16

Plane wave scattering by an object in an unbounded, homogeneous, isotropic, lossless embedding

In this chapter, the simplest scattering configuration is investigated in more detail. It consists of an unbounded, homogeneous, isotropic, lossless embedding in which a plane wave is incident upon a scattering object of bounded extent. First, the reciprocity properties of the amplitudes of the scattered waves in the far-field region are investigated. Next, an energy theorem ("extinction cross-section theorem") is derived that relates the sum of the energies carried by the scattered waves and the energy absorbed by the scattering object to the amplitude of the scattered waves in the far-field region when observed in the forward scattering direction. Finally, the first term in the Neumann solution to the relevant system of integral equations (the so-called "Rayleigh–Gans–Born approximation") is determined for penetrable, homogeneous scatterers of different shapes. The analysis is carried out in the time domain as well as in the complex frequency domain.

16.1 The scattering configuration, the incident plane waves and the far-field scattering amplitudes

The scattering configuration consists of a homogeneous, isotropic, lossless *embedding* that occupies the whole of $\mathcal{R}^3$. The elastodynamic properties of the embedding are characterised by either its volume density of mass ρ and its Lamé coefficients λ and μ, or its volume density of mass ρ, its compressional wave speed $c_P = [(\lambda + 2\mu)/\rho]^{1/2}$ and its shear wave speed $c_S = (\mu/\rho)^{1/2}$. Here, ρ, μ, c_P and c_S are positive constants and λ is a constant satisfying the condition $\lambda > -(2\mu/3)$. The related stiffness is

$$C_{p,q,i,j} = \lambda\delta_{p,q}\delta_{i,j} + \mu(\delta_{p,i}\delta_{q,j} + \delta_{p,j}\delta_{q,i})$$

and the compliance is

$$S_{i,j,p,q} = C^{-1}_{i,j,p,q} \,.$$

In the embedding, an elastic *scatterer* is present that occupies the bounded domain $\mathcal{D}^s$. The boundary surface of $\mathcal{D}^s$ is denoted by $\partial\mathcal{D}^s$ and ν is the unit vector along the normal to $\partial\mathcal{D}^s$ oriented away from $\mathcal{D}^s$. The complement of $\mathcal{D}^s \cup \partial\mathcal{D}^s$ in $\mathcal{R}^3$ is denoted by $\mathcal{D}^{s\prime}$ (Figure 16.1-1).

16.1-1(a)

16.1-1(b)

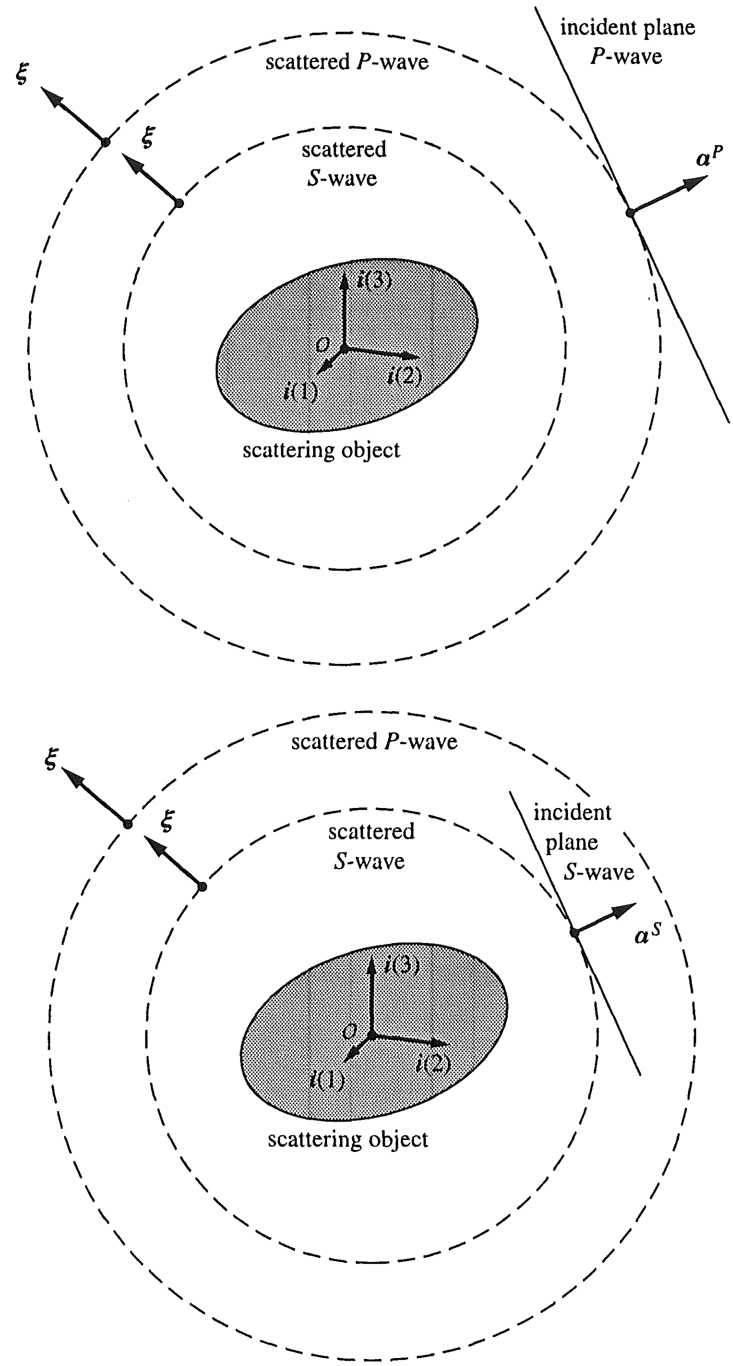


Figure 16.1-1 Scattering object occupying the bounded domain $\mathcal{D}^s$ in an unbounded elastodynamically homogeneous, isotropic, lossless embedding with volume density of mass ρ and compressional wavespeed c_P and shear wavespeed c_S: (a) incident plane P-wave; (b) incident plane S-wave.

Time-domain analysis

In the time-domain analysis of the problem, the elastodynamic properties of the scatterer are, if the scatterer is an elastodynamically penetrable object, characterised by the relaxation functions

$$\{\mu^{s}_{k,r},\chi^{s}_{i,j,p,q}\} = \{\mu^{s}_{k,r},\chi^{s}_{i,j,p,q}\}(\boldsymbol{x},t)\,,$$

which are causal functions of time. The equivalent contrast volume source densities of deformation rate and force are then given by (see Equations (15.9-18) and (15.9-19))

$$f^{s}_{k} = -\partial_t C_t(\mu^{s}_{k,r} - \rho\delta_{k,r}\delta(t), v_r; \boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}, \tag{16.1-1}$$

$$h^{s}_{i,j} = \partial_t C_t(\chi^{s}_{i,j,p,q} - S_{i,j,p,q}\delta(t), \tau_{p,q}; \boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}, \tag{16.1-2}$$

in which the *total elastic wave field* $\{\tau_{p,q},v_r\}$ is the sum of the *incident wave field* $\{\tau^{i}_{p,q},v^{i}_{r}\}$ and the *scattered wave field* $\{\tau^{i}_{p,q},v^{i}_{r}\}$ (see Equation (15.9-5)). If the scatterer is *elastodynamically impenetrable*, either of the two boundary conditions

$$\lim_{h\downarrow 0}\Delta^{+}_{k,m,p,q}\nu_m\tau_{p,q}(\boldsymbol{x}+h\boldsymbol{\nu},t) = 0 \qquad \text{for } \boldsymbol{x}\in\partial\mathcal{D}^{s} \tag{16.1-3}$$

or

$$\lim_{h\downarrow 0} v_r(\boldsymbol{x}+h\boldsymbol{\nu},t) = 0 \qquad \text{for } \boldsymbol{x}\in\partial\mathcal{D}^{s} \tag{16.1-4}$$

applies.

For the incident wave we now take a *uniform plane wave*. This can be either a uniform plane P-wave or a uniform plane S-wave. For the incident plane P-wave propagating in the direction of the unit vector $\boldsymbol{\alpha}^{P}$ (i.e. $\alpha^{P}_{s}\alpha^{P}_{s} = 1$) we have, on account of Equations (14.4-7), (14.4-13) and (14.4-14),

$$\{\tau_{p,q},v_r\} = \{T^{P}_{p,q},V^{P}_{r}\}a^{P}(t - \alpha^{P}_{s}x_s/c_P)\,, \tag{16.1-5}$$

with

$$c_P = [(\lambda + 2\mu)/\rho]^{1/2}, \tag{16.1-6}$$

$$V^{P}_{r} = (\alpha^{P}_{k}V^{P}_{k})\alpha^{P}_{r}\,, \tag{16.1-7}$$

and (see Equation (14.4-10))

$$T^{P}_{p,q} = -c_P^{-1}\left[\lambda\delta_{p,q}(\alpha^{P}_{k}V^{P}_{k}) + 2\mu(\alpha^{P}_{k}V^{P}_{k})\alpha^{P}_{p}\alpha^{P}_{q}\right], \tag{16.1-8}$$

where $a^{P}(t)$ denotes the normalised pulse shape.

For the incident plane S-wave propagating in the direction of the unit vector $\boldsymbol{\alpha}^{S}$ (i.e. $\alpha^{S}_{s}\alpha^{S}_{s} = 1$) we have, on account of Equations (14.4-7), (14.4-15) and (14.4-16),

$$\{\tau_{p,q},v_r\} = \{T^{S}_{p,q},V^{S}_{r}\}a^{S}(t - \alpha^{S}_{s}x_s/c_S)\,, \tag{16.1-9}$$

with

$$c_S = (\mu/\rho)^{1/2}, \tag{16.1-10}$$

$$\alpha^{S}_{k}V^{S}_{k} = 0, \tag{16.1-11}$$

and (see Equation (14.4-10))

$$T^S_{p,q} = -c_S^{-1}\,\mu(a^S_p V^S_q) + a^S_q V^S_p)\,, \tag{16.1-12}$$

where $a^S(t)$ denotes the normalised pulse shape.

For an *elastodynamically penetrable scatterer* we use for the scattered wave the constrast volume source integral representations (see Equations (15.9-20) and (15.9-21))

$$-\tau^{\mathrm{s}}_{p,q}(x',t) = \int_{x\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G^{\tau h}_{p,q,i,j}, h^{\mathrm{s}}_{i,j}; x', x, t) + \mathrm{C}_t(G^{\tau f}_{p,q,k}, f^{\mathrm{s}}_k; x', x, t)\right]\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \tag{16.1-13}$$

and

$$v^{\mathrm{s}}_r(x',t) = \int_{x\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G^{vh}_{r,i,j}, h^{\mathrm{s}}_{i,j}; x', x, t) + \mathrm{C}_t(G^{vf}_{r,k}, f^{\mathrm{s}}_k; x', x, t)\right]\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \tag{16.1-14}$$

in which (see Exercise 15.8-9, with x and x' interchanged)

$$G^{\tau h}_{p,q,i,j}(x',x,t) = -C_{p,q,i,j}H(t)\delta(x'-x) - \rho^{-1}C_{p,q,n,r}C_{k,m,i,j}\partial'_n\partial'_m \mathrm{I}_t G_{r,k}(x',x,t)\,, \tag{16.1-15}$$

$$G^{\tau f}_{p,q,k}(x',x,t) = \rho^{-1}C_{p,q,n,r}\partial'_n G_{r,k}(x',x,t)\,, \tag{16.1-16}$$

$$G^{vh}_{r,i,j}(x',x,t) = -\rho^{-1}C_{k,m,i,j}\partial'_m G_{r,k}(x',x,t)\,, \tag{16.1-17}$$

$$G^{vf}_{r,k}(x',x,t) = \rho^{-1}\partial_t G_{r,k}(x',x,t)\,, \tag{16.1-18}$$

with

$$\begin{aligned} G_{r,k}(x',x,t) = {} & \frac{\delta(t - |x'-x|/c_S)}{4\pi c_S^2\,|x'-x|}\,\delta_{r,k} \\ & + \partial'_r\partial'_k\left[\frac{(t-|x'-x|/c_P)H(t-|x'-x|/c_P)}{4\pi\,|x'-x|}\right. \\ & \left. - \frac{(t-|x'-x|/c_S)H(t-|x'-x|/c_S)}{4\pi\,|x'-x|}\right] \quad \text{for } |x'-x| \neq 0\,. \end{aligned} \tag{16.1-19}$$

Here, ∂'_m denotes differentiation with respect to x'_m.

In the *far-field region*, the expansion

$$\begin{aligned} \{\tau^{\mathrm{s}}_{p,q}, v^{\mathrm{s}}_r\}(x',t) = {} & \left[\frac{\{\tau^{\mathrm{s};P,\infty}_{p,q}, v^{\mathrm{s};P,\infty}_r\}(\xi, t-|x'|/c_P)}{4\pi c_P^2\,|x'|} + \frac{\{\tau^{\mathrm{s};S,\infty}_{p,q}, v^{\mathrm{s};S,\infty}_r\}(\xi, t-|x'|/c_S)}{4\pi c_S^2\,|x'|}\right] \\ & \times [1 + O(|x'|^{-1})] \quad \text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi \end{aligned} \tag{16.1-20}$$

holds, where (see Equations (13.8-7)–(13.8-9))

$$v^{\mathrm{s};P,\infty}_r(\xi,t) = \rho^{-1}\partial_t\Phi^{f^{\mathrm{s}};P,\infty}_r(\xi,t) + (\rho c_P)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi^{h^{\mathrm{s}};P,\infty}_{r,k,i,j}(\xi,t)\,, \tag{16.1-21}$$

$$v^{\mathrm{s};S,\infty}_r(\xi,t) = \rho^{-1}\partial_t\Phi^{f^{\mathrm{s}};S,\infty}_r(\xi,t) + (\rho c_S)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi^{h^{\mathrm{s}};S,\infty}_{r,k,i,j}(\xi,t)\,, \tag{16.1-22}$$

in which (see Equations (13.8-2), (13.8-3) and (13.8-5), (13.8-6))

$$\Phi^{f^{\mathrm{s}};P,\infty}_r(\xi,t) = \xi_r\xi_k\int_{x\in\mathcal{D}^{\mathrm{s}}} f^{\mathrm{s}}_k(x, t+\xi_s x_s/c_P)\,\mathrm{d}V\,, \tag{16.1-23}$$

$$\Phi_r^{f^s;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\mathcal{D}^s} f_k^s(x,t+\xi_s x_s/c_S)\,\mathrm{d}V\,, \tag{16.1-24}$$

and

$$\Phi_{r,k,i,j}^{h^s;P,\infty}(\xi,t) = \xi_r\xi_k\int_{x\in\mathcal{D}^s} h_{i,j}^s(x,t+\xi_s x_s/c_P)\,\mathrm{d}V\,, \tag{16.1-25}$$

$$\Phi_{r,k,i,j}^{h^s;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\mathcal{D}^s} h_{i,j}^s(x,t+\xi_s x_s/c_S)\,\mathrm{d}V\,, \tag{16.1-26}$$

while (see Equations (13.8-13)–(13.8-15))

$$\tau_{p,q}^{s;P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{s;P,\infty}) + 2\mu(\xi_k v_k^{s;P,\infty})\xi_p\xi_q\right], \tag{16.1-27}$$

$$\tau_{p,q}^{s;S,\infty} = -c_S^{-1}\mu(\xi_p v_q^{s;S,\infty} + \xi_q v_p^{s;S,\infty}). \tag{16.1-28}$$

For an *elastodynamically impenetrable scatterer* the elastic wave field is not defined in the interior $\mathcal{D}^s$ of the scatterer and we have to resort to an equivalent surface source integral representation that expresses the scattered wave field in the exterior $\mathcal{D}^{s\prime}$ of the scatterer in terms of the wave field on the boundary surface $\partial\mathcal{D}^s$ of $\mathcal{D}^s$. This representation is, on account of Equations (15.12-38) and (15.12-39),

$$-\tau_{p,q}^s(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = \int_{x\in\partial\mathcal{D}^s}\left[\mathrm{C}_t(G_{p,q,i,j}^{\tau h},\Delta_{i,j,n,r}^{+}\nu_m v_r^s;x',x,t)\right.$$
$$\left.+\,\mathrm{C}_t(G_{p,q,k}^{\tau f},-\Delta_{k,m,p',q'}^{+}\nu_n\tau_{p',q'}^s;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{16.1-29}$$

and

$$v_r^s(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = \int_{x\in\partial\mathcal{D}^s}\left[\mathrm{C}_t(G_{r,i,j}^{vh},\Delta_{i,j,n,r'}^{+}\nu_n v_{r'}^s;x',x,t)\right.$$
$$\left.+\,\mathrm{C}_t(G_{r,k}^{vf},-\Delta_{k,m,p,q}^{+}\nu_m\tau_{p,q}^s;x',x,t)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3\,. \tag{16.1-30}$$

Note that these expressions have resulted from applying Equations (15.12-38) and (15.12-39) to the domain $\mathcal{D}^{s\prime}$ exterior to the scatterer and that the unit vector along the normal is oriented towards $\mathcal{D}^{s\prime}$.

In the *far-field region*, the expansion given in Equation (16.1-20) holds, where, based upon Equations (16.1-29) and (16.1-30), we have

$$v_r^{s;P,\infty}(\xi,t) = \rho^{-1}\partial_t\Phi_r^{\partial f^s;P,\infty}(\xi,t) + (\rho c_P)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi_{r,k,i,j}^{\partial h^s;P,\infty}(\xi,t)\,, \tag{16.1-31}$$

$$v_r^{s;S,\infty}(\xi,t) = \rho^{-1}\partial_t\Phi_r^{\partial f^s;S,\infty}(\xi,t) + (\rho c_S)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi_{r,k,i,j}^{\partial h^s;S,\infty}(\xi,t)\,, \tag{16.1-32}$$

in which

$$\Phi_r^{\partial f^s;P,\infty}(\xi,t) = \xi_r\xi_k\int_{x\in\partial\mathcal{D}^s}\partial f_k^s(x,t+\xi_s x_s/c_P)\,\mathrm{d}A\,, \tag{16.1-33}$$

$$\Phi_r^{\partial f^s;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\partial\mathcal{D}^s}\partial f_k^s(x,t+\xi_s x_s/c_S)\,\mathrm{d}A\,, \tag{16.1-34}$$

with

$$\partial f_k^{\mathrm{s}} = -\Delta^+_{k,m,p,q}\nu_m\tau^{\mathrm{s}}_{p,q} \quad , \tag{16.1-35}$$

and

$$\Phi^{\partial h^{\mathrm{s}};P,\infty}_{r,k,i,j}(\xi,t) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial h^{\mathrm{s}}_{i,j}(x,t+\xi_s x_s/c_P)\,\mathrm{d}A \,, \tag{16.1-36}$$

$$\Phi^{\partial h^{\mathrm{s}};S,\infty}_{r,k,i,j}(\xi,t) = (\delta_{r,k}-\xi_r\xi_k) \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial h^{\mathrm{s}}_{i,j}(x,t+\xi_s x_s/c_S)\,\mathrm{d}A \,, \tag{16.1-37}$$

with

$$\partial h^{\mathrm{s}}_{i,j} = \Delta^+_{i,j,n,r}\nu_n v^{\mathrm{s}}_r \,, \tag{16.1-38}$$

while Equations (16.1-27) and (16.1-28) yield the far-field scattering amplitude for the dynamic stress.

However, upon applying Equations (15.12-12) and (15.12-19) to the incident wave field $\{\tau^{\mathrm{i}}_{p,q}, v^{\mathrm{i}}_r\}$ and to the domain $\mathcal{D}^{\mathrm{s}}$, we have (note that the incident wave field is source-free in $\mathcal{D}^{\mathrm{s}}$)

$$-\tau^{\mathrm{i}}_{p,q}(x',t)\chi_{\mathcal{D}^{\mathrm{s}}}(x') = -\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \Big[\mathrm{C}_t(G^{\tau h}_{p,q,i,j}, \Delta^+_{i,j,n,r}\nu_n v^{\mathrm{i}}_r; x',x,t)$$
$$+\,\mathrm{C}_t(G^{\tau f}_{p,q,k}, -\Delta^+_{k,m,p',q'}\nu_m\tau^{\mathrm{i}}_{p',q'}; x',x,t)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{16.1-39}$$

and

$$v^{\mathrm{i}}_r(x,t)\chi_{\mathcal{D}^{\mathrm{s}}}(x') = -\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \Big[\mathrm{C}_t(G^{vh}_{r,i,j}, \Delta^+_{i,j,n,r'}\nu_n v^{\mathrm{i}}_{r'}; x',x,t)$$
$$+\,\mathrm{C}_t(G^{vf}_{r,k}, -\Delta^+_{k,m,p,q}\nu_m\tau^{\mathrm{i}}_{p,q}; x',x,t)\Big] \quad \text{for } x'\in\mathcal{R}^3 \,. \tag{16.1-40}$$

Subtraction of Equation (16.1-39) from Equation (16.1-29) and of Equation (16.1-40) from Equation (16.1-30) leads to

$$-\tau^{\mathrm{s}}_{p,q}(x',t)\chi_{\mathcal{D}^{\mathrm{s}'}}(x') + \tau^{\mathrm{i}}_{p,q}(x',t)\chi_{\mathcal{D}^{\mathrm{s}}}(x')$$
$$= \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \Big[\mathrm{C}_t(G^{\tau h}_{p,q,i,j}, \Delta^+_{i,j,n,r}\nu_r; x',x,t)$$
$$+\,\mathrm{C}_t(G^{\tau f}_{p,q,k}, -\Delta^+_{k,m,p',q'}\nu_m\tau_{p',q'}; x',x,t)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{16.1-41}$$

and

$$v^{\mathrm{s}}_r(x',t)\chi_{\mathcal{D}^{\mathrm{s}'}}(x') - v^{\mathrm{i}}_r(x',t)\chi_{\mathcal{D}^{\mathrm{s}}}(x')$$
$$= \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \Big[\mathrm{C}_t(G^{vh}_{r,i,j}, \Delta^+_{i,j,n,r'}\nu_n v_{r'}; x',x,t)$$
$$+\,\mathrm{C}_t(G^{vf}_{r,k}, -\Delta^+_{k,m,p,q}\nu_m\tau_{p,q}; x',x,t)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \,. \tag{16.1-42}$$

In the *far-field region*, again the expansion given in Equation (16.1-20) with Equations

(16.1-27) and (16.1-28) holds (note that $\chi_{\mathcal{D}^s}(x') = 0$ for $x' \in \mathcal{D}^{s\,\prime}$ and hence in the far-field region), in which, based upon Equations (16.1-41) and (16.1-42), we now have

$$v_r^{s;P,\infty}(\xi,t) = \rho^{-1}\partial_t \Phi_r^{\partial f;P,\infty}(\xi,t) + (\rho c_P)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h;P,\infty}(\xi,t) \ , \tag{16.1-43}$$

$$v_r^{s;S,\infty}(\xi,t) = \rho^{-1}\partial_t \Phi_r^{\partial f;S,\infty}(\xi,t) + (\rho c_S)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h;S,\infty}(\xi,t) \ , \tag{16.1-44}$$

in which

$$\Phi_r^{\partial f;P,\infty}(\xi,t) = \xi_r \xi_k \int_{x\in\partial\mathcal{D}^s} \partial f_k(x, t + \xi_s x_s/c_P)\,\mathrm{d}A \ , \tag{16.1-45}$$

$$\Phi_r^{\partial f;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r \xi_k) \int_{x\in\partial\mathcal{D}^s} \partial f_k(x, t + \xi_s x_s/c_S)\,\mathrm{d}A \ , \tag{16.1-46}$$

with

$$\partial f_k = -\overset{+}{\Delta}_{k,m,p,q} \nu_m \tau_{p,q} \ , \tag{16.1-47}$$

and

$$\Phi_{r,k,i,j}^{\partial h;P,\infty}(\xi,t) = \xi_r \xi_k \int_{x\in\partial\mathcal{D}^s} \partial h_{i,j}(x, t + \xi_s x_s/c_P)\,\mathrm{d}A \ , \tag{16.1-48}$$

$$\Phi_{r,k,i,j}^{\partial h;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r \xi_k) \int_{x\in\partial\mathcal{D}^s} \partial h_{i,j}(x, t + \xi_s x_s/c_S)\,\mathrm{d}A \ , \tag{16.1-49}$$

with

$$\partial h_{i,j} = \overset{+}{\Delta}_{i,j,n,r} \nu_n v_r \ , \tag{16.1-50}$$

Of course, the equivalent surface source representations also apply to the case of an elastodynamically penetrable scatterer. For $x' \in \mathcal{D}^{s\,\prime}$ (i.e. outside the scatterer), Equations (16.1-13), (16.1-14) and (16.1-29), (16.1-30) and (16.1-41), (16.1-42) must then all yield the same result. Similarly, in the far-field region, Equations (16.1-21)–(16.1-26), (16.1-31)–(16.1-38) and (16.1-43)–(16.1-50) must all yield the same result. Note, however, that for $x' \in \mathcal{D}^s$ (i.e. in the interior of the scatterer) the results of the different representations differ.

Equations (16.1-13) and (16.1-14), when taken for $x' \in \mathcal{D}^s$, provide the basis for the *time-domain domain integral equation method* to solve problems of the scattering by penetrable objects. For solving problems of the scattering by impenetrable objects, Equations (16.1-41) and (16.1-42) provide, when taken for $x' \in \partial\mathcal{D}^s$, the basis for the *time-domain boundary integral equation method* and, when taken for $x' \in \mathcal{D}^s$, the basis for the *time-domain null-field method.* For general scatterers, all three methods need numerical implementation.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the elastodynamic properties of the scatterer are, if the scatterer is an elastodynamically penetrable object, characterised by the functions

$$\{\hat{\xi}^{s}_{k,r},\hat{\eta}^{s}_{i,j,p,q}\} = \{\hat{\xi}^{s}_{k,r},\hat{\eta}^{s}_{i,j,p,q}\}(\boldsymbol{x},s).$$

The equivalent contrast volume source densities of deformation rate and force are then given by (see Equations (15.9-41) and (15.9-42))

$$\hat{f}^{s}_{k} = -(\hat{\xi}^{s}_{k,r} - s\rho\delta_{k,r})\hat{v}_r \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}, \tag{16.1-51}$$

$$\hat{h}^{s}_{i,j} = (\hat{\eta}^{s}_{i,j,p,q} - sS_{i,j,p,q})\hat{\tau}_{p,q} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{s}, \tag{16.1-52}$$

in which the *total elastic wave field* $\{\hat{\tau}_{p,q},\hat{v}_r\}$ is the sum of the *incident wave field* $\{\hat{\tau}^{i}_{p,q},\hat{v}^{i}_r\}$ and the *scattered wave field* $\{\hat{\tau}^{s}_{p,q},\hat{v}^{s}_r\}$ (see Equation (15.9-28)). If the scatterer is *elastodynamically impenetrable*, either of the two boundary conditions

$$\lim_{h\downarrow 0}\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}_{p,q}(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0 \qquad \text{for } \boldsymbol{x}\in\partial\mathcal{D}^{s} \tag{16.1-53}$$

or

$$\lim_{h\downarrow 0}\hat{v}_r(\boldsymbol{x}+h\boldsymbol{\nu},s) = 0 \qquad \text{for } \boldsymbol{x}\in\partial\mathcal{D}^{s} \tag{16.1-54}$$

applies.

For the incident wave we now take a *uniform plane wave*. This can be either a uniform plane P-wave or a uniform plane S-wave. For the *incident plane P-wave* propagating in the direction of the unit vector $\boldsymbol{\alpha}^{P}$ (i.e. $\alpha^{P}_{s}\alpha^{P}_{s} = 1$) we have, on account of Equations (14.1-3), (14.2-19) and (14.2-20)

$$\{\hat{\tau}_{p,q},\hat{v}_r\} = \{T^{P}_{p,q},V^{P}_r\}\hat{a}^{P}(s)\exp(-s\alpha^{P}_{s}x_s/c_P)\,, \tag{16.1-55}$$

with

$$c_P = [(\lambda + 2\mu)/\rho]^{1/2}, \tag{16.1-56}$$

$$V^{P}_r = (\alpha^{P}_k V^{P}_k)\alpha^{P}_r\,, \tag{16.1-57}$$

and (see Equation (14.2-10))

$$T^{P}_{p,q} = -c_P^{-1}\left[\lambda\delta_{p,q}(\alpha^{P}_k V^{P}_k) + 2\mu(\alpha^{P}_k V^{P}_k)\alpha^{P}_p\alpha^{P}_q\right], \tag{16.1-58}$$

where $\hat{a}^{P}(s)$ denotes the normalised pulse shape.

For the *incident plane S-wave* propagating in the direction of the unit vector $\boldsymbol{\alpha}^{S}$ (i.e. $\alpha^{S}_{s}\alpha^{S}_{s} = 1$) we have, on account of Equations (14.1-3), (14.2-22), and (14.2-23)

$$\{\hat{\tau}_{p,q},\hat{v}_r\} = \{T^{S}_{p,q},V^{S}_r\}\hat{a}^{S}(s)\exp(-s\alpha^{S}_{s}x_s/c_S)\,, \tag{16.1-59}$$

with

$$c_S = (\mu/\rho)^{1/2}, \tag{16.1-60}$$

$$\alpha^{S}_k V^{S}_k = 0, \tag{16.1-61}$$

and (see Equation (14.2-10))

$$T^{S}_{p,q} = -c_S^{-1}\mu(\alpha^{S}_p V^{S}_q + \alpha^{S}_q V^{S}_p)\,, \tag{16.1-62}$$

where $\hat{a}^{S}(s)$ denotes the normalised pulse shape.

For an *elastodynamically penetrable scatterer* we use for the scattered wave the contrast volume source representations (see Equations (15.9-43) and (15.9-44))

$$-\hat{\tau}^{s}_{p,q}(x',s) = \int_{x\in\mathcal{D}^s} \left[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\hat{h}^{s}_{i,j}(x,s) + \hat{G}^{\tau f}_{p,q,k}(x',x,s)\hat{f}^{s}_{k}(x,s)\right] \mathrm{d}V$$

$$\text{for } x'\in\mathcal{R}^3 \tag{16.1-63}$$

and

$$\hat{v}^{s}_{r}(x',s) = \int_{x\in\mathcal{D}^s} \left[\hat{G}^{vh}_{r,i,j}(x',x,s)\hat{h}^{s}_{i,j}(x,s) + \hat{G}^{vf}_{r,k}(x',x,s)\hat{f}^{s}_{k}(x,s)\right] \mathrm{d}V$$

$$\text{for } x'\in\mathcal{R}^3 , \tag{16.1-64}$$

in which (see Exercise 15.8-10 with $\mathbf{x}$ and x' interchanged)

$$\hat{G}^{\tau h}_{p,q,i,j}(x',x,s) = -s^{-1}C_{p,q,i,j}\delta(x'-x) - (s\rho)^{-1}C_{p,q,n,r}C_{k,m,i,j}\partial'_n\partial'_m\hat{G}_{r,k}(x',x,s) , \tag{16.1-65}$$

$$\hat{G}^{\tau f}_{p,q,k}(x',x,s) = \rho^{-1}C_{p,q,n,r}\partial'_n\hat{G}_{r,k}(x',x,s) , \tag{16.1-66}$$

$$\hat{G}^{vh}_{r,i,j}(x',x,s) = -\rho^{-1}C_{k,m,i,j}\partial'_m\hat{G}_{r,k}(x',x,s) , \tag{16.1-67}$$

$$\hat{G}^{vf}_{r,k}(x',x,s) = s\rho^{-1}\hat{G}_{r,k}(x',x,s) , \tag{16.1-68}$$

with

$$\hat{G}_{r,k}(x',x,s) = c_S^{-2}\hat{G}_S(x',x,s)\delta_{r,k} + s^{-2}\partial'_r\partial'_k\left[\hat{G}_P(x',x,s) - \hat{G}_S(x',x,s)\right] \tag{16.1-69}$$

and

$$\hat{G}_{P,S}(x',x,s) = \frac{\exp(-s|x'-x|/c_{P,S})}{4\pi|x'-x|} \quad \text{for } |x'-x| \neq 0 . \tag{16.1-70}$$

Here, ∂'_m denotes differentiation with respect to x'_m.

