}}$, and show that $G_{p,q,i,j}^{\tau h;A}(\boldsymbol{x}'',\boldsymbol{x}',t) = G_{i,j,p,q}^{\tau h;\mathrm{B}}(\boldsymbol{x}',\boldsymbol{x}'',t)$.

Exercise 15.8-2

Let $\{\tau_{p,q}^{\mathrm{A}},v_r^{\mathrm{A}}\} = \{\tau_{p,q}^{\mathrm{A}},v_r^{\mathrm{A}}\}(\boldsymbol{x},\boldsymbol{x}',t)$ be the elastic wave field at $\boldsymbol{x}$ that is causally radiated by the point source at $\boldsymbol{x}'$ with volume source density $\{h_{i,j}^{\mathrm{A}},f_k^{\mathrm{A}}\} = \{a_{i,j}^{\mathrm{A}}\delta(\boldsymbol{x}-\boldsymbol{x}',t),0\}$ and let $\{\tau_{i,j}^{\mathrm{B}},v_k^{\mathrm{B}}\} =$

$\{\tau^{B}_{i,j},v^{B}_{k}\}(x,x'',t)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{h^{B}_{i,j},f^{B}_{r}\} = \{0,b^{B}_{r}\delta(x - x'',t)\}$ with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.2-7)) to the domain $\mathcal{D}$. (b) Write

$$-\tau^{A}_{p,q} = G^{\tau h;A}_{p,q,i,j}(x,x',t)a^{A}_{i,j}, \qquad v^{A}_{r} = G^{vh;A}_{i,j,r}(x,x',t)a^{A}_{i,j},$$

$$-\tau^{B}_{i,j} = G^{\tau f;B}_{i,j,r}(x,x'',t)b^{B}_{r}, \qquad v^{B}_{k} = G^{vf;B}_{k,r}(x,x'',t)b^{B}_{r}\,,$$

invoke the condition that the result should hold for arbitrary $a^{A}_{i,j}$ and b^{B}_{r}, and show that $G^{vh;A}_{r,i,j}(x'',x',t) = -G^{\tau f;B}_{i,j,r}(x',x'',t)$.

Exercise 15.8-3

Let $\{\tau^{A}_{p,q},v^{A}_{r}\} = \{\tau^{A}_{p,q},v^{A}_{r}\}(x,x',t)$ be the elastic wave field at x that is causally radiated by the point source at x' with volume source density $\{h^{A}_{i,j},f^{A}_{k}\} = \{0,b^{A}_{k}\delta(x - x',t)\}$ and let $\{\tau^{B}_{i,j},v^{B}_{k}\} = \{\tau^{B}_{i,j},v^{B}_{k}\}(x,x'',t)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{h^{B}_{p,q},f^{B}_{r}\} = \{a^{B}_{p,q}\delta(x - x'',t),0\}$ with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.2-7)) to the domain $\mathcal{D}$. (b) Write

$$-\tau^{A}_{p,q} = G^{\tau f;A}_{p,q,k}(x,x',t)b^{A}_{k}, \qquad v^{A}_{r} = G^{vf;A}_{r,k}(x,x',t)b^{A}_{k},$$

$$-\tau^{B}_{i,j} = G^{\tau h;B}_{i,j,p,q}(x,x'',t)a^{B}_{p,q}, \qquad v^{B}_{k} = G^{vh;B}_{k,p,q}(x,x'',t)a^{B}_{p,q}\,,$$

invoke the condition that the result should hold for arbitrary b^{A}_{k} and $a^{B}_{p,q}$, and show that $G^{\tau f;A}_{p,q,k}(x'',x',t) = -G^{vh;B}_{k,p,q}(x',x'',t)$. (Note that this result is consistent with the result of Exercise 15.8-2.)

Exercise 15.8-4

Let $\{\tau^{A}_{p,q},v^{A}_{r}\} = \{\tau^{A}_{p,q},v^{A}_{r}\}(x,x',t)$ be the elastic wave field at x that is causally radiated by the point source at x' with volume source density $\{h^{A}_{i,j},f^{A}_{k}\} = \{0,b^{A}_{k}\delta(x - x',t)\}$ and let $\{\tau^{B}_{i,j},v^{B}_{k}\} = \{\tau^{B}_{i,j},v^{B}_{k}\}(x,x'',t)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{h^{B}_{p,q},f^{B}_{r}\} = \{0,b^{B}_{r}\delta(x - x'',t)\}$ with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.2-7)) to the domain $\mathcal{D}$. (b) Write

$$-\tau^{A}_{p,q} = G^{\tau f;A}_{p,q,k}(x,x',t)b^{A}_{k}, \qquad v^{A}_{r} = G^{vf;A}_{r,k}(x,x',t)b^{A}_{k},$$

$$-\tau^{B}_{i,j} = G^{\tau f;B}_{i,j,r}(x,x'',t)b^{B}_{r}, \qquad v^{B}_{k} = G^{vf;B}_{k,r}(x,x'',t)b^{B}_{r}\,,$$

invoke the condition that the result should hold for arbitrary b_k^{A} and b_r^{B}, and show that $G_{r,k}^{vf;\mathrm{A}}(x'',x',t) = G_{k,r}^{vf;\mathrm{B}}(x',x'',t)$.

Exercise 15.8-5

Let $\{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(x,x',s)$ be the elastic wave field at x that is causally radiated by the point source at x' with volume source density $\{\hat{h}_{i,j}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\} = \{\hat{a}_{i,j}^{\mathrm{A}}(s)\delta(x-x'),0\}$ and let $\{\hat{\tau}_{i,j}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\} = \{\hat{\tau}_{i,j}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(x,x'',s)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{h}_{p,q}^{\mathrm{B}},\hat{f}_r^{\mathrm{B}}\} = \{\hat{a}_{p,q}^{\mathrm{B}}(s)\delta(x-x''),0\}$, with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.4-7)) to the domain $\mathcal{D}$. (b) Write

$$-\hat{\tau}_{p,q}^{\mathrm{A}} = \hat{G}_{p,q,i,j}^{\tau h;A}(x,x',s)\hat{a}_{i,j}^{\mathrm{A}}(s), \qquad \hat{v}_r^{\mathrm{A}} = G_{r,i,j}^{vh}(x,x',s)\hat{a}_{i,j}^{\mathrm{A}}(s),$$
$$-\hat{\tau}_{i,j}^{\mathrm{B}} = \hat{G}_{i,j,p,q}^{\tau h;\mathrm{B}}(x,x'',s)\hat{a}_{p,q}^{\mathrm{B}}(s), \qquad \hat{v}_k^{\mathrm{B}} = \hat{G}_{k,p,q}^{vh;\mathrm{B}}(x,x'',s)\hat{a}_{p,q}^{\mathrm{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}_{i,j}^{\mathrm{A}}(s)$ and $\hat{a}_{p,q}^{\mathrm{B}}(s)$, and show that $\hat{G}_{p,q,i,j}^{\tau h;A}(x'',x',s) = \hat{G}_{i,j,p,q}^{\tau h;\mathrm{B}}(x',x'',s)$.

Exercise 15.8-6

Let $\{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(x,x',s)$ be the elastic wave field at x that is causally radiated by the point source at x' with volume source density $\{\hat{h}_{i,j}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\} = \{\hat{a}_{i,j}^{\mathrm{A}}(s)\delta(x-x'),0\}$ and let $\{\hat{\tau}_{i,j}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\} = \{\hat{\tau}_{i,j}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(x,x'',s)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{h}_{p,q}^{\mathrm{B}},\hat{f}_r^{\mathrm{B}}\} = \{0,\hat{b}_r^{\mathrm{B}}(s)\delta(x-x'')\}$, with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.4-7)) to the domain $\mathcal{D}$. (b) Write

$$-\hat{\tau}_{p,q}^{\mathrm{A}} = \hat{G}_{p,q,i,j}^{\tau h;A}(x,x',s)\hat{a}_{i,j}^{\mathrm{A}}(s), \qquad \hat{v}_r^{\mathrm{A}} = \hat{G}_{r,i,j}^{vh}(x,x',s)\,\hat{a}_{i,j}^{\mathrm{A}}(s),$$
$$-\hat{\tau}_{i,j}^{\mathrm{B}} = \hat{G}_{i,j,r}^{\tau f;\mathrm{B}}(x,x'',s)\hat{b}_r^{\mathrm{B}}(s), \qquad \hat{v}_k^{\mathrm{B}} = \hat{G}_{k,r}^{vf;\mathrm{B}}(x,x'',s)\hat{b}_r^{\mathrm{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}_{i,j}^{\mathrm{A}}(s)$ and $\hat{b}_r^{\mathrm{B}}(s)$, and show that $\hat{G}_{r,i,j}^{vh;A}(x'',x',s) = -\hat{G}_{i,j,r}^{\tau f;\mathrm{B}}(x',x'',s)$.

Exercise 15.8-7

Let $\{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\}(x,x',s)$ be the elastic wave field at x that is causally radiated by the point source at x' with volume source density $\{\hat{h}_{i,j}^{\mathrm{A}},\hat{f}_k^{\mathrm{A}}\} = \{0,\hat{b}_k^{\mathrm{A}}(s)\delta(x-x')\}$ and let $\{\hat{\tau}_{i,j}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\} = \{\hat{\tau}_{i,j}^{\mathrm{B}},\hat{v}_k^{\mathrm{B}}\}(x,x'',s)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{h}_{p,q}^{\mathrm{B}},\hat{f}_r^{\mathrm{B}}\} = \{\hat{a}_{p,q}^{\mathrm{B}}(s)\delta(x-x''),0\}$, with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.4-7)) to the domain $\mathcal{D}$. (b) Write

$$-\hat{\tau}^{A}_{p,q} = \hat{G}^{\tau f;A}_{p,q,k}(x,x',s)\hat{b}^{A}_{k}(s), \qquad \hat{v}^{A}_{r} = \hat{G}^{vf;A}_{r,k}(x,x',s)\hat{b}^{A}_{k}(s),$$
$$-\hat{\tau}^{B}_{i,j} = \hat{G}^{\tau h;B}_{i,j,p,q}(x,x'',s)\hat{a}^{B}_{p,q}(s), \qquad \hat{v}^{B}_{k} = \hat{G}^{vh;B}_{k,p,q}(x,x'',s)\hat{a}^{B}_{p,q}(s)\,,$$

invoke the condition that the result should hold for arbitrary $\hat{b}^{A}_{k}(s)$ and $\hat{a}^{B}_{p,q}(s)$, and show that $\hat{G}^{\tau f;A}_{p,q,k}(x'',x',s) = -\hat{G}^{vh;B}_{k,p,q}(x',x'',s)$. (Note that this result is consistent with the result of Exercise 15.8-6.)

Exercise 15.8-8

Let $\{\hat{\tau}^{A}_{p,q},\hat{v}^{A}_{r}\} = \{\hat{\tau}^{A}_{p,q},\hat{v}^{A}_{r}\}(x,x',s)$ be the elastic wave field at x that is causally radiated by the point source at x' with volume source density $\{\hat{h}^{A}_{i,j},\hat{f}^{A}_{k}\} = \{0,\hat{b}^{A}_{k}(s)\delta(x-x')\}$ and let $\{\hat{\tau}^{B}_{i,j},\hat{v}^{B}_{k}\} = \{\hat{\tau}^{B}_{i,j},\hat{v}^{B}_{k}\}(x,x'',s)$ be the elastic wave field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{h}^{B}_{p,q},\hat{f}^{B}_{r}\} = \{0,\hat{b}^{B}_{r}(s)\delta(x-x'')\}$, with $x' \neq x''$. The two sources radiate in adjoint solids occupying the domain $\mathcal{D}$. If $\mathcal{D}$ is bounded, its boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable; if $\mathcal{D}$ is unbounded, the standard provisions given in Section 15.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (15.4-7)) to the domain $\mathcal{D}$. (b) Write

$$-\hat{\tau}^{A}_{p,q} = \hat{G}^{\tau f;A}_{p,q,k}(x,x',s)\hat{b}^{A}_{k}(s), \qquad \hat{v}^{A}_{r} = \hat{G}^{vf;A}_{r,k}(x,x',s)\hat{b}^{A}_{k}(s)\,,$$
$$-\hat{\tau}^{B}_{i,j} = \hat{G}^{\tau f;B}_{i,j,r}(x,x'',s)\hat{b}^{B}_{r}(s), \qquad \hat{v}^{B}_{k} = \hat{G}^{vf;B}_{k,r}(x,x'',s)\hat{b}^{B}_{r}(s)\,,$$

invoke the condition that the result should hold for arbitrary $\hat{b}^{A}_{k}(s)$ and $\hat{b}^{B}_{r}(s)$, and show that $\hat{G}^{vf;A}_{r,k}(x'',x',s) = \hat{G}^{vf;B}_{k,r}(x',x'',s)$.

Exercise 15.8-9

Give the expressions for the time-domain Green's functions (a) $G^{\tau h}_{p,q,i,j}(x,x',t)$, (b) $G^{\tau f}_{p,q,k}(x,x',t)$, (c) $G^{vh}_{r,i,j}(x,x',t)$, (d) $G^{vf}_{r,k}(x,x',t)$ for a homogeneous isotropic, lossless solid with volume density of mass ρ and stiffness $C_{p,q,i,j} = \lambda\delta_{p,q}\delta_{i,j} + \mu(\delta_{p,i}\delta_{q,j} + \delta_{p,j}\delta_{q,i})$ that occupies the entire $\mathcal{R}^3$. (*Hint*: Use Equations (13.5-1)–(13.5-7).)

Answers:

(a) $G^{\tau h}_{p,q,i,j} = -C_{p,q,i,j}H(t)\delta(x-x') - \rho^{-1}C_{p,q,n,r}C_{k,m,i,j}\partial_n\partial_m \mathrm{I}_t G_{r,k}(x,x',t)\,,$

(b) $G^{\tau f}_{p,q,k} = \rho^{-1}C_{p,q,n,r}\partial_n G_{r,k}(x,x',t)\,,$

(c) $G^{vh}_{r,i,j} = -\rho^{-1}C_{k,m,i,j}\partial_m \mathrm{I}_t G_{r,k}(x,x',t)\,,$

(d) $G^{vf}_{r,k} = \rho^{-1}\partial_t G_{r,k}(x,x',t)\,,$

in which

$$G_{r,k}(x,x',t) = \frac{\delta(t-|x-x'|/c_S)}{4\pi c_S^2|x-x'|}\delta_{r,k} + \partial_r\partial_k\left[\frac{(t-|x-x'|/c_P)H(t-|x-x'|/c_P)}{4\pi|x-x'|}\right.$$
$$\left. - \frac{(t-|x-x'|/c_S)H(t-|x-x'|/c_S)}{4\pi|x-x'|}\right] \qquad \text{for} \quad |x-x'| \neq 0,$$

$$c_P = [(\lambda+2\mu)/\rho]^{1/2}, \quad c_S = (\mu/\rho)^{1/2}.$$

Exercise 15.8-10

Give the expressions for the complex frequency-domain Green's functions (a) $\hat{G}^{\tau h}_{p,q,i,j}(x,x',s)$, (b) $\hat{G}^{\tau f}_{p,q,k}(x,x',s)$, (c) $\hat{G}^{vh}_{r,i,j}(x,x',s)$, (d) $\hat{G}^{vf}_{r,k}(x,x',s)$ for a homogeneous, isotropic, lossless solid with volume density of mass ρ and stiffness $C_{p,q,i,j} = \lambda\delta_{p,q}\delta_{i,j} + \mu(\delta_{p,i}\delta_{q,j} + \delta_{p,j}\delta_{q,i})$ that occupies the entire $\mathcal{R}^3$. (*Hint*: Use Equations (13.4-5)–(13.4-10).)