In the *far-field region*, the expansion

$$\{\hat{\tau}^{s}_{p,q},\hat{v}^{s}_{r}\} = \left[\{\hat{\tau}^{s;P,\infty}_{p,q},\hat{v}^{s;P,\infty}_{r}\}(\xi,s)\frac{\exp[-s|x'|/c_P]}{4\pi c_P^2|x'|} + \{\hat{\tau}^{s;S,\infty}_{p,q},\hat{v}^{s;S,\infty}_{r}\}(\xi,s)\frac{\exp[-s|x'|/c_S]}{4\pi c_S^2|x'|}\right]$$

$$\times[1+O(|x'|^{-1})] \quad \text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi \tag{16.1-71}$$

holds, where (see Equations (13.7-18) and (13.7-19))

$$\hat{v}^{s;P,\infty}_{r} = s\rho^{-1}\hat{\Phi}^{f^s;P,\infty}_{r} + s(\rho c_P)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}^{h^s;P,\infty}_{r,k,i,j} , \tag{16.1-72}$$

$$\hat{v}^{s;S,\infty}_{r} = s\rho^{-1}\hat{\Phi}^{f^s;S,\infty}_{r} + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}^{h^s;S,\infty}_{r,k,i,j} , \tag{16.1-73}$$

in which (see Equations (13.7-12), (13.7-13) and (13.7-15), (13.7-16))

$$\hat{\Phi}^{f^s;P,\infty}_{r}(\xi,s) = \xi_r\xi_k\int_{x\in\mathcal{D}^s}\exp(s\xi_s x_s/c_P)\hat{f}^{s}_{k}(x,s)\,\mathrm{d}V , \tag{16.1-74}$$

$$\hat{\Phi}^{f^s;S,\infty}_{r}(\xi,s) = (\delta_{r,k}-\xi_r\xi_k)\int_{x\in\mathcal{D}^s}\exp(s\xi_s x_s/c_S)\hat{f}^{s}_{k}(x,s)\,\mathrm{d}V , \tag{16.1-75}$$

and

$$\hat{\Phi}^{h^s;P,\infty}_{r,k,i,j}(\boldsymbol{\xi},s) = \xi_r\xi_k \int_{\boldsymbol{x}\in\mathcal{D}^s} \exp(s\xi_s x_s/c_P)\hat{h}^s_{i,j}(\boldsymbol{x},s)\,\mathrm{d}V, \tag{16.1-76}$$

$$\hat{\Phi}^{h^s;S,\infty}_{r,k,i,j}(\boldsymbol{\xi},s) = (\delta_{r,k} - \xi_r\xi_k) \int_{\boldsymbol{x}\in\mathcal{D}^s} \exp(s\xi_s x_s/c_S)\hat{h}^s_{i,j}(\boldsymbol{x},s)\,\mathrm{d}V, \tag{16.1-77}$$

while (see Equations (13.7-25)–(13.7-27))

$$\hat{\tau}^{s;P,\infty}_{p,q} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k\hat{v}^{s;P,\infty}_k) + 2\mu(\xi_k\hat{v}^{s;P,\infty}_k)\xi_p\xi_q\right], \tag{16.1-78}$$

$$\hat{\tau}^{s;S,\infty}_{p,q} = -c_S^{-1}\mu(\xi_p\hat{v}^{s;S,\infty}_q + \xi_q\hat{v}^{s;S,\infty}_p). \tag{16.1-79}$$

For an *elastodynamically impenetrable scatterer* the elastic wave field is not defined in the interior $\mathcal{D}^s$ of the scatterer and we have to resort to an equivalent surface source integral representation that expresses the scattered wave field in the exterior $\mathcal{D}^{s\prime}$ of the scatterer in terms of the wave field on the boundary surface $\partial\mathcal{D}^s$ of $\mathcal{D}^s$. This representation is, on account of Equations (15.12-40) and (15.12-41),

$$-\hat{\tau}^s_{p,q}(\boldsymbol{x}',s)\chi_{\mathcal{D}^{s\prime}}(\boldsymbol{x}') = \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \Big[\hat{G}^{\tau h}_{p,q,i,j}(\boldsymbol{x}',\boldsymbol{x},s)\Delta^+_{i,j,n,r}\nu_n\hat{v}^s_r(\boldsymbol{x},s)$$
$$- \hat{G}^{\tau f}_{p,q,k}(\boldsymbol{x}',\boldsymbol{x},s)\Delta^+_{k,m,p',q'}\nu_m\hat{\tau}^s_{p',q'}(\boldsymbol{x},s)\Big]\,\mathrm{d}A \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3 \tag{16.1-80}$$

and

$$\hat{v}^s_r(\boldsymbol{x}',s)\chi_{\mathcal{D}^{s\prime}}(\boldsymbol{x}') = \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \Big[\hat{G}^{vh}_{r,i,j}(\boldsymbol{x}',\boldsymbol{x},s)\Delta^+_{i,j,n,r'}\nu_n\hat{v}^s_{r'}(\boldsymbol{x},s)$$
$$- \hat{G}^{vf}_{r,k}(\boldsymbol{x}',\boldsymbol{x},s)\Delta^+_{k,m,p,q}\nu_m\hat{\tau}^s_{p,q}(\boldsymbol{x},s)\Big]\,\mathrm{d}A \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3. \tag{16.1-81}$$

Note that these expressions have resulted from applying Equations (15.12-40) and (15.12-41) to the domain $\mathcal{D}^{s\prime}$ exterior to the scatterer and that the unit vector along the normal to $\partial\mathcal{D}^s$ is oriented towards $\mathcal{D}^{s\prime}$.

In the *far-field region* the expansion given in Equation (16.1-71) holds, where, based upon Equations (16.1-80) and (16.1-81), we have

$$\hat{v}^{s;P,\infty}_r = s\rho^{-1}\hat{\Phi}^{\partial f^s;P,\infty}_r + s(\rho c_P)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}^{\partial h^s;P,\infty}_{r,k,i,j}, \tag{16.1-82}$$

$$\hat{v}^{s;S,\infty}_r = s\rho^{-1}\hat{\Phi}^{\partial f^s;S,\infty}_r + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}^{\partial h^s;S,\infty}_{r,k,i,j}, \tag{16.1-83}$$

in which

$$\hat{\Phi}^{\partial f^s;P,\infty}_r(\boldsymbol{\xi},s) = \xi_r\xi_k \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_P)\partial\hat{f}^s_k(\boldsymbol{x},s)\,\mathrm{d}A, \tag{16.1-84}$$

$$\hat{\Phi}^{\partial f^s;S,\infty}_r(\boldsymbol{\xi},s) = (\delta_{r,k} - \xi_r\xi_k) \int_{\boldsymbol{x}\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_S)\partial\hat{f}^s_k(\boldsymbol{x},s)\,\mathrm{d}A, \tag{16.1-85}$$

with

$$\partial\hat{f}^s_k = -\Delta^+_{k,m,p,q}\nu_m\hat{\tau}^s_{p,q}, \tag{16.1-86}$$

and

$$\hat{\Phi}^{\partial h^{s};P,\infty}_{r,k,i,j}(\xi,s) = \xi_r \xi_k \int_{x \in \partial\mathcal{D}^s} \exp(s\xi_s x_s / c_P)\partial\hat{h}^{s}_{i,j}(x,s)\, dA\,, \tag{16.1-87}$$

$$\hat{\Phi}^{\partial h^{s};S,\infty}_{r,k,i,j}(\xi,s) = (\delta_{r,k} - \xi_r \xi_k) \int_{x \in \partial\mathcal{D}^s} \exp(s\xi_s x_s / c_S)\partial\hat{h}^{s}_{i,j}(x,s)\, dA\,, \tag{16.1-88}$$

with

$$\partial\hat{h}^{s}_{i,j} = \Delta^{+}_{i,j,n,r}\nu_n \hat{v}^{s}_r\,, \tag{16.1-89}$$

while Equations (16.1-78) and (16.1-79) yield the far-field scattering amplitude of the dynamic stress.

However, upon applying Equations (15.12-30) and (15.12-37) to the incident wave field $\{\hat{\tau}^{i}_{p,q}, \hat{v}^{i}_{r}\}$ and to the domain $\mathcal{D}^s$, we have (note that the incident wave field is source-free in $\mathcal{D}^s$)

$$-\hat{\tau}^{i}_{p,q}(x',s)\chi_{\mathcal{D}^s}(x') = -\int_{x \in \partial\mathcal{D}^s} \Big[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\Delta^{+}_{i,j,n,r}\nu_n \hat{v}^{i}_r(x,s)$$
$$- \hat{G}^{\tau f}_{p,q,k}(x',x,s)\Delta^{+}_{k,m,p',q'}\nu_m \hat{\tau}^{i}_{p',q'}(x,s)\Big]\, dA \quad \text{for } x' \in \mathcal{R}^3 \tag{16.1-90}$$

and

$$\hat{v}^{i}_r(x',s)\chi_{\mathcal{D}^s}(x') = -\int_{x \in \partial\mathcal{D}^s} \Big[\hat{G}^{vh}_{r,i,j}(x',x,s)\Delta^{+}_{i,j,n,r'}\nu_n \hat{v}^{i}_{r'}(x,s)$$
$$- \hat{G}^{vf}_{r,k}(x',x,s)\Delta^{+}_{k,m,p,q}\nu_m \hat{\tau}^{i}_{p,q}(x,s)\Big]\, dA \quad \text{for } x' \in \mathcal{R}^3\,. \tag{16.1-91}$$

Subtraction of Equation (16.1-90) from Equation (16.1-80) and of Equation (16.1-91) from Equation (16.1-81) leads to

$$-\hat{\tau}^{s}_{p,q}(x',s)\chi_{\mathcal{D}^{s'}}(x') + \hat{\tau}^{i}_{p,q}(x',s)\chi_{\mathcal{D}^s}(x')$$
$$= \int_{x \in \partial\mathcal{D}^s} \Big[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\Delta^{+}_{i,j,n,r}\nu_n \hat{v}_r(x,s)$$
$$- \hat{G}^{\tau f}_{p,q,k}(x',x,s)\Delta^{+}_{k,m,p',q'}\nu_m \hat{\tau}_{p',q'}(x,s)\Big]\, dA \quad \text{for } x' \in \mathcal{R}^3 \tag{16.1-92}$$

and

$$-\hat{v}^{s}_r(x',s)\chi_{\mathcal{D}^{s'}}(x') - \hat{v}^{i}_r(x',s)\chi_{\mathcal{D}^s}(x')$$
$$= \int_{x \in \partial\mathcal{D}^s} \Big[\hat{G}^{vh}_{r,i,j}(x',x,s)\Delta^{+}_{i,j,n,r'}\nu_n \hat{v}_{r'}(x,s)$$
$$- \hat{G}^{vf}_{r,k}(x',x,s)\Delta^{+}_{k,m,p,q}\nu_m \hat{\tau}_{p,q}(x,s)\Big]\, dA \quad \text{for } x' \in \mathcal{R}^3\,. \tag{16.1-93}$$

In the *far-field region* again the expansion given in Equation (16.1-71) with Equations (16.1-78) and (16.1-79) holds (note that $\chi_{\mathcal{D}^s}(x') = 0$ for $x' \in \mathcal{D}^{s\,\prime}$ and hence in the far-field region), in which, based upon Equations (16.1-92) and (16.1-93), we now have

$$\hat{v}_r^{s;P,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f;P,\infty} + s(\rho c_P)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h;P,\infty} , \tag{16.1-94}$$

$$\hat{v}_r^{s;S,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f;S,\infty} + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h;S,\infty} , \tag{16.1-95}$$

in which

$$\hat{\Phi}_r^{\partial f;P,\infty}(\xi,s) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_P)\partial\hat{f}_k(x,s)\,dA , \tag{16.1-96}$$

$$\hat{\Phi}_r^{\partial f;S,\infty}(\xi,s) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_S)\partial\hat{f}_k(x,s)\,dA , \tag{16.1-97}$$

with

$$\partial\hat{f}_k = -\Delta^+_{k,m,p,q}\nu_m\hat{\tau}_{p,q} , \tag{16.1-98}$$

and

$$\hat{\Phi}_{r,k,i,j}^{\partial h;P,\infty}(\xi,s) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_P)\partial\hat{h}_{i,j}(x,s)\,dA , \tag{16.1-99}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h;S,\infty}(\xi,s) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_S)\partial\hat{h}_{i,j}(x,s)\,dA , \tag{16.1-100}$$

with

$$\partial\hat{h}_{i,j} = \Delta^+_{i,j,n,r}\nu_n\hat{v}_r . \tag{16.1-101}$$

Of course, the equivalent surface source representations also apply to the case of an elastodynamically penetrable scatterer. For $x'\in\mathcal{D}^{s\prime}$ (i.e. outside the scatterer), Equations (16.1-63) and (16.1-64), (16.1-80) and (16.1-81), and (16.1-92) and (16.1-93) must then all yield the same result. Similarly, in the far-field region, Equations (16.1-72)–(16.1-77), (16.1-82)–(16.1-89) and (16.1-94)–(16.1-101) must all yield the same result. Note, however, that for $x'\in\mathcal{D}^s$ (i.e. in the interior of the scatterer) the results of the different representations differ.

Equations (16.1-63) and (16.1-64), when taken for $x'\in\mathcal{D}^s$, provide the basis for the *complex frequency-domain domain integral equation method* to solve problems of the scattering by penetrable objects. For solving problems of the scattering by impenetrable objects, Equations (16.1-92) and (16.1-93) provide, when taken for $x'\in\partial\mathcal{D}^s$, the basis for the *complex frequency-domain boundary integral equation method* and, when taken for $x'\in\mathcal{D}^s$, the basis for the *complex frequency-domain null-field method.* For general scatterers, all three methods need numerical implementation.

The representations in this section will be needed in the remainder of this chapter.

Exercises

Exercise 16.1-1

Show that from Equations (16.1-41) and (16.1-42) it follows that

$$-\tau_{p,q}(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = -\tau^{i}_{p,q}(x',t) + \int_{x\in\partial\mathcal{D}^s} \Big[C_t(G^{\tau h}_{p,q,i,j},\Delta^{+}_{i,j,n,r}\nu_n v_r;x',x,t)$$

$$+ C_t(G^{\tau f}_{p,q,k},-\Delta^{+}_{k,m,p',q'}\nu_m\tau_{p',q'};x',x,t)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \quad (16.1\text{-}102)$$

and

$$v_r(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = v^{i}_r(x',t) + \int_{x\in\partial\mathcal{D}^s} \Big[C_t(G^{vh}_{r,i,j},\Delta^{+}_{i,j,n,r'}\nu_n v_{r'};x',x,t)$$

$$+ C_t(G^{vf}_{r,k},-\Delta^{+}_{k,m,p,q}\nu_m\tau_{p,q};x',x,t)\Big] \quad \text{for } x'\in\mathcal{R}^3. \quad (16.1\text{-}103)$$

(*Hint*: Consider the cases $x'\in\mathcal{D}^{s\prime}$, $x'=\partial\mathcal{D}^s$ and $x'\in\mathcal{D}^s$.)

Exercise 16.1-2

Show that from Equations (16.1-92) and (16.1-93) it follows that

$$-\hat{\tau}_{p,q}(x',s)\chi_{\mathcal{D}^{s\prime}}(x') = -\hat{\tau}^{i}_{p,q}(x',s) + \int_{x\in\partial\mathcal{D}^s} \Big[\hat{G}^{\tau h}_{p,q,i,j}(x',x,s)\Delta^{+}_{i,j,n,r}\nu_n\hat{v}_r(x,s)$$

$$- \hat{G}^{\tau f}_{p,q,k}(x',x,s)\Delta^{+}_{k,m,p',q'}\nu_m\hat{\tau}_{p',q'}(x,s)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \quad (16.1\text{-}104)$$

and

$$\hat{v}_r(x',s)\chi_{\mathcal{D}^{s\prime}}(x') = \hat{v}^{i}_r(x',s) + \int_{x\in\partial\mathcal{D}^s} \Big[\hat{G}^{vh}_{r,i,j}(x',x,s)\Delta^{+}_{i,j,n,r'}\nu_n\hat{v}_{r'}(x,s)$$

$$- \hat{G}^{vf}_{r,k}(x',x,s)\Delta^{+}_{k,m,p,q}\nu_m\hat{\tau}_{p,q}(x,s)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3. \quad (16.1\text{-}105)$$

(*Hint*: Consider the cases $x'\in\mathcal{D}^{s\prime}$, $x'=\partial\mathcal{D}^s$ and $x'\in\mathcal{D}^s$.)

16.2 Far-field scattered wave amplitudes reciprocity of the time convolution type

In this section we investigate the reciprocity relations of the time convolution type that apply to the far-field scattered wave amplitudes at plane wave incidence upon an elastodynamically penetrable or impenetrable object. The scattering configuration of Figure 16.1-1 applies. Two states in this configuration are considered; they are denoted as state A and state B, respectively. In state A, either a uniform plane P-wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}^P$ or a uniform plane S-wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}^S$ is incident upon the scattering object; in state B, either a uniform plane P-wave that propagates in the direction of the unit vector $\boldsymbol{\beta}^P$ or a uniform plane S-wave that propagates in the direction of the unit vector $\boldsymbol{\beta}^S$ is incident upon the scattering object. It will be shown that the far-field scattered P- and S-wave amplitudes in state A when observed in the direction of observation $\boldsymbol{\xi}=-\boldsymbol{\beta}^{P,S}$

are related, via reciprocity, to the far-field scattered P- and S-wave amplitudes in state B when observed in the direction of observation $\boldsymbol{\xi} = -\boldsymbol{\alpha}^{P,S}$ (Figure 16.2-1).

The corresponding relationships in the time domain and in the complex frequency domain will be derived separately below.

Time-domain analysis

In the time-domain analysis, the incident wave in state A is taken either as the uniform plane P-wave

$$\{\tau_{p,q}^{\mathrm{i;A};P}, v_r^{\mathrm{i;A};P}\} = \{T_{p,q}^{\mathrm{A};P}, V_r^{\mathrm{A};P}\}a^P(t - \alpha_s^P x_s/c_P) , \tag{16.2-1}$$

with

$$T_{p,q}^{\mathrm{A};P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\alpha_k^P V_k^{\mathrm{A};P}) + 2\mu(\alpha_k^P V_k^{\mathrm{A};P})\alpha_p^P\alpha_q^P\right], \tag{16.2-2}$$

or as the uniform plane S-wave

$$\{\tau_{p,q}^{\mathrm{i;A};S}, v_r^{\mathrm{i;A};S}\} = \{T_{p,q}^{\mathrm{A};S}, V_r^{\mathrm{A};S}\}a^S(t - \alpha_s^S x_s/c_S) , \tag{16.2-3}$$

with

$$T_{p,q}^{\mathrm{A};S} = -c_S^{-1}\mu(\alpha_p^S V_q^{\mathrm{A};S}) + \alpha_q^S V_p^{\mathrm{A};S}) . \tag{16.2-4}$$

In the far-field region, the scattered wave in state A is represented as

$$\{\tau_{p,q}^{\mathrm{s;A}}, v_r^{\mathrm{s;A}}\}(\boldsymbol{x}',t) = \left[\frac{\{\tau_{p,q}^{\mathrm{s;A};P,\infty}, v_r^{\mathrm{s;A};P,\infty}\}(\boldsymbol{\xi},t - |\boldsymbol{x}'|/c_P)}{4\pi c_P^2\,|\boldsymbol{x}'|} + \frac{\{\tau_{p,q}^{\mathrm{s;A};S,\infty}, v_r^{\mathrm{s;A};S,\infty}\}(\boldsymbol{\xi},t - |\boldsymbol{x}'|/c_S)}{4\pi c_S^2\,|\boldsymbol{x}'|}\right]$$
$$\times [1 + O(|\boldsymbol{x}'|^{-1})] \quad \text{as} \quad |\boldsymbol{x}'| \to \infty \quad \text{with} \quad \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi} , \tag{16.2-5}$$

in which, on account of Equations (16.1-27), (16.1-28) and (16.1-31)–(16.1-38) (note that the surface source representation for the far-field scattered wave amplitudes is used),

$$v_r^{\mathrm{s;A};P,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t\Phi_r^{\partial f^{\mathrm{s}};\mathrm{A};P,\infty}(\boldsymbol{\xi},t) + (\rho c_P)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};P,\infty}(\boldsymbol{\xi},t) , \tag{16.2-6}$$

$$\tau_{p,q}^{\mathrm{s;A};P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{\mathrm{s;A};P,\infty}) + 2\mu(\xi_k v_k^{\mathrm{s;A};P,\infty})\xi_p\xi_q\right], \tag{16.2-7}$$

with

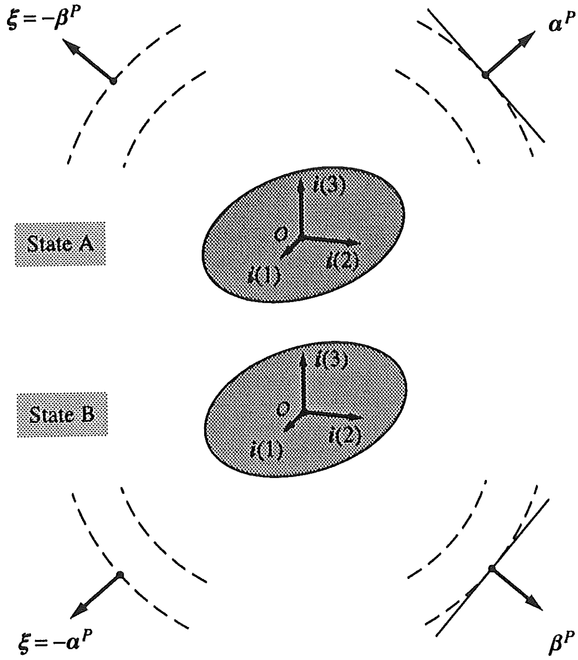
$$\Phi_r^{\partial f^{\mathrm{s}};\mathrm{A};P,\infty}(\boldsymbol{\xi},t) = \xi_r\xi_k\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial f_k^{\mathrm{s;A}}(\boldsymbol{x},t + \xi_s x_s/c_P)\,\mathrm{d}A , \tag{16.2-8}$$

$$\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};P,\infty}(\boldsymbol{\xi},t) = \xi_r\xi_k\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial h_{i,j}^{\mathrm{s;A}}(\boldsymbol{x},t + \xi_s x_s/c_P)\,\mathrm{d}A , \tag{16.2-9}$$

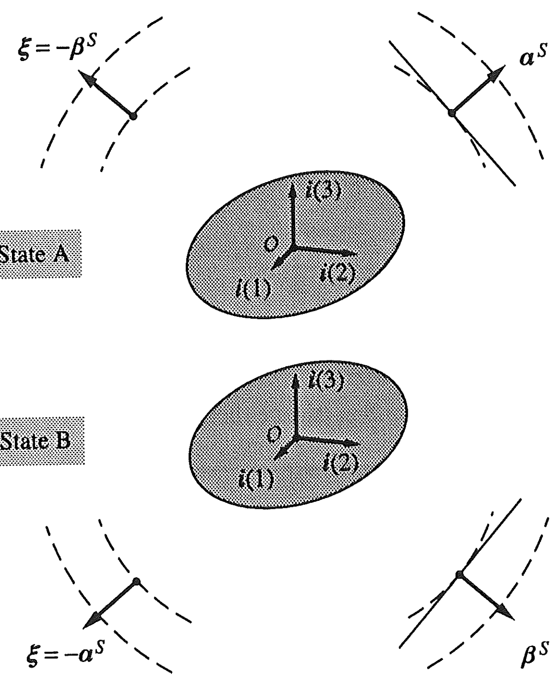
and

$$v_r^{\mathrm{s;A};S,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t\Phi_r^{\partial f^{\mathrm{s}};\mathrm{A};S,\infty}(\boldsymbol{\xi},t) + (\rho c_S)^{-1}C_{k,m,i,j}\xi_m\partial_t\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};S,\infty}(\boldsymbol{\xi},t) , \tag{16.2-10}$$

16.2-1(a)


$\xi = -\beta^P$
α^P
$i(3)$
State A
O
$i(2)$
$i(1)$
$i(3)$
State B
O
$i(2)$
$i(1)$
$\xi = -\alpha^P$
β^P

16.2-1(b)


$\xi = -\beta^S$
α^S
$i(3)$
State A
O
$i(2)$
$i(1)$
$i(3)$
State B
O
$i(2)$
$i(1)$
$\xi = -\alpha^S$
β^S

16.2-1(c)

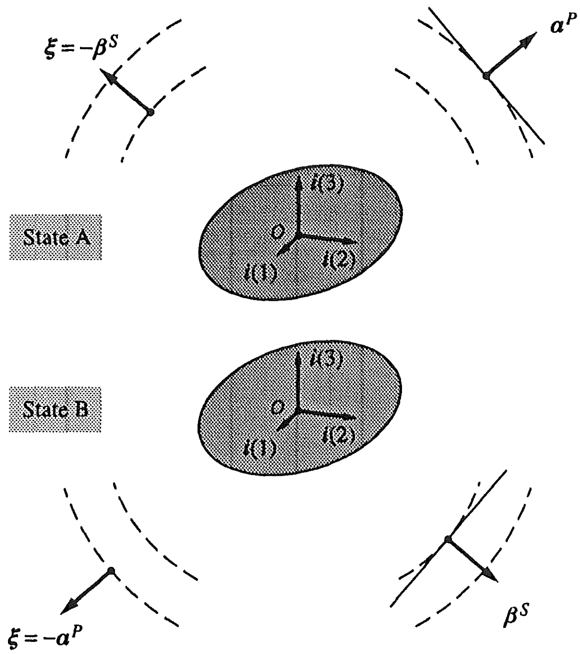


Figure 16.2-1 Configuration for the far-field scattered wave amplitudes reciprocity of the time convolution type: (a) two incident plane P-waves; (b) two incident plane S-waves; (c) an incident plane P-wave and an incident plane S-wave.

$$\tau_{p,q}^{s;A;S,\infty} = -c_S^{-1}\mu(\xi_p v_q^{s;A;S,\infty} + \xi_q v_p^{s;A;S,\infty}) , \tag{16.2-11}$$

with

$$\Phi_r^{\partial f^s;A;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \partial f_k^{s;A}(x,t+\xi_s x_s/c_S)\,\mathrm{d}A , \tag{16.2-12}$$

$$\Phi_{r,k,i,j}^{\partial h^s;A;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \partial h_{i,j}^{s;A}(x,t+\xi_s x_s/c_S)\,\mathrm{d}A . \tag{16.2-13}$$

Similarly, the incident wave in state B is taken either as the uniform plane P-wave

$$\{\tau_{p,q}^{i;B;P}, v_r^{i;B;P}\} = \{T_{p,q}^{B;P}, V_r^{B;P}\} b^P(t - \beta_s^P x_s/c_P) , \tag{16.2-14}$$

with

$$T_{p,q}^{B;P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\beta_k^P V_k^{B;P}) + 2\mu(\beta_k^P V_k^{B;P})\beta_p^P\beta_q^P\right] , \tag{16.2-15}$$

or as the uniform plane S-wave

$$\{\tau_{p,q}^{i;B;S}, v_r^{i;B;S}\} = \{T_{p,q}^{B;S}, V_r^{B;S}\} b^S(t - \beta_s^S x_s/c_S) , \tag{16.2-16}$$

with

$$T_{p,q}^{\mathrm{B};S} = -c_S^{-1}\mu(\beta_p^S V_q^{\mathrm{B};S} + \beta_q^S V_p^{\mathrm{B};S}) \,. \tag{16.2-17}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\tau_{p,q}^{\mathrm{s;B}}, v_r^{\mathrm{s;B}}\}(x',t) = \left[\frac{\{\tau_{p,q}^{\mathrm{s;B};P,\infty}, v_r^{\mathrm{s;B};P,\infty}\}(\xi, t - |x'|/c_P)}{4\pi c_P^2 |x'|} + \frac{\{\tau_{p,q}^{\mathrm{s;B};S,\infty}, v_r^{\mathrm{s;B};S,\infty}\}(\xi, t - |x'|/c_S)}{4\pi c_S^2 |x'|}\right]$$

$$\times [1 + O(|x'|^{-1})] \quad \text{as } |x'| \to \infty \quad \text{with } x' = |x'|\xi \,, \tag{16.2-18}$$

in which, on account of Equations (16.1-27), (16.1-28) and (16.1-31)–(16.1-38) (note that the surface source representation for the far-field scattered wave amplitudes is used),

$$v_r^{\mathrm{s;B};P,\infty}(\xi,t) = \rho^{-1}\partial_t \Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};P,\infty}(\xi,t) + (\rho c_P)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};P,\infty}(\xi,t) \,, \tag{16.2-19}$$

$$\tau_{p,q}^{\mathrm{s;B};P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{\mathrm{s;B};P,\infty}) + 2\mu(\xi_k v_k^{\mathrm{s;B};P,\infty})\xi_p\xi_q\right], \tag{16.2-20}$$

with

$$\Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};P,\infty}(\xi,t) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial f_k^{\mathrm{s;B}}(x, t + \xi_s x_s/c_P)\,\mathrm{d}A \,, \tag{16.2-21}$$

$$\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};P,\infty}(\xi,t) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial h_{i,j}^{\mathrm{s;B}}(x, t + \xi_s x_s/c_P)\,\mathrm{d}A \,, \tag{16.2-22}$$

and

$$v_r^{\mathrm{s;B};S,\infty}(\xi,t) = \rho^{-1}\partial_t \Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};S,\infty}(\xi,t) + (\rho c_S)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};S,\infty}(\xi,t) \,, \tag{16.2-23}$$

$$\tau_{p,q}^{\mathrm{s;B};S,\infty} = -c_S^{-1}\mu(\xi_p v_q^{\mathrm{s;B};S,\infty} + \xi_q v_p^{\mathrm{s;B};S,\infty}) \,, \tag{16.2-24}$$

with

$$\Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial f_k^{\mathrm{s;B}}(x, t + \xi_s x_s/c_S)\,\mathrm{d}A \,, \tag{16.2-25}$$

$$\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial h_{i,j}^{\mathrm{s;B}}(x, t + \xi_s x_s/c_S)\,\mathrm{d}A \,, \tag{16.2-26}$$

If the scatterer is penetrable, its elastodynamic properties in state B are assumed to be the adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equation (16.1-3) or Equation (16.1-4) applies. These boundary conditions apply to both state A and state B, and are, therefore, self-adjoint.

To establish the desired reciprocity relation, we first apply the time-domain reciprocity theorem of the time convolution type Equation (15.2-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{A}}, v_r^{\mathrm{B}}; x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{B}}, v_r^{\mathrm{A}}; x,t)\right]\mathrm{d}A = 0 \,, \tag{16.2-27}$$

since in the interior of the scatterer the total wave fields are source-free. For an impenetrable scatterer, Equation (16.2-27) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\mathrm{s}}$ via $\mathcal{D}^{\mathrm{s}'}$. In Equation (16.2-27) we substitute

$$\{\tau^{\mathrm{A}}_{p,q},v^{\mathrm{A}}_r\} = \{\tau^{\mathrm{i;A}}_{p,q} + \tau^{\mathrm{s;A}}_{p,q}, v^{\mathrm{i;A}}_r + v^{\mathrm{s;A}}_r\} \tag{16.2-28}$$

and

$$\{\tau^{\mathrm{B}}_{p,q},v^{\mathrm{B}}_r\} = \{\tau^{\mathrm{i;B}}_{p,q} + \tau^{\mathrm{s;B}}_{p,q}, v^{\mathrm{i;B}}_r + v^{\mathrm{s;B}}_r\}\ . \tag{16.2-29}$$

Next, the time-domain reciprocity theorem of the time convolution type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^{\mathrm{s}}$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is self-adjoint in its elastodynamic properties, this leads to

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{i;A}}_{p,q},v^{\mathrm{i;B}}_r;x,t) - \mathrm{C}_t(-\tau^{\mathrm{i;B}}_{p,q},v^{\mathrm{i;A}}_r;x,t)\right]\mathrm{d}A = 0\ . \tag{16.2-30}$$

Finally, the time-domain reciprocity theorem of the time convolution type is applied to the scattered wave fields in the states A and B and to the domain $\mathcal{D}^{\mathrm{s}'}$. Since the embedding is self-adjoint in its elastodynamic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},v^{\mathrm{s;B}}_r;x,t) - \mathrm{C}_t(-\tau^{\mathrm{s;B}}_{p,q},v^{\mathrm{s;A}}_r;x,t)\right]\mathrm{d}A = 0\ . \tag{16.2-31}$$

From Equations (16.2-27)–(16.2-31) we conclude that

$$\begin{aligned}\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\Big[&\mathrm{C}_t(-\tau^{\mathrm{i;A}}_{p,q},v^{\mathrm{s;B}}_r;x,t) + \mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},v^{\mathrm{i;B}}_r;x,t)\\ &- \mathrm{C}_t(-\tau^{\mathrm{i;B}}_{p,q},v^{\mathrm{s;A}}_r;x,t) - \mathrm{C}_t(-\tau^{\mathrm{s;B}}_{p,q},v^{\mathrm{i;A}}_r;x,t)\Big]\mathrm{d}A = 0\ .\end{aligned} \tag{16.2-32}$$

Equation (16.2-32) holds for both incident P- and incident S-waves. The ensuing reciprocity properties have to be discussed for the two types of incident waves separately.