Answers:

(a) $\hat{G}^{\tau h}_{p,q,i,j} = -s^{-1}C_{p,q,i,j}\delta(x - x') - (s\rho)^{-1}C_{p,q,n,r}C_{k,m,i,j}\partial_n\partial_m\hat{G}_{r,k}(x,x',s)$,

(b) $\hat{G}^{\tau f}_{p,q,k} = \rho^{-1}C_{p,q,n,r}\partial_n\hat{G}_{r,k}(x,x',s)$,

(c) $\hat{G}^{vh}_{r,i,j} = -\rho^{-1}C_{k,m,i,j}\partial_m\hat{G}_{r,k}(x,x',s)$,

(d) $\hat{G}^{vf}_{r,k} = s\rho^{-1}\hat{G}_{r,k}(x,x',s)$,

in which

$$\hat{G}_{r,k} = c_S^{-2}\hat{G}_S(x,x',s)\delta_{r,k} + s^{-2}\partial_r\partial_k\left[\hat{G}_P(x,x',s) - \hat{G}_S(x,x',s)\right],$$

with

$$\hat{G}_{P,S}(x,x',s) = \exp(-s|x - x'|/c_{P,S})/4\pi|x - x'| \qquad \text{for} \quad |x| \neq 0,$$

$$c_P = [(\lambda + 2\mu/\rho]^{1/2},\ c_S = (\mu/\rho)^{1/2}.$$

Exercise 15.8-11

Show that Equation (15.8-38) follows from Equation (15.8-14) and Equation (15.8-48) from Equation (15.8-24) by taking the Laplace transform with respect to time.

15.9 The direct (forward) scattering problem

The configuration in an elastodynamic scattering problem generally consists of a background solid with known elastodynamic properties, occupying the domain $\mathcal{D}$ (the *embedding*), in which, in principle, the radiation from given, arbitrarily distributed elastodynamic sources can be calculated with the aid of the theory developed in Section 15.8. In the embedding, an elastodynamically penetrable object of bounded support $\mathcal{D}^S$ (the *scatterer*) is present, whose known elastodynamic properties differ from those of the embedding (Figure 15.9-1).

The scatterer is elastodynamically irradiated by given sources located in the embedding, in a subdomain outside the scatterer. The problem is to determine the total elastic wave field in the configuration. The standard procedure is to calculate first the so-called *incident* elastic wave field, i.e. the wave field that would be present in the entire configuration if the object showed *no contrast* with respect to its embedding. (This can be done by employing the representations derived in Section 15.8.) Next, the total wave field is written as the superposition of the incident wave field and the *scattered* wave field, and, through a particular reasoning, the problem of determining the scattered wave field is reduced to calculating its *equivalent contrast source distributions*, whose common support will be shown to be the domain $\mathcal{D}^S$ occupied by the

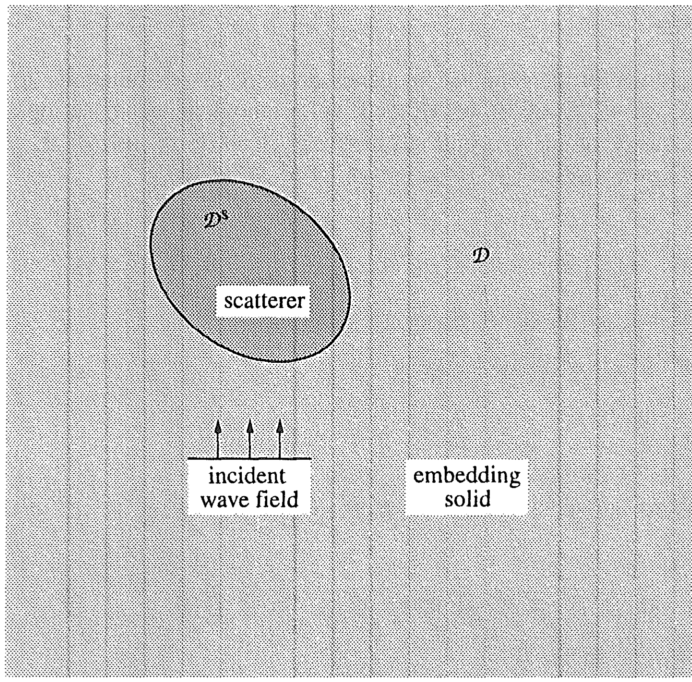


Figure 15.9-1 Scattering configuration with embedding $\mathcal{D}$ and scatterer $\mathcal{D}^{\rm s}$.

scatterer. In case the embedding $\mathcal{D}$ is a bounded domain, the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ is assumed to be elastodynamically impenetrable. If $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain are made. Both the incident wave field and the scattered wave field are causally related to the action of their respective sources.

Time-domain analysis

In the time-domain analysis of the problem, the elastodynamic properties of the embedding solid are characterised by the relaxation functions $\{\mu_{k,r},\chi_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t)$ which are causal functions of time. The elastodynamic properties of the scatterer are characterised by the relaxation functions $\{\mu^{\rm s}_{k,r},\chi^{\rm s}_{i,j,p,q}\} = \{\mu^{\rm s}_{k,r},\chi^{\rm s}_{i,j,p,q}\}(\boldsymbol{x},t)$ which are causal functions of time as well. The cases of an instantaneously reacting embedding and/or an instantaneously reacting scatterer easily follow from the more general cases for solids with relaxation. The contrast in the medium properties only differs from zero in $\mathcal{D}^{\rm s}$, and hence

$$\{\mu^{\rm s}_{k,r}-\mu_{k,r},\chi^{\rm s}_{i,j,p,q}-\chi_{i,j,p,q}\} = \{0,0\} \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\rm s\,\prime}, \tag{15.9-1}$$

where $\mathcal{D}^{\rm s\,\prime}$ is the complement of $\mathcal{D}^{\rm s}\cup\partial\mathcal{D}^{\rm s}$ in $\mathcal{D}$, i.e. the part of $\mathcal{D}$ that is exterior to $\mathcal{D}^{\rm s}$. The *incident wave field* is denoted by

$$\{\tau^{\rm i}_{p,q},v^{\rm i}_r\} = \{\tau^{\rm i}_{p,q},v^{\rm i}_r\}(\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.9-2}$$

and is considered to be known. (Once its generating sources are given, the expressions of the type derived in Section 15.8 yield the wave field values at any $x \in \mathcal{D}$.) The *total wave field* is denoted by

$$\{\tau_{p,q}, v_r\} = \{\tau_{p,q}, v_r\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}, \tag{15.9-3}$$

and the *scattered wave field* by

$$\{\tau^{\mathrm{s}}_{p,q}, v^{\mathrm{s}}_r\} = \{\tau^{\mathrm{s}}_{p,q}, v^{\mathrm{s}}_r\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}. \tag{15.9-4}$$

Then,

$$\{\tau_{p,q}, v_r\} = \{\tau^{\mathrm{i}}_{p,q} + \tau^{\mathrm{s}}_{p,q}, v^{\mathrm{i}}_r + v^{\mathrm{s}}_r\} \qquad \text{for} \quad x \in \mathcal{D}. \tag{15.9-5}$$

First, we investigate the structure of the elastic wave equations in the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. Since the sources that generate the total wave field are located in the domain exterior to the scatterer, the total wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$-\Delta^{+}_{k,m,p,q}\partial_m \tau_{p,q} + \partial_t C_t(\mu^{\mathrm{s}}_{k,r}, v_r; x,t) = 0 \qquad \text{for} \quad x \in \mathcal{D}^{\mathrm{s}}, \tag{15.9-6}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v_r - \partial_t C_t(\chi^{\mathrm{s}}_{i,j,p,q}, \tau_{p,q}; x,t) = 0 \qquad \text{for} \quad x \in \mathcal{D}^{\mathrm{s}}. \tag{15.9-7}$$

Since the sources that generate the total wave field would also generate the incident wave field, also this part of the wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$-\Delta^{+}_{k,m,p,q}\partial_m \tau^{\mathrm{i}}_{p,q} + \partial_t C_t(\mu_{k,r}, v^{\mathrm{i}}_r; x,t) = 0 \qquad \text{for} \quad x \in \mathcal{D}^{\mathrm{s}}, \tag{15.9-8}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{i}}_r - \partial_t C_t(\chi_{i,j,p,q}, \tau^{\mathrm{i}}_{p,q}; x,t) = 0 \qquad \text{for} \quad x \in \mathcal{D}^{\mathrm{s}}. \tag{15.9-9}$$

In view of Equation (15.9-5), Equations (15.9-6)–(15.9-9) lead to equations with the scattered wave field on the left-hand side that can alternatively be written as

$$-\Delta^{+}_{k,m,p,q}\partial_m \tau^{\mathrm{s}}_{p,q} + \partial_t C_t(\mu^{\mathrm{s}}_{k,r}, v^{\mathrm{s}}_r; x,t) = -\partial_t C_t(\mu^{\mathrm{s}}_{k,r} - \mu_{k,r}, v^{\mathrm{i}}_r; x,t) \qquad \text{for} \quad x \in \mathcal{D}^{\mathrm{s}}, \tag{15.9-10}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{s}}_r - \partial_t C_t(\chi^{\mathrm{s}}_{i,j,p,q}, \tau^{\mathrm{s}}_{p,q}; x,t) = \partial_t C_t(\chi^{\mathrm{s}}_{i,j,p,q} - \chi_{i,j,p,q}, \tau^{\mathrm{i}}_{p,q}; x,t) \quad \text{for} \ x \in \mathcal{D}^{\mathrm{s}}, \tag{15.9-11}$$

or as

$$-\Delta^{+}_{k,m,p,q}\partial_m \tau^{\mathrm{s}}_{p,q} + \partial_t C_t(\mu_{k,r}, v^{\mathrm{s}}_r; x,t) = -\partial_t C_t(\mu^{\mathrm{s}}_{k,r} - \mu_{k,r}, v_r; x,t) \qquad \text{for} \quad x \in \mathcal{D}^{\mathrm{s}}, \tag{15.9-12}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{s}}_r - \partial_t C_t(\chi_{i,j,p,q}, \tau^{\mathrm{s}}_{p,q}; x,t) = \partial_t C_t(\chi^{\mathrm{s}}_{i,j,p,q} - \chi_{i,j,p,q}, \tau_{p,q}; x,t) \quad \text{for} \ x \in \mathcal{D}^{\mathrm{s}}. \tag{15.9-13}$$

Equations (15.9-10) and (15.9-11) express that the scattered wave field in $\mathcal{D}^{\mathrm{s}}$ can be envisaged as being excited through both the presence of a contrast in the medium properties and the presence of an incident wave field. If either of the two is absent, the scattered wave field vanishes in $\mathcal{D}^{\mathrm{s}}$. This system of equations customarily serves as the starting point for the wave-field computation via a numerical discretisation procedure applied to the pertaining differential equations (finite-difference or finite-element techniques).

Equations (15.9-12) and (15.9-13) express that the scattered wave field can be envisaged as being generated by contrast sources (with support $\mathcal{D}^{\mathrm{s}}$) radiating into the embedding. This system of equations customarily serves as the starting point for the wave-field computation via an integral equation approach. This aspect, for which we also need the elastic wave equations that govern the wave field in $\mathcal{D}^{\mathrm{s}'}$, will be further discussed below. Now, in $\mathcal{D}^{\mathrm{s}'}$ the scattered wave field is source-free since the (actual) total wave field and the (calculated) incident wave field are assumed to be generated by the same source distributions. Consequently (note that in $\mathcal{D}^{\mathrm{s}'}$ the medium parameters are those of the embedding),

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau^{\mathrm{s}}_{p,q} + \mathrm{C}_t(\mu_{k,r},v^{\mathrm{s}}_r;\boldsymbol{x},t) = 0 \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}\prime}, \tag{15.9-14}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{s}}_r - \mathrm{C}_t(\chi_{i,j,p,q},\tau^{\mathrm{s}}_{p,q};\boldsymbol{x},t) = 0 \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}\prime}. \tag{15.9-15}$$

Equations (15.9-12), (15.9-13) and (15.9-14), (15.9-15) can be combined to

$$-\Delta^{+}_{k,m,p,q}\partial_m\tau^{\mathrm{s}}_{p,q} + \mathrm{C}_t(\mu_{k,r},v^{\mathrm{s}}_r;\boldsymbol{x},t) = \{f^{\mathrm{s}}_k,0\} \qquad \text{for} \quad \boldsymbol{x}\in\{\mathcal{D}^{\mathrm{s}},\mathcal{D}^{\mathrm{s}\prime}\}, \tag{15.9-16}$$

$$\Delta^{+}_{i,j,n,r}\partial_n v^{\mathrm{s}}_r - \mathrm{C}_t(\chi_{i,j,p,q},\tau^{\mathrm{s}}_{p,q};\boldsymbol{x},t) = \{h^{\mathrm{s}}_{i,j},0\} \qquad \text{for} \quad \boldsymbol{x}\in\{\mathcal{D}^{\mathrm{s}},\mathcal{D}^{\mathrm{s}\prime}\}, \tag{15.9-17}$$

where

$$f^{\mathrm{s}}_k = -\partial_t\mathrm{C}_t(\mu^{\mathrm{s}}_{k,r} - \mu_{k,r},v_r;\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}} \tag{15.9-18}$$

is the *equivalent contrast volume source density of force* and

$$h^{\mathrm{s}}_{i,j} = \partial_t\mathrm{C}_t(\chi^{\mathrm{s}}_{i,j,p,q} - \chi_{i,j,p,q},\tau_{p,q};\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}} \tag{15.9-19}$$

is the *equivalent contrast volume source density of deformation rate*. If the contrast volume source densities f^{s}_k and $h^{\mathrm{s}}_{i,j}$ were known, Equations (15.9-16) and (15.9-17) would constitute a direct (forward) source problem in the embedding of the type discussed in Section 15.8. As yet, however, these contrast source densities are unknown.

To construct a system of equations from which the scattering problem can be solved, we employ the source type integral representations for the scattered wave field (see Equations (15.8-14) and (15.8-24)), viz.

$$-\tau^{\mathrm{s}}_{p,q}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G^{\tau h}_{p,q,i,j},h^{\mathrm{s}}_{i,j};\boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G^{\tau f}_{p,q,k},f^{\mathrm{s}}_k;\boldsymbol{x}',\boldsymbol{x},t)\right]\mathrm{d}V \quad \text{for} \ \boldsymbol{x}'\in\mathcal{D}, \tag{15.9-20}$$

$$v^{\mathrm{s}}_r(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G^{vh}_{r,i,j},h^{\mathrm{s}}_{i,j};\boldsymbol{x}',\boldsymbol{x},t) + \mathrm{C}_t(G^{vf}_{r,k},f^{\mathrm{s}}_k;\boldsymbol{x}',\boldsymbol{x},t)\right]\mathrm{d}V \qquad \text{for} \quad \boldsymbol{x}'\in\mathcal{D}, \tag{15.9-21}$$

in which the Green's functions apply to a solid with the same elastodynamic properties as the embedding. Writing Equations (15.9-18) and (15.9-19) with the aid of Equation (15.9-5) as

$$f^{\mathrm{s}}_k = -\partial_t\mathrm{C}_t(\mu^{\mathrm{s}}_{k,r} - \mu_{k,r},v^{\mathrm{i}}_r + v^{\mathrm{s}}_r;\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-22}$$

$$h^{\mathrm{s}}_{i,j} = \partial_t\mathrm{C}_t(\chi^{\mathrm{s}}_{i,j,p,q} - \chi_{i,j,p,q},\tau^{\mathrm{i}}_{p,q} + \tau^{\mathrm{s}}_{p,q};\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}'\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-23}$$

and invoking Equations (15.9-20) and (15.9-21) for $\boldsymbol{x}'\in\mathcal{D}^{\mathrm{s}}$, a system of integral equations results from which f^{s}_k and $h^{\mathrm{s}}_{i,j}$ can be solved. Once these quantities have been determined, the scattered wave field can be calculated in the entire configuration by reusing Equations (15.9-20) and (15.9-21) for all $\boldsymbol{x}\in\mathcal{D}$, and since the incident wave field was presumably known already, the total wave field follows.

Except for some simple geometries where analytic methods can be employed, the integral equations for the scattering of elastic waves have to be solved with the aid of numerical methods. The circumstance that the Green's tensors are singular when $\boldsymbol{x}' = \boldsymbol{x}$ presents difficulties, in the sense that in the neighbourhood of $\boldsymbol{x}'$ the integrations with respect to $\boldsymbol{x}$ cannot be evaluated by a simple numerical formula (such as the tetrahedral formula, which is the three-dimensional equivalent of the one-dimensional trapezoidal formula), but have to be evaluated by a limiting analytic procedure. For the rest, the application of numerical methods to the relevant integral equations presents no essential difficulties.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the elastodynamic properties of the embedding solid are characterised by the functions $\{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s)$ and the elastodynamic properties of the scatterer are characterised by the functions $\{\hat{\zeta}^{\mathrm{s}}_{k,r},\hat{\eta}^{\mathrm{s}}_{i,j,p,q}\} = \{\hat{\zeta}^{\mathrm{s}}_{k,r},\hat{\eta}^{\mathrm{s}}_{i,j,p,q}\}(\boldsymbol{x},s)$. The contrast in the medium properties only differs from zero in $\mathcal{D}^{\mathrm{s}}$, and hence

$$\{\hat{\zeta}^{\mathrm{s}}_{k,r}-\hat{\zeta}_{k,r},\hat{\eta}^{\mathrm{s}}_{i,j,p,q}-\hat{\eta}_{i,j,p,q}\} = \{0,0\} \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}\prime}, \tag{15.9-24}$$

where $\mathcal{D}^{\mathrm{s}\prime}$ is the complement of $\mathcal{D}^{\mathrm{s}}\cup\partial\mathcal{D}^{\mathrm{s}}$ in $\mathcal{D}$, i.e. the part of $\mathcal{D}$ that is exterior to $\partial\mathcal{D}^{\mathrm{s}}$. The *incident wave field* is denoted by

$$\{\hat{\tau}^{\mathrm{i}}_{p,q},\hat{v}^{\mathrm{i}}_{r}\} = \{\hat{\tau}^{\mathrm{i}}_{p,q},\hat{v}^{\mathrm{i}}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.9-25}$$

and is considered to be known. (Once its generating sources are given, the expressions of the type derived in Section 15.8 yield the wave-field values at any $\boldsymbol{x}\in\mathcal{D}$.) The *total wave field* is denoted by

$$\{\hat{\tau}_{p,q},\hat{v}_{r}\} = \{\hat{\tau}_{p,q},\hat{v}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.9-26}$$

and the *scattered wave field* by

$$\{\hat{\tau}^{\mathrm{s}}_{p,q},\hat{v}^{\mathrm{s}}_{r}\} = \{\hat{\tau}^{\mathrm{s}}_{p,q},\hat{v}^{\mathrm{s}}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.9-27}$$

Then,

$$\{\hat{\tau}_{p,q},\hat{v}_{r}\} = \{\hat{\tau}^{\mathrm{i}}_{p,q}+\hat{\tau}^{\mathrm{s}}_{p,q},\hat{v}^{\mathrm{i}}_{r}+\hat{v}^{\mathrm{s}}_{r}\} \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.9-28}$$

First, we investigate the structure of the complex frequency-domain elastic wave equations in the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. Since the sources that generate the total wave field are located in the domain exterior to the scatterer, the total wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}_{p,q}+\hat{\zeta}^{\mathrm{s}}_{k,r}\hat{v}_r = 0 \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-29}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}_r-\hat{\eta}^{\mathrm{s}}_{i,j,p,q}\hat{\tau}_{p,q} = 0 \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}. \tag{15.9-30}$$

Since the sources that generate the total wave field would also generate the incident wave field, also this part of the wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{\mathrm{i}}_{p,q}+\hat{\zeta}_{k,r}\hat{v}^{\mathrm{i}}_r = 0 \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-31}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{\mathrm{i}}_r-\hat{\eta}_{i,j,p,q}\hat{\tau}^{\mathrm{i}}_{p,q} = 0 \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}. \tag{15.9-32}$$

In view of Equation (15.9-28), Equations (15.9-29)–(15.9-32) lead to equations with the scattered wave field on the left-hand side that can, alternatively, be written as

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{\mathrm{s}}_{p,q}+\hat{\zeta}^{\mathrm{s}}_{k,r}\hat{v}^{\mathrm{s}}_r = -(\hat{\zeta}^{\mathrm{s}}_{k,r}-\hat{\zeta}_{k,r})\hat{v}^{\mathrm{i}}_r \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-33}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{\mathrm{s}}_r-\hat{\eta}^{\mathrm{s}}_{i,j,p,q}\hat{\tau}^{\mathrm{s}}_{p,q} = (\hat{\eta}^{\mathrm{s}}_{i,j,p,q}-\hat{\eta}_{i,j,p,q})\hat{\tau}^{\mathrm{i}}_{p,q} \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-34}$$

or as

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{\mathrm{s}}_{p,q}+\hat{\zeta}_{k,r}\hat{v}^{\mathrm{s}}_r = -(\hat{\zeta}^{\mathrm{s}}_{k,r}-\hat{\zeta}_{k,r})\hat{v}_r \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.9-35}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{\mathrm{s}}_r-\hat{\eta}_{i,j,p,q}\hat{\tau}^{\mathrm{s}}_{p,q} = (\hat{\eta}^{\mathrm{s}}_{i,j,p,q}-\hat{\eta}_{i,j,p,q})\hat{\tau}_{p,q} \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}. \tag{15.9-36}$$

Equations (15.9-33) and (15.9-34) express that the scattered wave field in $\mathcal{D}^{s}$ can be envisaged as being excited through both the presence of a contrast in the medium properties and the presence of an incident wave field. If either of the two is absent, the scattered wave field vanishes in $\mathcal{D}^{s}$. This system of equations customarily serves as the starting point for the wave-field computation via a numerical discretisation procedure applied to the pertaining differential equations (finite-difference or finite-element techniques).

Equations (15.9-35) and (15.9-36) express that the scattered wave field can be envisaged as to be generated by contrast sources (with support $\mathcal{D}^{s}$) radiating into the embedding. This system of equations customarily serves as the starting point for the wave-field computation via an integral equation approach. This aspect, for which we also need the elastic wave equations that govern the wave field in $\mathcal{D}^{s'}$, will be further discussed below. Now, in $\mathcal{D}^{s'}$ the scattered wave field is source-free since the (actual) total wave field and the (calculated) incident wave field are assumed to be generated by the same source distributions. Consequently (note that in $\mathcal{D}^{s'}$ the medium parameters are those of the embedding),

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{s}_{p,q} + \hat{\zeta}_{k,r}\hat{v}^{s}_{r} = 0 \qquad \text{for} \quad x\in\mathcal{D}^{s'}, \tag{15.9-37}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{s}_{r} - \hat{\eta}_{i,j,p,q}\hat{\tau}^{s}_{p,q} = 0 \qquad \text{for} \quad x\in\mathcal{D}^{s'}. \tag{15.9-38}$$

Equations (15.9-35), (15.9-36) and (15.9-37), (15.9-38) can be combined to give

$$-\Delta^{+}_{k,m,p,q}\partial_m\hat{\tau}^{s}_{p,q} + \hat{\zeta}_{k,r}\hat{v}^{s}_{r} = \{\hat{f}^{s}_{k},0\} \qquad \text{for} \quad x\in\{\mathcal{D}^{s},\mathcal{D}^{s'}\}, \tag{15.9-39}$$

$$\Delta^{+}_{i,j,n,r}\partial_n\hat{v}^{s}_{r} - \hat{\eta}_{i,j,p,q}\hat{\tau}^{s}_{p,q} = \{\hat{h}^{s}_{i,j},0\} \qquad \text{for} \quad x\in\{\mathcal{D}^{s},\mathcal{D}^{s'}\}, \tag{15.9-40}$$

where

$$\hat{f}^{s}_{k} = -(\hat{\zeta}^{s}_{k,r} - \hat{\zeta}_{k,r})\hat{v}_{r} \qquad \text{for} \quad x\in\mathcal{D}^{s} \tag{15.9-41}$$

is the *equivalent contrast volume source density of force* and

$$\hat{h}^{s}_{i,j} = (\hat{\eta}^{s}_{i,j,p,q} - \hat{\eta}_{i,j,p,q})\hat{\tau}_{p,q} \qquad \text{for} \quad x\in\mathcal{D}^{s} \tag{15.9-42}$$

is the *equivalent contrast volume source density of deformation rate*. If the contrast source densities $\hat{f}^{s}_{k}$ and $\hat{h}^{s}_{i,j}$ were known, Equations (15.9-39) and (15.9-40) would constitute a direct (forward) source problem in the embedding of the type discussed in Section 15.8. As yet, however, these contrast source densities are unknown.

To construct a system of equations from which the scattering problem can be solved, we employ the source type integral representations for the scattered wave field (see Equations (15.8-38) and (15.8-48)), viz.

$$-\hat{\tau}^{s}_{p,q}(x',s) = \int_{x\in\mathcal{D}^{s}} \left[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\hat{h}^{s}_{i,j}(x,s) + \hat{G}^{\tau f}_{p,q,k}(x',x,s)\,\hat{f}^{s}_{k}(x,s)\right]\mathrm{d}V$$

$$\text{for} \quad x'\in\mathcal{D}, \tag{15.9-43}$$

$$\hat{v}^{s}_{r}(x',s) = \int_{x\in\mathcal{D}^{s}} \left[\hat{G}^{vh}_{r,i,j}(x',x,s)\hat{h}^{s}_{i,j}(x,s) + \hat{G}^{vf}_{r,k}(x',x,s)\,\hat{f}^{s}_{k}(x,s)\right]\mathrm{d}V \qquad \text{for} \quad x'\in\mathcal{D}, \tag{15.9-44}$$

in which the Green's functions apply to a solid with the same elastodynamic properties as the embedding. Writing Equations (15.9-41) and (15.9-42) with the aid of Equation (15.9-28) as

$$\hat{f}_k^{\,s} = -(\hat{\xi}_{k,r}^{\,s} - \hat{\xi}_{k,r})(\hat{v}_r^{\,i} + \hat{v}_r^{\,s}) \qquad \text{for} \quad x \in \mathcal{D}^s, \tag{15.9-45}$$

$$\hat{h}_{i,j}^{\,s} = (\hat{\eta}_{i,j,p,q}^{\,s} - \hat{\eta}_{i,j,p,q})(\hat{\tau}_{p,q}^{\,i} + \hat{\tau}_{p,q}^{\,s}) \qquad \text{for} \quad x \in \mathcal{D}^s, \tag{15.9-46}$$

and invoking Equations (15.9-43) and (15.9-44) for $x' \in \mathcal{D}^s$, a system of integral equations results from which $\hat{f}_k^{\,s}$ and $\hat{h}_{i,j}^{\,s}$ can be solved. Once these quantities have been determined, the scattered wave field can be calculated in the entire configuration by reusing Equations (15.9-43) and (15.9-44) for all $x \in \mathcal{D}$, and since the incident wave field was presumably known already, the total wave field follows.

Except for some simple geometries (see, for example, Bowman *et al.*, 1969), where analytic methods can be employed, the complex frequency-domain integral equations for the scattering of elastic waves have to be solved with the aid of numerical methods. The circumstance that the Green's tensors are singular when $x' = x$ presents difficulties, in the sense that in the neighbourhood of x' the integrations with respect to x cannot be evaluated by a simple numerical formula (such as the tetrahedral formula, which is the three-dimensional equivalent of the one-dimensional trapezoidal formula), but have to be evaluated by a limiting analytic procedure. For the rest, the application of numerical methods to the relevant integral equations presents no essential difficulties. Recent advances on this subject can be found in Van den Berg (1991), and in Fokkema and Van den Berg (1993).

15.10 The inverse source problem

The configuration in an elastodynamic inverse source problem generally consists of a background solid with known elastodynamic properties, occupying the domain $\mathcal{D}$ (the *embedding*), in which, in principle, the radiation from given, arbitrarily distributed elastodynamic sources can be calculated with the aid of the theory developed in Section 15.8. In the embedding an either known or guessed bounded domain $\mathcal{D}^T$ is present in which elastically radiating sources of unknown nature and unknown spatial distribution are present. The presence of these sources manifests itself in the entire embedding. In some bounded subdomain $\mathcal{D}^\Omega$ of $\mathcal{D}$, and exterior to $\mathcal{D}^T$, the radiated elastic wave field is accessible to measurement (Figure 15.10-1).