Two incident P-waves

In the case of two incident P-waves we take $\{\tau^{\mathrm{i;A}}_{p,q},v^{\mathrm{i;A}}_r\} = \{\tau^{\mathrm{i;A};P}_{p,q},v^{\mathrm{i;A};P}_r\}$ (Equations (16.2-1) and (16.2-2)) and $\{\tau^{\mathrm{i;B}}_{p,q},v^{\mathrm{i;B}}_r\} = \{\tau^{\mathrm{i;B};P}_{p,q},v^{\mathrm{i;B};P}_r\}$ (Equations (16.2-14) and (16.2-15)). Then, on account of Equations (16.2-6)–(16.2-13) we have

$$\begin{aligned}&\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},v^{\mathrm{i;B};P}_r;x,t) - \mathrm{C}_t(-\tau^{\mathrm{i;B};P}_{p,q},v^{\mathrm{s;A}}_r;x,t)\right]\mathrm{d}A\\ &= \Delta^+_{m,r,p,q}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau^{\mathrm{s;A}}_{p,q}(x,t')V^{\mathrm{B};P}_r + T^{\mathrm{B};P}_{p,q}v^{\mathrm{s;A}}_r(x,t')\right]b^P(t-\beta^P_s x_s/c_P - t')\,\mathrm{d}A\\ &= \int_{t''\in\mathcal{R}} b^P(t-t'')\,\mathrm{d}t''\,\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau^{\mathrm{s;A}}_{p,q}(x,t''-\beta^P_s x_s/c_P)V^{\mathrm{B};P}_r\right.\end{aligned}$$

$$+ T_{p,q}^{\mathrm{B};P} v_r^{\mathrm{s;A}}(x,t'' - \beta_s^P x_s/c_P)\Big] \mathrm{d}A$$

$$= \rho V_r^{\mathrm{B};P} \int_{t''\in\mathcal{R}} b^P(t - t'')\mathrm{I}_t v_r^{\mathrm{s;A};P,\infty}(-\boldsymbol{\beta}^P, t'')\,\mathrm{d}t'' \tag{16.2-33}$$

and on account of Equations (16.2-19)–(16.2-26)

$$\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m \Big[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{s;B}}, v_r^{\mathrm{i;A};P}; x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{i;A};P}, v_r^{\mathrm{s;B}}; x,t)\Big] \mathrm{d}A$$

$$= \Delta_{m,r,p,q}^{+} \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x\in\partial\mathcal{D}^s} \nu_m \Big[-\tau_{p,q}^{\mathrm{s;B}}(x,t')V_r^{\mathrm{A};P} + T_{p,q}^{\mathrm{A};P} v_r^{\mathrm{s;B}}(x,t')\Big] a^P(t - \alpha_s^P x_s/c_P - t')\,\mathrm{d}A$$

$$= \int_{t''\in\mathcal{R}} a^P(t - t'')\,\mathrm{d}t''\, \Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m \Big[-\tau_{p,q}^{\mathrm{s;B}}(x,t'' - \alpha_s^P x_s/c_P)V_r^{\mathrm{A};P}$$

$$+ T_{p,q}^{\mathrm{A};P} v_r^{\mathrm{s;B}}(x,t'' - \alpha_s^P x_s/c_P)\Big] \mathrm{d}A$$

$$= \rho V_r^{\mathrm{A};P} \int_{t''\in\mathcal{R}} a^P(t - t'')\mathrm{I}_t v_r^{\mathrm{s;B};P,\infty}(-\boldsymbol{\alpha}^P, t'')\,\mathrm{d}t''\,. \tag{16.2-34}$$

Equations (16.2-32), (16.2-33) and (16.2-34) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$V_r^{\mathrm{B};P} \int_{t''\in\mathcal{R}} b^P(t - t'')\mathrm{I}_t v_r^{\mathrm{s;A};P,\infty}(-\boldsymbol{\beta}^P, t'')\,\mathrm{d}t''$$

$$= V_r^{\mathrm{A};P} \int_{t''\in\mathcal{R}} a^P(t - t'')\mathrm{I}_t v_r^{\mathrm{s;B};P,\infty}(-\boldsymbol{\alpha}^P, t'')\,\mathrm{d}t''\,. \tag{16.2-35}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered P-wave amplitude and the incident P-wave amplitude, both in state A and state B. To this end, we write

$$v_r^{\mathrm{s;A};P,\infty}(\boldsymbol{\xi},t) = V_k^{\mathrm{A};P} \int_{t'\in\mathcal{R}} a^P(t')S_{r,k}^{\mathrm{A};P,P}(\boldsymbol{\xi},\boldsymbol{\alpha}^P, t - t')\,\mathrm{d}t' \tag{16.2-36}$$

and

$$v_k^{\mathrm{s;B};P,\infty}(\boldsymbol{\xi},t) = V_r^{\mathrm{B};P} \int_{t'\in\mathcal{R}} b^P(t')S_{k,r}^{\mathrm{B};P,P}(\boldsymbol{\xi},\boldsymbol{\beta}^P, t - t')\,\mathrm{d}t'\,, \tag{16.2-37}$$

where $S_{r,k}^{\mathrm{A};P,P}$ and $S_{k,r}^{\mathrm{B};P,P}$ are the configurational time-domain particle velocity *far-field $P \to P$ scattering tensors*. Substitution of Equations (16.2-36) and (16.2-37) in Equation (16.2-35) and rewriting the convolutions, we obtain

$$V_r^{\mathrm{B};P} V_k^{\mathrm{A};P} \mathrm{I}_t \int_{t''\in\mathcal{R}} b^P(t'')\,\mathrm{d}t'' \int_{t'\in\mathcal{R}} a^P(t')S_{r,k}^{\mathrm{A};P,P}(-\boldsymbol{\beta}^P,\boldsymbol{\alpha}^P, t - t'' - t')\,\mathrm{d}t'$$

$$= V_k^{\mathrm{A};P} V_r^{\mathrm{B};P} \mathrm{I}_t \int_{t''\in\mathcal{R}} a^P(t'')\,\mathrm{d}t'' \int_{t'\in\mathcal{R}} b^P(t')S_{k,r}^{\mathrm{B};P,P}(-\boldsymbol{\alpha}^P,\boldsymbol{\beta}^P, t - t'' - t')\,\mathrm{d}t'\,, \tag{16.2-38}$$

where, in accordance with the rules applying to the time convolution, the operator I_t has been brought in front of the integral signs. Taking into account that Equation (16.2-38) has to hold for arbitrary values of $V_k^{\mathrm{A};P}$, $V_r^{\mathrm{B};P}$, $a^P(t)$ and $b^P(t)$, and using the causality of the scattered waves, we end up with

$$S_{r,k}^{\mathrm{A};P,P}(-\beta^P,\alpha^P,t) = S_{k,r}^{\mathrm{B};P,P}(-\alpha^P,\beta^P,t) \tag{16.2-39}$$

as the final expression of the time-domain reciprocity property under consideration.

Two incident S-waves

In the case of two incident S-waves we take $\{\tau_{p,q}^{\mathrm{i;A}},v_r^{\mathrm{i;A}}\} = \{\tau_{p,q}^{\mathrm{i;A};S},v_r^{\mathrm{i;A};S}\}$ (Equations (16.2-3) and (16.2-4)) and $\{\tau_{p,q}^{\mathrm{i;B}},v_r^{\mathrm{i;B}}\} = \{\tau_{p,q}^{\mathrm{i;B};S},v_r^{\mathrm{i;B};S}\}$ (Equations (16.2-16) and (16.2-17)). Then, on account of Equations (16.2-6)–(16.2-13) we have

$$\begin{aligned}
&\Delta_{m,r,p,q}^{+}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{s;A}},v_r^{\mathrm{i;B};S};x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{i;B};S},v_r^{\mathrm{s;A}};x,t)\right]\mathrm{d}A \\
&= \Delta_{m,r,p,q}^{+}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau_{p,q}^{\mathrm{s;A}}(x,t')V_r^{\mathrm{B};S} + T_{p,q}^{\mathrm{B};S}v_r^{\mathrm{s;A}}(x,t')\right]b^S(t-\beta_s^S x_s/c_S - t')\,\mathrm{d}A \\
&= \int_{t''\in\mathcal{R}} b^S(t-t'')\,\mathrm{d}t''\,\Delta_{m,r,p,q}^{+}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau_{p,q}^{\mathrm{s;A}}(x,t''-\beta_s^S x_s/c_S)V_r^{\mathrm{B};S}\right. \\
&\quad \left. + T_{p,q}^{\mathrm{B};S}v_r^{\mathrm{s;A}}(x,t''-\beta_s^S x_s/c_S)\right]\mathrm{d}A \\
&= \rho V_r^{\mathrm{B};S}\int_{t''\in\mathcal{R}} b^S(t-t'')\mathrm{I}_t v_r^{\mathrm{s;A};S,\infty}(-\beta^S,t'')\,\mathrm{d}t''
\end{aligned} \tag{16.2-40}$$

and on account of Equations (16.2-19)–(16.2-26)

$$\begin{aligned}
&\Delta_{m,r,p,q}^{+}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{s;B}},v_r^{\mathrm{i;A};S};x,t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{i;A};S},v_r^{\mathrm{s;B}};x,t)\right]\mathrm{d}A \\
&= \Delta_{m,r,p,q}^{+}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau_{p,q}^{\mathrm{s;B}}(x,t')V_r^{\mathrm{A};S} + T_{p,q}^{\mathrm{A};S}v_r^{\mathrm{s;B}}(x,t')\right]a^S(t-\alpha_s^S x_s/c_S - t')\,\mathrm{d}A \\
&= \int_{t''\in\mathcal{R}} a^S(t-t'')\,\mathrm{d}t''\,\Delta_{m,r,p,q}^{+}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau_{p,q}^{\mathrm{s;B}}(x,t''-\alpha_s^S x_s/c_S)V_r^{\mathrm{A};S}\right. \\
&\quad \left. + T_{p,q}^{\mathrm{A};S}v_r^{\mathrm{s;B}}(x,t''-\alpha_s^S x_s/c_S)\right]\mathrm{d}A \\
&= \rho V_r^{\mathrm{A};S}\int_{t''\in\mathcal{R}} a^S(t-t'')\mathrm{I}_t v_r^{\mathrm{s;B};S,\infty}(-\alpha^S,t'')\,\mathrm{d}t''\,.
\end{aligned} \tag{16.2-41}$$

Equations (16.2-32), (16.2-40) and (16.2-41) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$V_r^{\mathrm{B};S} \int_{t''\in\mathcal{R}} b^S(t-t'')\mathrm{I}_t v_r^{\mathrm{s;A};S,\infty}(-\boldsymbol{\beta}^S,t'')\,\mathrm{d}t''$$

$$= V_r^{\mathrm{A};S} \int_{t''\in\mathcal{R}} a^S(t-t'')\mathrm{I}_t v_r^{\mathrm{s;B};S,\infty}(-\boldsymbol{\alpha}^S,t'')\,\mathrm{d}t'' . \tag{16.2-42}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered S-wave amplitude and the incident S-wave amplitude, both in state A and state B. To this end, we write

$$v_r^{\mathrm{s;A};S,\infty}(\boldsymbol{\xi},t) = V_k^{\mathrm{A};S} \int_{t'\in\mathcal{R}} a^S(t')S_{r,k}^{\mathrm{A};S,S}(\boldsymbol{\xi},\boldsymbol{\alpha}^S,t-t')\,\mathrm{d}t' \tag{16.2-43}$$

and

$$v_k^{\mathrm{s;B};S,\infty}(\boldsymbol{\xi},t) = V_r^{\mathrm{B};S} \int_{t'\in\mathcal{R}} b^S(t')S_{k,r}^{\mathrm{B};S,S}(\boldsymbol{\xi},\boldsymbol{\beta}^S,t-t')\,\mathrm{d}t' , \tag{16.2-44}$$

where $S_{r,k}^{\mathrm{A};S,S}$ and $S_{k,r}^{\mathrm{B};S,S}$ are the configurational time-domain particle velocity *far-field $S\to S$ scattering tensors*. Substituting Equations (16.2-43) and (16.2-44) in Equation (16.2-42) and rewriting the convolutions, we obtain

$$V_r^{\mathrm{B};S}V_k^{\mathrm{A};S}\mathrm{I}_t \int_{t''\in\mathcal{R}} b^S(t'')\,\mathrm{d}t'' \int_{t'\in\mathcal{R}} a^S(t')S_{r,k}^{\mathrm{A};S,S}(-\boldsymbol{\beta}^S,\boldsymbol{\alpha}^S,t-t''-t')\,\mathrm{d}t'$$

$$= V_k^{\mathrm{A};S}V_r^{\mathrm{B};S}\mathrm{I}_t \int_{t''\in\mathcal{R}} a^S(t'')\,\mathrm{d}t'' \int_{t'\in\mathcal{R}} b^S(t')S_{k,r}^{\mathrm{B};S,S}(-\boldsymbol{\alpha}^S,\boldsymbol{\beta}^S,t-t''-t')\,\mathrm{d}t' , \tag{16.2-45}$$

where, in accordance with the rules applying to the time convolution, the operator I_t has been brought in front of the integral signs. Taking into account that Equation (16.2-45) has to hold for arbitrary values of $V_k^{\mathrm{A};S}$, $V_r^{\mathrm{B};S}$, $a^S(t)$ and $b^S(t)$, and using the causality of the scattered waves, we end up with

$$S_{r,k}^{\mathrm{A};S,S}(-\boldsymbol{\beta}^S,\boldsymbol{\alpha}^S,t) = S_{k,r}^{\mathrm{B};S,S}(-\boldsymbol{\alpha}^S,\boldsymbol{\beta}^S,t) \tag{16.2-46}$$

as the final expression of the time-domain reciprocity property under consideration.

An incident P-wave and an incident S-wave

In the case of an incident P-wave and an incident S-wave we take $\{\tau_{p,q}^{\mathrm{i;A}},v_r^{\mathrm{i;A}}\} = \{\tau_{p,q}^{\mathrm{i;A};P},v_r^{\mathrm{i;A};P}\}$ (Equations (16.2-1) and (16.2-2)) and $\{\tau_{p,q}^{\mathrm{i;B}},v_r^{\mathrm{i;B}}\} = \{\tau_{p,q}^{\mathrm{i;B};S},v_r^{\mathrm{i;B};S}\}$ (Equations (16.2-16) and (16.2-17)). Then, on account of Equations (16.2-6)–(16.2-13) we have

$$\Delta_{m,r,p,q}^{+} \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{s;A}},v_r^{\mathrm{i;B};S};\boldsymbol{x},t) - \mathrm{C}_t(-\tau_{p,q}^{\mathrm{i;B};S},v_r^{\mathrm{s;A}};\boldsymbol{x},t)\right]\mathrm{d}A$$

$$= \Delta_{m,r,p,q}^{+} \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau_{p,q}^{\mathrm{s;A}}(\boldsymbol{x},t')V_r^{\mathrm{B};S} + T_{p,q}^{\mathrm{B};S}v_r^{\mathrm{s;A}}(\boldsymbol{x},t')\right] b^S(t-\beta_s^S x_s/c_S - t')\,\mathrm{d}A$$

$$= \int_{t'' \in \mathcal{R}} b^S(t-t'')\,\mathrm{d}t''\,\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \Big[-\tau^{s;A}_{p,q}(x,t'' - \beta^S_s x_s/c_S) V^{B;S}_r$$

$$+ T^{B;S}_{p,q} v^{s;A}_r(x,t'' - \beta^S_s x_s/c_S) \Big]\,\mathrm{d}A$$

$$= \rho V^{B;S}_r \int_{t'' \in \mathcal{R}} b^S(t-t'')\mathrm{I}_t v^{s;A;S,\infty}_r(-\beta^S, t'')\,\mathrm{d}t'' \tag{16.2-47}$$

and on account of Equations (16.2-19)–(16.2-26)

$$\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \Big[\mathrm{C}_t(-\tau^{s;B}_{p,q}, v^{i;A;P}_r; x,t) - \mathrm{C}_t(-\tau^{i;A;P}_{p,q}, v^{s;B}_r; x,t) \Big]\,\mathrm{d}A$$

$$= \Delta^+_{m,r,p,q} \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{x \in \partial\mathcal{D}^s} \nu_m \Big[-\tau^{s;B}_{p,q}(x,t') V^{A;P}_r + T^{A;P}_{p,q} v^{s;B}_r(x,t') \Big]\, a^P(t - \alpha^P_s x_s/c_P - t')\,\mathrm{d}A$$

$$= \int_{t'' \in \mathcal{R}} a^P(t-t'')\,\mathrm{d}t''\,\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \Big[-\tau^{s;B}_{p,q}(x,t'' - \alpha^P_s x_s/c_P) V^{A;P}_r$$

$$+ T^{A;P}_{p,q} v^{s;B}_r(x,t'' - \alpha^P_s x_s/c_P) \Big]\,\mathrm{d}A$$

$$= \rho V^{A;P}_r \int_{t'' \in \mathcal{R}} a^P(t-t'')\mathrm{I}_t v^{s;B;P,\infty}_r(-\alpha^P, t'')\,\mathrm{d}t'' \,. \tag{16.2-48}$$

Equations (16.2-32), (16.2-47) and (16.2-48) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$V^{B;S}_r \int_{t'' \in \mathcal{R}} b^S(t-t'')\mathrm{I}_t v^{s;A;S,\infty}_r(-\beta^S, t'')\,\mathrm{d}t''$$

$$= V^{A;P}_r \int_{t'' \in \mathcal{R}} a^P(t-t'')\mathrm{I}_t v^{s;B;P,\infty}_r(-\alpha^P, t'')\,\mathrm{d}t'' \,. \tag{16.2-49}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered S-wave amplitude and the incident P-wave amplitude in state A and the far-field scattered P-wave amplitude and the incident S-wave amplitude in state B. To this end, we write

$$v^{s;A;S,\infty}_r(\xi,t) = V^{A;P}_k \int_{t' \in \mathcal{R}} a^P(t') S^{A;S,P}_{r,k}(\xi, \alpha^P, t-t')\,\mathrm{d}t' \tag{16.2-50}$$

and

$$v^{s;B;P,\infty}_k(\xi,t) = V^{B;S}_r \int_{t' \in \mathcal{R}} b^S(t') S^{B;P,S}_{k,r}(\xi, \beta^S, t-t')\,\mathrm{d}t' \,, \tag{16.2-51}$$

where $S^{A;S,P}_{r,k}$ and $S^{B;P,S}_{k,r}$ are the configurational time-domain particle velocity *far-field P→S and S→P scattering tensors*, respectively. Substituting Equations (16.2-50) and (16.2-51) in Equation (16.2-49) and rewriting the convolutions, we obtain

$$V_r^{\mathrm{B};S} V_k^{\mathrm{A};P} \mathrm{I}_t \int_{t'' \in \mathcal{R}} b^S(t'')\,\mathrm{d}t'' \int_{t' \in \mathcal{R}} a^P(t') S_{r,k}^{\mathrm{A};S,P}(-\boldsymbol{\beta}^S, \boldsymbol{\alpha}^P, t - t'' - t'))\,\mathrm{d}t'$$

$$= V_k^{\mathrm{A};P} V_r^{\mathrm{B};S} \mathrm{I}_t \int_{t'' \in \mathcal{R}} a^P(t'')\,\mathrm{d}t'' \int_{t' \in \mathcal{R}} b^S(t') S_{k,r}^{\mathrm{B};P,S}(-\boldsymbol{\alpha}^P, \boldsymbol{\beta}^S, t - t'' - t'))\,\mathrm{d}t' , \qquad (16.2\text{-}52)$$

where, in accordance with the rules applying to the time convolution, the operator I_t has been brought in front of the integral signs. Taking into account that Equation (16.2-52) has to hold for arbitrary values of $V_r^{\mathrm{A};P}$, $V_r^{\mathrm{B};S}$, $a^P(t)$ and $b^S(t)$, and using the causality of the scattered waves, we end up with

$$S_{r,k}^{\mathrm{A};S,P}(-\boldsymbol{\beta}^S, \boldsymbol{\alpha}^P, t) = S_{k,r}^{\mathrm{B};P,S}(-\boldsymbol{\alpha}^P, \boldsymbol{\beta}^S, t) \qquad (16.2\text{-}53)$$

as the final expression of the time-domain reciprocity property under consideration.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the incident wave in state A is taken either as the uniform plane P-wave

$$\{\hat{\tau}_{p,q}^{\mathrm{i};\mathrm{A};P}, \hat{v}_r^{\mathrm{i};\mathrm{A};P}\} = \{T_{p,q}^{\mathrm{A};P}, V_r^{\mathrm{A};P}\}\hat{a}^P(s)\exp(-s\alpha_s^P x_s/c_P) , \qquad (16.2\text{-}54)$$

with

$$T_{p,q}^{\mathrm{A};P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\alpha_k^P V_k^{\mathrm{A};P}) + 2\mu(\alpha_k^P V_k^{\mathrm{A};P})\alpha_p^P \alpha_q^P\right], \qquad (16.2\text{-}55)$$

or as the uniform plane S-wave

$$\{\hat{\tau}_{p,q}^{\mathrm{i};\mathrm{A};S}, \hat{v}_r^{\mathrm{i};\mathrm{A};S}\} = \{T_{p,q}^{\mathrm{A};S}, V_r^{\mathrm{A};S}\}\hat{a}^S(s)\exp(-s\alpha_s^S x_s/c_S) , \qquad (16.2\text{-}56)$$

with

$$T_{p,q}^{\mathrm{A};S} = -c_S^{-1}\mu(\alpha_p^S V_q^{\mathrm{A};S} + \alpha_q^S V_p^{\mathrm{A};S}) . \qquad (16.2\text{-}57)$$

In the far-field region, the scattered wave in state A is represented as

$$\{\hat{\tau}_{p,q}^{\mathrm{s};\mathrm{A}}, \hat{v}_r^{\mathrm{s};\mathrm{A}}\}(\boldsymbol{x}', s) = \left[\{\hat{\tau}_{p,q}^{\mathrm{s};\mathrm{A};P,\infty}, \hat{v}_r^{\mathrm{s};\mathrm{A};P,\infty}\}(\boldsymbol{\xi}, s)\frac{\exp[-s|\boldsymbol{x}'|/c_P]}{4\pi c_P^2|\boldsymbol{x}'|}\right.$$

$$\left. + \{\hat{\tau}_{p,q}^{\mathrm{s};\mathrm{A};S,\infty}, \hat{v}_r^{\mathrm{s};\mathrm{A};S,\infty}\}(\boldsymbol{\xi}, s)\frac{\exp[-s|\boldsymbol{x}'|/c_S]}{4\pi c_S^2|\boldsymbol{x}'|}\right]$$

$$\times[1 + O(|\boldsymbol{x}'|^{-1})] \quad \text{as} \quad |\boldsymbol{x}'| \to \infty \quad \text{with} \quad \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi} , \qquad (16.2\text{-}58)$$

in which, on account of Equations (16.1-76), (16.1-77) and (16.1-80)–(16.1-87) (note that the surface source representation for the far-field scattered wave is used),

$$\hat{v}_r^{\mathrm{s};\mathrm{A};P,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^{\mathrm{s}};\mathrm{A};P,\infty} + s(\rho c_P)^{-1} C_{k,m,i,j}\xi_m \hat{\Phi}_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};P,\infty} , \qquad (16.2\text{-}59)$$

$$\hat{\tau}_{p,q}^{\mathrm{s};\mathrm{A};P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k \hat{v}_k^{\mathrm{s};\mathrm{A};P,\infty}) + 2\mu(\xi_k \hat{v}_k^{\mathrm{s};\mathrm{A};P,\infty})\xi_p \xi_q\right], \qquad (16.2\text{-}60)$$

with

$$\hat{\Phi}_r^{\partial f^s;A;P,\infty} = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_P)\partial\hat{f}_k^{s;A}(x,s)\,dA\,, \tag{16.2-61}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^s;A;P,\infty} = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_P)\partial\hat{h}_{i,j}^{s;A}(x,s)\,dA\,, \tag{16.2-62}$$

and

$$\hat{v}_r^{s;A;S,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^s;A;S,\infty} + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h^s;A;S,\infty}\,, \tag{16.2-63}$$

$$\hat{\tau}_{p,q}^{s;A;S,\infty} = -c_S^{-1}\mu(\xi_p\hat{v}_q^{s;A;S,\infty} + \xi_q\hat{v}_p^{s;A;S,\infty})\,, \tag{16.2-64}$$

with

$$\hat{\Phi}_{r,k,i,j}^{\partial f^s;A;S,\infty} = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_S)\partial\hat{f}_k^{s;A}(x,s)\,dA\,, \tag{16.2-65}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^s;A;S,\infty} = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \exp(s\xi_s x_s/c_S)\partial\hat{h}_{i,j}^{s;A}(x,s)\,dA\,. \tag{16.2-66}$$

Similarly, the incident wave in state B is taken either as the uniform plane P-wave

$$\{\hat{\tau}_{p,q}^{i;B;P},\hat{v}_r^{i;B;P}\} = \{T_{p,q}^{B;P},V_r^{B;P}\}\hat{b}^P(s)\exp(-s\beta_s^P x_s/c_P)\,, \tag{16.2-67}$$

with

$$T_{p,q}^{B;P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\beta_k^P V_k^{B;P}) + 2\mu(\beta_k^P V_k^{B;P})\beta_p^P\beta_q^P\right], \tag{16.2-68}$$

or as the uniform plane S-wave

$$\{\hat{\tau}_{p,q}^{i;B;S},\hat{v}_r^{i;B;S}\} = \{T_{p,q}^{B;S},V_r^{B;S}\}\hat{b}^S(s)\exp(-s\beta_s^S x_s/c_S)\,, \tag{16.2-69}$$

with

$$T_{p,q}^{B;S} = -c_S^{-1}\mu(\beta_p^S V_q^{B;S}) + \beta_q^S V_p^{B;S})\,. \tag{16.2-70}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\hat{\tau}_{p,q}^{s;B},\hat{v}_r^{s;B}\}(x',s) = \left[\{\hat{\tau}_{p,q}^{s;B;P,\infty},\hat{v}_r^{s;B;P,\infty}\}(\xi,s)\frac{\exp[-s|x'|/c_P]}{4\pi c_P^2|x'|} + \{\hat{\tau}_{p,q}^{s;B;S,\infty},\hat{v}_r^{s;B;S,\infty}\}(\xi,s)\frac{\exp[-s|x'|/c_S]}{4\pi c_S^2|x'|}\right] \times[1+O(|x'|^{-1})] \quad \text{as } |x'|\to\infty \quad \text{with } x'=|x'|\xi\,, \tag{16.2-71}$$

in which, on account of Equations (16.1-76), (16.1-77) and (16.1-80)–(16.1-87) (note that the surface source representation for the far-field scattered wave is used),

$$\hat{v}_r^{s;B;P,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^s;B;P,\infty} + s(\rho c_P)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h^s;B;P,\infty}\,, \tag{16.2-72}$$

$$\hat{\tau}_{p,q}^{s;B;P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k\hat{v}_k^{s;B;P,\infty}) + 2\mu(\xi_k\hat{v}_k^{s;B;P,\infty})\xi_p\xi_q\right], \tag{16.2-73}$$

with

$$\hat{\Phi}_r^{\partial f^{\rm s};{\rm B};P,\infty} = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^{\rm s}} \exp(s\xi_s x_s/c_P)\partial\hat{f}_k^{\rm s;B}(x,s)\,{\rm d}A\,, \tag{16.2-74}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^{\rm s};{\rm B};P,\infty} = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^{\rm s}} \exp(s\xi_s x_s/c_P)\partial\hat{h}_{i,j}^{\rm s;B}(x,s)\,{\rm d}A\,, \tag{16.2-75}$$

and

$$\hat{v}_r^{\rm s;B;\mathit{S},\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^{\rm s};{\rm B};S,\infty} + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h^{\rm s};{\rm B};S,\infty}\,, \tag{16.2-76}$$

$$\hat{\tau}_{p,q}^{\rm s;B;\mathit{S},\infty} = -c_S^{-1}\mu(\xi_p\hat{v}_q^{\rm s;B;\mathit{S},\infty} + \xi_q\hat{v}_p^{\rm s;B;\mathit{S},\infty})\,, \tag{16.2-77}$$

with

$$\hat{\Phi}_{r,k,i,j}^{\partial f^{\rm s};{\rm B};S,\infty} = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^{\rm s}} \exp(s\xi_s x_s/c_S)\partial\hat{f}_k^{\rm s;B}(x,s)\,{\rm d}A\,, \tag{16.2-78}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^{\rm s};{\rm B};S,\infty} = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^{\rm s}} \exp(s\xi_s x_s/c_S)\partial\hat{h}_{i,j}^{\rm s;B}(x,s)\,{\rm d}A\,. \tag{16.2-79}$$

If the scatterer is penetrable, its elastodynamic properties in state B are assumed to be the adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equation (16.1-51) or Equation (16.1-52) applies. These boundary conditions apply to both state A and state B, and are, therefore, self-adjoint.

To establish the desired reciprocity relation, we first apply the complex frequency-domain reciprocity theorem of the time convolution type Equation (15.4-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\rm s}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\Delta^+_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\rm s}} \nu_m\left[-\hat{\tau}_{p,q}^{\rm A}(x,s)\hat{v}_r^{\rm B}(x,s) + \hat{\tau}_{p,q}^{\rm B}(x,s)\hat{v}_r^{\rm A}(x,s)\right]{\rm d}A = 0\,, \tag{16.2-80}$$

since in the interior of the scatterer the total wave fields are source-free. For an impenetrable scatterer, Equation (16.2-80) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\rm s}$ via $\mathcal{D}^{\rm s'}$. In Equation (16.2-80) we substitute

$$\{\hat{\tau}_{p,q}^{\rm A},\hat{v}_r^{\rm A}\} = \{\hat{\tau}_{p,q}^{\rm i;A} + \hat{\tau}_{p,q}^{\rm s;A},\hat{v}_r^{\rm i;A} + \hat{v}_r^{\rm s;A}\} \tag{16.2-81}$$

and

$$\{\hat{\tau}_{p,q}^{\rm B},\hat{v}_r^{\rm B}\} = \{\hat{\tau}_{p,q}^{\rm i;B} + \hat{\tau}_{p,q}^{\rm s;B},\hat{v}_r^{\rm i;B} + \hat{v}_r^{\rm s;B}\}\,. \tag{16.2-82}$$

Next, the complex frequency-domain reciprocity theorem of the time convolution type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^{\rm s}$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is self-adjoint in its elastodynamic properties, this leads to

$$\Delta^+_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\rm s}} \nu_m\left[-\hat{\tau}_{p,q}^{\rm i;A}(x,s)\hat{v}_r^{\rm i;B}(x,s) + \hat{\tau}_{p,q}^{\rm i;B}(x,s)\hat{v}_r^{\rm i;A}(x,s)\right]{\rm d}A = 0\,. \tag{16.2-83}$$

Finally, the complex frequency-domain reciprocity theorem of the time convolution type is applied to the scattered wave fields in the states A and B and to the domain $\mathcal{D}^{\rm s'}$. Since the embedding is self-adjoint in its elastodynamic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{s;B}_r(x,s) + \hat{\tau}^{s;B}_{p,q}(x,s)\hat{v}^{s;A}_r(x,s) \right] \mathrm{d}A = 0 \,. \tag{16.2-84}$$

From Equations (16.2-80)–(16.2-84) we conclude that

$$\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \left[-\hat{\tau}^{i;A}_{p,q}(x,s)\hat{v}^{s;B}_r(x,s) - \hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{i;B}_r(x,s) \right.$$
$$\left. + \hat{\tau}^{i;B}_{p,q}(x,s)\hat{v}^{s;A}_r(x,s) + \hat{\tau}^{s;B}_{p,q}(x,s)\hat{v}^{i;A}_r(x,s) \right] \mathrm{d}A = 0 \,. \tag{16.2-85}$$

Equation (16.2-85) holds for both incident P-waves and incident S-waves. The ensuing reciprocity properties have to be discussed for the two types of incident waves separately.