We assume that the action of the radiating sources can be modelled by volume source densities of deformation rate and force. The objective is to reconstruct these volume source densities with support $\mathcal{D}^T$ from (a set of) measured values of the dynamic stress and/or the particle velocity in $\mathcal{D}^\Omega$. Since the inverse source problem is, by necessity, a remote sensing problem, the global reciprocity theorems of Sections 15.2–15.5 can be expected to provide a means for interrelating the known, measured wave-field data with the unknown source distributions. In case the embedding $\mathcal{D}$ is a bounded domain, the boundary surface $\partial\mathcal{D}$ is assumed to be elastodynamically impenetrable. If $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain are made. The radiated wave field is, by its nature, causally related to the sources by which it is generated. For gathering maximum information, the reciprocity theorems are applied to the domain interior to a closed surface $\mathcal{S}^\Omega$ that completely surrounds both $\mathcal{D}^T$ and $\mathcal{D}^\Omega$. If necessary, measurement on $\mathcal{S}^\Omega$ can also be carried out.

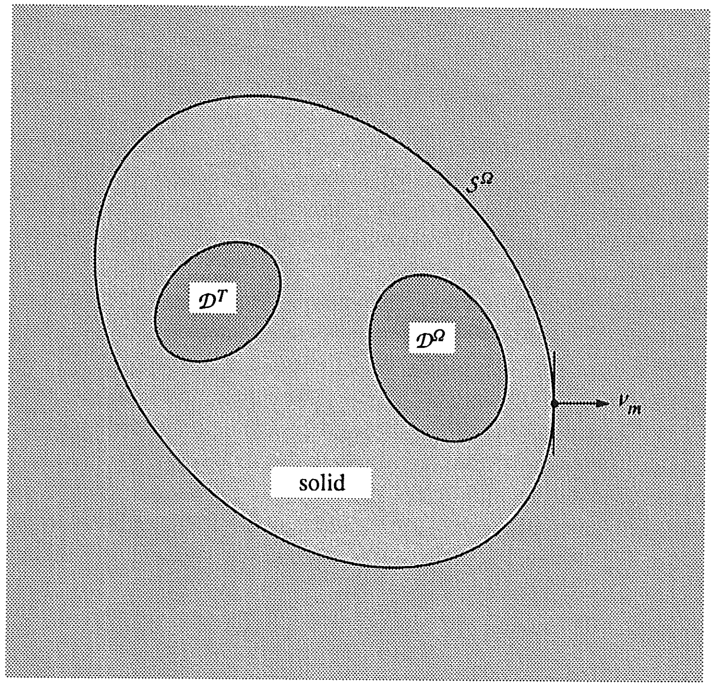


Figure 15.10-1 Configuration of the inverse source problem: $\mathcal{D}^T$ is the support of the unknown radiating sources; on $\mathcal{D}^{\Omega}$ and $\mathcal{S}^{\Omega}$ the transmitted wave field is accessible to measurement.

Time-domain analysis

In the time-domain analysis of the problem, the elastodynamic properties of the embedding solid are characterised by the relaxation functions $\{\mu_{k,r},\chi_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t)$ which are causal functions of time. The case of an instantaneously reacting embedding solid easily follows from the more general case of a solid with relaxation. The causally radiated elastodynamic wave field is denoted by $\{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\} = \{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}(\boldsymbol{x},t)$.

First, the measured elastic wave-field data are interrelated with the unknown source distributions $\{h^{\mathrm{T}}_{i,j},f^{\mathrm{T}}_k\} = \{h^{\mathrm{T}}_{i,j},f^{\mathrm{T}}_k\}(\boldsymbol{x},t)$, via the global time-domain reciprocity theorem of the *convolution type*, Equation (15.2-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\tau^{\mathrm{A}}_{p,q},v^{\mathrm{A}}_r\}(\boldsymbol{x},t) = \{\tau^{\mathrm{T}}_{p,q},v^{\mathrm{T}}_r\}(\boldsymbol{x},t) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.10-1}$$

$$\{h^{\mathrm{A}}_{i,j},f^{\mathrm{A}}_k\}(\boldsymbol{x},t) = \{h^{\mathrm{T}}_{i,j},f^{\mathrm{T}}_k\}(\boldsymbol{x},t) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^T, \tag{15.10-2}$$

and

$$\{\mu^{\mathrm{A}}_{k,r},\chi^{\mathrm{A}}_{i,j,p,q}\}(\boldsymbol{x},t) = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.10-3}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\tau_{p,q}^{\rm B},v_k^{\rm B}\}(x,t) = \{\tau_{i,j}^{\Omega},v_k^{\Omega}\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}, \tag{15.10-4}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{h_{p,q}^{\rm B},f_r^{\rm B}\}(x,t) = \{h_{p,q}^{\Omega},f_r^{\Omega}\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}^{\Omega}. \tag{15.10-5}$$

Furthermore, the solid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\mu_{r,k}^{\rm B},\chi_{p,q,i,j}^{\rm B}\}(x,t) = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}. \tag{15.10-6}$$

Then, application of Equation (15.2-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{x\in\mathcal{D}^T} \left[\mathrm{C}_t(-\tau_{i,j}^{\Omega},h_{i,j}^{\rm T};x,t) - \mathrm{C}_t(v_k^{\Omega},f_k^{\rm T};x,t)\right] \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^{\Omega}} \left[\mathrm{C}_t(-\tau_{p,q}^{\rm T},h_{p,q}^{\Omega};x,t) - \mathrm{C}_t(v_r^{\rm T},f_r^{\Omega};x,t)\right] \mathrm{d}V$$

$$+ \Delta_{m,r,p,q}^{+} \int_{x\in\mathcal{S}^{\Omega}} \nu_m \left[-\mathrm{C}_t(-\tau_{p,q}^{\rm T},v_r^{\Omega};x,t) + \mathrm{C}_t(v_r^{\rm T},-\tau_{p,q}^{\Omega};x,t)\right] \mathrm{d}A\,. \tag{15.10-7}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over $\partial\mathcal{D}$ does and in between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ no sources of the radiated or the computational wave fields are present (see Exercise 15.2-2). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are also causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (because the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are not causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.2-2).

Secondly, the measured elastic wave-field data are interrelated with the unknown source distributions $\{h_{i,j}^{\rm T},f_k^{\rm T}\} = \{h_{i,j}^{\rm T},f_k^{\rm T}\}(x,t)$, via the global time-domain reciprocity theorem of the *correlation type*, Equation (15.3-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\tau_{p,q}^{\rm A},v_r^{\rm A}\}(x,t) = \{\tau_{p,q}^{\rm T},v_r^{\rm T}\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}, \tag{15.10-8}$$

$$\{h_{i,j}^{\rm A},f_k^{\rm A}\}(x,t) = \{h_{i,j}^{\rm T},f_k^{\rm T}\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}^T, \tag{15.10-9}$$

and

$$\{\mu_{k,r}^{\rm A},\chi_{i,j,p,q}^{\rm A}\}(x,t) = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}. \tag{15.10-10}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\tau_{i,j}^{\mathrm{B}},v_k^{\mathrm{B}}\}(x,t) = \{\tau_{i,j}^{\Omega},v_k^{\Omega}\}(x,t) \qquad \text{for} \quad x\in\mathcal{D}, \tag{15.10-11}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{h_{p,q}^{\mathrm{B}},f_r^{\mathrm{B}}\}(x,t) = \{h_{p,q}^{\Omega},f_r^{\Omega}\}(x,t) \qquad \text{for} \quad x\in\mathcal{D}^{\Omega}. \tag{15.10-12}$$

Furthermore, the solid properties in state B will be taken to be the time-reverse adjoint of ones in state A, i.e.

$$\{\mu_{r,k}^{\mathrm{B}},\chi_{p,q,i,j}^{\mathrm{B}}\}(x,t) = \{\mathrm{J}_t(\mu_{k,r}),\mathrm{J}_t(\chi_{i,j,p,q}\}(x,t) \qquad \text{for} \quad x\in\mathcal{D}. \tag{15.10-13}$$

Then, application of Equation (15.3-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(\mathrm{J}_t(-\tau_{i,j}^{\Omega}),h_{i,j}^{\mathrm{T}};x,t) + \mathrm{C}_t(\mathrm{J}_t(v_k^{\Omega}),f_k^{\mathrm{T}};x,t)\right]\mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{T}},\mathrm{J}_t(h_{p,q}^{\Omega});x,t) + \mathrm{C}_t(v_r^{\mathrm{T}},\mathrm{J}_t(f_r^{\Omega});x,t)\right]\mathrm{d}V$$

$$+ \Delta_{m,r,p,q}^{+}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{T}},\mathrm{J}_t(v_r^{\Omega});x,t) + \mathrm{C}_t(v_r^{\mathrm{T}},\mathrm{J}_t(-\tau_{p,q}^{\Omega});x,t)\right]\mathrm{d}A\,. \tag{15.10-14}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.3-2)). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are non-causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.3-2). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are causal and the surface integral over $\mathcal{S}^{\Omega}$ does vanish (since the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). For additional literature on the subject, see De Hoop (1988).

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the elastodynamic properties of the embedding solid are characterised by the functions $\{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(x,s)$. The causally radiated elastic wave field is denoted by $\{\hat{\tau}_{p,q}^{\mathrm{T}},\hat{v}_r^{\mathrm{T}}\} = \{\hat{\tau}_{p,q}^{\mathrm{T}},\hat{v}_r^{\mathrm{T}}\}(x,s)$.

First, the measured elastic wave-field data are interrelated with the unknown source distributions $\{\hat{h}^{\mathrm{T}}_{i,j}, \hat{f}^{\mathrm{T}}_k\} = \{\hat{h}^{\mathrm{T}}_{i,j}, \hat{f}^{\mathrm{T}}_k\}(\boldsymbol{x},s)$, via the global complex frequency-domain reciprocity theorem of the *time convolution type*, Equation (15.4-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\hat{\tau}^{\mathrm{A}}_{p,q},\hat{v}^{\mathrm{A}}_r\}(\boldsymbol{x},s) = \{\hat{\tau}^{\mathrm{T}}_{p,q},\hat{v}^{\mathrm{T}}_r\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.10-15}$$

$$\{\hat{h}^{\mathrm{A}}_{i,j}, \hat{f}^{\mathrm{A}}_k\}(\boldsymbol{x},s) = \{\hat{h}^{\mathrm{T}}_{i,j}, \hat{f}^{\mathrm{T}}_k\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{T}, \tag{15.10-16}$$

and

$$\{\hat{\zeta}^{\mathrm{A}}_{k,r},\hat{\eta}^{\mathrm{A}}_{i,j,p,q}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.10-17}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{\tau}^{\mathrm{B}}_{i,j},\hat{v}^{\mathrm{B}}_k\}(\boldsymbol{x},s) = \{\hat{\tau}^{\Omega}_{i,j},\hat{v}^{\Omega}_k\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.10-18}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{h}^{\mathrm{B}}_{p,q}, \hat{f}^{\mathrm{B}}_r\}(\boldsymbol{x},s) = \{\hat{h}^{\Omega}_{p,q}, \hat{f}^{\Omega}_r\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{15.10-19}$$

Furthermore, the solid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\hat{\zeta}^{\mathrm{B}}_{r,k},\hat{\eta}^{\mathrm{B}}_{p,q,i,j}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.10-20}$$

Then, application of Equation (15.4-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^T}\left[-\hat{\tau}^{\Omega}_{i,j}(\boldsymbol{x},s)\hat{h}^{\mathrm{T}}_{i,j}(\boldsymbol{x},s) - \hat{v}^{\Omega}_k(\boldsymbol{x},s)\,\hat{f}^{\mathrm{T}}_k(\boldsymbol{x},s)\right]\mathrm{d}V \\
&= \int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}}\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(\boldsymbol{x},s)\hat{h}^{\Omega}_{p,q}(\boldsymbol{x},s) - \hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\,\hat{f}^{\Omega}_r(\boldsymbol{x},s)\right]\mathrm{d}V \\
&\quad + \Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}}\nu_m\left[\hat{\tau}^{\mathrm{T}}_{p,q}(\boldsymbol{x},s)\hat{v}^{\Omega}_r(\boldsymbol{x},s) - \hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\hat{\tau}^{\Omega}_{p,q}(\boldsymbol{x},s)\right]\mathrm{d}A\,.
\end{aligned} \tag{15.10-21}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state T and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.4-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.4-4).

Secondly, the measured elastic wave-field data are interrelated with the unknown source distributions $\{\hat{h}^{\mathrm{T}}_{i,j}, \hat{f}^{\mathrm{T}}_k\} = \{\hat{h}^{\mathrm{T}}_{i,j}, \hat{f}^{\mathrm{T}}_k\}(\boldsymbol{x},s)$, via the global complex frequency-domain reciprocity theorem of the *time correlation type*, Equation (15.5-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\hat{\tau}^{\mathrm{A}}_{p,q},\hat{v}^{\mathrm{A}}_r\}(\boldsymbol{x},s) = \{\hat{\tau}^{\mathrm{T}}_{p,q},\hat{v}^{\mathrm{T}}_r\}(\boldsymbol{x},s) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.10-22}$$

$$\{\hat{h}^{\mathrm{A}}_{i,j}, \hat{f}^{\mathrm{A}}_k\}(\boldsymbol{x},s) = \{\hat{h}^{\mathrm{T}}_{i,j}, \hat{f}^{\mathrm{T}}_k\}(\boldsymbol{x},s) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{T}, \tag{15.10-23}$$

and

$$\{\hat{\zeta}^{\mathrm{A}}_{k,r},\hat{\eta}^{\mathrm{A}}_{i,j,p,q}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.10-24}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{\tau}^{\mathrm{B}}_{i,j},\hat{v}^{\mathrm{B}}_k\}(\boldsymbol{x},s) = \{\hat{\tau}^{\Omega}_{i,j},\hat{v}^{\Omega}_k\}(\boldsymbol{x},s) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.10-25}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{h}^{\mathrm{B}}_{p,q}, \hat{f}^{\mathrm{B}}_r\}(\boldsymbol{x},s) = \{\hat{h}^{\Omega}_{p,q}, \hat{f}^{\Omega}_r\}(\boldsymbol{x},s) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{15.10-26}$$

Furthermore, the solid properties in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\hat{\zeta}^{\mathrm{B}}_{r,k},\hat{\eta}^{\mathrm{B}}_{p,q,i,j}\}(\boldsymbol{x},s) = \{-\hat{\zeta}_{k,r},-\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},-s) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.10-27}$$

Then, application of Equation (15.5-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{T}} \left[-\hat{\tau}^{\Omega}_{i,j}(\boldsymbol{x},-s)\hat{h}^{\mathrm{T}}_{i,j}(\boldsymbol{x},s) + \hat{v}^{\Omega}_k(\boldsymbol{x},-s)\,\hat{f}^{\mathrm{T}}_k(\boldsymbol{x},s)\right]\mathrm{d}V \\
&= -\int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[-\hat{\tau}^{\mathrm{T}}_{p,q}(\boldsymbol{x},s)\hat{h}^{\Omega}_{p,q}(\boldsymbol{x},-s) + \hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\,\hat{f}^{\Omega}_r(\boldsymbol{x},-s)\right]\mathrm{d}V \\
&\quad + \Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m\left[-\hat{\tau}^{\mathrm{T}}_{p,q}(\boldsymbol{x},s)\hat{v}^{\Omega}_r(\boldsymbol{x},-s) - \hat{v}^{\mathrm{T}}_r(\boldsymbol{x},s)\hat{\tau}^{\Omega}_{p,q}(\boldsymbol{x},-s)\right]\mathrm{d}A\,.
\end{aligned} \tag{15.10-28}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state T and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.5-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.5-4). If, however, $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the radiated or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere).

A solution to the inverse source problem is commonly constructed as follows. For the source distributions in the computational state Ω we take a sequence of M linearly independent spatial distributions with the common spatial support $\mathcal{D}^{\Omega}$. The corresponding sequence of elastic wave-field distributions (in the medium adjoint, or time-reverse adjoint, to the actual one) is computed. Next, the unknown source distributions are expanded into an appropriate sequence of N expansion functions with the common spatial support $\mathcal{D}^{T}$ or a subset of it; the corresponding expansion coefficients are unknown. Substitution of the results in Equations (15.10-7), (15.10-14), (15.10-21) or (15.10-28) and evaluation of the relevant integrals lead to a system of M linear algebraic equations with N unknowns. When $M < N$, the system is underdetermined and cannot be solved. When $M = N$, the system can be solved, unless the pertaining matrix of coefficients is singular. However, even if this matrix is non-singular, it turns out to be ill-conditioned in most practical cases. Therefore, one usually takes $M > N$, and a best fit of the expanded source distributions to the measured data is obtained by the application of minimisation techniques (for example, least-squares minimisation). Note that each of the Equations (15.10-7), (15.10-14), (15.10-21) or (15.10-28) leads to an associated inversion algorithm.

The computational state Ω is representative for the manner in which the measured data are processed in the inversion algorithms. Since a computational state does not have to meet the physical condition of causality, there is no objection to its being anti-causal. Which of the two possibilities (causal or anti-causal) leads to the best results, as far as accuracy and amount of computational effort are concerned, is difficult to judge. Research on this aspect is still in full progress (see Fokkema and Van den Berg, 1993).

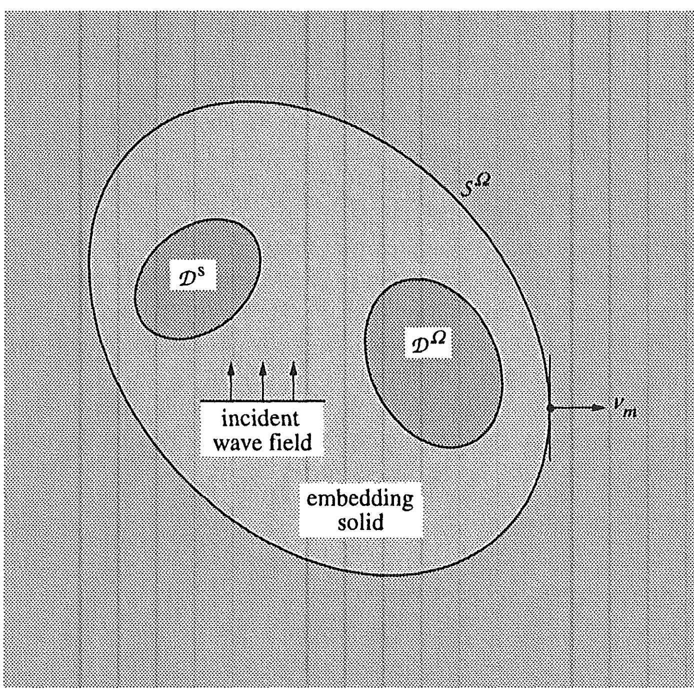
It is to be noted that the solution to an inverse source problem is not unique because of the existence of non-radiating source distributions (i.e. non-zero distributions with the support $\mathcal{D}^{T}$ that yield a vanishing wave field in the domain exterior to $\mathcal{D}^{T}$). Therefore, a numerically constructed solution to an inverse source problem is always *a* solution (and *not the* solution) that depends on the solution method employed.

Examples of elastodynamic inverse source problems are found in the detection of (spontaneous) elastodynamic emission during crack formation in a solid and the investigation into earthquake mechanisms.

15.11 The inverse scattering problem

The configuration in an elastodynamic inverse scattering problem generally consists of a background solid with known elastodynamic properties, occupying the domain $\mathcal{D}$ (the *embedding*), in which, in principle, the radiation from given, arbitrarily distributed elastodynamic sources can be calculated with the aid of the theory developed in Section 15.8. In the embedding an either known or guessed bounded domain $\mathcal{D}^{\mathrm{s}}$ (the *scatterer*) is present, in which the solid properties show an unknown contrast with those of the embedding. The contrasting domain is irradiated by an incident elastic wave field that is generated by sources in some subdomain $\mathcal{D}^{\mathrm{i}}$ of $\mathcal{D}$ and that propagates in the embedding. The presence of the contrasting domain manifests itself through the presence of a non-vanishing scattered wave field in the entire embedding. In some bounded subdomain $\mathcal{D}^{\Omega}$ of $\mathcal{D}$, and exterior to $\mathcal{D}^{\mathrm{s}}$, the scattered elastic wave field is accessible to measurement (Figure 15.11-1).

The objective is to reconstruct the medium parameters (or their contrasts with those of the embedding) from (a set of) measured values of the dynamic stress and/or the particle velocity in $\mathcal{D}^{\Omega}$. Since the inverse scattering problem is, by necessity, a remote sensing problem, the global reciprocity theorems of Sections 15.2–15.5 can be expected to provide a means for interrelating the known, measured wave-field data with the unknown medium properties in the scattering region. In case the embedding $\mathcal{D}$ is a bounded domain, the boundary surface is assumed to be elastodynamically impenetrable. If $\mathcal{D}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain are made. The scattered wave field is, by its nature, causally related to the contrast sources by which it is generated. For gathering maximum information, the reciprocity theorems are applied to the domain interior to a closed surface $\mathcal{S}^{\Omega}$ that completely surrounds both $\mathcal{D}^{\mathrm{s}}$ and $\mathcal{D}^{\Omega}$. If necessary, measurements on $\mathcal{S}^{\Omega}$ can also be carried out. In general, $\mathcal{D}^{\mathrm{s}}$ and $\mathcal{D}^{\mathrm{i}}$ are disjoint, as are $\mathcal{D}^{\mathrm{s}}$ and $\mathcal{D}^{\Omega}$. This need not be the case for $\mathcal{D}^{\mathrm{i}}$ and $\mathcal{D}^{\Omega}$; these domains may have a non-empty cross-section.



15.11-1 Configuration of the inverse scattering problem: $\mathcal{D}^{\mathrm{s}}$ is the support of the unknown contrast in solid properties; on $\mathcal{D}^{\Omega}$ and $\mathcal{S}^{\Omega}$ the scattered wave field is accessible to measurement.

The incident, scattered and total wave fields are introduced as in Section 15.9. Now, the easiest way to address the inverse scattering problem is to consider it partly as an inverse source problem with the contrast volume source densities as the unknowns, where the non-uniqueness of the contrast volume source distributions is to be removed by invoking the remaining conditions to be satisfied. In the latter, the condition that the reconstructed contrast-in-medium parameters must be independent of the incident wave field plays a crucial role. Once the contrast volume source distributions have been determined, the scattered wave field is, following the procedures of Section 15.9, calculated in the domain $\mathcal{D}^{\mathrm{s}}$, and since the incident wave field and the medium parameters of the embedding are known, the parameters of the solid in $\mathcal{D}^{\mathrm{s}}$ follow.

Time-domain analysis

In the time-domain analysis of the problem, the elastodynamic properties of the embedding solid are characterised by the relaxation functions $\{\mu_{k,r},\chi_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t)$, which are causal functions of time. The case of an instantaneously reacting embedding solid easily follows from the more general case of a solid with relaxation. The unknown elastodynamic properties of the scatterer are characterised by the relaxation functions $\{\mu^{\mathrm{s}}_{k,r},\chi^{\mathrm{s}}_{i,j,p,q}\} = \{\mu^{\mathrm{s}}_{k,r},\chi^{\mathrm{s}}_{i,j,p,q}\}(x,t)$, which are causal functions of time as well. The incident wave field is $\{\tau^{\mathrm{i}}_{p,q},v^{\mathrm{i}}_r\} = \{\tau^{\mathrm{i}}_{p,q},v^{\mathrm{i}}_r\}(x,t)$, the scattered wave field is $\{\tau^{\mathrm{s}}_{p,q},v^{\mathrm{s}}_r\} = \{\tau^{\mathrm{s}}_{p,q},v^{\mathrm{s}}_r\}(x,t)$, and the total wave field is $\{\tau_{p,q},v_r\} = \{\tau_{p,q},v_r\}(x,t)$ with $\{\tau_{p,q},v_r\} = \{\tau^{\mathrm{i}}_{p,q}+\tau^{\mathrm{s}}_{p,q},\ v^{\mathrm{i}}_r+v^{\mathrm{s}}_r\}$. The equivalent contrast volume source distributions that generate the scattered wave field are then (see Equations (15.9-18) and (15.9-19))

$$f^{\mathrm{s}}_k = -C_t(\mu^{\mathrm{s}}_{k,r}-\mu_{k,r},v_r;x,t) \quad \text{for} \quad x\in\mathcal{D}^{\mathrm{s}}, \tag{15.11-1}$$

$$h^{\mathrm{s}}_{i,j} = C_t(\chi^{\mathrm{s}}_{i,j,p,q}-\chi_{i,j,p,q},\tau_{p,q};x,t) \quad \text{for} \quad x\in\mathcal{D}^{\mathrm{s}}. \tag{15.11-2}$$

First, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{h^{\mathrm{s}}_{i,j},f^{\mathrm{s}}_k\} = \{h^{\mathrm{s}}_{i,j},f^{\mathrm{s}}_k\}(x,t)$, via the global time-domain reciprocity theorem of the *convolution type*, Equation (15.2-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\tau^{\mathrm{A}}_{p,q},v^{\mathrm{A}}_r\}(x,t) = \{\tau^{\mathrm{s}}_{p,q},v^{\mathrm{s}}_r\}(x,t) \quad \text{for} \quad x\in\mathcal{D}, \tag{15.11-3}$$

$$\{h^{\mathrm{A}}_{i,j},f^{\mathrm{A}}_k\}(x,t) = \{h^{\mathrm{s}}_{i,j},f^{\mathrm{s}}_k\}(x,t) \quad \text{for} \quad x\in\mathcal{D}^{\mathrm{s}}, \tag{15.11-4}$$

and

$$\{\mu^{\mathrm{A}}_{k,r},\chi^{\mathrm{A}}_{i,j,p,q}\}(x,t) = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t) \quad \text{for} \quad x\in\mathcal{D}. \tag{15.11-5}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\tau^{\mathrm{B}}_{i,j},v^{\mathrm{B}}_k\}(x,t) = \{\tau^{\Omega}_{i,j},v^{\Omega}_k\}(x,t) \quad \text{for} \quad x\in\mathcal{D}, \tag{15.11-6}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{h^{\mathrm{B}}_{p,q},f^{\mathrm{B}}_r\}(x,t) = \{h^{\Omega}_{p,q},f^{\Omega}_r\}(x,t) \quad \text{for} \quad x\in\mathcal{D}^{\Omega}. \tag{15.11-7}$$

Furthermore, the solid properties in state B will be taken to be the adjoint of those in state A, i.e.

$$\{\mu^{\mathrm{B}}_{r,k},\chi^{\mathrm{B}}_{p,q,i,j}\}(\boldsymbol{x},t) = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.11-8}$$

Then, application of Equation (15.2-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(-\tau^{\Omega}_{i,j},h^{\mathrm{s}}_{i,j};\boldsymbol{x},t) - \mathrm{C}_t(v^{\Omega}_k,f^{\mathrm{s}}_k;\boldsymbol{x},t)\right]\mathrm{d}V$$

$$= \int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[\mathrm{C}_t(-\tau^{\mathrm{s}}_{p,q},h^{\Omega}_{p,q};\boldsymbol{x},t) - \mathrm{C}_t(v^{\mathrm{s}}_r,f^{\Omega}_r;\boldsymbol{x},t)\right]\mathrm{d}V$$

$$+ \Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_m\left[-\mathrm{C}_t(-\tau^{\mathrm{s}}_{p,q},v^{\Omega}_r;\boldsymbol{x},t) + \mathrm{C}_t(v^{\mathrm{s}}_r,-\tau^{\Omega}_{p,q};\boldsymbol{x},t)\right]\mathrm{d}A\,. \tag{15.11-9}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state s and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.2-2). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are also causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are not causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.2-2).

Secondly, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{h^{\mathrm{s}}_{i,j},f^{\mathrm{s}}_k\} = \{h^{\mathrm{s}}_{i,j},f^{\mathrm{s}}_k\}(\boldsymbol{x},t)$, via the global time-domain reciprocity theorem of the *correlation type*, Equation (15.3-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\tau^{\mathrm{A}}_{p,q},v^{\mathrm{A}}_r\}(\boldsymbol{x},t) = \{\tau^{\mathrm{s}}_{p,q},v^{\mathrm{s}}_r\}(\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.11-10}$$

$$\{h^{\mathrm{A}}_{i,j},f^{\mathrm{A}}_k\}(\boldsymbol{x},t) = \{h^{\mathrm{s}}_{i,j},f^{\mathrm{s}}_k\}(\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{15.11-11}$$

and

$$\{\mu^{\mathrm{A}}_{k,r},\chi^{\mathrm{A}}_{i,j,p,q}\}(\boldsymbol{x},t) = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.11-12}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\tau^{\mathrm{B}}_{i,j},v^{\mathrm{B}}_k\}(\boldsymbol{x},t) = \{\tau^{\Omega}_{i,j},v^{\Omega}_k\}(\boldsymbol{x},t) \qquad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.11-13}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{h^{\mathrm{B}}_{p,q}, f^{\mathrm{B}}_r\}(x,t) = \{h^{\Omega}_{p,q}, f^{\Omega}_r\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}^{\Omega}. \qquad (15.11\text{-}14)$$

Furthermore, the solid properties in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\mu^{\mathrm{B}}_{r,k}, \chi^{\mathrm{B}}_{p,q,i,j}\}(x,t) = \{\mathrm{J}_t(\mu_{k,r}), \mathrm{J}_t(\chi_{i,j,p,q})\}(x,t) \qquad \text{for} \quad x \in \mathcal{D}. \qquad (15.11\text{-}15)$$

Then, application of Equation (15.3-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{x \in \mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(\mathrm{J}_t(-\tau^{\Omega}_{i,j}), h^{\mathrm{s}}_{i,j}; x,t) + \mathrm{C}_t(\mathrm{J}_t(v^{\Omega}_k), f^{\mathrm{s}}_k; x,t) \right] \mathrm{d}V$$

$$= -\int_{x \in \mathcal{D}^{\Omega}} \left[\mathrm{C}_t(-\tau^{\mathrm{s}}_{p,q}, \mathrm{J}_t(h^{\Omega}_{p,q}); x,t) + \mathrm{C}_t(v^{\mathrm{s}}_r, \mathrm{J}_t(f^{\Omega}_r); x,t) \right] \mathrm{d}V$$

$$+ \Delta^{+}_{m,r,p,q} \int_{x \in \mathcal{S}^{\Omega}} \nu_m \left[\mathrm{C}_t(-\tau^{\mathrm{s}}_{p,q}, \mathrm{J}_t(v^{\Omega}_r); x,t) + \mathrm{C}_t(v^{\mathrm{s}}_r, \mathrm{J}_t(-\tau^{\Omega}_{p,q}); x,t) \right] \mathrm{d}A \,. \qquad (15.11\text{-}16)$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.3-2)). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. In case the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are non-causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.3-2). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere).