Two incident P-waves

In the case of two incident P-waves we take $\{\hat{\tau}^{i;A}_{p,q},\hat{v}^{i;A}_r\} = \{\hat{\tau}^{i;A;P}_{p,q},\hat{v}^{i;A;P}_r\}$ (Equations (16.2-54) and (16.2-55)) and $\{\hat{\tau}^{i;B}_{p,q},\hat{v}^{i;B}_r\} = \{\hat{\tau}^{i;B;P}_{p,q},\hat{v}^{i;B;P}_r\}$ (Equations (16.2-67) and (16.2-68)). Then, on account of Equations (16.2-59)–(16.2-66) we have

$$\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{i;B;P}_r(x,s) + \hat{\tau}^{i;B;P}_{p,q}(x,s)\hat{v}^{s;A}_r(x,s) \right] \mathrm{d}A$$
$$= \Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;A}_{p,q}(x,s) V^{B;P}_r + T^{B;P}_{p,q}\hat{v}^{s;A}_r(x,s) \right] \hat{b}^P(s) \exp(-s\beta^P_s x_s/c_P)\, \mathrm{d}A$$
$$= s^{-1}\rho V^{B;P}_r \hat{b}^P(s)\hat{v}^{s;A;P,\infty}_r(-\beta^P,s) \tag{16.2-86}$$

and on account of Equations (16.2-72)–(16.2-79)

$$\Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;B}_{p,q}(x,s)\hat{v}^{i;A;P}_r(x,s) + \hat{\tau}^{i;A;P}_{p,q}(x,s)\hat{v}^{s;B}_r(x,s) \right] \mathrm{d}A$$
$$= \Delta^+_{m,r,p,q} \int_{x \in \partial\mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;B}_{p,q}(x,s) V^{A;P}_r + T^{A;P}_{p,q}\hat{v}^{s;B}_r(x,s) \right] \hat{a}^P(s) \exp(-s\alpha^P_s x_s/c_P)\, \mathrm{d}A$$
$$= s^{-1}\rho V^{A;P}_r \hat{a}^P(s)\hat{v}^{s;B;P,\infty}_r(-\alpha^P,s) \,. \tag{16.2-87}$$

Equations (16.2-85)–(16.2-87) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$V^{B;P}_r \hat{b}^P(s)\hat{v}^{s;A;P,\infty}_r(-\beta^P,s) = V^{A;P}_r \hat{a}^P(s)\hat{v}^{s;B;P,\infty}_r(-\alpha^P,s) \,. \tag{16.2-88}$$

At this point it is, again, elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident P-wave amplitude, both in state A and in state B. To this end, we write, in accordance with Equations (16.2-36) and (16.2-37)

$$\hat{v}^{s;A;P,\infty}_r(\xi,s) = V^{A;P}_k \hat{a}^P(s)\hat{S}^{A;P,P}_{r,k}(\xi,\alpha^P,s) \tag{16.2-89}$$

and

$$\hat{v}_k^{\mathrm{s;B};P,\infty}(\boldsymbol{\xi},s) = V_r^{\mathrm{B};P}\hat{b}^P(s)\hat{S}_{k,r}^{\mathrm{B};P,P}(\boldsymbol{\xi},\boldsymbol{\beta}^P,s) \qquad (16.2\text{-}90)$$

where $\hat{S}_{r,k}^{\mathrm{A};P,P}$ and $\hat{S}_{k,r}^{\mathrm{B};P,P}$ are the configurational complex frequency-domain particle velocity *far-field scattering tensors*. Substitution of Equations (16.2-89) and (16.2-90) in Equation (16.2-88) yields

$$V_r^{\mathrm{B};P}V_k^{\mathrm{A};P}\hat{b}^P(s)\hat{a}^P(s)\hat{S}_{r,k}^{\mathrm{A};P,P}(-\boldsymbol{\beta}^P,\boldsymbol{\alpha}^P,s)$$
$$= V_k^{\mathrm{A};P}V_r^{\mathrm{B};P}\hat{a}^P(s)\hat{b}^P(s)\hat{S}_{k,r}^{\mathrm{B};P,P}(-\boldsymbol{\alpha}^P,\boldsymbol{\beta}^P,s)\,. \qquad (16.2\text{-}91)$$

Taking into account that Equation (16.2-91) has to hold for arbitrary values of $V_k^{\mathrm{A};P}$, $V_r^{\mathrm{B};P}$, $\hat{a}^P(s)$ and $\hat{b}^P(s)$, we end up with

$$\hat{S}_{r,k}^{\mathrm{A};P,P}(-\boldsymbol{\beta}^P,\boldsymbol{\alpha}^P,s) = \hat{S}_{k,r}^{\mathrm{B};P,P}(-\boldsymbol{\alpha}^P,\boldsymbol{\beta}^P,s) \qquad (16.2\text{-}92)$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

Two incident S-waves

In the case of two incident S-waves we take $\{\hat{\tau}_{p,q}^{\mathrm{i;A}},\hat{v}_r^{\mathrm{i;A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;A};S},\hat{v}_r^{\mathrm{i;A};S}\}$ (Equations (16.2-56) and (16.2-57)) and $\{\hat{\tau}_{p,q}^{\mathrm{i;B}},\hat{v}_r^{\mathrm{i;B}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;B};S},\hat{v}_r^{\mathrm{i;B};S}\}$ (Equations (16.2-69) and (16.2-70)). Then, on account of Equations (16.2-59)–(16.2-66) we have

$$\Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{i;B};S}(\boldsymbol{x},s) + \hat{\tau}_{p,q}^{\mathrm{i;B};S}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{s;A}}(\boldsymbol{x},s)\right]\mathrm{d}A$$
$$= \Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(\boldsymbol{x},s)V_r^{\mathrm{B};S} + T_{p,q}^{\mathrm{B};S}\hat{v}_r^{\mathrm{s;A}}(\boldsymbol{x},s)\right]\hat{b}^S(s)\exp(-s\beta_s^S x_s/c_S)\,\mathrm{d}A$$
$$= s^{-1}\rho V_r^{\mathrm{B};S}\hat{b}^S(s)\hat{v}_r^{\mathrm{s;A};S,\infty}(-\boldsymbol{\beta}^S,s) \qquad (16.2\text{-}93)$$

and on account of Equations (16.2-72)–(16.2-79)

$$\Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{i;A};S}(\boldsymbol{x},s) + \hat{\tau}_{p,q}^{\mathrm{i;A};S}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{s;B}}(\boldsymbol{x},s)\right]\mathrm{d}A$$
$$= \Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;B}}(\boldsymbol{x},s)V_r^{\mathrm{A};S} + T_{p,q}^{\mathrm{A};S}\hat{v}_r^{\mathrm{s;B}}(\boldsymbol{x},s)\right]\hat{a}^S(s)\exp(-s\alpha_s^S x_s/c_S)\,\mathrm{d}A$$
$$= s^{-1}\rho V_r^{\mathrm{A};S}\hat{a}^S(s)\hat{v}_r^{\mathrm{s;B};S,\infty}(-\boldsymbol{\alpha}^S,s)\,. \qquad (16.2\text{-}94)$$

Equations (16.2-85), (16.2-93) and (16.2-94) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$V_r^{\mathrm{B};S}\hat{b}^S(s)\hat{v}_r^{\mathrm{s;A};S,\infty}(-\boldsymbol{\beta}^S,s) = V_r^{\mathrm{A};S}\hat{a}^S(s)\hat{v}_r^{\mathrm{s;B};S,\infty}(-\boldsymbol{\alpha}^S,s)\,. \qquad (16.2\text{-}95)$$

At this point it is, again, elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident S-wave amplitude, both in state A and in state B. To this end, we write, in accordance with Equations (16.2-43) and (16.2-44)

$$\hat{v}_r^{\mathrm{s;A};S,\infty}(\boldsymbol{\xi},s) = V_k^{\mathrm{A};S}\hat{a}^S(s)\hat{S}_{r,k}^{\mathrm{A};S,S}(\boldsymbol{\xi},\boldsymbol{\alpha}^S,s) \tag{16.2-96}$$

and

$$\hat{v}_k^{\mathrm{s;B};S,\infty}(\boldsymbol{\xi},s) = V_r^{\mathrm{B};S}\hat{b}^S(s)\hat{S}_{k,r}^{\mathrm{B};S,S}(\boldsymbol{\xi},\boldsymbol{\beta}^S,s)\,, \tag{16.2-97}$$

where $\hat{S}_{r,k}^{\mathrm{A};S,S}$ and $\hat{S}_{k,r}^{\mathrm{B};S,S}$ are the configurational complex frequency-domain particle velocity *far-field* $S\to S$ *scattering tensors*. Substitution of Equations (16.2-96) and (16.2-97) in Equation (16.2-95) yields

$$V_r^{\mathrm{B};S}V_k^{\mathrm{A};S}\hat{b}^S(s)\hat{a}^S(s)\hat{S}_{r,k}^{\mathrm{A};S,S}(-\boldsymbol{\beta}^S,\boldsymbol{\alpha}^S,s)$$
$$= V_k^{\mathrm{A};S}V_r^{\mathrm{B};S}\hat{a}^S(s)\hat{b}^S(s)\hat{S}_{k,r}^{\mathrm{B};S,S}(-\boldsymbol{\alpha}^S,\boldsymbol{\beta}^S,s)\,. \tag{16.2-98}$$

Taking into account that Equation (16.2-98) has to hold for arbitrary values of $V_k^{\mathrm{A};S}$, $V_r^{\mathrm{B};S}$, $\hat{a}^S(s)$ and $\hat{b}^S(s)$, we end up with

$$\hat{S}_{r,k}^{\mathrm{A};S,S}(-\boldsymbol{\beta}^S,\boldsymbol{\alpha}^S,s) = \hat{S}_{k,r}^{\mathrm{B};S,S}(-\boldsymbol{\alpha}^S,\boldsymbol{\beta}^S,s) \tag{16.2-99}$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

An incident P-wave and an incident S-wave

In the case of an incident P-wave and an incident S-wave we take $\{\hat{\tau}_{p,q}^{\mathrm{i;A}},\hat{v}_r^{\mathrm{i;A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;A};P},\hat{v}_r^{\mathrm{i;A};P}\}$ (Equations (16.2-54) and (16.2-55)) and $\{\hat{\tau}_{p,q}^{\mathrm{i;B}},\hat{v}_r^{\mathrm{i;B}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;B};S},\hat{v}_r^{\mathrm{i;B};S}\}$ (Equations (16.2-69) and (16.2-70)). Then, on account of Equations (16.2-59)–(16.2-66) we have

$$\Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{i;B};S}(\boldsymbol{x},s) + \hat{\tau}_{p,q}^{\mathrm{i;B};S}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{s;A}}(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(\boldsymbol{x},s)V_r^{\mathrm{B};S} + T_{p,q}^{\mathrm{B};S}\hat{v}_r^{\mathrm{s;A}}(\boldsymbol{x},s)\right]\hat{b}^S(s)\exp(-s\beta_s^S x_s/c_S)\,\mathrm{d}A$$

$$= s^{-1}\rho V_r^{\mathrm{B};S}\hat{b}^S(s)\hat{v}_r^{\mathrm{s;A};S,\infty}(-\boldsymbol{\beta}^S,s) \tag{16.2-100}$$

and on account of Equations (16.2-72)–(16.2-79)

$$\Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;B}}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{i;A};P}(\boldsymbol{x},s) + \hat{\tau}_{p,q}^{\mathrm{i;A};P}(\boldsymbol{x},s)\hat{v}_r^{\mathrm{s;B}}(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \Delta_{m,r,p,q}^{+}\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{s;B}}(\boldsymbol{x},s)V_r^{\mathrm{A};P} + T_{p,q}^{\mathrm{A};P}\hat{v}_r^{\mathrm{s;B}}(\boldsymbol{x},s)\right]\hat{a}^P(s)\exp(-s\alpha_s^P x_s/c_P)\,\mathrm{d}A$$

$$= s^{-1}\rho V_r^{\mathrm{A};P}\hat{a}^P(s)\hat{v}_r^{\mathrm{s;B};P,\infty}(-\boldsymbol{\alpha}^P,s)\,. \tag{16.2-101}$$

Equations (16.2-85), (16.2-100) and (16.2-101) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$V_r^{\mathrm{B};S}\hat{b}^S(s)\hat{v}_r^{\mathrm{s;A};S,\infty}(-\beta^S,s) = V_r^{\mathrm{A};P}\hat{a}^P(s)\hat{v}_r^{\mathrm{s;B};P,\infty}(-\alpha^P,s)\,. \tag{16.2-102}$$

At this point it is, again, elegant to express the linear relationship that exists between the far-field scattered S-wave amplitude and the incident P-wave amplitude in state A and the far-field scattered P-wave amplitude and the incident S-wave amplitude in state B. To this end, we write, in accordance with Equations (16.2-50) and (16.2-51),

$$\hat{v}_r^{\mathrm{s;A};S,\infty}(\xi,s) = V_k^{\mathrm{A};P}\hat{a}^P(s)\hat{S}_{r,k}^{\mathrm{A};S,P}(\xi,\alpha^P,s) \tag{16.2-103}$$

and

$$\hat{v}_k^{\mathrm{s;B};P,\infty}(\xi,s) = V_r^{\mathrm{B};S}\hat{b}^S(s)\hat{S}_{k,r}^{\mathrm{B};P,S}(\xi,\beta^S,s)\,, \tag{16.2-104}$$

where $\hat{S}_{r,k}^{\mathrm{A};S,P}$ and $\hat{S}_{k,r}^{\mathrm{B};P,S}$ are the configurational complex frequency-domain particle velocity *far-field P→S and S→P scattering tensors*, respectively. Substitution of Equations (16.2-103) and (16.2-104) in Equation (16.2-102) yields

$$V_r^{\mathrm{B};S}V_k^{\mathrm{A};P}\hat{b}^S(s)\hat{a}^P(s)\hat{S}_{r,k}^{\mathrm{A};S,P}(-\beta^S,\alpha^P,s)$$
$$= V_r^{\mathrm{A};P}V_r^{\mathrm{B};S}\hat{a}^P(s)\hat{b}^S(s)\hat{S}_{k,r}^{\mathrm{B};P,S}(-\alpha^P,\beta^S,s)\,. \tag{16.2-105}$$

Taking into account that Equation (16.2-105) has to hold for arbitrary values of $V_k^{\mathrm{A};P}$, $V_r^{\mathrm{B};S}$, $\hat{a}^P(s)$ and $\hat{b}^S(s)$, we end up with

$$\hat{S}_{r,k}^{\mathrm{A};S,P}(-\beta^S,\alpha^P,s) = \hat{S}_{k,r}^{\mathrm{B};P,S}(-\alpha^P,\beta^S,s) \tag{16.2-106}$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

In a theoretical analysis, the reciprocity relations derived in this section serve as an important check on the correctness of the analytic solutions as well as on the accuracy of numerical solutions to scattering problems. Note, however, that the reciprocity relations are necessary conditions to be satisfied by the scattered wave field (in the far-field region), but their satisfaction does not guarantee the correctness of a total analytic solution nor the accuracy of a total numerical solution. In a physical experiment, the redundancy induced by the reciprocity relations can be exploited to reduce the influence of noise on the quality of the observed data.

References to the earlier literature on the reciprocity relations of the type discussed in this section can be found in Tan (1977).

Exercises

Exercise 16.2-1

Show by taking the Laplace transform with respect to time that Equation (16.2-92) follows from Equation (16.2-39).

Exercise 16.2-2

Show by taking the Laplace transform with respect to time that Equation (16.2-99) follows from Equation (16.2-46).

Exercise 16.2-3

Show by taking the Laplace transform with respect to time that Equation (16.2-106) follows from Equation (16.2-53).

16.3 Far-field scattered wave amplitudes reciprocity of the time correlation type

In this section we investigate the reciprocity relations of the time correlation type that apply to the far-field scattered wave amplitudes at plane wave incidence upon an elastodynamically penetrable or impenetrable object. The scattering configuration of Figure 16.3-1 applies.

Two states in this configuration are considered; they are denoted as state A and state B, respectively. In state A, either a uniform plane P-wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}^P$ or a uniform plane S-wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}^S$ is incident upon the scattering object; in state B, either a uniform plane P-wave that propagates in the direction of the unit vector $\boldsymbol{\beta}^P$ or a uniform plane S-wave that propagates in the direction of the unit vector $\boldsymbol{\beta}^S$ is incident upon the scattering object. It will be shown that certain relations exist between the far-field scattered wave amplitudes in states A and B. The corresponding relationships in the time domain and in the complex frequency domain will be derived separately below.

Time-domain analysis

In the time-domain analysis, the incident wave in state A is either taken as the uniform plane P-wave

$$\{\tau_{p,q}^{\mathrm{i;A};P}, v_r^{\mathrm{i;A};P}\} = \{T_{p,q}^{\mathrm{A};P}, V_r^{\mathrm{A};P}\} a^P(t - \alpha_s^P x_s / c_P)\,, \tag{16.3-1}$$

with

$$T_{p,q}^{\mathrm{A};P} = -c_P^{-1}\left[\lambda \delta_{p,q}(\alpha_k^P V_k^{\mathrm{A};P}) + 2\mu(\alpha_k^P V_k^{\mathrm{A};P})\alpha_p^P \alpha_q^P\right], \tag{16.3-2}$$

or as the uniform plane S-wave

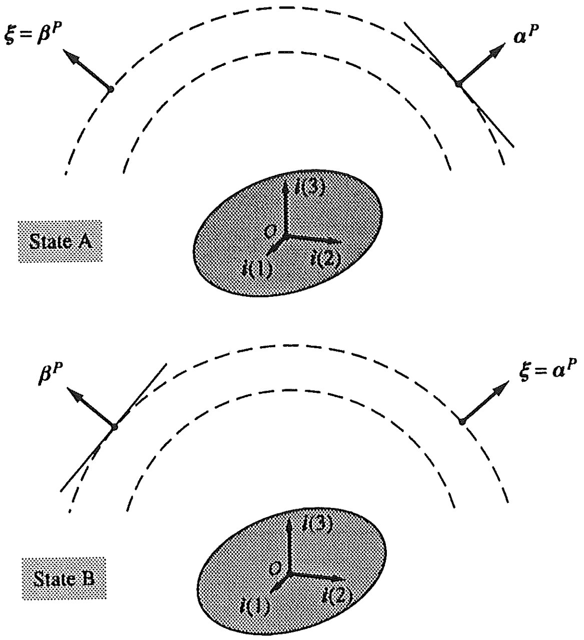
$$\{\tau_{p,q}^{\mathrm{i;A};S}, v_r^{\mathrm{i;A};S}\} = \{T_{p,q}^{\mathrm{A};S}, V_r^{\mathrm{A};S}\} a^S(t - \alpha_s^S x_s / c_S)\,, \tag{16.3-3}$$

with

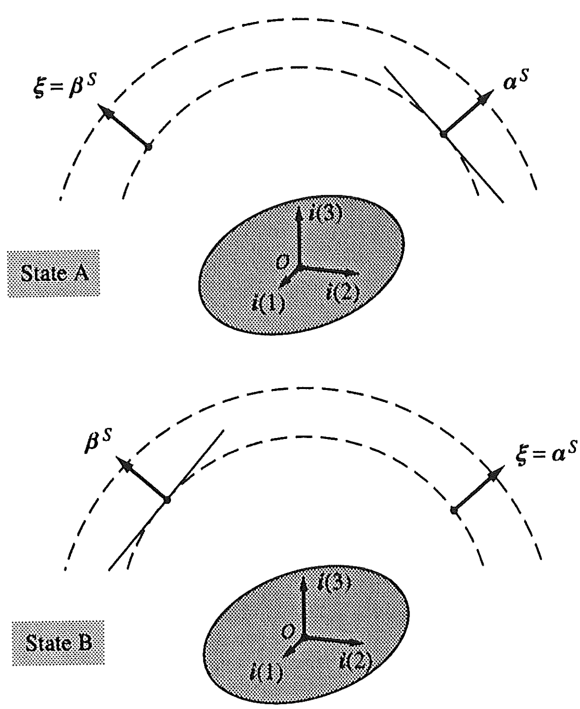
$$T_{p,q}^{\mathrm{A};S} = -c_S^{-1}\mu(\alpha_p^S V_q^{\mathrm{A};S} + \alpha_q^S V_p^{\mathrm{A};S})\,. \tag{16.3-4}$$

In the far-field region, the scattered wave in state A is represented as

16.3-1(a)


$\xi = \beta^P$
α^P
$i(3)$
State A
O
$i(1)$
$i(2)$
β^P
$\xi = \alpha^P$
$i(3)$
State B
O
$i(1)$
$i(2)$

16.3-1(b)


$\xi = \beta^S$
α^S
$i(3)$
State A
O
$i(1)$
$i(2)$
β^S
$\xi = \alpha^S$
$i(3)$
State B
O
$i(1)$
$i(2)$

16.3-1(c)

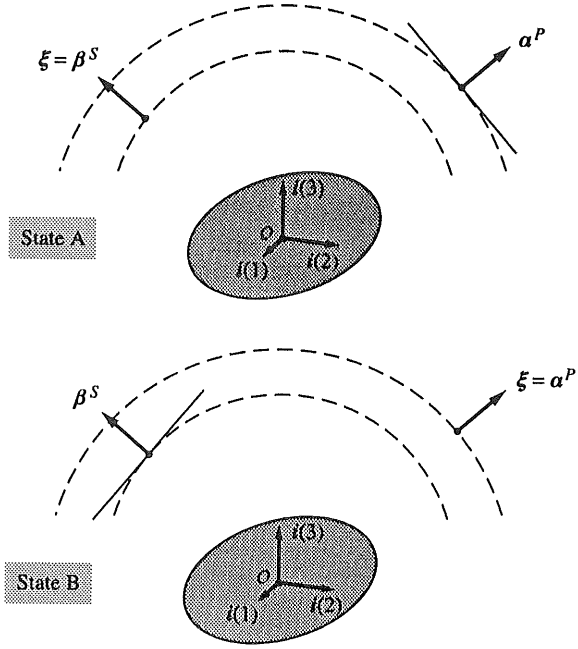


Figure 16.3-1 Configuration for the far-field scattered wave amplitudes reciprocity of the time correlation type: (a) two incident plane P-waves; (b) two incident plane S-waves; (c) an incident plane P-wave and an incident plane S-wave.

$$\{\tau_{p,q}^{s;A}, v_r^{s;A}\}(x',t) = \left[\frac{\{\tau_{p,q}^{s;A;P,\infty}, v_r^{s;A;P,\infty}\}(\xi, t - |x'|/c_P)}{4\pi c_P^2 |x'|} + \frac{\{\tau_{p,q}^{s;A;S,\infty}, v_r^{s;A;S,\infty}\}(\xi, t - |x'|/c_S)}{4\pi c_S^2 |x'|}\right]$$

$$\times [1 + O(|x'|^{-1})] \quad \text{as} \quad |x'| \to \infty \quad \text{with} \quad x' = |x'|\xi\,, \tag{16.3-5}$$

in which, on account of Equations (16.1-27), (16.1-28) and (16.1-31)–(16.1-38) (note that the surface source representation for the far-field scattered wave amplitudes is used),

$$v_r^{s;A;P,\infty}(\xi,t) = \rho^{-1}\partial_t \Phi_r^{\partial f^s;A;P,\infty}(\xi,t) + (\rho c_P)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h^s;A;P,\infty}(\xi,t)\,, \tag{16.3-6}$$

$$\tau_{p,q}^{s;A;P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{s;A;P,\infty}) + 2\mu(\xi_k v_k^{s;A;P,\infty})\xi_p\xi_q\right], \tag{16.3-7}$$

with

$$\Phi_r^{\partial f^s;A;P,\infty}(\xi,t) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^s} \partial f_k^{s;A}(x, t + \xi_s x_s/c_P)\,\mathrm{d}A\,, \tag{16.3-8}$$

$$\Phi_{r,k,i,j}^{\partial h^s;A;P,\infty}(\xi,t) = \xi_r\xi_k \int_{x\in\partial\mathcal{D}^s} \partial h_{i,j}^{s;A}(x, t + \xi_s x_s/c_P)\,\mathrm{d}A\,, \tag{16.3-9}$$

and

$$v_r^{\mathrm{s;A};S,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t \Phi_r^{\partial f^{\mathrm{s}};\mathrm{A};S,\infty}(\boldsymbol{\xi},t) + (\rho c_S)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};S,\infty}(\boldsymbol{\xi},t) \ , \tag{16.3-10}$$

$$\tau_{p,q}^{\mathrm{s;A};S,\infty} = -c_S^{-1}\mu(\xi_p v_q^{\mathrm{s;A};S,\infty} + \xi_q v_p^{\mathrm{s;A};S,\infty}) \ , \tag{16.3-11}$$

with

$$\Phi_r^{\partial f^{\mathrm{s}};\mathrm{A};S,\infty}(\boldsymbol{\xi},t) = (\delta_{r,k} - \xi_r\xi_k)\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial f_k^{\mathrm{s;A}}(\boldsymbol{x},t+\xi_s x_s/c_S)\,\mathrm{d}A \ , \tag{16.3-12}$$

$$\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};S,\infty}(\boldsymbol{\xi},t) = (\delta_{r,k} - \xi_r\xi_k)\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial h_{i,j}^{\mathrm{s;A}}(\boldsymbol{x},t+\xi_s x_s/c_S)\,\mathrm{d}A \ . \tag{16.3-13}$$

Similarly, the incident wave in state B is taken either as the uniform plane P-wave

$$\{\tau_{p,q}^{\mathrm{i;B};P}, v_r^{\mathrm{i;B};P}\} = \{T_{p,q}^{\mathrm{B};P}, V_r^{\mathrm{B};P}\} b^P(t - \beta_s^P x_s/c_P) \ , \tag{16.3-14}$$

with

$$T_{p,q}^{\mathrm{B};P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\beta_k^P V_k^{\mathrm{B};P}) + 2\mu(\beta_k^P V_k^{\mathrm{B};P})\beta_p^P\beta_q^P\right], \tag{16.3-15}$$

or as the uniform plane S-wave

$$\{\tau_{p,q}^{\mathrm{i;B};S}, v_r^{\mathrm{i;B};S}\} = \{T_{p,q}^{\mathrm{B};S}, V_r^{\mathrm{B};S}\} b^S(t - \beta_s^S x_s/c_S) \ , \tag{16.3-16}$$

with

$$T_{p,q}^{\mathrm{B};S} = -c_S^{-1}\mu(\beta_p^S V_q^{\mathrm{B};S} + \beta_q^S V_p^{\mathrm{B};S}) \ . \tag{16.3-17}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\tau_{p,q}^{\mathrm{s;B}}, v_r^{\mathrm{s;B}}\}(\boldsymbol{x}',t) = \left[\frac{\{\tau_{p,q}^{\mathrm{s;B};P,\infty}, v_r^{\mathrm{s;B};P,\infty}\}(\boldsymbol{\xi},t-|\boldsymbol{x}'|/c_P)}{4\pi c_P^2\,|\boldsymbol{x}'|} + \frac{\{\tau_{p,q}^{\mathrm{s;B};S,\infty}, v_r^{\mathrm{s;B};S,\infty}\}(\boldsymbol{\xi},t-|\boldsymbol{x}'|/c_S)}{4\pi c_S^2\,|\boldsymbol{x}'|}\right]$$

$$\times[1 + O(|\boldsymbol{x}'|^{-1})] \quad \text{as} \quad |\boldsymbol{x}'|\to\infty \quad \text{with} \quad \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi} \ , \tag{16.3-18}$$

in which, on account of Equations (16.1-27), (16.1-28) and (16.1-31)–(16.1-38) (note that the surface source representation for the far-field scattered wave amplitudes is used),

$$v_r^{\mathrm{s;B};P,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t \Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};P,\infty}(\boldsymbol{\xi},t) + (\rho c_P)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};P,\infty}(\boldsymbol{\xi},t) \ , \tag{16.3-19}$$

$$\tau_{p,q}^{\mathrm{s;B};P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{\mathrm{s;B};P,\infty}) + 2\mu(\xi_k v_k^{\mathrm{s;B};P,\infty})\xi_p\xi_q\right], \tag{16.3-20}$$

with

$$\Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};P,\infty}(\boldsymbol{\xi},t) = \xi_r\xi_k\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial f_k^{\mathrm{s;B}}(\boldsymbol{x},t+\xi_s x_s/c_P)\,\mathrm{d}A \ , \tag{16.3-21}$$

$$\Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};P,\infty}(\boldsymbol{\xi},t) = \xi_r\xi_k\int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial h_{i,j}^{\mathrm{s;B}}(\boldsymbol{x},t+\xi_s x_s/c_P)\,\mathrm{d}A \ , \tag{16.3-22}$$

and

$$v_r^{\mathrm{s;B};S,\infty}(\boldsymbol{\xi},t) = \rho^{-1}\partial_t \Phi_r^{\partial f^{\mathrm{s}};\mathrm{B};S,\infty}(\boldsymbol{\xi},t) + (\rho c_S)^{-1} C_{k,m,i,j}\xi_m \partial_t \Phi_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};S,\infty}(\boldsymbol{\xi},t) \ , \tag{16.3-23}$$

$$\tau_{p,q}^{s;B;S,\infty} = -c_S^{-1}\mu(\xi_p v_q^{s;B;S,\infty} + \xi_q v_p^{s;B;S,\infty}) , \tag{16.3-24}$$

with

$$\Phi_r^{\partial f^s;B;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \partial f_k^{s;B}(x,t + \xi_s x_s/c_S)\, dA , \tag{16.3-25}$$

$$\Phi_{r,k,i,j}^{\partial h^s;B;S,\infty}(\xi,t) = (\delta_{r,k} - \xi_r\xi_k) \int_{x\in\partial\mathcal{D}^s} \partial h_{i,j}^{s;B}(x,t + \xi_s x_s/c_S)\, dA . \tag{16.3-26}$$

If the scatterer is penetrable, its elastodynamic properties in state B are assumed to be the time-reverse adjoint of those pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equation (16.1-3) or Equation (16.1-4) applies. These boundary conditions apply to both state A and state B, and are, therefore, time-reverse self-adjoint.

To establish the desired reciprocity relation, we first apply the time-domain reciprocity theorem of the time correlation type, Equation (15.3-7), to the total wave fields in the states A and B, and to the domain $\mathcal{D}^s$ occupied by the scatterer. For a penetrable scatterer this yields

$$\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m \left[C_t(-\tau_{p,q}^{A},J_t(v_r^{B});x,t) + C_t(-J_t(\tau_{p,q}^{B}),v_r^{A};x,t)\right] dA = 0 , \tag{16.3-27}$$

since in the interior of the scatterer the total wave fields are source-free. For an impenetrable scatterer, Equation (16.3-27) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^s$ via $\mathcal{D}^{s'}$. In Equation (16.3-27) we substitute

$$\{\tau_{p,q}^{A},v_r^{A}\} = \{\tau_{p,q}^{i;A} + \tau_{p,q}^{s;A}, v_r^{i;A} + v_r^{s;A}\} \tag{16.3-28}$$

and

$$\{\tau_{p,q}^{B},v_r^{B}\} = \{\tau_{p,q}^{i;B} + \tau_{p,q}^{s;B}, v_r^{i;B} + v_r^{s;B}\} . \tag{16.3-29}$$

Next, the time-domain reciprocity theorem of the time correlation type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^s$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is time-reverse self-adjoint in its elastodynamic properties, this leads to

$$\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m \left[C_t(-\tau_{p,q}^{i;A},J_t(v_r^{i;B});x,t) + C_t(-J_t(\tau_{p,q}^{i;B}),v_r^{i;A};x,t)\right] dA = 0 . \tag{16.3-30}$$

Finally, the time-domain reciprocity theorem of the time correlation type is applied to the scattered wave fields in the states A and B and to the domain $\mathcal{D}^{s'}$. Since the embedding is time-reverse self-adjoint in its elastodynamic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m \left[C_t(-\tau_{p,q}^{s;A},J_t(v_r^{s;B});x,t) + C_t(-J_t(\tau_{p,q}^{s;B}),v_r^{s;A};x,t)\right] dA$$

$$= \lim_{\Delta\to\infty} \Delta_{m,r,p,q}^{+} \int_{x\in\mathcal{S}(O,\Delta)} \left[C_t(-\tau_{p,q}^{s;A},J_t(v_r^{s;B});x,t) + C_t(-J_t(\tau_{p,q}^{s;B}),v_r^{s;A};x,t)\right] dA , \tag{16.3-31}$$

where $\mathcal{S}(O,\Delta)$ is the sphere of radius Δ with centre at origin O of the chosen reference frame. From Equations (16.3-27)–(16.3-31) we conclude that

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[\mathrm{C}_t(-\tau^{\mathrm{i;A}}_{p,q},\mathrm{J}_t(v^{\mathrm{s;B}}_r);x,t)+\mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},\mathrm{J}_t(v^{\mathrm{i;B}}_r);x,t)$$
$$+\mathrm{C}_t(-\mathrm{J}_t(\tau^{\mathrm{i;B}}_{p,q}),v^{\mathrm{s;A}}_r;x,t)+\mathrm{C}_t(-\mathrm{J}_t(\tau^{\mathrm{s;B}}_{p,q}),v^{\mathrm{i;A}}_r;x,t)\Big]\,\mathrm{d}A$$
$$+\lim_{\Delta\to\infty}\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)}\Big[\mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},\mathrm{J}_t(v^{\mathrm{s;B}}_r);x,t)$$
$$+\mathrm{C}_t(-\mathrm{J}_t(\tau^{\mathrm{s;B}}_{p,q}),v^{\mathrm{s;A}}_r;x,t)\Big]\,\mathrm{d}A=0\,. \tag{16.3-32}$$

Equation (16.3-32) holds for both incident P- and incident S-waves. The ensuing reciprocity properties have to be discussed for the two types of incident waves separately.