A solution to the time-domain inverse scattering problem is commonly constructed as follows. First, the contrast-in-medium parameters are discretised by writing them as a linear combination of M expansion functions with unknown expansion coefficients. Each of the expansion functions has $\mathcal{D}^{\mathrm{s}}$, or a subset of it, as its support. Next, for each given incident wave field, the scattered wave field is measured in N subdomains of $\mathcal{D}^{\Omega}$. The latter discretisation induces the choice of the N supports of the source distributions of the observational state. Finally, a number I of different incident wave fields is selected, where "different" may involve different choices in temporal behaviour, in location in space, or in both. With $NI \geqslant M$, the non-linear problem of evaluating the M expansion coefficients of the contrast-in-medium discretisation is solved by some iterative procedure (for example, by iterative minimisation of the global error over all domains where equality signs should pointwise hold, and all time intervals involved). In this procedure, Equations (15.11-1), (15.11-2), (15.11-9) or (15.11-16) and the source type integral representations (15.9-20) and (15.9-21) are used simultaneously.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the elastodynamic properties of the embedding solid are characterised by the relaxation functions $\{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s)$. The unknown elastodynamic properties of the scatterer are characterised by the relaxation functions $\{\hat{\zeta}^{\rm s}_{k,r},\hat{\eta}^{\rm s}_{i,j,p,q}\} = \{\hat{\zeta}^{\rm s}_{k,r},\hat{\eta}^{\rm s}_{i,j,p,q}\}(\boldsymbol{x},s)$. The incident wave field is $\{\hat{\tau}^{\rm i}_{p,q},\hat{v}^{\rm i}_{r}\} = \{\hat{\tau}^{\rm i}_{p,q},\hat{v}^{\rm i}_{r}\}(\boldsymbol{x},s)$, the scattered wave field is $\{\hat{\tau}^{\rm s}_{p,q},\hat{v}^{\rm s}_{r}\} = \{\hat{\tau}^{\rm s}_{p,q},\hat{v}^{\rm s}_{r}\}(\boldsymbol{x},s)$ and the total wave field is $\{\hat{\tau}_{p,q},\hat{v}_{r}\} = \{\hat{\tau}_{p,q},\hat{v}_{r}\}(\boldsymbol{x},s)$, with $\{\hat{\tau}_{p,q},\hat{v}_{r}\} = \{\hat{\tau}^{\rm i}_{p,q}+\hat{\tau}^{\rm s}_{p,q},\hat{v}^{\rm i}_{r}+\hat{v}^{\rm s}_{r}\}$. The equivalent contrast volume source distributions that generate the scattered wave field are then (see Equations (15.9-41) and (15.9-42))

$$\hat{f}^{\rm s}_{k} = -(\hat{\zeta}^{\rm s}_{k,r} - \hat{\zeta}_{k,r})\hat{v}_{r} \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\rm s}, \tag{15.11-17}$$

$$\hat{h}^{\rm s}_{i,j} = (\hat{\eta}^{\rm s}_{i,j,p,q} - \hat{\eta}_{i,j,p,q})\hat{\tau}_{p,q} \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\rm s}. \tag{15.11-18}$$

First, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{\hat{h}^{\rm s}_{i,j},\hat{f}^{\rm s}_{k}\}$, via the global complex frequency-domain reciprocity theorem of the *time convolution type*, Equation (15.4-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\hat{\tau}^{\rm A}_{p,q},\hat{v}^{\rm A}_{r}\}(\boldsymbol{x},s) = \{\hat{\tau}^{\rm s}_{p,q},\hat{v}^{\rm s}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.11-19}$$

$$\{\hat{h}^{\rm A}_{i,j},\hat{f}^{\rm A}_{k}\}(\boldsymbol{x},s) = \{\hat{h}^{\rm s}_{i,j},\hat{f}^{\rm s}_{k}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\rm s}, \tag{15.11-20}$$

and

$$\{\hat{\zeta}^{\rm A}_{k,r},\hat{\eta}^{\rm A}_{i,j,p,q}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.11-21}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{\tau}^{\rm B}_{i,j},\hat{v}^{\rm B}_{k}\}(\boldsymbol{x},s) = \{\hat{\tau}^{\Omega}_{i,j},\hat{v}^{\Omega}_{k}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.11-22}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{h}^{\rm B}_{p,q},\hat{f}^{\rm B}_{r}\}(\boldsymbol{x},s) = \{\hat{h}^{\Omega}_{p,q},\hat{f}^{\Omega}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{15.11-23}$$

Furthermore, the solid properties in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\hat{\zeta}^{\rm B}_{r,k},\hat{\eta}^{\rm B}_{p,q,i,j}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.11-24}$$

Then, application of Equation (15.4-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \left[-\hat{\tau}^{\Omega}_{i,j}(\boldsymbol{x},s)\hat{h}^{\rm s}_{i,j}(\boldsymbol{x},s) - \hat{v}^{\Omega}_{k}(\boldsymbol{x},s)\,\hat{f}^{\rm s}_{k}(\boldsymbol{x},s)\right]{\rm d}V \\
&= \int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[-\hat{\tau}^{\rm s}_{p,q}(\boldsymbol{x},s)\hat{h}^{\Omega}_{p,q}(\boldsymbol{x},s) - \hat{v}^{\rm s}_{r}(\boldsymbol{x},s)\,\hat{f}^{\Omega}_{r}(\boldsymbol{x},s)\right]{\rm d}V \\
&\quad + \Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_{m}\left[\hat{\tau}^{\rm s}_{p,q}(\boldsymbol{x},s)\hat{v}^{\Omega}_{k}(\boldsymbol{x},s) - \hat{v}^{\rm s}_{k}(\boldsymbol{x},s)\hat{\tau}^{\Omega}_{p,q}(\boldsymbol{x},s)\right]{\rm d}A\,.
\end{aligned} \tag{15.11-25}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to state s and the wave-field evaluations pertaining to state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the scattered or the computatinal wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.4-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.4-4).

Secondly, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{\hat{h}^{s}_{i,j},\hat{f}^{s}_{k}\} = \{\hat{h}^{s}_{i,j},\hat{f}^{s}_{k}\}(\boldsymbol{x},s)$, via the global complex frequency-domain reciprocity theorem of the *time correlation type*, Equation (15.5-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\hat{\tau}^{A}_{p,q},\hat{v}^{A}_{r}\}(\boldsymbol{x},s) = \{\hat{\tau}^{s}_{p,q},\hat{v}^{s}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.11-26}$$

$$\{\hat{h}^{A}_{i,j},\hat{f}^{A}_{k}\}(\boldsymbol{x},s) = \{\hat{h}^{s}_{i,j},\hat{f}^{s}_{k}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{s}, \tag{15.11-27}$$

and

$$\{\hat{\zeta}^{A}_{k,r},\hat{\eta}^{A}_{i,j,p,q}\}(\boldsymbol{x},s) = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.11-28}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{\tau}^{B}_{i,j},\hat{v}^{B}_{k}\}(\boldsymbol{x},s) = \{\hat{\tau}^{\Omega}_{i,j},\hat{v}^{\Omega}_{k}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{15.11-29}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{h}^{B}_{p,q},\hat{f}^{B}_{r}\}(\boldsymbol{x},s) = \{\hat{h}^{\Omega}_{p,q},\hat{f}^{\Omega}_{r}\}(\boldsymbol{x},s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{15.11-30}$$

Furthermore, the solid properties in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\hat{\zeta}^{B}_{r,k},\hat{\eta}^{B}_{p,q,i,j}\}(\boldsymbol{x},s) = \{-\hat{\zeta}_{k,r},-\hat{\eta}_{i,j,p,q}\}(\boldsymbol{x},-s) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}. \tag{15.11-31}$$

Then, application of Equation (15.5-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{D}^{s}} \left[-\hat{\tau}^{\Omega}_{i,j}(\boldsymbol{x},-s)\hat{h}^{s}_{i,j}(\boldsymbol{x},s) + \hat{v}^{\Omega}_{k}(\boldsymbol{x},-s)\hat{f}^{s}_{k}(\boldsymbol{x},s)\right]\mathrm{d}V \\
&= -\int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}} \left[-\hat{\tau}^{s}_{p,q}(\boldsymbol{x},s)\hat{h}^{\Omega}_{p,q}(\boldsymbol{x},-s) + \hat{v}^{s}_{r}(\boldsymbol{x},s)\hat{f}^{\Omega}_{r}(\boldsymbol{x},-s)\right]\mathrm{d}V \\
&\quad + \Delta^{+}_{m,r,p,q}\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}} \nu_{m}\left[-\hat{\tau}^{s}_{p,q}(\boldsymbol{x},s)\hat{v}^{\Omega}_{r}(\boldsymbol{x},-s) - \hat{v}^{s}_{r}(\boldsymbol{x},s)\hat{\tau}^{\Omega}_{p,q}(\boldsymbol{x},-s)\right]\mathrm{d}A\,.
\end{aligned} \tag{15.11-32}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This can make a difference for the surface contribution over $\mathcal{S}^{\Omega}$. If the domain $\mathcal{D}$ occupied by the configuration is bounded, $\partial\mathcal{D}$ is impenetrable and the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the one over $\partial\mathcal{D}$ does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and $\partial\mathcal{D}$ (see Exercise 15.5-4). This conclusion holds for both causal and anti-causal generation of the wave field in state Ω. When the domain $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 15.5-4). However, if $\mathcal{D}$ is unbounded and the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes (since the integral over a sphere with an infinitely large radius does, and no sources of the scattered or the computational wave fields are present between $\mathcal{S}^{\Omega}$ and that sphere).

A solution to the complex frequency-domain inverse scattering problem is commonly constructed as follows. First, the contrast-in-medium parameters are discretised by writing them as a linear combination of M expansion functions with unknown expansion coefficients. Each of the expansion functions has $\mathcal{D}^{s}$, or a subset of it, as its support. Next, for each given incident wave field, the scattered wave field is measured in N subdomains of $\mathcal{D}^{\Omega}$. The latter discretisation induces the choice of the N supports of the source distributions of the observational state. Finally, a number I of different incident wave fields is selected, where "different" may involve different choices in complex frequency content, in location in space, or in both. With $NI \geqslant M$, the non-linear problem of evaluating the M expansion coefficients of the contrast-in-medium discretisation is solved by some iterative procedure (for example, by iterative minimisation of the global error over all domains where equality signs should pointwise hold, and all complex frequency values involved). In this procedure, Equations (15.11-17), (15.11-18), (15.11-25) or (15.11-32) and the source type integral representations (15.9-43) and (15.9-44) are used simultaneously.

The computational state Ω is representative for the manner in which the measured data are processed in the inversion algorithms. Since a computational state does not have to meet the physical condition of causality, there is no objection to its being anti-causal. Which of the two possibilities (causal or anti-causal) leads to the best results as far as accuracy and amount of computational effort are concerned, is difficult to say. Research on this aspect is still in full progress (see Fokkema and Van den Berg, 1993).

Examples of elastodynamic inverse scattering problems are found in the non-destructive evaluation of mechanical structures and exploration geophysics.

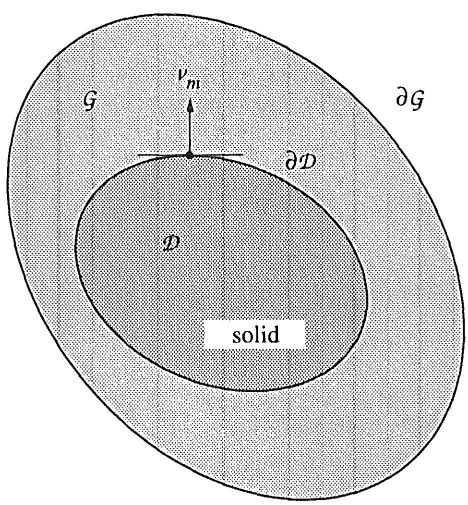
15.12 Elastic wave-field representations in a subdomain of the configuration space; equivalent surface sources; Huygens' principle and the Ewald–Oseen extinction theorem

In Section 15.8, wave-field representations have been derived that express the dynamic stress and the particle velocity at any point of a configuration in terms of the volume source distributions of deformation rate and force that generate the wave field. In them, the point-source solutions (Green's functions) to the radiation problem play a crucial role. In a number of cases we are, however, only interested in the values of the wave-field quantities in some subdomain of the configuration, and a wave-field representation pertaining to that subdomain would suffice. In the present section it is shown how the reciprocity theorem of the time convolution type leads to the desired expressions, although now, in addition to the volume integrals over the volume source distributions (as far as present in the subdomain of interest), surface integrals over the boundary surface of this subdomain occur. In these representations, again the point-source solution (Green's functions) are the intervening kernels.

Let $\mathcal{G}$ be the domain for which the Green's functions introduced in Section 15.8 are defined. If $\mathcal{G}$ is bounded, its boundary surface $\partial\mathcal{G}$ is assumed to be elastodynamically impenetrable. If $\mathcal{G}$ is unbounded, the standard provisions of Section 15.1 for handling an unbounded domain are made. Furthermore, let $\mathcal{D}$ be the subdomain of $\mathcal{G}$ in which expressions for the generated elastic wave field are to be found. The boundary surface of $\mathcal{D}$ is $\partial\mathcal{D}$ and the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{G}$ is denoted by $\mathcal{D}'$ (Figure 15.12-1). In fact, the relevant wave field need only be defined in $\mathcal{D}$ and on $\partial\mathcal{D}$. The constitutive properties must, however, be defined in $\mathcal{G}$ in order that the necessary point-source solutions can be defined in $\mathcal{G}$. In this sense, $\mathcal{G}$ serves as an embedding of $\mathcal{D}$. Since the generated elastic wave field is a physical wave field, it is causally related to its source distributions.

Time-domain analysis

For the time-domain analysis of the problem the elastodynamic properties of the solid present in $\mathcal{G}$ are characterised by the relaxation functions $\{\mu_{k,r},\chi_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(\boldsymbol{x},t)$, which are causal functions of time, and the global reciprocity theorem of the time convolution type,



15.12-1 Configuration for the wave field representations in the subdomain $\mathcal{D}$ of the configuration space $\mathcal{G}$ for which the Green's functions are defined. $\partial\mathcal{D}$ is the (smooth) boundary surface of $\mathcal{D}$. (a) If $\mathcal{G}$ is bounded, $\partial\mathcal{G}$ is impenetrable; (b) for unbounded $\mathcal{G}$ the Green's functions satisfy causality conditions at infinity.