Two incident P-waves

In the case of two incident P-waves we take $\{\tau^{\mathrm{i;A}}_{p,q},v^{\mathrm{i;A}}_r\}=\{\tau^{\mathrm{i;A};P}_{p,q},v^{\mathrm{i;A};P}_r\}$ (Equations (16.3-1) and (16.3-2)) and $\{\tau^{\mathrm{i;B}}_{p,q},v^{\mathrm{i;B}}_r\}=\{\tau^{\mathrm{i;B};P}_{p,q},v^{\mathrm{i;B};P}_r\}$ (Equations (16.3-14) and (16.3-15)). Then, on account of Equations (16.3-6)–(16.3-9) we have

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[\mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},\mathrm{J}_t(v^{\mathrm{i;B};P}_r);x,t)+\mathrm{C}_t(-\mathrm{J}_t(\tau^{\mathrm{i;B};P}_{p,q}),v^{\mathrm{s;A}}_r;x,t)\Big]\,\mathrm{d}A$$
$$=\Delta^{+}_{m,r,p,q}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\tau^{\mathrm{s;A}}_{p,q}(x,t')V^{\mathrm{B};P}_r-T^{\mathrm{B};P}_{p,q}v^{\mathrm{s;A}}_r(x,t')\Big]\,b^{P}(t'-\beta^{P}_s x_s/c_P-t)\,\mathrm{d}A$$
$$=\int_{t''\in\mathcal{R}}b^{P}(t''-t)\,\mathrm{d}t''\,\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\tau^{\mathrm{s;A}}_{p,q}(x,t''-\beta^{P}_s x_s/c_P)V^{\mathrm{B};P}_r$$
$$-T^{\mathrm{B};P}_{p,q}v^{\mathrm{s;A}}_r(x,t''-\beta^{P}_s x_s/c_P)\Big]\,\mathrm{d}A$$
$$=\rho V^{\mathrm{B};P}_r\int_{t''\in\mathcal{R}}b^{P}(t''-t)\mathrm{I}_t v^{\mathrm{s;A};P,\infty}_r(\beta^{P},t'')\,\mathrm{d}t'' \tag{16.3-33}$$

and on account of Equations (16.3-19)–(16.3-22)

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[\mathrm{C}_t(-\mathrm{J}_t(\tau^{\mathrm{s;B}}_{p,q}),v^{\mathrm{i;A};P}_r;x,t)+\mathrm{C}_t(-\tau^{\mathrm{i;A};P}_{p,q},\mathrm{J}_t(v^{\mathrm{s;B}}_r);x,t)\Big]\,\mathrm{d}A$$
$$=\Delta^{+}_{m,r,p,q}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\tau^{\mathrm{s;B}}_{p,q}(x,t')V^{\mathrm{A};P}_r-T^{\mathrm{A};P}_{p,q}v^{\mathrm{s;B}}_r(x,t')\Big]\,a^{P}(t'-\alpha^{P}_s x_s/c_P-t)\,\mathrm{d}A$$
$$=\int_{t''\in\mathcal{R}}a^{P}(t''-t)\,\mathrm{d}t''\,\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\tau^{\mathrm{s;B}}_{p,q}(x,t''+\alpha^{P}_s x_s/c_P)V^{\mathrm{A};P}_r$$
$$-T^{\mathrm{A};P}_{p,q}v^{\mathrm{s;B}}_r(x,t''+\alpha^{P}_s x_s/c_P)\Big]\,\mathrm{d}A$$
$$=\rho V^{\mathrm{A};P}_r\int_{t''\in\mathcal{R}}a^{P}(t''-t)\mathrm{I}_t v^{\mathrm{s;B};P,\infty}_r(\alpha^{P},t'')\,\mathrm{d}t''\,. \tag{16.3-34}$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[\mathrm{C}_t(-\tau^{\mathrm{s;A}}_{p,q},\mathrm{J}_t(v^{\mathrm{s;B}}_r);x,t)+\mathrm{C}_t(-\mathrm{J}_t(\tau^{\mathrm{s;B}}_{p,q}),v^{\mathrm{s;A}}_r;x,t)\right]\mathrm{d}A$$

$$=\int_{t'\in\mathcal{R}}\mathrm{d}t'\,\Delta^{+}_{m,r,p,q}\int_{\xi\in\Omega}\xi_m\left[\left(-\frac{\tau^{\mathrm{s;A};P,\infty}_{p,q}(\xi,t'-|x|/c_P)}{4\pi c_P^2}-\frac{\tau^{\mathrm{s;A};S,\infty}_{p,q}(\xi,t'-|x|/c_S)}{4\pi c_S^2}\right)\right.$$

$$\times\left(\frac{v^{\mathrm{s;B};P,\infty}_r(\xi,t'-|x|/c_P-t)}{4\pi c_P^2}+\frac{v^{\mathrm{s;B};S,\infty}_r(\xi,t'-|x|/c_S-t)}{4\pi c_S^2}\right)$$

$$+\left(-\frac{\tau^{\mathrm{s;B};P,\infty}_{p,q}(\xi,t'-|x|/c_P-t)}{4\pi c_P^2}-\frac{\tau^{\mathrm{s;B};S,\infty}_{p,q}(\xi,t'-|x|/c_S-t)}{4\pi c_S^2}\right)$$

$$\left.\times\left(\frac{v^{\mathrm{s;A};P,\infty}_r(\xi,t'-|x|/c_P)}{4\pi c_P^2}+\frac{v^{\mathrm{s;A};S,\infty}_r(\xi,t'-|x|/c_S)}{4\pi c_S^2}\right)\right]\mathrm{d}A$$

$$=\frac{1}{8\pi^2}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{\xi\in\Omega}\left[\rho c_P^{-3}v^{\mathrm{s;A};P,\infty}_r(\xi,t')v^{\mathrm{s;B};P,\infty}_r(\xi,t'-t)\right.$$

$$\left.+\rho c_S^{-3}v^{\mathrm{s;A};S,\infty}_r(\xi,t')v^{\mathrm{s;B};S,\infty}_r(\xi,t'-t)\right]\mathrm{d}A\,, \tag{16.3-35}$$

where Ω is the sphere of unit radius and centre at O and the properties have been used that in the far-field region the radial tractions and the particle velocities of the scattered P- and S-waves are proportional, with the P- and S-wave impedances as their respective proportionality factors, while the particle velocity of the scattered P-wave is perpendicular to the particle velocity of the scattered S-wave. Equations (16.3-32)–(16.3-35) lead to the desired reciprocity relation for the far-field scattered wave amplitudes

$$\rho V^{\mathrm{B};P}_r\int_{t''\in\mathcal{R}}b^P(t''-t)\mathrm{I}_t v^{\mathrm{s;A};P,\infty}_r(\beta^P,t'')\,\mathrm{d}t''+\rho V^{\mathrm{A};P}_r\int_{t''\in\mathcal{R}}a^P(t''-t)\mathrm{I}_t v^{\mathrm{s;B};P,\infty}_r(\alpha^P,t'')\,\mathrm{d}t''$$

$$=-\frac{1}{8\pi^2}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{\xi\in\Omega}\left[\rho c_P^{-3}v^{\mathrm{s;A};P,\infty}_r(\xi,t')v^{\mathrm{s;B};P,\infty}_r(\xi,t'-t)\right.$$

$$\left.+\rho c_S^{-3}v^{\mathrm{s;A};S,\infty}_r(\xi,t')v^{\mathrm{s;B};S,\infty}_r(\xi,t'-t)\right]\mathrm{d}A\,. \tag{16.3-36}$$

Two incident S-waves

In the case of two incident S-waves we take $\{\tau^{\mathrm{i;A}}_{p,q},v^{\mathrm{i;A}}_r\}=\{\tau^{\mathrm{i;A};S}_{p,q},v^{\mathrm{i;A};S}_r\}$ (Equations (16.3-3) and (16.3-4)) and $\{\tau^{\mathrm{i;B}}_{p,q},v^{\mathrm{i;B}}_r\}=\{\tau^{\mathrm{i;B};S}_{p,q},v^{\mathrm{i;B};S}_r\}$ (Equations (16.3-16) and (16.3-17)). Then, on account of Equations (16.3-10)–(16.3-13) we have

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[C_t(-\tau^{s;A}_{p,q},J_t(v^{i;B;S}_r);x,t)+C_t(-J_t(\tau^{i;B;S}_{p,q}),v^{s;A}_r;x,t)\right]dA$$

$$=\Delta^+_{m,r,p,q}\int_{t'\in\mathcal{R}}dt'\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\tau^{s;A}_{p,q}(x,t')V^{B;S}_r-T^{B;S}_{p,q}v^{s;A}_r(x,t')\right]b^S(t'-\beta^S_s x_s/c_S-t)\,dA$$

$$=\int_{t''\in\mathcal{R}}b^S(t''-t)\,dt''\,\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\tau^{s;A}_{p,q}(x,t''+\beta^S_s x_s/c_S)V^{B;S}_r\right.$$

$$\left.-T^{B;S}_{p,q}v^{s;A}_r(x,t''+\beta^S_s x_s/c_S)\right]dA$$

$$=\rho V^{B;S}_r\int_{t''\in\mathcal{R}}b^S(t''-t)I_t v^{s;A;S,\infty}_r(\beta^S,t'')\,dt'' \qquad (16.3\text{-}37)$$

and on account of Equations (16.3-23)–(16.3-26)

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[C_t(-J_t(\tau^{s;B}_{p,q}),v^{i;A;S}_r;x,t)+C_t(-\tau^{i;A;S}_{p,q},J_t(v^{s;B}_r);x,t)\right]dA$$

$$=\Delta^+_{m,r,p,q}\int_{t'\in\mathcal{R}}dt'\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\tau^{s;B}_{p,q}(x,t')V^{A;S}_r-T^{A;S}_{p,q}v^{s;B}_r(x,t')\right]a^S(t'-\alpha^S_s x_s/c_S-t)\,dA$$

$$=\int_{t''\in\mathcal{R}}a^S(t''-t)\,dt''\,\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\tau^{s;B}_{p,q}(x,t''+\alpha^S_s x_s/c_S)V^{A;S}_r\right.$$

$$\left.-T^{A;S}_{p,q}v^{s;B}_r(x,t''+\alpha^S_s x_s/c_S)\right]dA$$

$$=\rho V^{A;S}_r\int_{t''\in\mathcal{R}}a^S(t''-t)I_t v^{s;B;S,\infty}_r(\alpha^S,t'')\,dt''\,. \qquad (16.3\text{-}38)$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\Delta^+_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\left[C_t(-\tau^{s;A}_{p,q},J_t(v^{s;B}_r);x,t)+C_t(-J_t(\tau^{s;B}_{p,q}),v^{s;A}_r;x,t)\right]dA$$

$$=\int_{t'\in\mathcal{R}}dt'\,\Delta^+_{m,r,p,q}\int_{\xi\in\Omega}\xi_m\left[\left(-\frac{\tau^{s;A;P,\infty}_{p,q}(\xi,t'-|x|/c_P)}{4\pi c_P^2}-\frac{\tau^{s;A;S,\infty}_{p,q}(\xi,t'-|x|/c_S)}{4\pi c_S^2}\right)\right.$$

$$\times\left(\frac{v^{s;B;P,\infty}_r(\xi,t'-|x|/c_P-t)}{4\pi c_P^2}+\frac{v^{s;B;S,\infty}_r(\xi,t'-|x|/c_S-t)}{4\pi c_S^2}\right)$$

$$+\left(-\frac{\tau^{s;B;P,\infty}_{p,q}(\xi,t'-|x|/c_P-t)}{4\pi c_P^2}-\frac{\tau^{s;B;S,\infty}_{p,q}(\xi,t'-|x|/c_S-t)}{4\pi c_S^2}\right)$$

$$\left.\times\left(\frac{v^{s;A;P,\infty}_r(\xi,t'-|x|/c_P)}{4\pi c_P^2}+\frac{v^{s;A;S,\infty}_r(\xi,t'-|x|/c_S)}{4\pi c_S^2}\right)\right]dA$$

$$= \frac{1}{8\pi^2}\int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{\xi\in\Omega} \Big[\rho c_P^{-3} v_r^{\mathrm{s;A};P,\infty}(\xi,t') v_r^{\mathrm{s;B};P,\infty}(\xi,t'-t)$$

$$+ \rho c_S^{-3} v_r^{\mathrm{s;A};S,\infty}(\xi,t') v_r^{\mathrm{s;B};S,\infty}(\xi,t'-t)\Big]\,\mathrm{d}A\,, \tag{16.3-39}$$

where Ω is the sphere of unit radius and centre at O and the properties have been used that in the far-field region the radial tractions and the particle velocities of the scattered P- and S-waves are proportional, with the P- and S-wave impedances as their respective proportionality factors, while the particle velocity of the scattered P-wave is perpendicular to the particle velocity of the scattered S-wave. Equations (16.3-32) and (16.3-37)–(16.3-39) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$\rho V_r^{\mathrm{B};S} \int_{t''\in\mathcal{R}} b^S(t''-t)\mathrm{I}_t v_r^{\mathrm{s;A};S,\infty}(\beta^S,t'')\,\mathrm{d}t''$$

$$+ \rho V_r^{\mathrm{A};S} \int_{t''\in\mathcal{R}} a^S(t''-t)\mathrm{I}_t v_r^{\mathrm{s;B};S,\infty}(\alpha^S,t'')\,\mathrm{d}t''$$

$$= -\frac{1}{8\pi^2}\int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{\xi\in\Omega} \Big[\rho c_P^{-3} v_r^{\mathrm{s;A};P,\infty}(\xi,t') v_r^{\mathrm{s;B};P,\infty}(\xi,t'-t)$$

$$+ \rho c_S^{-3} v_r^{\mathrm{s;A};S,\infty}(\xi,t') v_r^{\mathrm{s;B};S,\infty}(\xi,t'-t)\Big]\,\mathrm{d}A\,. \tag{16.3-40}$$

An incident *P*-wave and an incident *S*-wave

In the case of an incident P-wave and an incident S-wave we take $\{\tau_{p,q}^{\mathrm{i;A}},v_r^{\mathrm{i;A}}\} = \{\tau_{p,q}^{\mathrm{i;A};P},v_r^{\mathrm{i;A};P}\}$ (Equations (16.3-1) and (16.3-2)) and $\{\tau_{p,q}^{\mathrm{i;B}},v_r^{\mathrm{i;B}}\} = \{\tau_{p,q}^{\mathrm{i;B};S},v_r^{\mathrm{i;B};S}\}$ (Equations (16.3-16) and (16.3-17)). Then, on account of Equations (16.3-10)–(16.3-13) we have

$$\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m\Big[\mathrm{C}_t(-\tau_{p,q}^{\mathrm{s;A}},\mathrm{J}_t(v_r^{\mathrm{i;B};S});x,t) + \mathrm{C}_t(-\mathrm{J}_t(\tau_{p,q}^{\mathrm{i;B};S}),v_r^{\mathrm{s;A}};x,t)\Big]\,\mathrm{d}A$$

$$= \Delta_{m,r,p,q}^{+} \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x\in\partial\mathcal{D}^s} \nu_m\Big[-\tau_{p,q}^{\mathrm{s;A}}(x,t')V_r^{\mathrm{B};S} - T_{p,q}^{\mathrm{B};S} v_r^{\mathrm{s;A}}(x,t')\Big]\, b^S(t'-\beta_s^S x_s/c_S - t)\,\mathrm{d}A$$

$$= \int_{t''\in\mathcal{R}} b^S(t''-t)\,\mathrm{d}t''\,\Delta_{m,r,p,q}^{+} \int_{x\in\partial\mathcal{D}^s} \nu_m\Big[-\tau_{p,q}^{\mathrm{s;A}}(x,t''+\beta_s^S x_s/c_S)V_r^{\mathrm{B};S}$$

$$- T_{p,q}^{\mathrm{B};S} v_r^{\mathrm{s;A}}(x,t''+\beta_s^S x_s/c_S)\Big]\,\mathrm{d}A$$

$$= \rho V_r^{\mathrm{B};S} \int_{t''\in\mathcal{R}} b^S(t''-t)\mathrm{I}_t v_r^{\mathrm{s;A};S,\infty}(\beta^S,t'')\,\mathrm{d}t'' \tag{16.3-41}$$

and on account of Equations (16.3-19)–(16.3-22)

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[C_t(-J_t(\tau^{s;B}_{p,q}),v^{i;A;P}_r;x,t)+C_t(-\tau^{i;A;P}_{p,q},J_t(v^{s;B}_r);x,t)\right]dA$$

$$=\Delta^+_{m,r,p,q}\int_{t'\in\mathcal{R}}dt'\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\tau^{s;B}_{p,q}(x,t')V^{A;P}_r-T^{A;P}_{p,q}v^{s;B}_r(x,t')\right]a^P(t'-\alpha^P_s x_s/c_P-t)\,dA$$

$$+\int_{t''\in\mathcal{R}}a^P(t''-t)\,dt''\,\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\tau^{s;B}_{p,q}(x,t''+\alpha^P_s x_s/c_P)V^{A;P}_r\right.$$

$$\left.-T^{A;P}_{p,q}v^{s;B}_r(x,t''+\alpha^P_s x_s/c_P)\right]dA$$

$$=\rho V^{A;P}_r\int_{t''\in\mathcal{R}}a^P(t''-t)\mathrm{I}_t v^{s;B;P,\infty}_r(\alpha^P,t'')\,dt''\,. \tag{16.3-42}$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\Delta^+_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\left[C_t(-\tau^{s;A}_{p,q},J_t(v^{s;B}_r);x,t)+C_t(-J_t(\tau^{s;B}_{p,q}),v^{s;A}_r;x,t)\right]dA$$

$$=\int_{t'\in\mathcal{R}}dt'\,\Delta^+_{m,r,p,q}\int_{\xi\in\Omega}\xi_m\left[\left(-\frac{\tau^{s;A;P,\infty}_{p,q}(\xi,t'-|x|/c_P)}{4\pi c_P^2}-\frac{\tau^{s;A;S,\infty}_{p,q}(\xi,t'-|x|/c_S)}{4\pi c_S^2}\right)\right.$$

$$\times\left(\frac{v^{s;B;P,\infty}_r(\xi,t'-|x|/c_P-t)}{4\pi c_P^2}+\frac{v^{s;B;S,\infty}_r(\xi,t'-|x|/c_S-t)}{4\pi c_S^2}\right)$$

$$+\left(-\frac{\tau^{s;B;P,\infty}_{p,q}(\xi,t'-|x|/c_P-t)}{4\pi c_P^2}-\frac{\tau^{s;B;S,\infty}_{p,q}(\xi,t'-|x|/c_S-t)}{4\pi c_S^2}\right)$$

$$\left.\times\left(\frac{v^{s;A;P,\infty}_r(\xi,t'-|x|/c_P)}{4\pi c_P^2}+\frac{v^{s;A;S,\infty}_r(\xi,t'-|x|/c_S)}{4\pi c_S^2}\right)\right]dA$$

$$=\frac{1}{8\pi^2}\int_{t'\in\mathcal{R}}dt'\int_{\xi\in\Omega}\left[\rho c_P^{-3}v^{s;A;P,\infty}_r(\xi,t')v^{s;B;P,\infty}_r(\xi,t'-t)\right.$$

$$\left.+\rho c_S^{-3}v^{s;A;S,\infty}_r(\xi,t')v^{s;B;S,\infty}_r(\xi,t'-t)\right]dA\,, \tag{16.3-43}$$

where Ω is the sphere of unit radius and centre at O and the properties have been used that in the far-field region the radial tractions and the particle velocities of the scattered P- and S-waves are proportional, with the P- and S-wave impedances as their respective proportionality factors, while the particle velocity of the scattered P-wave is perpendicular to the particle velocity of the scattered S-wave. Equations (16.3-32) and (16.3-41)–(16.3-43) lead to the desired reciprocity relation for the far-field scattered wave amplitudes

$$\rho V_r^{\mathrm{B};S} \int_{t'' \in \mathcal{R}} b^S(t''-t) \mathrm{I}_t v_r^{\mathrm{s;A};S,\infty}(\beta^S,t'')\,\mathrm{d}t'' + \rho V_r^{\mathrm{A};P} \int_{t'' \in \mathcal{R}} a^P(t''-t) \mathrm{I}_t v_r^{\mathrm{s;B};P,\infty}(\alpha^P,t'')\,\mathrm{d}t''$$
$$= -\frac{1}{8\pi^2} \int_{t' \in \mathcal{R}} \mathrm{d}t' \int_{\xi \in \Omega} \Big[\rho c_P^{-3} v_r^{\mathrm{s;A};P,\infty}(\xi,t') v_r^{\mathrm{s;B};P,\infty}(\xi,t'-t)$$
$$+ \rho c_S^{-3} v_r^{\mathrm{s;A};S,\infty}(\xi,t') v_r^{\mathrm{s;B};S,\infty}(\xi,t'-t)\Big]\,\mathrm{d}A\,. \tag{16.3-44}$$

Complex frequency-domain analysis

In the complex frequency-domain analysis, the incident wave in state A is taken either as the uniform plane P-wave

$$\{\hat{\tau}_{p,q}^{\mathrm{i;A};P}, \hat{v}_r^{\mathrm{i;A};P}\} = \{T_{p,q}^{\mathrm{A};P}, V_r^{\mathrm{A};P}\} \hat{a}^P(s) \exp(-s\alpha_s^P x_s/c_P)\,, \tag{16.3-45}$$

with

$$T_{p,q}^{\mathrm{A};P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\alpha_k^P V_k^{\mathrm{A};P}) + 2\mu(\alpha_k^P V_k^{\mathrm{A};P})\alpha_p^P \alpha_q^P\right], \tag{16.3-46}$$

or as the uniform plane S-wave

$$\{\hat{\tau}_{p,q}^{\mathrm{i;A};S}, \hat{v}_r^{\mathrm{i;A};S}\} = \{T_{p,q}^{\mathrm{A};S}, V_r^{\mathrm{A};S}\} \hat{a}^S(s) \exp(-s\alpha_s^S x_s/c_S)\,, \tag{16.3-47}$$

with

$$T_{p,q}^{\mathrm{A};S} = -c_S^{-1}\,\mu(\alpha_p^S V_q^{\mathrm{A};S} + \alpha_q^S V_p^{\mathrm{A};S})\,. \tag{16.3-48}$$

In the far-field region, the scattered wave in state A is represented as

$$\{\hat{\tau}_{p,q}^{\mathrm{s;A}}, \hat{v}_r^{\mathrm{s;A}}\}(x',s) = \left[\{\hat{\tau}_{p,q}^{\mathrm{s;A};P,\infty}, \hat{v}_r^{\mathrm{s;A};P,\infty}\}(\xi,s) \frac{\exp(-s|x'|/c_P)}{4\pi c_P^2 |x'|}\right.$$
$$\left. + \{\hat{\tau}_{p,q}^{\mathrm{s;A};S,\infty}, \hat{v}_r^{\mathrm{s;A};S,\infty}\}(\xi,s) \frac{\exp(-s|x'|/c_S)}{4\pi c_S^2 |x'|}\right]$$
$$\times [1 + O(|x'|^{-1})] \quad \text{as } |x'| \to \infty \quad \text{with } x' = |x'|\xi\,, \tag{16.3-49}$$

in which, on account of Equations (16.1-76) and (16.1-77) and (16.1-80)–(16.1-87) (note that the surface source representation for the far-field scattered wave is used),

$$\hat{v}_r^{\mathrm{s;A};P,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^{\mathrm{s}};\mathrm{A};P,\infty} + s(\rho c_P)^{-1} C_{k,m,i,j}\xi_m \hat{\Phi}_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};P,\infty}\,, \tag{16.3-50}$$

$$\hat{\tau}_{p,q}^{\mathrm{s;A};P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k \hat{v}_k^{\mathrm{s;A};P,\infty}) + 2\mu(\xi_k \hat{v}_k^{\mathrm{s;A};P,\infty})\xi_p\xi_q\right], \tag{16.3-51}$$

with

$$\hat{\Phi}_r^{\partial f^{\mathrm{s}};\mathrm{A};P,\infty} = \xi_r\xi_k \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \exp(s\xi_s x_s/c_P)\partial\hat{f}_k^{\mathrm{s;A}}(x,s)\,\mathrm{d}A\,, \tag{16.3-52}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{A};P,\infty} = \xi_r\xi_k \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \exp(s\xi_s x_s/c_P)\partial\hat{h}_{i,j}^{\mathrm{s;A}}(x,s)\,\mathrm{d}A\,, \tag{16.3-53}$$

and

$$\hat{v}_r^{\text{s;A};S,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^{\text{s}};\text{A};S,\infty} + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h^{\text{s}};\text{A};S,\infty}, \tag{16.3-54}$$

$$\hat{\tau}_{p,q}^{\text{s;A};S,\infty} = -c_S^{-1}\mu(\xi_p\hat{v}_q^{\text{s;A};S,\infty}) + \xi_q\hat{v}_p^{\text{s;A};S,\infty}), \tag{16.3-55}$$

with

$$\hat{\Phi}_{r,k,i,j}^{\partial f^{\text{s}};\text{A};S,\infty} = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\partial\mathcal{D}^{\text{s}}} \exp(s\xi_s x_s/c_S)\partial\hat{f}_k^{\text{s;A}}(x,s)\,\mathrm{d}A, \tag{16.3-56}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^{\text{s}};\text{A};S,\infty} = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\partial\mathcal{D}^{\text{s}}} \exp(s\xi_s x_s/c_S)\partial\hat{h}_{i,j}^{\text{s;A}}(x,s)\,\mathrm{d}A. \tag{16.3-57}$$

Similarly, the incident in state B is taken either as the uniform plane P-wave

$$\{\hat{\tau}_{p,q}^{\text{i;B};P},\hat{v}_r^{\text{i;B};P}\} = \{T_{p,q}^{\text{B};P},V_r^{\text{B};P}\}\hat{b}^P(s)\exp(-s\beta_s^P x_s/c_P), \tag{16.3-58}$$

with

$$T_{p,q}^{\text{B};P} = -c_P^{-1}\left[\lambda\delta_{p,q}(\beta_k^P V_k^{\text{B};P}) + 2\mu(\beta_k^P V_k^{\text{B};P})\beta_p^P\beta_q^P\right], \tag{16.3-59}$$

or as the uniform plane S-wave

$$\{\hat{\tau}_{p,q}^{\text{i;B};S},\hat{v}_r^{\text{i;B};S}\} = \{T_{p,q}^{\text{B};S},V_r^{\text{B};S}\}\hat{b}^S(s)\exp(-s\beta_s^S x_s/c_S), \tag{16.3-60}$$

with

$$T_{p,q}^{\text{B};S} = -c_S^{-1}\mu(\beta_p^S V_q^{\text{B};S}) + \beta_q^S V_p^{\text{B};S}). \tag{16.3-61}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\hat{\tau}_{p,q}^{\text{s;B}},\hat{v}_r^{\text{s;B}}\}(x',s) = \left[\{\hat{\tau}_{p,q}^{\text{s;B};P,\infty},\hat{v}_r^{\text{s;B};P,\infty}\}(\xi,s)\frac{\exp(-s|x'|/c_P)}{4\pi c_P^2\,|x'|} + \{\hat{\tau}_{p,q}^{\text{s;B};S,\infty},\hat{v}_r^{\text{s;B};S,\infty}\}(\xi,s)\frac{\exp(-s|x'|/c_S)}{4\pi c_S^2\,|x'|}\right]$$
$$\times[1 + O(|x'|^{-1})] \quad \text{as} \quad |x'|\to\infty \quad \text{with} \quad x' = |x'|\xi, \tag{16.3-62}$$

in which, on account of Equations (16.1-76), (16.1-77) and (16.1-80)–(16.1-87) (note that the surface source representation for the far-field scattered wave is used),

$$\hat{v}_r^{\text{s;B};P,\infty} = s\rho^{-1}\hat{\Phi}_r^{\partial f^{\text{s}};\text{B};P,\infty} + s(\rho c_P)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h^{\text{s}};\text{B};P,\infty}, \tag{16.3-63}$$

$$\hat{\tau}_{p,q}^{\text{s;B};P,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k\hat{v}_k^{\text{s;B};P,\infty}) + 2\mu(\xi_k\hat{v}_k^{\text{s;B};P,\infty})\xi_p\xi_q\right], \tag{16.3-64}$$

with

$$\hat{\Phi}_r^{\partial f^{\text{s}};\text{B};P,\infty} = \xi_r\xi_k\int_{x\in\partial\mathcal{D}^{\text{s}}} \exp(s\xi_s x_s/c_P)\partial\hat{f}_k^{\text{s;B}}(x,s)\,\mathrm{d}A, \tag{16.3-65}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^{\text{s}};\text{B};P,\infty} = \xi_r\xi_k\int_{x\in\partial\mathcal{D}^{\text{s}}} \exp(s\xi_s x_s/c_P)\partial\hat{h}_{i,j}^{\text{s;B}}(x,s)\,\mathrm{d}A, \tag{16.3-66}$$

and

$$\hat{v}_r^{\mathrm{s;B};S,\infty} = sp^{-1}\hat{\Phi}_r^{\partial f^{\mathrm{s}};\mathrm{B};S,\infty} + s(\rho c_S)^{-1}C_{k,m,i,j}\xi_m\hat{\Phi}_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};S,\infty}, \tag{16.3-67}$$

$$\hat{\tau}_{p,q}^{\mathrm{s;B};S,\infty} = -c_S^{-1}\mu(\xi_p\hat{v}_q^{\mathrm{s;B};S,\infty} + \xi_q\hat{v}_p^{\mathrm{s;B};S,\infty}), \tag{16.3-68}$$

with

$$\hat{\Phi}_{r,k,i,j}^{\partial f^{\mathrm{s}};\mathrm{B};S,\infty} = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \exp(s\xi_s x_s/c_S)\partial\hat{f}_k^{\mathrm{s;B}}(x,s)\,\mathrm{d}A, \tag{16.2-69}$$

$$\hat{\Phi}_{r,k,i,j}^{\partial h^{\mathrm{s}};\mathrm{B};S,\infty} = (\delta_{r,k} - \xi_r\xi_k)\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \exp(s\xi_s x_s/c_S)\partial\hat{h}_{i,j}^{\mathrm{s;B}}(x,s)\,\mathrm{d}A. \tag{16.3-70}$$

If the scatterer is penetrable, its elastodynamic properties in state B are assumed to be the time-reverse adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equation (16.1-51) or Equation (16.1-52) applies. These boundary conditions apply to both state A and state B, and are, therefore, time-reverse self-adjoint.

To establish the desired reciprocity relation, we first apply the complex frequency-domain reciprocity theorem of the time correlation type, Equation (15.5-7), to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\Delta_{m,r,p,q}^{+}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{A}}(x,s)\hat{v}_r^{\mathrm{B}}(x,-s) - \hat{\tau}_{p,q}^{\mathrm{B}}(x,-s)\hat{v}_r^{\mathrm{A}}(x,s)\right]\mathrm{d}A = 0, \tag{16.3-71}$$

since in the interior of the scatterer the total wave fields are source-free. For an impenetrable scatterer, Equation (16.3-71) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\mathrm{s}}$ via $\mathcal{D}^{\mathrm{s}'}$. In Equation (16.3-71) we substitute

$$\{\hat{\tau}_{p,q}^{\mathrm{A}},\hat{v}_r^{\mathrm{A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;A}} + \hat{\tau}_{p,q}^{\mathrm{s;A}},\hat{v}_r^{\mathrm{i;A}} + \hat{v}_r^{\mathrm{s;A}}\} \tag{16.3-72}$$

and

$$\{\hat{\tau}_{p,q}^{\mathrm{B}},\hat{v}_r^{\mathrm{B}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;B}} + \hat{\tau}_{p,q}^{\mathrm{s;B}},\hat{v}_r^{\mathrm{i;B}} + \hat{v}_r^{\mathrm{s;B}}\}. \tag{16.3-73}$$

Next, the complex frequency-domain reciprocity theorem of the time correlation type is applied to the incident wave fields in the states A and B and to the domain $\mathcal{D}^{\mathrm{s}}$. Since the incident wave fields are source-free in the interior of the scatterer and the embedding is time-reverse self-adjoint in its elastodynamic properties, this leads to

$$\Delta_{m,r,p,q}^{+}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\hat{\tau}_{p,q}^{\mathrm{i;A}}(x,s)\hat{v}_r^{\mathrm{i;B}}(x,-s) - \hat{\tau}_{p,q}^{\mathrm{i;B}}(x,-s)\hat{v}_r^{\mathrm{i;A}}(x,s)\right]\mathrm{d}A = 0. \tag{16.3-74}$$

Finally, the complex frequency-domain reciprocity theorem of the time correlation type is applied to the scattered wave fields in the states A and B and to the domain $\mathcal{D}^{\mathrm{s}'}$. Since the embedding is time-reverse self-adjoint in its elastodynamic properties and the scattered wave fields are source-free in the exterior of the scatterer and satisfy the condition of causality, this leads to

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{s;B}_r(x,-s)-\hat{\tau}^{s;B}_{p,q}(x,-s)\hat{v}^{s;A}_r(x,s)\right]dA$$

$$=\lim_{\Delta\to\infty}\Delta^+_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)}\left[-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{s;B}_r(x,-s)-\hat{\tau}^{s;B}_{p,q}(x,-s)\hat{v}^{s;A}_r(x,s)\right]dA\,, \quad (16.3\text{-}75)$$

where $\mathcal{S}(O,\Delta)$ is the sphere of radius Δ with centre at the origin O of the chosen reference frame. From Equations (16.3-71)–(16.3-75) we conclude that

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\hat{\tau}^{i;A}_{p,q}(x,s)\hat{v}^{s;B}_r(x,-s)-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{i;B}_r(x,-s)\right.$$
$$\left.-\hat{\tau}^{i;B}_{p,q}(x,-s)\hat{v}^{s;A}_r(x,s)-\hat{\tau}^{s;B}_{p,q}(x,-s)\hat{v}^{i;A}_r(x,s)\right]dA$$
$$+\lim_{\Delta\to\infty}\Delta^+_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)}\left[-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{s;B}_r(x,-s)\right.$$
$$\left.-\hat{\tau}^{s;B}_{p,q}(x,-s)\hat{v}^{s;A}_r(x,s)\right]dA=0\,. \quad (16.3\text{-}76)$$

Equation (16.3-76) holds for both incident P-waves and incident S-waves. The ensuing reciprocity properties have to be discussed for the two types of incident waves separately.

Two incident P-waves

In the case of two incident P-waves we take $\{\hat{\tau}^{i;A}_{p,q},\hat{v}^{i;A}_r\}=\{\hat{\tau}^{i;A;P}_{p,q},\hat{v}^{i;A;P}_r\}$ (Equations (16.3-45) and (16.3-46)) and $\{\hat{\tau}^{i;B}_{p,q},\hat{v}^{i;B}_r\}=\{\hat{\tau}^{i;B;P}_{p,q},\hat{v}^{i;B;P}_r\}$ (Equations (16.3-58) and (16.3-59)). Then, on account of Equations (16.3-50)–(16.3-53) we have

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\hat{\tau}^{s;A}_{p,q}(x,s)\hat{v}^{i;B;P}_r(x,-s)-\hat{\tau}^{i;B;P}_{p,q}(x,-s)\hat{v}^{s;A}_r(x,s)\right]dA$$
$$=\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\hat{\tau}^{s;A}_{p,q}(x,s)V^{B;P}_r-T^{B;P}_{p,q}\hat{v}^{s;A}_r(x,s)\right]\hat{b}^P(-s)\exp(s\beta^P_s x_s/c_P)\,dA$$
$$=s^{-1}\rho V^{B;P}_r\hat{b}^P(-s)\hat{v}^{s;A;P,\infty}_r(\beta^P,s) \quad (16.3\text{-}77)$$

and on account of Equations (16.3-63)–(16.3-66)

$$\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\hat{\tau}^{s;B}_{p,q}(x,-s)\hat{v}^{i;A;P}_r(x,s)-\hat{\tau}^{i;A;P}_{p,q}(x,s)\hat{v}^{s;B}_r(x,-s)\right]dA$$
$$=\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[-\hat{\tau}^{s;B}_{p,q}(x,-s)V^{A;P}_r-T^{A;P}_{p,q}\hat{v}^{s;B}_r(x,-s)\right]\hat{a}^P(s)\exp(-s\alpha^P_s x_s/c_P)\,dA$$
$$=s^{-1}\rho V^{A;P}_r\hat{a}^P(s)\hat{v}^{s;B;P,\infty}_r(\alpha^P,-s)\,. \quad (16.3\text{-}78)$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[-\hat{\tau}^{\mathrm{s;A}}_{p,q}(x,s)\hat{v}^{\mathrm{s;B}}_r(x,-s) - \hat{\tau}^{\mathrm{s;B}}_{p,q}(x,-s)\hat{v}^{\mathrm{s;A}}_r(x,s)\right]\mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q}\int_{\xi\in\Omega}\xi_m\left[\left(-\frac{\hat{\tau}^{\mathrm{s;A};P,\infty}_{p,q}(\xi,s)}{4\pi c_P^2}\exp(-s|x|/c_P) - \frac{\hat{\tau}^{\mathrm{s;A};S,\infty}_{p,q}(\xi,s)}{4\pi c_S^2}\exp(-s|x|/c_S)\right)\right.$$

$$\times\left(\frac{\hat{v}^{\mathrm{s;B};P,\infty}_r(\xi,-s)}{4\pi c_P^2}\exp(s|x|/c_P) + \frac{\hat{v}^{\mathrm{s;B};S,\infty}_r(\xi,-s)}{4\pi c_S^2}\exp(s|x|/c_S)\right)$$

$$+\left(-\frac{\hat{\tau}^{\mathrm{s;B};P,\infty}_{p,q}(\xi,-s)}{4\pi c_P^2}\exp(s|x|/c_P) - \frac{\hat{\tau}^{\mathrm{s;B};S,\infty}_{p,q}(\xi,-s)}{4\pi c_S^2}\exp(s|x|/c_S)\right)$$

$$\left.\times\left(\frac{\hat{v}^{\mathrm{s;A};P,\infty}_r(\xi,s)}{4\pi c_P^2}\exp(-s|x|/c_P) + \frac{\hat{v}^{\mathrm{s;A};S,\infty}_r(\xi,s)}{4\pi c_S^2}\exp(-s|x|/c_S)\right)\right]\mathrm{d}A$$

$$= \frac{1}{8\pi^2}\int_{\xi\in\Omega}\left[\rho c_P^{-3}\hat{v}^{\mathrm{s;A};P,\infty}_r(\xi,s)\hat{v}^{\mathrm{s;B};P,\infty}_r(\xi,-s)\right.$$

$$\left. + \rho c_S^{-3}\hat{v}^{\mathrm{s;A};S,\infty}_r(\xi,s)\hat{v}^{\mathrm{s;B};S,\infty}_r(\xi,-s)\right]\mathrm{d}A\,, \qquad (16.3\text{-}79)$$

where Ω is the sphere of unit radius and centre at O and the properties have been used that in the far-field region the radial tractions and the particle velocities of the scattered P- and S-waves are proportional, with the P- and S-wave impedances as their respective proportionality factors, while the particle velocity of the scattered P-wave is perpendicular to the particle velocity of the scattered S-wave. Equations (16.3-76)–(16.3-79) lead to the desired reciprocity relation for the far-field scattered wave amplitudes

$$s^{-1}\rho V_r^{\mathrm{B};P}\,\hat{b}^{P}(-s)\hat{v}^{\mathrm{s;A};P,\infty}_r(\beta^{P},s) + s^{-1}\rho V_r^{\mathrm{A};P}\,\hat{a}^{P}(s)\hat{v}^{\mathrm{s;B};P,\infty}_r(\alpha^{P},-s)$$

$$= -\frac{1}{8\pi^2}\int_{\xi\in\Omega}\left[\rho c_P^{-3}\hat{v}^{\mathrm{s;A};P,\infty}_r(\xi,s)\hat{v}^{\mathrm{s;B};P,\infty}_r(\xi,-s)\right.$$

$$\left. + \rho c_S^{-3}\hat{v}^{\mathrm{s;A};S,\infty}_r(\xi,s)\hat{v}^{\mathrm{s;B};S,\infty}_r(\xi,-s)\right]\mathrm{d}A\,. \qquad (16.3\text{-}80)$$

Two incident S-waves

In the case of two incident S-waves we take $\{\hat{\tau}^{\mathrm{i;A}}_{p,q},\hat{v}^{\mathrm{i;A}}_r\} = \{\hat{\tau}^{\mathrm{i;A};S}_{p,q},\hat{v}^{\mathrm{i;A};S}_r\}$ (Equations (16.3-47) and (16.3-48)) and $\{\hat{\tau}^{\mathrm{i;B}}_{p,q},\hat{v}^{\mathrm{i;B}}_r\} = \{\hat{\tau}^{\mathrm{i;B};S}_{p,q},\hat{v}^{\mathrm{i;B};S}_r\}$ (Equations (16.3-60) and (16.3-61)). Then, on account of Equations (16.3-54)–(16.3-57) we have

$$\Delta^+_{m,r,p,q} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;A}_{p,q}(x,s) \hat{v}^{i;B;S}_r(x,-s) - \hat{\tau}^{i;B;S}_{p,q}(x,-s) \hat{v}^{s;A}_r(x,s) \right] dA$$

$$= \Delta^+_{m,r,p,q} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;A}_{p,q}(x,s) V^{B;S}_r - T^{B;S}_{p,q} \hat{v}^{s;A}_r(x,s) \right] \hat{b}^S(-s) \exp(s\beta^S_s x_s / c_S) \, dA$$

$$= s^{-1} \rho V^{B;S}_r \hat{b}^S(-s) \hat{v}^{s;A;S,\infty}_r(\beta^S, s) \tag{16.3-81}$$

and on account of Equations (16.3-67)–(16.3-70)

$$\Delta^+_{m,r,p,q} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;B}_{p,q}(x,-s) \hat{v}^{i;A;S}_r(x,s) - \hat{\tau}^{i;A;S}_{p,q}(x,s) \hat{v}^{s;B}_r(x,-s) \right] dA$$

$$= \Delta^+_{m,r,p,q} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[-\hat{\tau}^{s;B}_{p,q}(x,-s) V^{A;S}_r - T^{A;S}_{p,q} \hat{v}^{s;B}_r(x,-s) \right] \hat{a}^S(s) \exp(-s\alpha^S_s x_s / c_S) \, dA$$

$$= s^{-1} \rho V^{A;S}_r \hat{a}^S(s) \hat{v}^{s;B;S,\infty}_r(\alpha^S, -s) \, . \tag{16.3-82}$$

Furthermore, we have

$$\lim_{\Delta \to \infty} \Delta^+_{m,r,p,q} \int_{x \in \mathcal{S}(O,\Delta)} \nu_m \left[-\hat{\tau}^{s;A}_{p,q}(x,s) \hat{v}^{s;B}_r(x,-s) - \hat{\tau}^{s;B}_{p,q}(x,-s) \hat{v}^{s;A}_r(x,s) \right] dA$$

$$= \Delta^+_{m,r,p,q} \int_{\xi \in \Omega} \xi_m \Bigg[\left(-\frac{\hat{\tau}^{s;A;P,\infty}_{p,q}(\xi,s)}{4\pi c_P^2} \exp(-s|x|/c_P) - \frac{\hat{\tau}^{s;A;S,\infty}_{p,q}(\xi,s)}{4\pi c_S^2} \exp(-s|x|/c_S) \right)$$

$$\times \left(\frac{\hat{v}^{s;B;P,\infty}_r(\xi,-s)}{4\pi c_P^2} \exp(s|x|/c_P) + \frac{\hat{v}^{s;B;S,\infty}_r(\xi,-s)}{4\pi c_S^2} \exp(s|x|/c_S) \right)$$

$$+ \left(-\frac{\hat{\tau}^{s;B;P,\infty}_{p,q}(\xi,-s)}{4\pi c_P^2} \exp(s|x|/c_P) - \frac{\hat{\tau}^{s;B;S,\infty}_{p,q}(\xi,-s)}{4\pi c_S^2} \exp(s|x|/c_S) \right)$$

$$\times \left(\frac{\hat{v}^{s;A;P,\infty}_r(\xi,s)}{4\pi c_P^2} \exp(-s|x|/c_P) + \frac{\hat{v}^{s;A;S,\infty}_r(\xi,s)}{4\pi c_S^2} \exp(-s|x|/c_S) \right) \Bigg] dA$$

$$= \frac{1}{8\pi^2} \int_{\xi \in \Omega} \Big[\rho c_P^{-3} \hat{v}^{s;A;P,\infty}_r(\xi,s) \hat{v}^{s;B;P,\infty}_r(\xi,-s)$$

$$+ \rho c_S^{-3} \hat{v}^{s;A;S,\infty}_r(\xi,s) \hat{v}^{s;B;S,\infty}_r(\xi,-s) \Big] dA \, , \tag{16.3-83}$$

where Ω is the sphere of unit radius and centre at O and the properties have been used that in the far-field region the radial tractions and the particle velocities of the scattered P- and S-waves are proportional, with the P- and S-wave impedances as their respective proportionality factors, while the particle velocity of the scattered P-wave is perpendicular to the particle velocity of the scattered S-wave. Equations (16.3-76) and (16.3-81)–(16.3-83) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$s^{-1}\rho V_r^{\mathrm{B};S}\,\hat{b}^{S}(-s)\hat{v}_r^{\mathrm{s;A};S,\infty}(\beta^{S},s) + s^{-1}\rho V_r^{\mathrm{A};S}\,\hat{a}^{S}(s)\hat{v}_r^{\mathrm{s;B};S,\infty}(\alpha^{S},-s)$$
$$= -\frac{1}{8\pi^2}\int_{\xi\in\Omega}\Big[\rho c_P^{-3}\hat{v}_r^{\mathrm{s;A};P,\infty}(\xi,s)\hat{v}_r^{\mathrm{s;B};P,\infty}(\xi,-s)$$
$$+\rho c_S^{-3}\hat{v}_r^{\mathrm{s;A};S,\infty}(\xi,s)\hat{v}_r^{\mathrm{s;B};S,\infty}(\xi,-s)\Big]\,\mathrm{d}A\,. \qquad (16.3\text{-}84)$$

An incident *P*-wave and an incident *S*-wave

In the case of an incident P-wave and an incident S-wave we take $\{\hat{\tau}_{p,q}^{\mathrm{i;A}},\hat{v}_r^{\mathrm{i;A}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;A};P},\hat{v}_r^{\mathrm{i;A};P}\}$ (Equations (16.3-45) and (16.3-46)) and $\{\hat{\tau}_{p,q}^{\mathrm{i;B}},\hat{v}_r^{\mathrm{i;B}}\} = \{\hat{\tau}_{p,q}^{\mathrm{i;B};S},\hat{v}_r^{\mathrm{i;B};S}\}$ (Equations (16.3-60) and (16.3-61)). Then, on account of Equations (16.3-56) and (16.3-57) we have

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(x,s)\hat{v}_r^{\mathrm{i;B};S}(x,-s) - \hat{\tau}_{p,q}^{\mathrm{i;B};S}(x,-s)\hat{v}_r^{\mathrm{s;A}}(x,s)\Big]\,\mathrm{d}A$$
$$= \Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(x,s)V_r^{\mathrm{B};S} - T_{p,q}^{\mathrm{B};S}\hat{v}_r^{\mathrm{s;A}}(x,s)\Big]\hat{b}^{S}(-s)\exp(s\beta_s^{S}x_s/c_S)\,\mathrm{d}A$$
$$= s^{-1}\rho V_r^{\mathrm{B};S}\hat{b}^{S}(-s)\hat{v}_r^{\mathrm{s;A};S,\infty}(\beta^{S},s) \qquad (16.3\text{-}85)$$

and on account of Equations (16.3-63)–(16.3-66)

$$\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\hat{\tau}_{p,q}^{\mathrm{s;B}}(x,-s)\hat{v}_r^{\mathrm{i;A};P}(x,s) - \hat{\tau}_{p,q}^{\mathrm{i;A};P}(x,s)\hat{v}_r^{\mathrm{s;B}}(x,-s)\Big]\,\mathrm{d}A$$
$$= \Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}}\nu_m\Big[-\hat{\tau}_{p,q}^{\mathrm{s;B}}(x,-s)V_r^{\mathrm{A};P} - T_{p,q}^{\mathrm{A};P}\hat{v}_r^{\mathrm{s;B}}(x,-s)\Big]\hat{a}^{P}(s)\exp(-s\alpha_s^{P}x_s/c_P)\,\mathrm{d}A$$
$$= s^{-1}\rho V_r^{\mathrm{A};P}\hat{a}^{P}(s)\hat{v}_r^{\mathrm{s;B};P,\infty}(\alpha^{P},-s)\,. \qquad (16.3\text{-}86)$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\Delta^{+}_{m,r,p,q}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\Big[-\hat{\tau}_{p,q}^{\mathrm{s;A}}(x,s)\hat{v}_r^{\mathrm{s;B}}(x,-s) - \hat{\tau}_{p,q}^{\mathrm{s;B}}(x,-s)\hat{v}_r^{\mathrm{s;A}}(x,s)\Big]\,\mathrm{d}A$$
$$= \Delta^{+}_{m,r,p,q}\int_{\xi\in\Omega}\xi_m\Bigg[\Bigg(-\frac{\hat{\tau}_{p,q}^{\mathrm{s;A};P,\infty}(\xi,s)}{4\pi c_P^2}\exp(-s|x|/c_P) - \frac{\hat{\tau}_{p,q}^{\mathrm{s;A};S,\infty}(\xi,s)}{4\pi c_S^2}\exp(-s|x|/c_S)\Bigg)$$
$$\times\Bigg(\frac{\hat{v}_r^{\mathrm{s;B};P,\infty}(\xi,-s)}{4\pi c_P^2}\exp(s|x|/c_P) + \frac{\hat{v}_r^{\mathrm{s;B};S,\infty}(\xi,-s)}{4\pi c_S^2}\exp(s|x|/c_S)\Bigg)$$
$$+\Bigg(-\frac{\hat{\tau}_{p,q}^{\mathrm{s;B};P,\infty}(\xi,-s)}{4\pi c_P^2}\exp(s|x|/c_P) - \frac{\hat{\tau}_{p,q}^{\mathrm{s;B};S,\infty}(\xi,-s)}{4\pi c_S^2}\exp(s|x|/c_S)\Bigg)$$

$$\times\left(\frac{\hat{v}_r^{\mathrm{s;A};P,\infty}(\xi,s)}{4\pi c_P^2}\exp(-s|x|/c_P)+\frac{\hat{v}_r^{\mathrm{s;A};S,\infty}(\xi,s)}{4\pi c_S^2}\exp(-s|x|/c_S)\right)\Bigg]\,\mathrm{d}A$$

$$=\frac{1}{8\pi^2}\int_{\xi\in\Omega}\Big[\rho c_P^{-3}\hat{v}_r^{\mathrm{s;A};P,\infty}(\xi,s)\hat{v}_r^{\mathrm{s;B};P,\infty}(\xi,-s)$$

$$+\rho c_S^{-3}\hat{v}_r^{\mathrm{s;A};S,\infty}(\xi,s)\hat{v}_r^{\mathrm{s;B};S,\infty}(\xi,-s)\Big]\,\mathrm{d}A\,, \tag{16.3-87}$$

where Ω is the sphere of unit radius and centre at O and the properties have been used that in the far-field region the radial tractions and the particle velocities of the scattered P- and S-waves are proportional, with the P- and S-wave impedances as their respective proportionality factors, while the particle velocity of the scattered P-wave is perpendicular to the particle velocity of the scattered S-wave. Equations (16.3-76) and (16.3-85)–(16.3-87) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$s^{-1}\rho V_r^{\mathrm{B};S}\,\hat{b}^S(-s)\hat{v}_r^{\mathrm{s;A};S,\infty}(\beta^S,s)+s^{-1}\rho V_r^{\mathrm{A};P}\,\hat{a}^P(s)\hat{v}_r^{\mathrm{s;B};P,\infty}(\alpha^P,-s)$$

$$=-\frac{1}{8\pi^2}\int_{\xi\in\Omega}\Big[\rho c_P^{-3}\hat{v}_r^{\mathrm{s;A};P,\infty}(\xi,s)\hat{v}_r^{\mathrm{s;B};P,\infty}(\xi,-s)$$

$$+\rho c_S^{-3}\hat{v}_r^{\mathrm{s;A};S,\infty}(\xi,s)\hat{v}_r^{\mathrm{s;B};S,\infty}(\xi,-s)\Big]\,\mathrm{d}A\,. \tag{16.3-88}$$

In a theoretical analysis, the reciprocity relations derived in this section serve as an important check on the correctness of the analytic solutions as well as on the accuracy of numerical solutions to scattering problems. Note, however, that the reciprocity relations are necessary conditions to be satisfied by the scattered wave field (in the far-field region), but their satisfaction does not guarantee the correctness of a total analytic solution or the accuracy of a total numerical solution. In a physical experiment, the redundancy induced by the reciprocity relations can be exploited to reduce the influence of noise on the quality of the observed data.

16.4 An energy theorem about the far-field forward scattered wave amplitudes

A special case arises when in the reciprocity relations of the time correlation type derived in Section 16.3, state A and state B are taken to be identical states. Since the superscripts A and B are then superfluous, they will be omitted in the present section.

Time-domain version of the energy theorem

In the time-domain version of the theorem we start from Equation (16.3-27), take state A identical to state B, and consider the result at zero correlation time shift. Furthermore, for the case of an elastodynamically penetrable scatterer, the solid occupying the scattering domain $\mathcal{D}^{\mathrm{s}}$ is no longer assumed to be time-reverse self-adjoint, i.e. it may have non-zero elastodynamic losses. Thus, we are led to consider the expression

$$\Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \mathrm{C}_t(-\tau_{p,q},\mathrm{J}_t(v_r);x,0)\,\mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \mathrm{C}_t(\mathrm{J}_t(-\tau_{p,q}),v_r;x,0)\,\mathrm{d}A$$

$$= \Delta^{+}_{m,r,p,q} \int_{t'\in\mathcal{R}} \mathrm{d}t' \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau_{p,q}(x,t')v_r(x,t')\right]\mathrm{d}A = -W^{\mathrm{a}}, \tag{16.4-1}$$

where

$$W^{\mathrm{a}} = \int_{t'\in\mathcal{R}} P^{\mathrm{a}}(t')\,\mathrm{d}t' \tag{16.4-2}$$

is the *total elastodynamic energy absorbed by the scatterer* and

$$P^{\mathrm{a}}(t') = -\Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau_{p,q}(x,t')v_r(x,t')\right]\mathrm{d}A \tag{16.4-3}$$

is the instantaneous elastodynamic power absorbed by the scatterer. (Note that the minus sign in front of the integral sign on the right-hand side of Equation (16.4-3) is due to the fact that power absorption by the scatterer is affected by an inward power flow, while ν_m points away from the scatterer.)

Next, we substitute in the right-hand side of Equation (16.4-3) the relation

$$\{\tau_{p,q},v_r\} = \{\tau^{\mathrm{i}}_{p,q} + \tau^{\mathrm{s}}_{p,q}, v^{\mathrm{i}}_r + v^{\mathrm{s}}_r\}\,, \tag{16.4-4}$$

and observe that the incident wave dissipates no net energy upon traversing the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer when this domain has the elastodynamic properties of the lossless embedding. Hence, with

$$P^{\mathrm{i}}(t') = -\Delta^{+}_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau^{\mathrm{i}}_{p,q}(x,t')v^{\mathrm{i}}_r(x,t')\right]\mathrm{d}A \tag{16.4-5}$$

as the instantaneous elastodynamic power that the incident wave carries across $\partial\mathcal{D}^{\mathrm{s}}$ towards the domain $\mathcal{D}^{\mathrm{s}}$, we have

$$W^{\mathrm{i}} = \int_{t'\in\mathcal{R}} P^{\mathrm{i}}(t')\,\mathrm{d}t' = 0\,. \tag{16.4-6}$$

Furthermore, the *total elastodynamic energy carried by the scattered wave* across $\partial\mathcal{D}^{\mathrm{s}}$ towards the embedding is introduced as

$$W^{\mathrm{s}} = \int_{t'\in\mathcal{R}} P^{\mathrm{s}}(t')\,\mathrm{d}t'\,, \tag{16.4-7}$$

where

$$P^{\mathrm{i}}(t') = \Delta^{+}_{m,r,p,q} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau^{\mathrm{s}}_{p,q}(x,t') v^{\mathrm{s}}_r(x,t') \right] \mathrm{d}A \tag{16.4-8}$$

is the instantaneous elastodynamic power that the scattered wave carries across $\partial\mathcal{D}^{\mathrm{s}}$ towards the embedding. Using Equations (16.4-4)–(16.4-8) in Equations (16.4-1)–(16.4-3), it follows that

$$-\int_{t' \in \mathcal{R}} \mathrm{d}t' \, \Delta^{+}_{m,r,p,q} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau^{\mathrm{i}}_{p,q}(x,t') v^{\mathrm{s}}_r(x,t') - \tau^{\mathrm{s}}_{p,q}(x,t') v^{\mathrm{i}}_r(x,t') \right] \mathrm{d}A$$

$$= W^{\mathrm{a}} + W^{\mathrm{s}}. \tag{16.4-9}$$

Equation (16.4-9) holds for an arbitrary incident wave field, in particular for both an incident plane P-wave and an incident plane S-wave. The ensuing energy theorem differs for the two kinds of incident waves and the two cases will, therefore, be discussed separately below.

Incident P-wave

First, the incident wave is taken to be the uniform plane P-wave

$$\{\tau^{\mathrm{i}}_{p,q}, v^{\mathrm{i}}_r\} = \{T^{P}_{p,q}, V^{P}_r\} a^{P}(t - \alpha^{P}_s x_s / c_P), \tag{16.4-10}$$

with

$$T^{P}_{p,q} = -c_P^{-1} \left[\lambda \delta_{p,q} (\alpha^{P}_k V^{P}_k) + 2\mu (\alpha^{P}_k V^{P}_k) \alpha^{P}_p \alpha^{P}_q \right]. \tag{16.4-11}$$

Using Equations (16.4-10) and (16.4-11) in Equations (16.1-31), (16.1-33) and (16.1-36) it follows that

$$-\int_{t' \in \mathcal{R}} \mathrm{d}t' \, \Delta^{+}_{m,r,p,q} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[-\tau^{\mathrm{i}}_{p,q}(x,t') v^{\mathrm{s}}_r(x,t') - \tau^{\mathrm{s}}_{p,q}(x,t') v^{\mathrm{i}}_r(x,t') \right] \mathrm{d}A$$

$$= -\rho V^{P}_r \int_{t' \in \mathcal{R}} a^{P}(t') \mathrm{I}_t v^{\mathrm{s};P,\infty}_r(\alpha^{P}, t') \, \mathrm{d}t'. \tag{16.4-12}$$

Substitution of Equation (16.4-12) in Equation (16.4-9) leads to

$$-\rho V^{P}_r \int_{t' \in \mathcal{R}} a^{P}(t') \mathrm{I}_t v^{\mathrm{s};P,\infty}_r(\alpha^{P}, t') \, \mathrm{d}t' = W^{\mathrm{a}} + W^{\mathrm{s}}. \tag{16.4-13}$$

Equation (16.4-13) is the desired *P-wave time-domain energy relation*. It relates the sum of the elastodynamic energies absorbed and scattered by the object to the scattered P-wave amplitude in the far-field region, for observation of this wave in the direction $\boldsymbol{\alpha}^{P}$ of propagation of the incident plane P-wave, i.e. in the "forward" direction, or "behind" the scatterer (Figure 16.4-1).

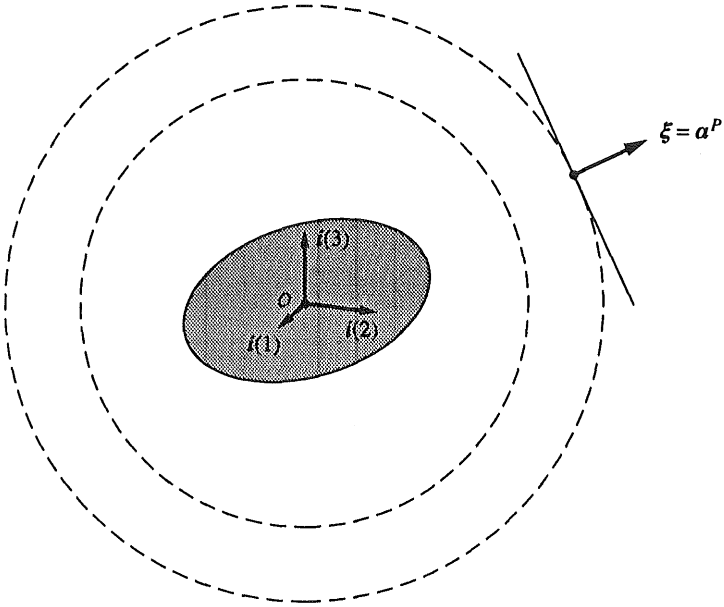


Figure 16.4-1 Elastodynamic scattering configuration for the energy theorem about the far-field forward scattered P-wave amplitude.

Incident S-wave

Secondly, the incident wave is taken to be the uniform plane S-wave

$$\{\tau^{\mathrm{i}}_{p,q}, v^{\mathrm{i}}_r\} = \{T^S_{p,q}, V^S_r\} a^S(t - \alpha^S_s x_s / c_S)\,, \tag{16.4-14}$$

with

$$T^S_{p,q} = -c_S^{-1}\mu(\alpha^S_p V^S_q + \alpha^S_q V^S_p)\,. \tag{16.4-15}$$

Using Equations (16.4-14) and (16.4-15) in Equations (16.1-32), (16.1-33) and (16.1-37) it follows that

$$-\int_{t'\in\mathcal{R}} \mathrm{d}t'\, \Delta^+_{m,r,p,q} \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[-\tau^{\mathrm{i}}_{p,q}(x,t')v^{\mathrm{s}}_r(x,t') - \tau^{\mathrm{s}}_{p,q}(x,t')v^{\mathrm{i}}_r(x,t')\right]\mathrm{d}A$$

$$= -\rho V^S_r \int_{t'\in\mathcal{R}} a^S(t')\mathrm{I}_t v^{\mathrm{s};S,\infty}_r(\alpha^S, t')\,\mathrm{d}t'. \tag{16.4-16}$$

Substitution of Equation (16.4-16) in Equation (16.4-9) leads to

$$-\rho V^S_r \int_{t'\in\mathcal{R}} a^S(t')\mathrm{I}_t v^{\mathrm{s};S,\infty}_r(\alpha^S, t')\,\mathrm{d}t' = W^{\mathrm{a}} + W^{\mathrm{s}}. \tag{16.4-17}$$

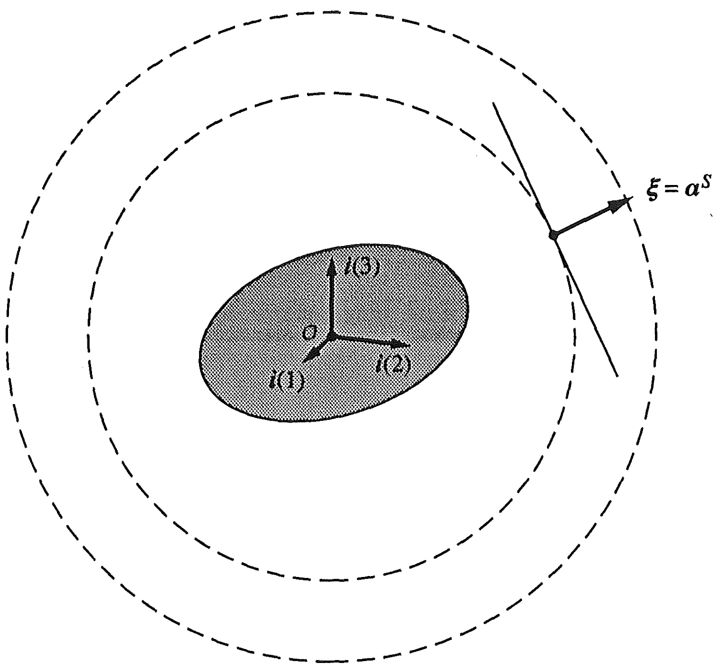


Figure 16.4-2 Elastodynamic scattering configuration for the energy theorem about the far-field forward scattered S-wave amplitude.

Equation (16.4-17) is the desired *S-wave time-domain energy relation.* It relates the sum of the elastodynamic energies absorbed and scattered by the object to the scattered S-wave amplitude in the far-field region, for observation of this wave in the direction α^S of propagation of the incident plane S-wave, i.e. in the "forward" direction, or "behind" the scatterer (Figure 16.4-2).

It is noted that for a lossless elastodynamically penetrable scatterer we have $W^a = 0$. Also, $W^a = 0$ for an impenetrable scatterer, since the right-hand side of Equation (16.4-3) then vanishes in view of the pertaining boundary conditions (Equation (16.1-3) or Equation (16.1-4)). Note also that in the derivation of the result we have nowhere used the linearity in the elastodynamic behaviour of the scatterer. Therefore, Equations (16.4-13) and (16.4-17) also hold for non-linear elastodynamic scatterers, subject to the condition, of course, that the embedding retains its linear elastodynamic properties.

Complex frequency-domain version of the energy theorem

In the complex frequency-domain version of the theorem we start from Equation (16.3-71) and take state A identical to state B. Furthermore, for the case of an elastodynamically penetrable scatterer the solid occupying the scattering domain $\mathcal{D}^s$ is no longer assumed to be time-reverse self-adjoint, i.e. it may have non-zero elastodynamic losses. Thus, we are led to consider the expression

$$\tfrac{1}{4}\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\left[-\hat{\tau}_{p,q}(x,s)\hat{v}_r(x,-s)-\hat{\tau}_{p,q}(x,-s)\hat{v}_r(x,s)\right]\mathrm{d}A=-\hat{P}^{\mathrm{a}}(s)\,, \tag{16.4-18}$$

where the symbol on the right-hand side has been chosen for reasons of equivalence with Equation (16.4-3) and the factor $\frac{1}{4}$ in the left-hand side has been included because of its occurrence in the time-averaged elastodynamic power flow of sinusoidally in time-varying wave fields. It must be stressed, however, that $\hat{P}^{\mathrm{a}}(s)$ is *not* the time Laplace transform of $P^{\mathrm{a}}(t)$.

In the left-hand side of Equation (16.4-18) we now substitute the relation

$$\{\hat{\tau}_{p,q},\hat{v}_r\}=\{\hat{\tau}^{\mathrm{i}}_{p,q}+\hat{\tau}^{\mathrm{s}}_{p,q},\hat{v}^{\mathrm{i}}_r+\hat{v}^{\mathrm{s}}_r\}\,, \tag{16.4-19}$$

and observe that

$$\tfrac{1}{4}\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\left[-\hat{\tau}^{\mathrm{i}}_{p,q}(x,s)\hat{v}^{\mathrm{i}}_r(x,-s)-\hat{\tau}^{\mathrm{i}}_{p,q}(x,-s)\hat{v}^{\mathrm{i}}_r(x,s)\right]\mathrm{d}A=0\,, \tag{16.4-20}$$

since the solid in the embedding has been assumed to be time-reverse self-adjoint.