Equation (15.2-7), is applied to the subdomain $\mathcal{D}$ of $\mathcal{G}$. In the theorem, state A is taken to be the generated elastodynamic wave field under consideration, i.e.

$$\{\tau^{\mathrm{A}}_{p,q},v^{\mathrm{A}}_r\} = \{\tau_{p,q},v_r\}(x,t) \qquad \text{for} \quad x\in\mathcal{D}, \tag{15.12-1}$$

$$\{h^{\mathrm{A}}_{i,j},f^{\mathrm{A}}_k\} = \{h_{i,j},f_k\}(x,t) \qquad \text{for} \quad x\in\mathcal{D}, \tag{15.12-2}$$

and

$$\{\mu^{\mathrm{A}}_{k,r},\chi^{\mathrm{A}}_{i,j,p,q}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t) \qquad \text{for} \quad x\in\mathcal{G}. \tag{15.12-3}$$

Next, state B is chosen such that the application of Equation (15.2-7) to the subdomain $\mathcal{D}$ leads to the values of $\{\tau_{p,q},v_r\}$ at some arbitrary point $x'\in\mathcal{D}$. Inspection of the right-hand side of Equation (15.2-7) reveals that this is accomplished if we take for the source distributions of state B a point source at x' of volume deformation rate in case we want an expression for the dynamic stress at x' and a point source of force at x' in case we want an expression for the particle velocity at x', while the solid in state B must be taken to be the adjoint of the one in state A, i.e.

$$\{\mu^{\mathrm{B}}_{r,k},\chi^{\mathrm{B}}_{p,q,i,j}\} = \{\mu_{k,r},\chi_{i,j,p,q}\}(x,t) \qquad \text{for all} \quad x\in\mathcal{G}. \tag{15.12-4}$$

The two choices for the source distributions will be discussed separately below.

First, we choose

$$h^{\mathrm{B}}_{p,q} = a_{p,q}\delta(x-x',t) \quad \text{and} \quad f^{\mathrm{B}}_r = 0\,, \tag{15.12-5}$$

where $\delta(x-x',t)$ represents the four-dimensional unit impulse (Dirac distribution) operative at the point $x = x'$ and at the instant $t = 0$, while $a_{p,q}$ is an arbitrary constant tensor of rank two. The elastic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is denoted by

$$\{\tau^{\mathrm{B}}_{i,j},v^{\mathrm{B}}_k\} = \{\tau^{h;\mathrm{B}}_{i,j},v^{h;\mathrm{B}}_k\}(x,x',t)\,, \tag{15.12-6}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (15.12-5) and the properties of $\delta(x-x',t)$, we have

$$\int_{x\in\mathcal{D}} \left[-\mathrm{C}_t(f^{\mathrm{B}}_r,v^{\mathrm{A}}_r;x,t) + \mathrm{C}_t(-\tau^{\mathrm{A}}_{p,q},h^{\mathrm{B}}_{p,q};x,t)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}} \mathrm{C}_t(-\tau_{p,q},a_{p,q}\delta(x-x',t);x,t)\,\mathrm{d}V = -a_{p,q}\tau_{p,q}(x',t)\chi_{\mathcal{D}}(x') \qquad \text{for} \quad x'\in\mathcal{G}, \tag{15.12-7}$$

where

$$\chi_{\mathcal{D}}(x') = \{1,½,0\} \qquad \text{for} \quad x'\in\{\mathcal{D},\partial\mathcal{D},\mathcal{D}'\} \tag{15.12-8}$$

is the characteristic function of the set $\mathcal{D}$. With this, we arrive at

$$-a_{p,q}\tau_{p,q}(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(-\tau^{h;\mathrm{B}}_{i,j},h_{i,j};x,x',t) - \mathrm{C}_t(v^{h;\mathrm{B}}_k,f_k;x,x',t)\right]\mathrm{d}V$$

$$-\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(-\tau^{h;\mathrm{B}}_{p,q},v_r;x,x',t) + \mathrm{C}_t(v^{h;\mathrm{B}}_r,\tau_{p,q};x,x',t)\right]\mathrm{d}A \quad \text{for} \quad x'\in\mathcal{G}, \tag{15.12-9}$$

where, in the second terms in the integrands, we have used the symmetry of the convolution in its functional arguments. From Equation (15.12-9) a representation for $\tau_{p,q}(x',t)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\tau_{i,j}^{h;\mathrm{B}}$ and $v_k^{h;\mathrm{B}}$ are linearly related to $a_{p,q}$. Introducing the Green's functions through

$$\{-\tau_{i,j}^{h;\mathrm{B}},v_k^{h;\mathrm{B}}\}(x,x',t) = \{G_{i,j,p,q}^{\tau h;\mathrm{B}},G_{k,p,q}^{vh;\mathrm{B}}\}(x,x',t)a_{p,q}\,, \tag{15.12-10}$$

using the reciprocity relations for these functions (see Exercises 15.8-1 and 15.8-3)

$$\{G_{i,j,p,q}^{\tau h;\mathrm{B}},G_{k,p,q}^{vh;\mathrm{B}}\}(x,x',t) = \{G_{p,q,i,j}^{\tau h},-G_{p,q,k}^{\tau f}\}(x',x,t)\,, \tag{15.12-11}$$

and invoking the condition that the resulting equation has to hold for arbitrary values of $a_{p,q}$, Equation (15.12-9) leads to the final result

$$\begin{aligned}-\tau_{p,q}(x',t)\chi_{\mathcal{D}}(x') = &\int_{x\in\mathcal{D}}\left[\mathrm{C}_t(G_{p,q,i,j}^{\tau h},h_{i,j};x',x,t) + \mathrm{C}_t(G_{p,q,k}^{\tau f},f_k;x',x,t)\right]\mathrm{d}V\\ &-\int_{x\in\partial\mathcal{D}}\left[\mathrm{C}_t(G_{p,q,i,j}^{\tau h},\Delta_{i,j,n,r}^{+}\nu_n v_r;x',x,t)\right.\\ &\left.+\,\mathrm{C}_t(G_{p,q,k}^{\tau f},-\Delta_{k,m,p',q'}\nu_m\tau_{p',q'};x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{G}\,. \end{aligned}\tag{15.12-12}$$

Equation (15.12-12) expresses, for $x'\in\mathcal{D}$, the dynamic stress $\tau_{p,q}$ of the generated elastic wave field at x' as the superposition of the contributions from the elementary volume sources $h_{i,j}\,\mathrm{d}V$ and $f_k\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the equivalent elementary surface sources $-\Delta_{i,j,n,r}^{+}\nu_n v_r\,\mathrm{d}A$ and $\Delta_{k,m,p',q'}^{+}\nu_m\tau_{p',q'}\,\mathrm{d}A$ at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Secondly, we choose

$$h_{p,q}^{\mathrm{B}} = 0 \quad \text{and} \quad f_r^{\mathrm{B}} = b_r\delta(x-x',t)\,, \tag{15.12-13}$$

where b_r is an arbitrary constant vector. The elastic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is denoted by

$$\{\tau_{i,j}^{\mathrm{B}},v_k^{\mathrm{B}}\} = \{\tau_{i,j}^{f;\mathrm{B}},v_k^{f;\mathrm{B}}\}(x,x',t)\,, \tag{15.12-14}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (15.12-13) and the properties of $\delta(x-x',t)$, we have

$$\begin{aligned}&\int_{x\in\mathcal{D}}\left[-\mathrm{C}_t(f_r^{\mathrm{B}},v_r^{\mathrm{A}};x,t) + \mathrm{C}_t(-\tau_{p,q}^{\mathrm{A}},h_{p,q}^{\mathrm{B}};x,t)\right]\mathrm{d}V\\ &= -\int_{x\in\mathcal{D}}\mathrm{C}_t(b_r\delta(x-x',t),v_r;x,t)\,\mathrm{d}V = -b_r v_r(x',t)\chi_{\mathcal{D}}(x') \quad \text{for } x'\in\mathcal{G}\,. \end{aligned}\tag{15.12-15}$$

With this, we arrive at

$$\begin{aligned}b_r v_r(x',t)\chi_{\mathcal{D}}(x') = &\int_{x\in\mathcal{D}}\left[\mathrm{C}_t(\tau_{i,j}^{f;\mathrm{B}},h_{i,j};x,x',t) + \mathrm{C}_t(v_k^{f;\mathrm{B}},f_k;x,x',t)\right]\mathrm{d}V\\ -\Delta_{m,r,p,q}^{+}&\int_{x\in\partial\mathcal{D}}\nu_m\left[\mathrm{C}_t(\tau_{p,q}^{f;\mathrm{B}},v_r;x,x',t) - \mathrm{C}_t(v_r^{f;\mathrm{B}},\tau_{p,q};x,x',t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{G}\,, \end{aligned}\tag{15.12-16}$$

where, in the second terms in the integrands, we have used the symmetry of the convolution in its functional arguments. From Equation (15.12-16) a representation for $v_r(x',t)\chi_{\mathcal{D}}(x')$ is obtained by taking into account the fact that $\tau_{i,j}^{f;B}$ and $v_k^{f;B}$ are linearly related to b_r. Introducing the Green's functions through

$$\{-\tau_{i,j}^{f;B},v_k^{f;B}\}(x,x',t) = \{G_{i,j,r}^{\tau f;B},G_{k,r}^{vf;B}\}(x,x',t)b_r\,, \tag{15.12-17}$$

using the reciprocity relations for these functions (see Exercises 15.8-2 and 15.8-4)

$$\{G_{i,j,r}^{\tau f;B},G_{k,r}^{vf;B}\}(x,x',t) = \{-G_{r,i,j}^{vh},G_{r,k}^{vf}\}(x',x,t)\,, \tag{15.12-18}$$

and invoking the condition the resulting equation has to hold for arbitrary values of b_r, Equation (15.12-16) leads to the final result

$$\begin{aligned} v_r(x',t)\chi_{\mathcal{D}}(x') = &\int_{x\in\mathcal{D}} \left[\mathrm{C}_t(G_{r,i,j}^{vh},h_{i,j};x',x,t) + \mathrm{C}_t(G_{r,k}^{vf},f_k;x',x,t)\right]\mathrm{d}V \\ &-\int_{x\in\partial\mathcal{D}} \left[\mathrm{C}_t(G_{r,i,j}^{vh},\Delta_{i,j,n,r'}^{+}\nu_n v_{r'};x',x,t)\right. \\ &\left.+\,\mathrm{C}_t(G_{r,k}^{vf},-\Delta_{k,m,p,q}^{+}\nu_m\tau_{p,q};x',x,t)\right]\mathrm{d}A \qquad \text{for } x'\in\mathcal{G}\,. \end{aligned} \tag{15.12-19}$$

Equation (15.12-19) expresses, for $x'\in\mathcal{D}$, the particle velocity v_r of the generated elastodynamic wave field at x' as the superposition of the contributions from the elementary volume sources $h_{i,j}\,\mathrm{d}V$ and $f_k\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions $-\Delta_{i,j,n,r'}^{+}\nu_n v_{r'}\,\mathrm{d}A$ and $\Delta_{k,m,p,q}^{+}\nu_m\tau_{p,q}\,\mathrm{d}A$ from the elementary equivalent surface sources at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Complex frequency-domain analysis

For the complex frequency-domain analysis of the problem the elastodynamic properties of the solid present in $\mathcal{G}$ are characterised by the functions $\{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(x,s)$ and the global complex frequency-domain reciprocity theorem of the time convolution type, Equation (15.4-7), is applied to the subdomain $\mathcal{D}$ of $\mathcal{G}$. In the theorem, state A is taken to be the generated elastic wave field under consideration, i.e.

$$\{\hat{\tau}_{p,q}^{A},\hat{v}_r^{A}\} = \{\hat{\tau}_{p,q},\hat{v}_r\}(x,s) \qquad \text{for } x\in\mathcal{D}\,, \tag{15.12-20}$$

$$\{\hat{h}_{i,j}^{A},\hat{f}_k^{A}\} = \{\hat{h}_{i,j},\hat{f}_k\}(x,s) \qquad \text{for } x\in\mathcal{D}\,, \tag{15.12-21}$$

and

$$\{\hat{\zeta}_{k,r}^{A},\hat{\eta}_{i,j,p,q}^{A}\} = \{\hat{\zeta}_{k,r},\hat{\eta}_{i,j,p,q}\}(x,s) \qquad \text{for } x\in\mathcal{G}\,. \tag{15.12-22}$$

Next, state B is chosen such that the application of Equation (15.4-7) to the subdomain $\mathcal{D}$ leads to the values of $\{\hat{\tau}_{p,q},\hat{v}_r\}$ at some arbitrary point $x'\in\mathcal{D}$. Inspection of the right-hand side of Equation (15.4-7) reveals that this is accomplished if we take for the source distributions of state B a point source at x' of volume deformation rate in case we want an expression for the dynamic stress at x' and a point source of force at x' in case we want an expression for the particle velocity at x', while the solid in state B must be taken to be the adjoint of the one in state A, i.e.

$$\{\hat{\xi}^{B}_{r,k},\hat{\eta}^{B}_{p,q,i,j}\} = \{\hat{\xi}_{k,r},\hat{\eta}_{i,j,p,q}\}(x,s) \qquad \text{for all } x\in\mathcal{G}. \tag{15.12-23}$$

The two choices for the source distributions will be discussed separately below.