Furthermore, we introduce, by analogy with Equation (16.4-8), the quantity

$$\hat{P}^{\mathrm{s}}(s)=\tfrac{1}{4}\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\left[-\hat{\tau}^{\mathrm{s}}_{p,q}(x,s)\hat{v}^{\mathrm{s}}_r(x,-s)-\hat{\tau}^{\mathrm{s}}_{p,q}(x,-s)\hat{v}^{\mathrm{s}}_r(x,s)\right]\mathrm{d}A \tag{16.4-21}$$

that is associated with the elastodynamic power carried by the scattered wave. Using Equations (16.4-19)–(16.4-21) in Equation (16.4-18), it follows that

$$\begin{aligned}-\tfrac{1}{4}\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\Big[&-\hat{\tau}^{\mathrm{i}}_{p,q}(x,s)\hat{v}^{\mathrm{s}}_r(x,-s)-\hat{\tau}^{\mathrm{s}}_{p,q}(x,s)\hat{v}^{\mathrm{i}}_r(x,-s)\\&-\hat{\tau}^{\mathrm{i}}_{p,q}(x,-s)\hat{v}^{\mathrm{s}}_r(x,s)-\hat{\tau}^{\mathrm{s}}_{p,q}(x,-s)\hat{v}^{\mathrm{i}}_r(x,s)\Big]\mathrm{d}A=\hat{P}^{\mathrm{a}}(s)+\hat{P}^{\mathrm{s}}(s)\,.\end{aligned} \tag{16.4-22}$$

Equation (16.4-22) holds for an arbitrary incident wave field, in particular for both an incident P-wave and an incident S-wave. The ensuing energy theorem differs for the two kinds of incident plane waves, and the two cases will, therefore, be discussed separately below.

Incident *P*-wave

First, the incident wave is taken to be the uniform plane P-wave

$$\{\hat{\tau}^{\mathrm{i}}_{p,q},\hat{v}^{\mathrm{i}}_r\}=\{T^P_{p,q},V^P_r\}\hat{a}^P(s)\exp(-s\alpha^P_s x_s/c_P)\,, \tag{16.4-23}$$

with

$$T^P_{p,q}=-c_P^{-1}\left[\lambda\delta_{p,q}(\alpha^P_k V^P_k)+2\mu(\alpha^P_k V^P_k)\alpha^P_p\alpha^P_q\right]. \tag{16.4-24}$$

Using Equations (16.4-23) and (16.4-24) in Equations (16.1-82), (16.1-84) and (16.1-87) it follows that

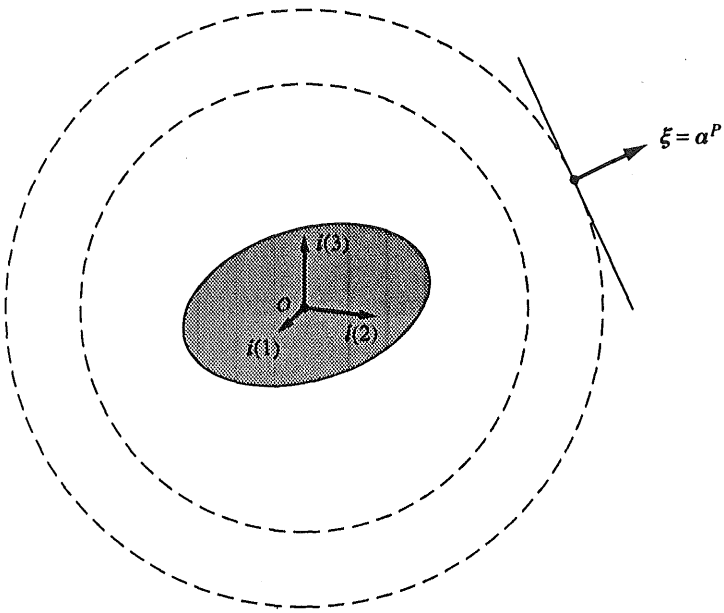


Figure 16.4-3 Elastodynamic scattering configuration for the energy theorem about the far-field forward scattered P-wave amplitude.

$$
\begin{aligned}
&-\tfrac{1}{4}\Delta^{+}_{m,r,p,q}\int_{x\in\partial\mathcal{D}^{s}} \nu_m\Big[-\hat{\tau}^{i}_{p,q}(x,s)\hat{v}^{s}_{r}(x,-s)-\hat{\tau}^{s}_{p,q}(x,s)\hat{v}^{i}_{r}(x,-s)\\
&\qquad -\hat{\tau}^{i}_{p,q}(x,-s)\hat{v}^{s}_{r}(x,s)-\hat{\tau}^{s}_{p,q}(x,-s)\hat{v}^{i}_{r}(x,s)\Big]\,\mathrm{d}A\\
&=-\tfrac{1}{4}s^{-1}\rho\Big[V^P_r\hat{a}^P(s)\hat{v}^{s;P,\infty}_{r}(\alpha^P,-s)-V^P_r\hat{a}^P(-s)\hat{v}^{s;P,\infty}_{r}(\alpha^P,s)\Big].
\end{aligned}
\tag{16.4-25}
$$

Substitution of Equation (16.4-25) in Equation (16.4-22) leads to

$$
=-\tfrac{1}{4}s^{-1}\rho\Big[V^P_r\hat{a}^P(s)\hat{v}^{s;P,\infty}_{r}(\alpha^P,-s)-V^P_r\hat{a}^P(-s)\hat{v}^{s;P,\infty}_{r}(\alpha^P,s)\Big]=\hat{P}^{a}(s)+\hat{P}^{s}(s)\,. \tag{16.4-26}
$$

Equation (16.4-26) is the desired complex frequency-domain P-wave energy relation. It relates the sum of the quantities $\hat{P}^{a}(s)$ and $\hat{P}^{s}(s)$ to the scattered P-wave amplitude in the far-field region for observation of this wave in the direction of propagation of the incident plane P-wave, i.e. in the "forward" direction, or "behind" the scatterer (Figure 16.4-3).

Incident S-wave

Secondly, the incident wave is taken to be the uniform plane S-wave

$$\{\hat{\tau}^{\mathrm{i}}_{p,q},\hat{v}^{\mathrm{i}}_r\} = \{T^S_{p,q},V^S_r\}\hat{a}^S(s)\exp(-s\alpha^S_s x_s/c_S)\,, \tag{16.4-27}$$

with

$$T^S_{p,q} = -c_S^{-1}\mu\left[\alpha^S_p V^S_q + \alpha^S_q V^S_p\right]. \tag{16.4-28}$$

Using Equations (16.4-27) and (16.4-28) in Equations (16.1-83), (16.1-85) and (16.1-88) it follows that

$$\begin{aligned}&-\tfrac{1}{4}\Delta^+_{m,r,p,q}\int_{x\in\partial\mathcal{D}^s} \nu_m\Big[-\hat{\tau}^{\mathrm{i}}_{p,q}(x,s)\hat{v}^{\mathrm{s}}_r(x,-s) - \hat{\tau}^{\mathrm{s}}_{p,q}(x,s)\hat{v}^{\mathrm{i}}_r(x,-s)\\&\qquad - \hat{\tau}^{\mathrm{i}}_{p,q}(x,-s)\hat{v}^{\mathrm{s}}_r(x,s) - \hat{\tau}^{\mathrm{s}}_{p,q}(x,-s)\hat{v}^{\mathrm{i}}_m(x,s)\Big]\,\mathrm{d}A\\&= -\tfrac{1}{4}s^{-1}\rho\left[V^S_r\hat{a}^S(s)\hat{v}^{\mathrm{s};S,\infty}_r(\alpha^S,-s) - V^S_r\hat{a}^S(-s)\hat{v}^{\mathrm{s};S,\infty}_r(\alpha^S,s)\right].\end{aligned} \tag{16.4-29}$$

Substitution of Equation (16.4-29) in Equation (16.4-22) leads to

$$-\tfrac{1}{4}s^{-1}\rho\left[V^S_r\hat{a}^S(s)\hat{v}^{\mathrm{s};S,\infty}_r(\alpha^S,-s) - V^S_r\hat{a}^S(-s)\hat{v}^{\mathrm{s};S,\infty}_r(\alpha^S,s)\right] = \hat{P}^{\mathrm{a}}(s) + \hat{P}^{\mathrm{s}}(s)\,. \tag{16.4-30}$$

Equation (16.4-30) is the desired complex frequency-domain S-wave energy relation. It relates the sum of the quantities $\hat{P}^{\mathrm{a}}(s)$ and $\hat{P}^{\mathrm{s}}(s)$ to the scattered S-wave amplitude in the far-field region for observation of this wave in the direction of propagation of the incident plane S-wave, i.e. in the "forward" direction, or "behind" the scatterer (Figure 16.4-4).

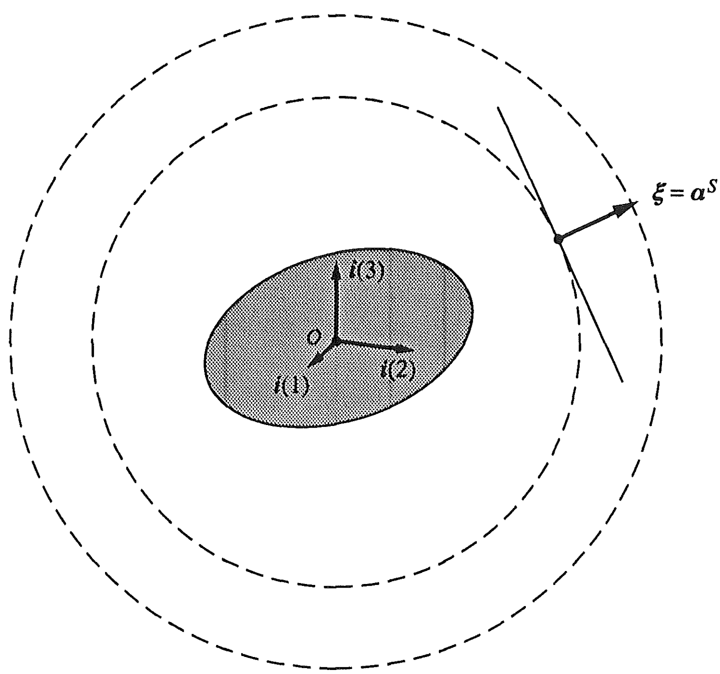


Figure 16.4-4 Elastodynamic scattering configuration for the energy theorem about the far-field forward scattered S-wave amplitude.

It is noted that for a lossless elastodynamically penetrable scatterer we have $\hat{P}^{\mathrm{a}} = 0$. Also, $\hat{P}^{\mathrm{a}} = 0$ for an impenetrable scatterer, since the left-hand side of Equation (16.4-18) vanishes in view of the pertaining boundary conditions (Equation (16.1-53) or Equation (16.1-54)). For imaginary values of s, i.e. $s = \mathrm{j}\omega$ with $\omega \in \mathcal{R}$, Equations (16.4-26) and (16.4-30) are known as the "*P*- and *S*-wave *extinction cross-section theorems*", respectively (see Exercises 16.4-1 and 16.4-2). Note that in the complex frequency-domain result (contrary to the corresponding time-domain result) the linearity in the elastodynamic behaviour of the scatterer has implicitly been used since the space–time wave quantities have been represented, through the Bromwich integral, as a (linear) superposition of exponential time functions.

References to the earlier literature on the subject can be found in De Hoop (1985), De Hoop (1959), and Tan (1976).

Exercises

Exercise 16.4-1

Consider, in the complex frequency-domain *P*-wave energy relation Equation (16.4-26), the case $s = \mathrm{j}\omega$. Observe that, since all time-domain wave quantities are real-valued, the quantities $\hat{P}^{\mathrm{s}}(s)$ as introduced in Equation (16.4-18) and $\hat{P}^{\mathrm{a}}(s)$ as introduced in Equation (16.4-21) have the property $\hat{P}^{\mathrm{a}}(s) = \hat{P}^{\mathrm{a}}(-s)$ and $\hat{P}^{\mathrm{s}}(s) = \hat{P}^{\mathrm{s}}(-s)$. As a consequence, $\hat{P}^{\mathrm{a}}(\mathrm{j}\omega)$ and $\hat{P}^{\mathrm{s}}(\mathrm{j}\omega)$ are real-valued. Next, introduce the quantity

$$\hat{S}^{\mathrm{i}}(s) = \tfrac{1}{2}\rho c_P V_r^P V_r^P \hat{a}^P(s)\hat{a}^P(-s) \tag{16.4-31}$$

that is associated with the elastodynamic power flow density in the incident *P*-wave. Also $\hat{S}^{\mathrm{i}}(s) = \hat{S}^{\mathrm{i}}(-s)$ and hence, $\hat{S}^{\mathrm{i}}(\mathrm{j}\omega)$ is real-valued. Furthermore, let

$$\hat{\sigma}^{\mathrm{a}}(s) = \hat{P}^{\mathrm{a}}(s)/\hat{S}^{\mathrm{i}}(s) \tag{16.4-32}$$

denote the complex frequency-domain *absorption cross-section* of the scattering object and

$$\hat{\sigma}^{\mathrm{s}}(s) = \hat{P}^{\mathrm{s}}(s)/\hat{S}^{\mathrm{i}}(s) \tag{16.4-33}$$

its *scattering cross-section*. Note that $\hat{\sigma}^{\mathrm{a}}(s) = \hat{\sigma}^{\mathrm{a}}(-s)$ and $\hat{\sigma}^{\mathrm{s}}(s) = \hat{\sigma}^{\mathrm{s}}(-s)$, which entails that $\hat{\sigma}^{\mathrm{a}}(\mathrm{j}\omega)$ and $\hat{\sigma}^{\mathrm{s}}(\mathrm{j}\omega)$ are real-valued. Show that, for $s = \mathrm{j}\omega$, Equation (16.4-26) leads to

$$\hat{\sigma}^{\mathrm{a}}(\mathrm{j}\omega) + \hat{\sigma}^{\mathrm{s}}(\mathrm{j}\omega) = \frac{1}{\omega c_P}\,\frac{\mathrm{Im}\left[V_r^P \hat{a}^P(-\mathrm{j}\omega)\hat{v}_r^{\mathrm{s};P,\infty}(\boldsymbol{\alpha}^P,\mathrm{j}\omega)\right]}{V_r^P V_r^P |\hat{a}^P(\mathrm{j}\omega)|^2} \tag{16.4-34}$$

for the uniform plane *P*-wave incidence. Equation (16.4-34) is known as the *extinction cross-section theorem* for the scattering of plane *P*-waves (see Tan, 1976). (*Note*: Extinction cross-section = absorption cross-section + scattering cross-section.)

Exercise 16.4-2

Consider, in the complex frequency-domain S-wave energy relation Equation (16.4-30), the case $s = \mathrm{j}\omega$. Observe that, since all time-domain wave quantities are real-valued, the quantities $\hat{P}^{\mathrm{a}}(s)$ as introduced in Equation (16.4-18) and $\hat{P}^{\mathrm{s}}(s)$ as introduced in Equation (16.4-21) have the property $\hat{P}^{\mathrm{a}}(s) = \hat{P}^{\mathrm{a}}(-s)$ and $\hat{P}^{\mathrm{s}}(s) = \hat{P}^{\mathrm{s}}(-s)$. As a consequence, $\hat{P}^{\mathrm{a}}(\mathrm{j}\omega)$ and $\hat{P}^{\mathrm{s}}(\mathrm{j}\omega)$ are real-valued. Next, introduce the quantity

$$\hat{S}^{\mathrm{i}}(s) = \tfrac{1}{2}\rho c_S V_r^S V_r^S \hat{a}^S(s)\hat{a}^S(-s) \tag{16.4-35}$$

that is associated with the elastodynamic power flow density in the incident S-wave. Also $\hat{S}^{\mathrm{i}}(s) = \hat{S}^{\mathrm{i}}(-s)$ and hence, $\hat{S}^{\mathrm{i}}(\mathrm{j}\omega)$ is real-valued. Furthermore, let

$$\hat{\sigma}^{\mathrm{a}}(s) = \hat{P}^{\mathrm{a}}(s)/\hat{S}^{\mathrm{i}}(s) \tag{16.4-36}$$

denote the complex frequency-domain *absorption cross-section* of the scattering object and

$$\hat{\sigma}^{\mathrm{s}}(s) = \hat{P}^{\mathrm{s}}(s)/\hat{S}^{\mathrm{i}}(s) \tag{16.4-37}$$

its *scattering cross-section*. Note that $\hat{\sigma}^{\mathrm{a}}(s) = \hat{\sigma}^{\mathrm{a}}(-s)$ and $\hat{\sigma}^{\mathrm{s}}(s) = \hat{\sigma}^{\mathrm{s}}(-s)$, which entails that $\hat{\sigma}^{\mathrm{a}}(\mathrm{j}\omega)$ and $\hat{\sigma}^{\mathrm{s}}(\mathrm{j}\omega)$ are real-valued. Show that, for $s = \mathrm{j}\omega$, Equation (16.4-30) leads to

$$\hat{\sigma}^{\mathrm{a}}(\mathrm{j}\omega) + \hat{\sigma}^{\mathrm{s}}(\mathrm{j}\omega) = \frac{1}{\omega c_S}\,\frac{\mathrm{Im}\left[V_r^S\hat{a}^S(-\mathrm{j}\omega)\hat{v}_r^{\mathrm{s};S,\infty}(\alpha^S,\mathrm{j}\omega)\right]}{V_r^S V_r^S|\hat{a}^S(\mathrm{j}\omega)|^2} \tag{16.4-38}$$

for the uniform plane S-wave incidence. Equation (16.4-38) is known as the *extinction cross-section theorem* for the scattering of plane S-waves (see Tan, 1976). (*Note*: Extinction cross-section = absorption cross-section + scattering cross-section.)

16.5 The Neumann expansion in the integral equation formulation of the scattering by a penetrable object

In this section we discuss the Neumann expansion in the integral equation formulation of the elastodynamic scattering problem. The expansion is an analytic procedure that applies to a *penetrable scatterer*. The procedure is *iterative* in nature and is expected to converge for sufficiently low contrast of the scatterer with respect to its embedding.

Time-domain analysis

In the time-domain presentation of the method we start from Equations (15.9-5) and (15.9-20)–(15.9-23), which, through combination of the time convolutions, we write for the present configuration as

$$\tau_{p,q}(x',t) = \tau^{\rm i}_{p,q}(x',t) - \int_{x\in\mathcal{D}^{\rm s}} \Big[\partial_t C_t(G^{\tau h}_{p,q,i',j'},\chi^{\rm s}_{i',j',p',q'} - S_{i',j',p',q'}\delta(t),\tau_{p',q'};x',x,t)$$
$$- \partial_t C_t(G^{\tau f}_{p,q,k'},\mu^{\rm s}_{k',r'} - \rho\delta(t)\delta_{k',r'},v_{r'};x',x,t)\Big]\,\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \tag{16.5-1}$$

and

$$v_r(x',t) = v^{\rm i}_r(x',t) - \int_{x\in\mathcal{D}^{\rm s}} \Big[-\partial_t C_t(G^{vh}_{r,i',j'},\chi^{\rm s}_{i',j',p',q'} - S_{i',j',p',q'}\delta(t),\tau_{p',q'};x',x,t)$$
$$+ \partial_t C_t(G^{vf}_{r,k'},\mu^{\rm s}_{k',r'} - \rho\delta(t)\delta_{k',r'},v_{r'};x',x,t)\Big]\,\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 . \tag{16.5-2}$$

For $x'\in\mathcal{D}^{\rm s}$, Equations (16.5-1) and (16.5-2) constitute a system of linear integral equations of the second kind to be solved for $\{\tau_{p,q},v_r\}$ for $x\in\mathcal{D}^{\rm s}$ and $t\in\mathcal{R}$, and with $\{\tau^{\rm i}_{p,q},v^{\rm i}_r\}$ as forcing terms. To solve these equations analytically, an iterative procedure known as the *Neumann expansion* is set up. The successive steps in this procedure will be labelled by integer superscripts enclosed by brackets ([...]). The procedure is *initialised* by putting

$$\tau^{[0]}_{p,q} = \tau^{\rm i}_{p,q} \qquad \text{for } x'\in\mathcal{R}^3, \tag{16.5-3}$$

$$v^{[0]}_r = v^{\rm i}_r \qquad \text{for } x'\in\mathcal{R}^3. \tag{16.5-4}$$

Next, the procedure is *updated* through

$$\tau^{[n+1]}_{p,q}(x',t) = - \int_{x\in\mathcal{D}^{\rm s}} \Big[-\partial_t C_t(G^{\tau h}_{p,q,i',j'},\chi^{\rm s}_{i',j',p',q'} - S_{i',j',p',q'}\delta(t),\tau^{[n]}_{p',q'};x',x,t)$$
$$- \partial_t C_t(G^{\tau f}_{p,q,k'},\mu^{\rm s}_{k',r'} - \rho\delta(t)\delta_{k',r'},v^{[n]}_{r'};x',x,t)\Big]\,\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{etc. }, \tag{16.5-5}$$

and

$$v^{[n+1]}_r(x',t) = - \int_{x\in\mathcal{D}^{\rm s}} \Big[-\partial_t C_t(G^{vh}_{r,i',j'},\chi^{\rm s}_{i',j',p',q'} - S_{i',j',p',q'}\delta(t),\tau^{[n]}_{p',q'};x',x,t)$$
$$+ \partial_t C_t(G^{vf}_{r,k'},\mu^{\rm s}_{k',r'} - \rho\delta(t)\delta_{k',r'},v^{[n]}_{r'};x',x,t)\Big]\,\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{etc} . \tag{16.5-6}$$

As can be inferred from these updating equations, the terms of order $[n + 1]$ can be expected to be "smaller" than their counterparts of order $[n]$, provided that the contrast quantities are "small enough". On account of this, it can be conjectured that for sufficiently small contrast of the scatterer with respect to its embedding the *procedure is convergent* and we can put

$$\tau_{p,q} = \sum_{n=0}^{\infty} \tau^{[n]}_{p,q} \qquad \text{for } x'\in\mathcal{R}^3, \tag{16.5-7}$$

$$v_r = \sum_{n=0}^{\infty} v^{[n]}_r \qquad \text{for } x'\in\mathcal{R}^3. \tag{16.5-8}$$

Assuming that the series on the right-hand sides of Equations (16.5-7) and (16.5-8) are uniformly convergent, it can easily be proved that $\{\tau_{p,q},v_r\}$ as defined by these equations indeed satisfy Equations (16.5-1) and (16.5-2). To this end we observe that

$$
\begin{aligned}
&-\int_{x\in\mathcal{D}^{\mathrm{s}}}\Big[\partial_t C_t(G^{\tau h}_{p,q,i',j'},\chi^{\mathrm{s}}_{i',j',p',q'}-S_{i',j',p',q'}\delta(t),\tau_{p',q'};x',x,t)\\
&\quad-\partial_t C_t(G^{\tau f}_{p,q,k'},\mu^{\mathrm{s}}_{k',r'}-\rho\delta(t)\delta_{k',r'},v_{r'};x',x,t)\Big]\,\mathrm{d}V\\
&=-\int_{x\in\mathcal{D}^{\mathrm{s}}}\Bigg[\partial_t C_t(G^{\tau h}_{p,q,i',j'},\chi^{\mathrm{s}}_{i',j',p',q'}-S_{i',j',p',q'}\delta(t),\sum_{n=0}^{\infty}\tau^{[n]}_{p',q'};x',x,t)\\
&\quad-\partial_t C_t(G^{\tau f}_{p,q,k'},\mu^{\mathrm{s}}_{k',r'}-\rho\delta(t)\delta_{k',r'},\sum_{n=0}^{\infty}v^{[n]}_{r'};x',x,t)\Bigg]\,\mathrm{d}V\\
&=-\sum_{n=0}^{\infty}\int_{x\in\mathcal{D}^{\mathrm{s}}}\Big[\partial_t C_t(G^{\tau h}_{p,q,i',j'},\chi^{\mathrm{s}}_{i',j',p',q'}-S_{i',j',p',q'}\delta(t),\tau^{[n]}_{p',q'};x',x,t)\\
&\quad-\partial_t C_t(G^{\tau f}_{p,q,k'},\mu^{\mathrm{s}}_{k',r'}-\rho\delta(t)\delta_{k',r'},v^{[n]}_{r'};x',x,t)\Big]\,\mathrm{d}V\\
&=\sum_{n=0}^{\infty}\tau^{[n+1]}_{p,q}(x',t)=\sum_{m=0}^{\infty}\tau^{[m]}_{p,q}(x',t)-\tau^{[0]}_{p,q}(x',t)\\
&=\tau_{p,q}(x',t)-\tau^{\mathrm{i}}_{p,q}(x',t)\qquad\text{for } x'\in\mathcal{R}^3
\end{aligned}
\tag{16.5-9}
$$

and

$$
\begin{aligned}
&-\int_{x\in\mathcal{D}^{\mathrm{s}}}\Big[-\partial_t C_t(G^{vh}_{r,i',j'},\chi^{\mathrm{s}}_{i',j',p',q'}-S_{i',j',p',q'}\delta(t),\tau_{p',q'};x',x,t)\\
&\quad+\partial_t C_t(G^{vf}_{r,k'},\mu^{\mathrm{s}}_{k',r'}-\rho\delta(t)\delta_{k',r'},v_{r'};x',x,t)\Big]\,\mathrm{d}V\\
&=-\int_{x\in\mathcal{D}^{\mathrm{s}}}\Bigg[-\partial_t C_t(G^{vh}_{r,i',j'},\chi^{\mathrm{s}}_{i',j',p',q'}-S_{i',j',p',q'}\delta(t),\sum_{n=0}^{\infty}\tau^{[n]}_{p',q'};x',x,t)\\
&\quad+\partial_t C_t(G^{vf}_{r,k'},\mu^{\mathrm{s}}_{k',r'}-\rho\delta(t)\delta_{k',r'},\sum_{n=0}^{\infty}v^{[n]}_{r'};x',x,t)\Bigg]\,\mathrm{d}V\\
&=-\sum_{n=0}^{\infty}\int_{x\in\mathcal{D}^{\mathrm{s}}}\Big[-\partial_t C_t(G^{vh}_{r,i',j'},\chi^{\mathrm{s}}_{i',j',p',q'}-S_{i',j',p',q'}\delta(t),\tau^{[n]}_{p',q'};x',x,t)\\
&\quad+\partial_t C_t(G^{vf}_{r,k'},\mu^{\mathrm{s}}_{k',r'}-\rho\delta(t)\delta_{k',r'},v^{[n]}_{r'};x',x,t)\Big]\,\mathrm{d}V\\
&=\sum_{n=0}^{\infty}v^{[n+1]}_{r}(x',t)=\sum_{m=0}^{\infty}v^{[m]}_{r}(x',t)-v^{[0]}_{r}(x',t)\\
&=v_r(x',t)-v^{\mathrm{i}}_r(x',t)\qquad\text{for } x'\in\mathcal{R}^3,
\end{aligned}
\tag{16.5-10}
$$

where Equations (16.5-3)–(16.5-8) have been used and the interchange of the summations with respect to n and the integration with respect to x are justified by the assumed uniform convergence of the series expansions. Equations (16.5-9) and (16.5-10) are evidently identical to Equations (16.5-1) and (16.5-2), and, hence, the expansions given in Equations (16.5-7) and (16.5-8) indeed solve the problem.

Complex frequency-domain analysis

In the complex frequency-domain presentation of the method we start from Equations (15.9-28) and (15.9-43)–(15.9-46), which are combined to

$$\hat{\tau}_{p,q}(\boldsymbol{x}',s) = \hat{\tau}^{\rm i}_{p,q}(\boldsymbol{x}',s) - \int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big\{\hat{G}^{\tau h}_{p,q,i',j'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\eta}^{\rm s}_{i',j',p',q'}(\boldsymbol{x},s) - sS_{i',j',p',q'}\right]\hat{\tau}_{p',q'}(\boldsymbol{x},s)$$
$$-\,(\hat{G}^{\tau f}_{p,q,k'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\xi}^{\rm s}_{k',r'}(\boldsymbol{x},s) - s\rho\delta_{k',r'}\right]\hat{v}_{r'}(\boldsymbol{x},s)\Big\}\,{\rm d}V \quad \text{for} \quad \boldsymbol{x}'\in\mathcal{R}^3 \qquad (16.5\text{-}11)$$

and

$$\hat{v}_r(\boldsymbol{x}',s) = \hat{v}^{\rm i}_r(\boldsymbol{x}',s) + \int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big\{\hat{G}^{vh}_{r,i',j'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\eta}^{\rm s}_{i',j',p',q'}(\boldsymbol{x},s) - sS_{i',j',p',q'}\right]\hat{\tau}_{p',q'}(\boldsymbol{x},s)$$
$$-\,(\hat{G}^{vf}_{r,k'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\xi}^{\rm s}_{k',r'}(\boldsymbol{x},s) - s\rho\delta_{k',r'}\right]\hat{v}_{r'}(\boldsymbol{x},s)\Big\}\,{\rm d}V \quad \text{for} \quad \boldsymbol{x}'\in\mathcal{R}^3. \qquad (16.5\text{-}12)$$

For $\boldsymbol{x}'\in\mathcal{D}^{\rm s}$, Equations (16.5-11) and (16.5-12) constitute a system of linear integral equations of the second kind to be solved for $\{\hat{\tau}_{p,q},\hat{v}_r\}$ for $\boldsymbol{x}\in\mathcal{D}^{\rm s}$, and with $\{\hat{\tau}^{\rm i}_{p,q},\hat{v}^{\rm i}_r\}$ as forcing terms. The Neumann procedure to solve these equations is *initialised* by putting

$$\hat{\tau}^{[0]}_{p,q} = \hat{\tau}^{\rm i}_{p,q} \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3, \qquad (16.5\text{-}13)$$

$$\hat{v}^{[0]}_r = \hat{v}^{\rm i}_r \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3. \qquad (16.5\text{-}14)$$

Next, the procedure is *updated* through

$$\hat{\tau}^{[n+1]}_{p,q}(\boldsymbol{x}',s) = -\int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big\{\hat{G}^{\tau h}_{p,q,i',j'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\eta}^{\rm s}_{i',j',p',q'}(\boldsymbol{x},s) - sS_{i',j',p',q'}\right]\hat{\tau}^{[n]}_{p',q'}(\boldsymbol{x},s)$$
$$-\,(\hat{G}^{\tau f}_{p,q,k'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\xi}^{\rm s}_{k',r'}(\boldsymbol{x},s) - s\rho\delta_{k',r'}\right]\hat{v}^{[n]}_{r'}(\boldsymbol{x},s)\Big\}\,{\rm d}V$$
$$\text{for} \quad \boldsymbol{x}'\in\mathcal{R}^3 \quad \text{and} \quad n = 0,1,2, \text{etc.}, \qquad (16.5\text{-}15)$$

and

$$\hat{v}^{[n+1]}_r(\boldsymbol{x}',s) = +\int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big\{\hat{G}^{vh}_{r,i',j'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\eta}^{\rm s}_{i',j',p',q'}(\boldsymbol{x},s) - sS_{i',j',p',q'}\right]\hat{\tau}^{[n]}_{p',q'}(\boldsymbol{x},s)$$
$$-\,(\hat{G}^{vf}_{r,k'}(\boldsymbol{x}',\boldsymbol{x},s)\left[\hat{\xi}^{\rm s}_{k',r'}(\boldsymbol{x},s) - s\rho\delta_{k',r'}\right]\hat{v}^{[n]}_{r'}(\boldsymbol{x},s)\Big\}\,{\rm d}V$$
$$\text{for} \quad \boldsymbol{x}'\in\mathcal{R}^3 \quad \text{and} \quad n = 0,1,2, \text{etc}. \qquad (16.5\text{-}16)$$

Assuming that the *procedure is convergent*, we can put

$$\hat{\tau}_{p,q} = \sum_{n=0}^{\infty} \hat{\tau}^{[n]}_{p,q} \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3, \qquad (16.5\text{-}17)$$

$$\hat{v}_r = \sum_{n=0}^{\infty} \hat{v}^{[n]}_r \qquad \text{for } \boldsymbol{x}'\in\mathcal{R}^3. \qquad (16.5\text{-}18)$$

Assuming that the series on the right-hand sides of Equations (16.5-17) and (16.5-18) are uniformly convergent, it can easily be proved that $\{\hat{\tau}_{p,q},\hat{v}_r\}$ as defined by these equations indeed satisfy Equations (16.5-11) and (16.5-12). To this end we observe that

$$
\begin{aligned}
&-\int_{x\in\mathcal{D}^s}\Big\{\hat{G}^{\tau h}_{p,q,i',j'}(x',x,s)\left[\hat{\eta}^{s}_{i',j',p',q'}(x,s)-sS_{i',j',p',q'}\right]\hat{\tau}_{p',q'}(x,s)\\
&\quad-(\hat{G}^{\tau f}_{p,q,k'}(x',x,s)\left[\hat{\xi}^{s}_{k',r'}(x,s)-s\rho\delta_{k',r'}\right]\hat{v}_{r'}(x,s)\Big\}\,\mathrm{d}V\\
&=-\int_{x\in\mathcal{D}^s}\left\{\hat{G}^{\tau h}_{p,q,i',j'}(x',x,s)\left[\hat{\eta}^{s}_{i',j',p',q'}(x,s)-sS_{i',j',p',q'}\right]\sum_{n=0}^{\infty}\hat{\tau}^{[n]}_{p',q'}(x,s)\right.\\
&\quad\left.-(\hat{G}^{\tau f}_{p,q,k'}(x',x,s)\left[\hat{\xi}^{s}_{k',r'}(x,s)-s\rho\delta_{k',r'}\right]\sum_{n=0}^{\infty}\hat{v}^{[n]}_{r'}(x,s)\right\}\mathrm{d}V\\
&=-\sum_{n=0}^{\infty}\int_{x\in\mathcal{D}^s}\Big\{\hat{G}^{\tau h}_{p,q,i',j'}(x',x,s)\left[\hat{\eta}^{s}_{i',j',p',q'}(x,s)-sS_{i',j',p',q'}\right]\hat{\tau}^{[n]}_{p',q'}(x,s)\\
&\qquad-(\hat{G}^{\tau f}_{p,q,k'}(x',x,s)\left[\hat{\xi}^{s}_{k',r'}(x,s)-s\rho\delta_{k',r'}\right]\hat{v}^{[n]}_{r'}(x,s)\Big\}\,\mathrm{d}V\\
&=\sum_{n=0}^{\infty}\hat{\tau}^{[n+1]}_{p,q}(x',s)=\sum_{m=0}^{\infty}\hat{\tau}^{[m]}_{p,q}(x',s)-\hat{\tau}^{[0]}_{p,q}(x',s)\\
&=\hat{\tau}_{p,q}(x',s)-\hat{\tau}^{\mathrm{i}}_{p,q}(x',s)\quad\text{for } x'\in\mathcal{R}^3
\end{aligned}
\tag{16.5-19}
$$

and

$$
\begin{aligned}
&\int_{x\in\mathcal{D}^s}\Big\{\hat{G}^{vh}_{r,i',j'}(x',x,s)\left[\hat{\eta}^{s}_{i',j',p',q'}(x,s)-sS_{i',j',p',q'}\right]\hat{\tau}_{p',q'}(x,s)\\
&\quad-(\hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\xi}^{s}_{k',r'}(x,s)-s\rho\delta_{k',r'}\right]\hat{v}_{r'}(x,s)\Big\}\,\mathrm{d}V\\
&=\int_{x\in\mathcal{D}^s}\left\{\hat{G}^{vh}_{r,i',j'}(x',x,s)\left[\hat{\eta}^{s}_{i',j',p',q'}(x,s)-sS_{i',j',p',q'}\right]\sum_{n=0}^{\infty}\hat{\tau}^{[n]}_{p',q'}(x,s)\right.\\
&\quad\left.-(\hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\xi}^{s}_{k',r'}(x,s)-s\rho\delta_{k',r'}\right]\sum_{n=0}^{\infty}\hat{v}^{[n]}_{r'}(x,s)\right\}\mathrm{d}V\\
&=\sum_{n=0}^{\infty}\int_{x\in\mathcal{D}^s}\Big\{\hat{G}^{vh}_{r,i',j'}(x',x,s)\left[\hat{\eta}^{s}_{i',j',p',q'}(x,s)-sS_{i',j',p',q'}\right]\hat{\tau}^{[n]}_{p',q'}(x,s)\\
&\qquad-(\hat{G}^{vf}_{r,k'}(x',x,s)\left[\hat{\xi}^{s}_{k',r'}(x,s)-s\rho\delta_{k',r'}\right]\hat{v}^{[n]}_{r'}(x,s)\Big\}\,\mathrm{d}V\\
&=\sum_{n=0}^{\infty}\hat{v}^{[n+1]}_{r}(x',s)=\sum_{m=0}^{\infty}\hat{v}^{[m]}_{r}(x',s)-\hat{v}^{[0]}_{r}(x',s)\\
&=\hat{v}_r(x',s)-\hat{v}^{\mathrm{i}}_r(x',s)\quad\text{for } x'\in\mathcal{R}^3,
\end{aligned}
\tag{16.5-20}
$$

where Equations (16.5-13)–(16.5-18) have been used and the interchange of the summations with respect to n and the integrations with respect to $\boldsymbol{x}$ is justified by the assumed uniform convergence of the series expansions. Equations (16.5-19) and (16.5-20) are evidently identical to Equations (16.5-11) and (16.5-12) and hence the expansions given in Equations (16.5-17) and (16.5-18) indeed solve the problem.