First, we choose

$$\hat{h}^{B}_{p,q} = \hat{a}_{p,q}(s)\delta(x-x') \quad \text{and} \quad \hat{f}^{B}_{r} = 0\,, \tag{15.12-24}$$

where $\delta(x-x')$ represents the three-dimensional unit impulse (Dirac distribution) operative at the point $x = x'$, while $\hat{a}_{p,q} = \hat{a}_{p,q}(s)$ is an arbitrary tensor function of rank two of s. The elastic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is denoted by

$$\{\hat{\tau}^{B}_{i,j},\hat{f}^{B}_{k}\} = \{\hat{\tau}^{h;B}_{i,j},\hat{v}^{h;B}_{k}\}(x,x',s)\,, \tag{15.12-25}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (15.12-24) and the properties of $\delta(x-x')$, we have

$$\int_{x\in\mathcal{D}} \left[-\hat{f}^{B}_{r}(x,s)\hat{v}^{A}_{r}(x,s) - \hat{\tau}^{A}_{p,q}(x,s)\hat{h}^{B}_{p,q}(x,s)\right] \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \hat{\tau}_{p,q}(x,s)\hat{a}_{p,q}(s)\delta(x-x')\,\mathrm{d}V = -\hat{a}_{p,q}(s)\hat{\tau}_{p,q}(x',s)\chi_{\mathcal{D}}(x') \qquad \text{for} \quad x'\in\mathcal{G}. \tag{15.12-26}$$

With this, we arrive at

$$-\hat{a}_{p,q}(s)\hat{\tau}_{p,q}(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[-\hat{\tau}^{h;B}_{i,j}(x,x',s)\hat{h}_{i,j}(x,s) - \hat{v}^{h;B}_{k}(x,x's)\hat{f}_{k}(x,s)\right]\mathrm{d}V$$

$$-\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[-\hat{\tau}^{h;B}_{p,q}(x,x',s)\hat{v}_r(x,s) + \hat{v}^{h;B}_{r}(x,x',s)\hat{\tau}_{p,q}(x,s)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{G}. \tag{15.12-27}$$

From Equation (15.12-27) a representation for $\hat{\tau}_{p,q}(x',s)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\hat{\tau}^{h;B}_{i,j}$ and $\hat{v}^{h;B}_{k}$ are linearly related to $\hat{a}_{p,q}(s)$. Introducing the Green's functions through

$$\{-\hat{\tau}^{h;B}_{i,j},\hat{v}^{h;B}_{k}\}(x,x',s) = \{\hat{G}^{\tau h;B}_{i,j,p,q},\hat{G}^{vh;B}_{k,p,q}\}(x,x',s)\hat{a}_{p,q}(s)\,, \tag{15.12-28}$$

using the reciprocity relations for these functions (see Exercises 15.8-5 and 15.8-7)

$$\{\hat{G}^{\tau h;B}_{i,j,p,q},\hat{G}^{vh;B}_{k,p,q}\}(x,x',s) = \{\hat{G}^{\tau h}_{p,q,i,j},-\hat{G}^{\tau f}_{p,q,k}\}(x',x,s)\,, \tag{15.12-29}$$

and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{a}_{p,q}(s)$, Equation (15.12-27) leads to the final result

$$-\hat{\tau}_{p,q}(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\hat{h}_{i,j}(x,s) + \hat{G}^{\tau f}_{p,q,k}(x',x,s)\hat{f}_{k}(x,s)\right]\mathrm{d}V$$

$$+\int_{x\in\partial\mathcal{D}} \Big\{\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\left[-\Delta^{+}_{i,j,n,r}\nu_n\hat{v}_r(x,s)\right]$$

$$+\,\hat{G}^{\tau f}_{p,q,k}(x',x,s)\left[\Delta^{+}_{k,m,p',q'}\nu_m\hat{\tau}_{p',q'}(x,s)\right]\Big\}\,\mathrm{d}A \qquad \text{for} \quad x'\in\mathcal{G}. \tag{15.12-30}$$

Equation (15.12-30) expresses, for $x' \in \mathcal{D}$, the dynamic stress $\hat{\tau}_{p,q}$ of the generated elastic wave field at x' as the superposition of the contributions from the elementary volume sources $\hat{h}_{i,j}\,\mathrm{d}V$ and $\hat{f}_k\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the elementary equivalent surface sources $-\Delta^{+}_{i,j,n,r}\nu_n\hat{v}_r\,\mathrm{d}A$ and $\Delta_{k,m,p',q'}\nu_m\hat{\tau}_{p',q'}\,\mathrm{d}A$ at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Secondly, we choose

$$\hat{h}^{\mathrm{B}}_{i,j} = 0 \quad \text{and} \quad f^{\mathrm{B}}_r = \hat{b}_r(s)\delta(x - x'), \tag{15.12-31}$$

where $\hat{b}_r = \hat{b}_r(s)$ is an arbitrary vector function of s. The elastic wave field that is, for the present application, causally radiated by this source and satisfies the proper boundary condition at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded, is denoted by

$$\{\hat{\tau}^{\mathrm{B}}_{i,j}, \hat{v}^{\mathrm{B}}_k\} = \{\hat{\tau}^{f;\mathrm{B}}_{i,j}, \hat{v}^{f;\mathrm{B}}_k\}(x,x',s), \tag{15.12-32}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (15.12-31) and the properties of $\delta(x - x')$, we have

$$\int_{x\in\mathcal{D}} \left[-\hat{f}^{\mathrm{B}}_r(x,s)\hat{v}^{\mathrm{A}}_r(x,s) - \hat{\tau}^{\mathrm{A}}_{p,q}(x,s)\hat{h}^{\mathrm{B}}_{p,q}(x,s)\right]\mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \hat{b}_r(s)\delta(x - x')\hat{v}_r(x,s)\,\mathrm{d}V = -\hat{b}_r(s)\hat{v}_r(x',s)\chi_{\mathcal{D}}(x') \qquad \text{for} \quad x' \in \mathcal{G}. \tag{15.12-33}$$

With this, we arrive at

$$\hat{b}_r(s)v_r(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{\tau}^{f;\mathrm{B}}_{i,j}(x,x',s)\hat{h}_{i,j}(x,s) + \hat{v}^{f;\mathrm{B}}_k(x,x's)\hat{f}_k(x,s)\right]\mathrm{d}V$$

$$-\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{\tau}^{f;\mathrm{B}}_{p,q}(x,x',s)\hat{v}_m(x,s) - \hat{v}^{f;\mathrm{B}}_r(x,x's)\hat{\tau}_{p,q}(x,s)\right]\mathrm{d}A \quad \text{for} \quad x' \in \mathcal{G}. \tag{15.12-34}$$

From Equation (15.12-34) a representation for $\hat{v}_r(x',s)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\hat{\tau}^{f;\mathrm{B}}_{i,j}$ and $\hat{v}^{f;\mathrm{B}}_k$ are linearly related to $\hat{b}_r(s)$. Introducing the Green's functions through

$$\{-\hat{\tau}^{f;\mathrm{B}}_{i,j}, \hat{v}^{f;\mathrm{B}}_k\}(x,x',s) = \{\hat{G}^{\tau f;\mathrm{B}}_{i,j,r}, \hat{G}^{vf;\mathrm{B}}_{k,r}\}(x,x',s)\hat{b}_r(s), \tag{15.12-35}$$

using the reciprocity relations for these functions (see Exercises 15.8-6 and 15.8-8)

$$\{\hat{G}^{\tau f;\mathrm{B}}_{i,j,r}, \hat{G}^{vf;\mathrm{B}}_{k,r}\}(x,x',s) = \{-\hat{G}^{vh}_{r,i,j}, \hat{G}^{vf}_{r,k}\}(x',x,s), \tag{15.12-36}$$

and invoking the condition the resulting equation has to hold for arbitrary values of $\hat{b}_r(s)$, Equation (15.12-34) leads to the final result

$$\hat{v}_r(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{G}^{vh}_{r,i,j}(x',x,s)\hat{h}_{i,j}(x,s) + \hat{G}^{vf}_{r,k}(x',x,s)\hat{f}_k(x,s)\right]\mathrm{d}V$$

$$+ \int_{x\in\partial\mathcal{D}} \Big\{\hat{G}^{vh}_{r,i,j}(x',x,s)\left[-\Delta^{+}_{i,j,n,r'}\nu_n\hat{v}_{r'}(x,s)\right]$$

$$+ \hat{G}^{vf}_{r,k}(x',x,s)\left[\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}_{p,q}(x,s)\right]\Big\}\,\mathrm{d}A \qquad \text{for} \quad x' \in \mathcal{G}. \tag{15.12-37}$$

Equation (15.12-37) expresses, for $x' \in \mathcal{D}$, the particle velocity $\hat{v}_r$ of the generated elastic wave field at x' as the superposition of the contributions from the elementary volume sources $\hat{h}_{i,j}\,\mathrm{d}V$ and $\hat{f}_k\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the elementary equivalent surface sources $-\Delta^{+}_{i,j,n,r'}\nu_n\hat{v}_{r'}\,\mathrm{d}A$ and $\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}_{p,q}\,\mathrm{d}A$ at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

For $x' \in \mathcal{D}$, Equations (15.12-12), (15.12-19), (15.12-30) and (15.12-37) express the values of the dynamic stress and the particle velocity in some point of $\mathcal{D}$ as the sum of the contributions from the volume sources of deformation rate and force as far as these are present in $\mathcal{D}$, and the equivalent surface sources on $\partial\mathcal{D}$. Evidently, the equivalent surface sources yield, in the interior of $\mathcal{D}$, the contribution to the wave field insofar that arises from (unspecified) sources that are located in $\mathcal{D}'$, i.e. in the exterior of $\mathcal{D}$. In particular, the surface integrals in these expressions vanish in case the wave field is not only defined in $\mathcal{D}$ and on $\partial\mathcal{D}$, but also in $\mathcal{D}'$ and no sources are located in between $\partial\mathcal{D}$ and $\partial\mathcal{G}$, and the wave fields in state B either satisfy the proper boundary conditions at $\partial\mathcal{G}$ if $\mathcal{G}$ is bounded (see Exercises 15.2-2, 15.3-2, 15.4-4 and 15.5-4), or are causally related to their point source excitations if $\mathcal{G}$ is unbounded. In the latter case Equations (15.12-12), (15.12-19), (15.12-30) and (15.12-37) reduce to Equations (15.8-14), (15.8-24), (15.8-38) and (15.8-48), respectively.

Another property of Equations (15.12-12), (15.12-19), (15.12-30) and (15.12-37) is that the wave field emitted by the volume sources in $\mathcal{D}$ and the wave field emitted by the equivalent surface sources on $\partial\mathcal{D}$ apparently cancel each other when $x' \in \mathcal{D}'$. This property is known as the Ewald-Oseen *extinction theorem* (Ewald, 1916; Oseen, 1915).

Another special case arises when Equations (15.12-12), (15.12-19), (15.12-30) and (15.12-37) are used in a domain in which no volume source distributions are present. Then, they express *Huygens' principle* (Huygens 1690) that states that an elastic wave field due to sources "behind" a closed surface, which divides the configuration space into two disjoint regions and "in front" of which no volume sources are present, can be represented as due to equivalent surface sources located on that surface, while that representation yields the value zero "behind" that surface. In particular, Huygens stated his principle for the case where the relevant surface is a wave front of the wave motion in space–time. A number of historical details about the development of the mathematical theory of Huygens' principle can be found in Baker and Copson (1950). Additional literature on the subject can be found in Blok *et al.*, (1992), and in De Hoop (1992).

Applications of the wave-field representations in a subdomain of space are found in the integral equation formulation of scattering problems while the Ewald–Oseen's extinction theorem is at the basis of the so-called "null-field method" to solve such problems.

Exercises

Exercise 15.12-1

Let $\mathcal{D}$ be a bounded subdomain of three-dimensional Euclidean space $\mathcal{R}^3$. Let $\partial\mathcal{D}$ be the closed boundary surface of $\mathcal{D}$ and denote by $\mathcal{D}'$ the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$ (i.e. towards $\mathcal{D}'$), is denoted by ν (Figure 15.12-2).

In the domain $\mathcal{D}'$ an elastic wave field $\{\tau_{p,q}, v_r\}$ is present whose sources are located in $\mathcal{D}$. Use Equations (15.12-12) and (15.12-19) to arrive at the equivalent surface source time-domain integral representations:

$$-\tau_{p,q}(x',t)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big[\mathrm{C}_t(G^{\tau h}_{p,q,i,j},\Delta^{+}_{i,j,n,r}\nu_n v_r;x',x,t)$$
$$+\,\mathrm{C}_t(G^{\tau f}_{p,q,k},-\Delta^{+}_{k,m,p',q'}\nu_m\tau_{p',q'};x',x,t)\Big]\,\mathrm{d}A \quad \text{for} \quad x'\in\mathcal{R}^3 \qquad (15.12\text{-}38)$$

and

$$v_r(x',t)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big[\mathrm{C}_t(G^{vh}_{r,i,j},\Delta^{+}_{i,j,n,r'}\nu_n v_{r'};x',x,t)$$
$$+\,\mathrm{C}_t(G^{vf}_{r,k},-\Delta^{+}_{k,m,p,q}\nu_m\tau_{p,q};x',x,t)\Big]\,\mathrm{d}A \qquad \text{for} \quad x'\in\mathcal{R}^3. \qquad (15.12\text{-}39)$$

Exercise 15.12-2

Let $\mathcal{D}$ be a bounded subdomain of three-dimensional Euclidean space $\mathcal{R}^3$. Let $\partial\mathcal{D}$ be the closed boundary surface of $\mathcal{D}$ and denote by $\mathcal{D}'$ the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$ (i.e. towards $\mathcal{D}'$), is denoted by ν (Figure 15.12-2).

In the domain $\mathcal{D}'$ an elastic wave field $\{\hat{\tau}_{p,q},\hat{v}_r\}$ is present whose sources are located in $\mathcal{D}$. Use Equations (15.12-30) and (15.12-37) to arrive at the equivalent surface source complex frequency-domain integral representations

$$-\hat{\tau}_{p,q}(x',s)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\Delta^{+}_{i,j,n,r}\nu_n\hat{v}_r(x,s)$$
$$-\,\hat{G}^{\tau f}_{p,q,k}(x',x,s)\Delta^{+}_{k,m,p',q'}\nu_m\hat{\tau}_{p',q'}(x,s)\Big]\,\mathrm{d}A \quad \text{for} \quad x'\in\mathcal{R}^3 \qquad (15.12\text{-}40)$$

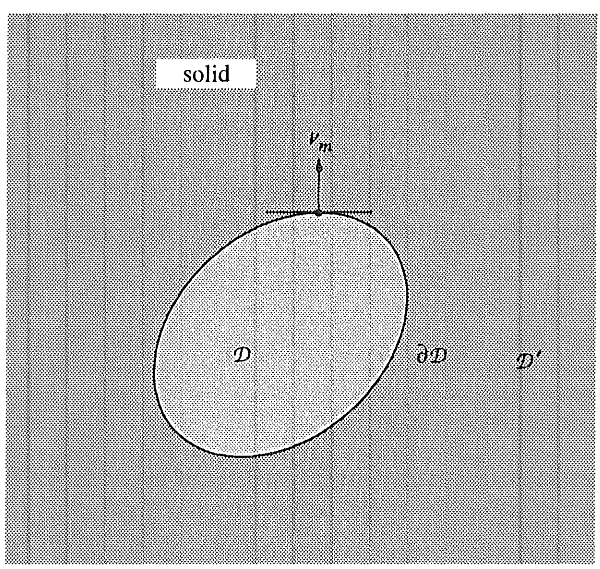


Figure 15.12-2 Configuration for the equivalent surface source integral representation for an elastic wave field in the source-free domain $\mathcal{D}'$ exterior to a bounded subdomain $\mathcal{D}$ of $\mathcal{R}^3$.

and

$$\hat{v}_r(x',s)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big[\hat{G}^{vh}_{r,i,j}(x',x,s)\Delta^{+}_{i,j,n,r'}\nu_n\hat{v}_{r'}(x,s)$$
$$- \hat{G}^{vf}_{r,k}(x',x,s)\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}_{p,q}(x,s)\Big]\,\mathrm{d}A \quad \text{for} \quad x'\in\mathcal{R}^3. \qquad (15.12\text{-}41)$$

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