The construction of convergence criteria for the Neumann expansion is complicated by the singularities of the Green's functions. For the simpler case of the scattering problem associated with the scalar wave equation, a convergence criterion has been derived (De Hoop, 1991).

The nth term in the Neumann expansion is also known as the nth *Rayleigh–Gans–Born approximation.*

16.6 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation; time-domain analysis and complex frequency-domain analysis for canonical geometries of the scattering object

In this section the far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation is further investigated. In particular, closed-form analytic expressions are derived for the far-field scattered P- and S-wave amplitudes associated with the incident uniform plane P-or S-wave scattering by a *homogeneous object* in the shape of an ellipsoid, a rectangular block, an elliptical cylinder of finite height, an elliptical cone of finite height, or a tetrahedron. A structure consisting of the union of the listed objects can, in the first-order Rayleigh–Gans–Born approximation, be dealt with by superposition. The cases of an incident plane P-wave and an incident plane S-wave will be dealt with separately.

Time-domain analysis

In the time-domain analysis, the expressions for the scattered wave amplitude in the far-field region in the first-order Rayleigh–Gans–Born approximation follow from Equations (16.1-1), (16.1-2), (16.1-21)–(16.1-28), (16.5-3), (16.5-4), (16.5-5) and (16.5-6) for $n = 0$.

Incident *P*-wave

For the incident uniform plane P-wave given by Equations (16.1-5)–(16.1-8), the far-field scattered P-wave amplitude is obtained as (Figure 16.6-1)

$$
\begin{aligned}
v_r^{s;P,P,\infty} = & -\xi_r \xi_k \Lambda_{k,r'}^{\mu;P}(\boldsymbol{\xi}/c_P - \boldsymbol{\alpha}^P/c_P, t) V_{r'}^P \\
& + (\rho c_P)^{-1} \xi_m \xi_r \xi_k \Lambda_{k,m,p',q'}^{\chi;P}(\boldsymbol{\xi}/c_P - \boldsymbol{\alpha}^P/c_P, t) T_{p',q'}^P ,
\end{aligned}
\tag{16.6-1}
$$

with

$$
\Lambda_{k,r'}^{\mu;P}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[\rho^{-1}\mu_{k,r'}(\boldsymbol{x},t') - \delta_{k,r'}\delta(t')\right] \partial_t^2 a^P(t - t' + u_s x_s)\, \mathrm{d}t'
\tag{16.6-2}
$$

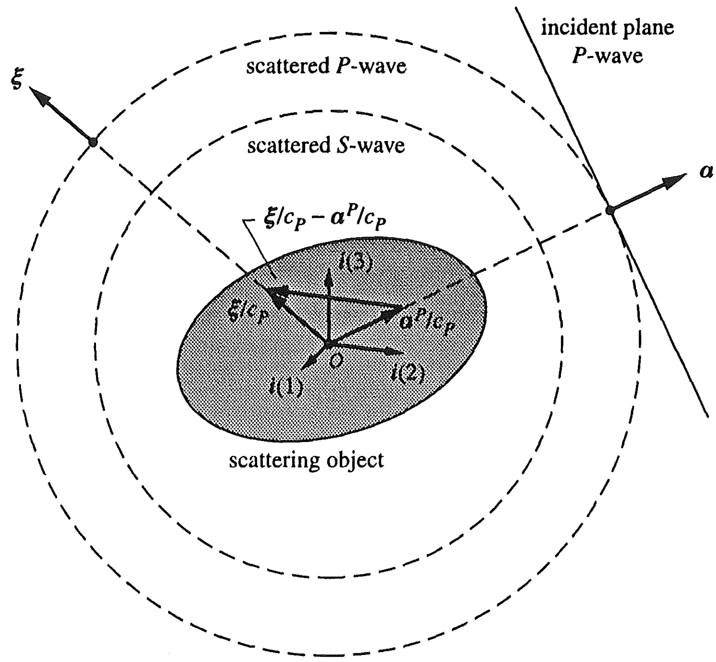


Figure 16.6-1 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident P-wave, scattered P-wave).

and

$$\Lambda^{\chi;P}_{k,m,p',q'}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[C_{k,m,i,j}\chi_{i,j,p',q'}(\boldsymbol{x},t') - \Delta^{+}_{k,m,p',q'}\delta(t')\right]$$
$$\times\, \partial_t^2 a^P(t - t' + u_s x_s)\,\mathrm{d}t' , \tag{16.6-3}$$

while

$$\tau^{s;P,P,\infty}_{p,q} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v^{s;P,P,\infty}_k) + 2\mu(\xi_k v^{s;P,P,\infty}_k)\xi_p\xi_q\right], \tag{16.6-4}$$

and the far-field scattered S-wave amplitudes are (Figure 16.6-2)

$$v^{s;S,P,\infty}_r = -(\delta_{r,k} - \xi_r\xi_k)\Lambda^{\mu;P}_{k,r'}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^P/c_P,t)V^P_{r'}$$
$$+ (\rho c_S)^{-1}\xi_m(\delta_{r,k} - \xi_r\xi_k)\Lambda^{\chi;P}_{k,m,p',q'}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^P/c_P,t)T^P_{p',q'} , \tag{16.6-5}$$

while

$$\tau^{s;S,P,\infty}_{p,q} = -c_S^{-1}\mu(\xi_p v^{s;S,P,\infty}_q + \xi_q v^{s;S,P,\infty}_p) . \tag{16.6-6}$$

For a *homogeneous object*, Equations (16.6-2) and (16.6-3) reduce to

$$\Lambda^{\mu;P}_{k,r'}(\boldsymbol{u},t) = \int_{t'=0}^{\infty} \left[\rho^{-1}\mu_{k,r'}(t') - \delta_{k,r'}\delta(t')\right]\hat{\Upsilon}^P(t - t' + u_s x_s)\,\mathrm{d}t' \tag{16.6-7}$$

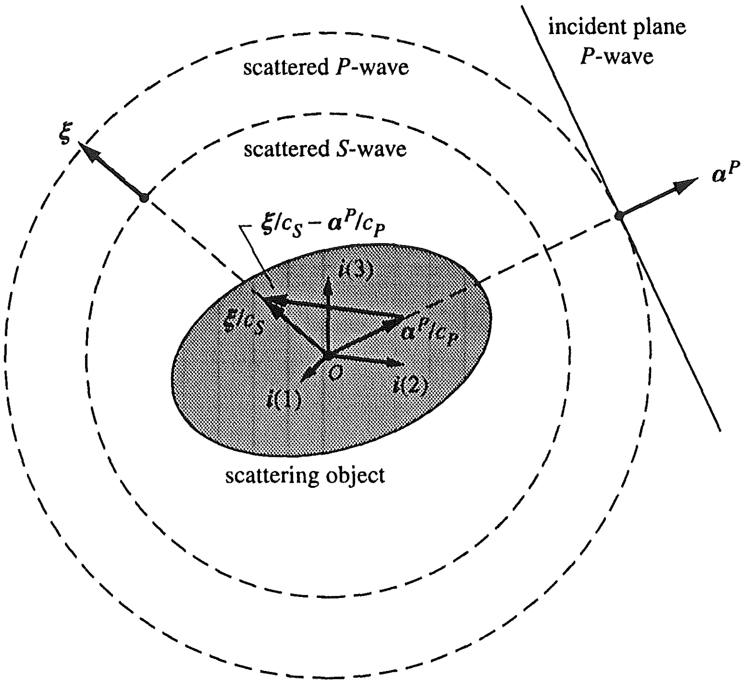


Figure 16.6-2 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident P-wave, scattered S-wave).

and

$$\Lambda^{\chi;P}_{k,m,p',q'}(\boldsymbol{u},t) = \int_{t'=0}^{\infty} \left[C_{k,m,i,j}\chi_{i,j,p',q'}(t') - \Delta^{+}_{k,m,p',q'}\delta(t')\right] \hat{\Upsilon}^{P}(t - t' + u_s x_s)\, \mathrm{d}t' , \quad (16.6\text{-}8)$$

respectively, in which

$$\hat{\Upsilon}^{P}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \partial_t^2 a^P(t + u_s x_s)\, \mathrm{d}V \quad (16.6\text{-}9)$$

is the *time-domain P-wave shape factor* of the object.

Incident S-wave

For the incident uniform plane S-wave given by Equations (16.1-9)–(16.1-12) the far-field scattered P-wave amplitude is obtained as (Figure 16.6-3)

$$\begin{aligned} v_r^{s;P,S,\infty} &= -\xi_r\xi_k\Lambda^{\mu;S}_{k,r'}(\boldsymbol{\xi}/c_P - \boldsymbol{\alpha}^S/c_S,t)V^S_{r'} \\ &\quad + (\rho c_P)^{-1}\xi_m\xi_r\xi_k\Lambda^{\chi;S}_{k,m,p',q'}(\boldsymbol{\xi}/c_P - \boldsymbol{\alpha}^S/c_S,t)T^S_{p',q'} , \end{aligned} \quad (16.6\text{-}10)$$

with

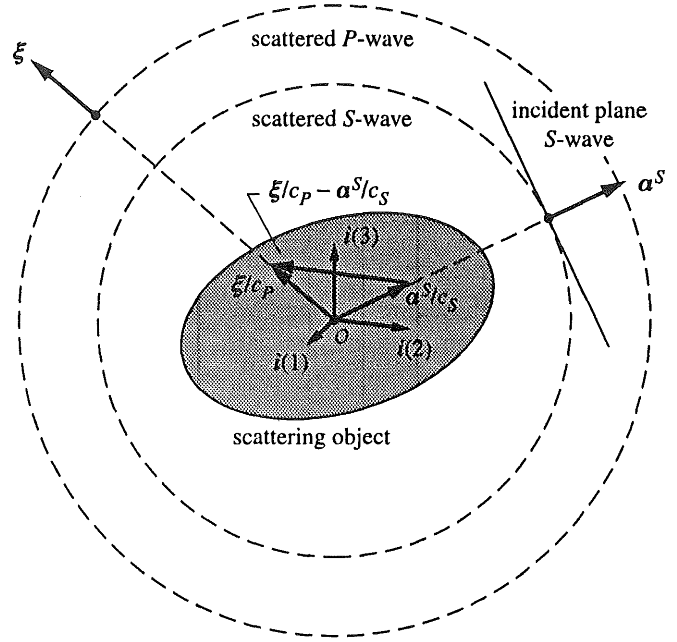


Figure 16.6-3 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident S-wave, scattered P-wave).

$$\Lambda^{\mu;S}_{k,r'}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[\rho^{-1}\mu_{k,r'}(\boldsymbol{x},t') - \delta_{k,r'}\delta(t')\right] \partial_t^2 a^S(t - t' + u_s x_s)\,\mathrm{d}t' \tag{16.6-11}$$

and

$$\Lambda^{\chi;S}_{k,m,p',q'}(\boldsymbol{u},t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[C_{k,m,i,j}\chi_{i,j,p',q'}(\boldsymbol{x},t') - \overset{+}{\Delta}_{k,m,p',q'}\delta(t')\right] \times \partial_t^2 a^S(t - t' + u_s x_s)\,\mathrm{d}t' , \tag{16.6-12}$$

while

$$\tau^{s;P,S,\infty}_{p,q} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{s;P,S,\infty}) + 2\mu(\xi_k v_k^{s;P,S,\infty})\xi_p\xi_q\right], \tag{16.6-13}$$

and the far-field scattered S-wave amplitudes as (Figure 16.6-4)

$$v_r^{s;S,S,\infty} = -(\delta_{r,k} - \xi_r\xi_k)\Lambda^{\mu;S}_{k,r'}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^S/c_S,t)V^S_{r'} + (\rho c_S)^{-1}\xi_m(\delta_{r,k} - \xi_r\xi_k)\Lambda^{\chi;S}_{k,m,p',q'}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^S/c_S,t)T^S_{p',q'} , \tag{16.6-14}$$

while

$$\tau^{s;S,S,\infty}_{p,q} = -c_S^{-1}\mu(\xi_p v_q^{s;S,S,\infty} + \xi_q v_p^{s;S,S,\infty}) . \tag{16.6-15}$$

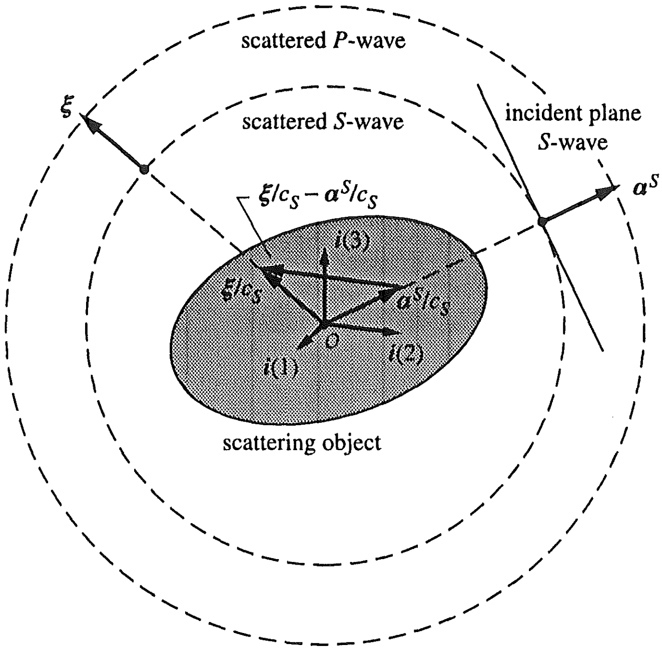


Figure 16.6-4 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident S-wave, scattered S-wave).

For a *homogeneous object*, Equations (16.6-11) and (16.6-12) reduce to

$$\mathcal{A}^{\mu;S}_{k,r'}(u,t) = \int_{t'=0}^{\infty} \left[\rho^{-1}\mu_{k,r'}(t') - \delta_{k,r'}\delta(t')\right]\hat{\Upsilon}^{S}(t - t' + u_s x_s)\,\mathrm{d}t' \tag{16.6-16}$$

and

$$\mathcal{A}^{\chi;S}_{k,m,p',q'}(u,t) = \int_{t'=0}^{\infty} \left[C_{k,m,i,j}\chi_{i,j,p',q'}(t') - \overset{+}{\Delta}_{k,m,p',q'}\delta(t')\right]\hat{\Upsilon}^{S}(t - t' + u_s x_s)\,\mathrm{d}t' \,, \tag{16.6-17}$$

respectively, in which

$$\hat{\Upsilon}^{S}(u,t) = \int_{x\in\mathcal{D}^s} \partial_t^2 a^S(t + u_s x_s)\,\mathrm{d}V \tag{16.6-18}$$

is the *time-domain S-wave shape factor* of the object.

From Equations (16.6-9) and (16.6-18) it immediately follows that for $u = \mathbf{0}$, we have

$$\hat{\Upsilon}^{P,S}(\mathbf{0},t) = V^{s}\partial_t^2 a^{P,S}(t) \,, \tag{16.6-19}$$

where V^s is the *volume of the scatterer*. Note that $u = \mathbf{0}$ occurs for P-wave/P-wave scattering when $\boldsymbol{\xi} = \boldsymbol{\alpha}^P$ and for S-wave/S-wave scattering for $\boldsymbol{\xi} = \boldsymbol{\alpha}^S$, i.e. in both cases, for observation "behind" the scatterer or in "forward scattering". Note, also, that $u = \mathbf{0}$ can never occur for P-wave/S-wave scattering or for S-wave/P-wave scattering.

Below we shall derive, for a number of canonical geometries of the scatterer, closed-form analytic expressions for the shape factor

$$\hat{\Upsilon}(u,t) = \int_{x\in\mathcal{D}^s} \partial_t^2 a(t + u_s x_s)\,\mathrm{d}V\,. \qquad (16.6\text{-}20)$$

Ellipsoid

Let the scattering ellipsoid be defined by (see Equation (A.9-21) and Figure 16.6-5)

$$\mathcal{D}^s = \left\{x\in\mathcal{R}^3; 0 \leqslant (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 < 1\right\}. \qquad (16.6\text{-}21)$$

Its volume is

$$V^s = (4\pi/3)a_1a_2a_3\,. \qquad (16.6\text{-}22)$$

In the integral on the right-hand side of Equation (16.6-20) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \qquad (16.6\text{-}23)$$

as the variables of integration. In y-space, the domain of integration is then the unit ball $\{y\in\mathcal{R}^3;\ 0\leqslant y_1^2 + y_2^2 + y_3^2 < 1\}$. The integration over this unit ball is carried out with the aid of spherical polar coordinates $\{r,\theta,\phi\}$, with $0\leqslant r<1$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, about the vector $u_1a_1\boldsymbol{i}(1) + u_2a_2\boldsymbol{i}(2) + u_3a_3\boldsymbol{i}(3)$ as polar axis. Then

$$u_sx_s = u_1x_1 + u_2x_2 + u_3x_3 = (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3a_3)y_3 = Ur\cos(\theta)\,, \qquad (16.6\text{-}24)$$

where

$$U = \left[(u_1a_1)^2 + (u_2a_2)^2 + (u_3a_3)^2\right]^{1/2} \geqslant 0\,, \qquad (16.6\text{-}25)$$

while

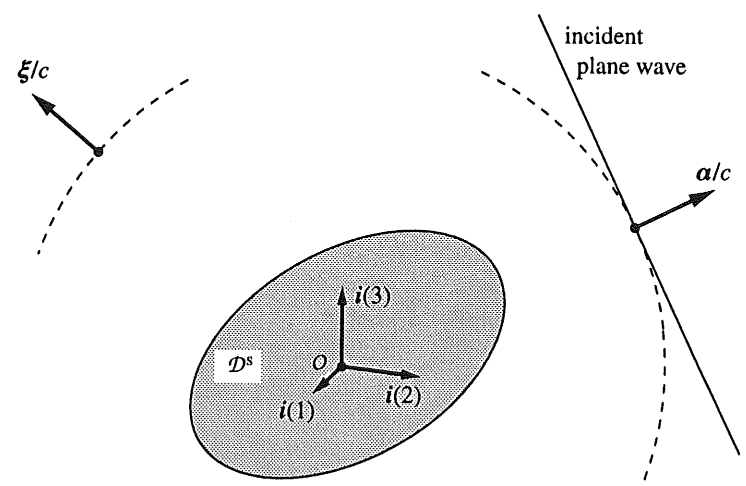


Figure 16.6-5 Scatterer in the shape of an ellipsoid.

$$\mathrm{d}V = a_1 a_2 a_3 r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{16.6-26}$$

The integration then runs as follows:

$$\begin{aligned}
\Upsilon(u,t) &= a_1 a_2 a_3 \int_{r=0}^{1} r^2\,\mathrm{d}r \int_{\theta=0}^{\pi} \sin(\theta)\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \partial_t^2 a\,[t + Ur\cos(\theta)]\,\mathrm{d}\phi \\
&= 2\pi a_1 a_2 a_3 \int_{r=0}^{1} r^2\,\mathrm{d}r \int_{\theta=0}^{\pi} \partial_t^2 a\,[t + Ur\cos(\theta)]\sin(\theta)\,\mathrm{d}\theta \\
&= 2\pi a_1 a_2 a_3 U^{-1} \int_{r=0}^{1} \left[\partial_t a\,(t+Ur) - \partial_t a(t-Ur)\right] r\,\mathrm{d}r \\
&= 2\pi a_1 a_2 a_3 \left\{U^{-2} a(t+U) - U^{-3}\left[\mathrm{I}_t a(t+U) - \mathrm{I}_t a(t)\right]\right. \\
&\quad \left. + U^{-2} a(t-U) + U^{-3}\left[\mathrm{I}_t a(t-U) - \mathrm{I}_t a(t)\right]\right\} \\
&= (3V^{\mathrm{s}}/2)\left\{U^{-2}\left[a(t+U) + a(t-U)\right] - U^{-3}\left[\mathrm{I}_t a(t+U) - \mathrm{I}_t a(t-U)\right]\right\}.
\end{aligned} \tag{16.6-27}$$

By using the Taylor expansion of the right-hand side about $U = 0$ and taking the limit $U \downarrow 0$, it can be verified that the result is in accordance with Equation (16.6-19).

Rectangular block

Let the scattering domain be the rectangular block defined by (see Equation (A.9-14) and Figure 16.6-6)

$$\mathcal{D}^{\mathrm{s}} = \left\{x \in \mathcal{R}^3 ;\ -a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2,\ -a_3 < x_3 < a_3\right\}. \tag{16.6-28}$$

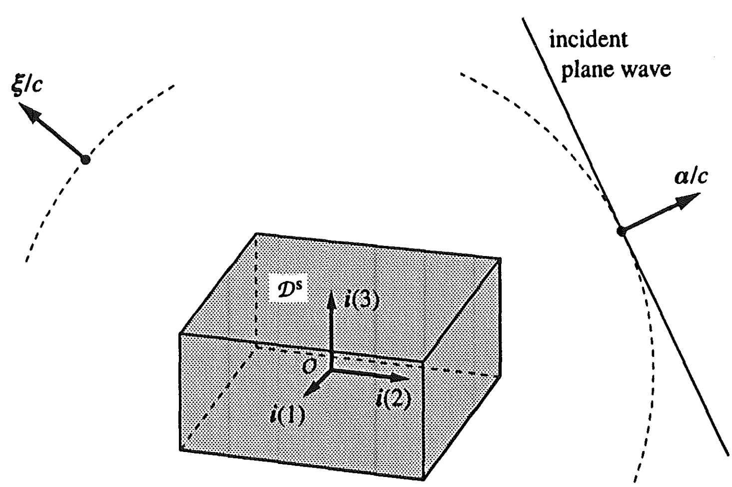


Figure 16.6-6 Scatterer in the shape of a rectangular block.

Its volume is given by

$$V^{s} = 8a_1a_2a_3 . \tag{16.6-29}$$

In the integral on the right-hand side of Equations (16.6-20) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{16.6-30}$$

as the variables of integration. In y-space the domain of integration is then the cube $\{y \in \mathcal{R}^3; -1<y_1<1,\ -1<y_2<1,\ -1<y_3<1\}$ with edge lengths 2. With

$$U_1 = u_1a_1,\ U_2 = u_2a_2,\ U_3 = u_3a_3 , \tag{16.6-31}$$

furthermore, we have

$$u_sx_s = u_1x_1 + u_2x_2 + u_3x_3$$

$$= (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3a_3)y_3 = U_1y_1 + U_2y_2 + U_3y_3 , \tag{16.6-32}$$

while

$$\mathrm{d}V = a_1a_2a_3\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3 . \tag{16.6-33}$$

The integration then runs as follows:

$$\Upsilon(u,t) = a_1a_2a_3 \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \mathrm{d}y_2 \int_{y_1=-1}^{1} \partial_t^2 a(t + U_1y_1 + U_2y_2 + U_3y_3)\, \mathrm{d}y_1$$

$$= a_1a_2a_3U_1^{-1} \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} [\partial_t a(t + U_1 + U_2y_2 + U_3y_3)$$

$$- \partial_t a(t - U_1 + U_2y_2 + U_3y_3)]\, \mathrm{d}y_2$$

$$= a_1a_2a_3(U_1U_2)^{-1} \int_{y_3=-1}^{1} [a(t + U_1 + U_2 + U_3y_3) - a(t + U_1 - U_2 + U_3y_3)$$

$$- a(t - U_1 + U_2 + U_3y_3) + a(t - U_1 - U_2 + U_3y_3)]\, \mathrm{d}y_3$$

$$= a_1a_2a_3(U_1U_2U_3)^{-1} [\mathrm{I}_t a(t + U_1 + U_2 + U_3) - \mathrm{I}_t a(t + U_1 + U_2 - U_3)$$

$$- \mathrm{I}_t a(t + U_1 - U_2 + U_3) + \mathrm{I}_t a(t + U_1 - U_2 - U_3) - \mathrm{I}_t a(t - U_1 + U_2 + U_3)$$

$$+ \mathrm{I}_t a(t - U_1 + U_2 - U_3) + \mathrm{I}_t a(t - U_1 - U_2 + U_3) - \mathrm{I}_t a(t - U_1 - U_2 - U_3)] . \tag{16.6-34}$$

Special cases occur for either $U_1 \to 0$, $U_2 \to 0$, and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (16.6-34) by using the pertaining Taylor expansions in the right-hand side. In particular, it can be verified that for $U_1 \to 0$ and $U_2 \to 0$ and $U_3 \to 0$ the result is in accordance with Equation (16.6-19).

Elliptical cylinder of finite height

Let the elliptical cylinder of finite height be defined by (Figure 16.6-7)

$$\mathcal{D}^s = \left\{x \in \mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\ -h < x_3 < h\right\}. \tag{16.6-35}$$

Its volume is

$$V^s = 2\pi a_1 a_2 h. \tag{16.6-36}$$

In the integral on the right-hand side of Equation (16.6-20) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{16.6-37}$$

as the variables of integration. In y-space, the domain of integration is then the Cartesian product of the unit disk $\Delta^2 = \{(y_1, y_2) \in \mathcal{R}^2;\ 0 \leqslant y_1^2 + y_2^2 < 1\}$ and the interval $\{y_3 \in \mathcal{R};\ -1 < y_3 < 1\}$ along the axis of the cylinder. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h, \tag{16.6-38}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3, \end{aligned} \tag{16.6-39}$$

while

$$\mathrm{d}V = a_1 a_2 h\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3. \tag{16.6-40}$$

The integration then runs as follows:

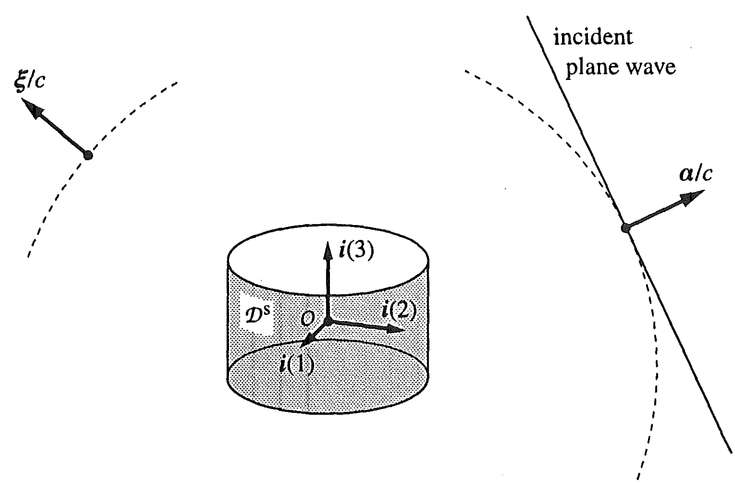


Figure 16.6-7 Scatterer in the shape of an elliptical cylinder of finite height.

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} \mathrm{d}y_1\,\mathrm{d}y_2 \int_{y_3=-1}^{1} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_3$$

$$= a_1 a_2 h U_3^{-1} \int_{(y_1,y_2)\in\Delta^2} [\partial_t a(t + U_1 y_1 + U_2 y_2 + U_3)$$

$$- \partial_t a(t + U_1 y_1 + U_2 y_2 - U_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2\,. \qquad (16.6\text{-}41)$$

Next, we observe that

$$\partial_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) = \partial_t^2 \mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)$$

$$= (U_1^2 + U_2^2)^{-1}(\partial_{y_1}^2 + \partial_{y_2}^2)\mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) \qquad \text{for } U_1^2 + U_2^2 \neq 0\,. \qquad (16.6\text{-}42)$$

Now, applying Gauss' divergence theorem to the integration over Δ^2, we obtain

$$\int_{(y_1,y_2)\in\Delta^2} (\partial_{y_1}^2 + \partial_{y_2}^2)\mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}y_1\,\mathrm{d}y_2$$

$$= \int_{(y_1,y_2)\in C^2} (y_1\partial_{y_1} + y_2\partial_{y_2})\mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}\sigma$$

$$= \int_{(y_1,y_2)\in C^2} (U_1 y_1 + U_2 y_2) a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}\sigma, \qquad (16.6\text{-}43)$$

where $\mathrm{d}\sigma$ is the elementary arc length along the unit circle C^2 that forms the closed boundary of the unit disk Δ^2, and where we have used the property that the unit vector along the normal to C^2 pointing away from Δ^2 is given by $\boldsymbol{\nu} = y_1\boldsymbol{i}(1) + y_2\boldsymbol{i}(2)$. In the integral on the right-hand side of Equation (16.6-43) we introduce the polar coordinates $\{r,\phi\}$, with $r = 1$ and $0 \leqslant \phi < 2\pi$, about the vector $U_1\boldsymbol{i}(1) + U_2\boldsymbol{i}(2)$ as polar axis, as the variables of integration. This yields

$$\int_{(y_1,y_2)\in C^2} (U_1 y_1 + U_2 y_2) a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}\sigma$$

$$= \int_{\phi=0}^{2\pi} U\cos(\phi) a\left[t + U\cos(\phi) \pm U_3\right]\mathrm{d}\phi\,, \qquad (16.6\text{-}44)$$

where

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0\,. \qquad (16.6\text{-}45)$$

Collecting the results, we end up with

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} U_3^{-1} \int_{\phi=0}^{2\pi} \cos(\phi) \Big\{ a\left[t + U\cos(\phi) + U_3\right]$$

$$- a\left[t + U\cos(\phi) - U_3\right]\Big\}\,\mathrm{d}\phi \qquad (16.6\text{-}46)$$

Special cases occur for $U \downarrow 0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (16.6-46) by using the pertaining Taylor expansions in the right-hand side. In particular, it can be verified that for $U\downarrow 0$ and $U_3 \to 0$ the result is in accordance with Equation (16.6-19).

Elliptical cone of finite height

Let the elliptical cone of finite height be defined by (Figure 16.6-8)

$$\mathcal{D}^{s} = \left\{ x \in \mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < x_3^2/h^2,\ 0 < x_3 < h \right\}. \qquad (16.6\text{-}47)$$

Its volume is

$$V^{s} = \pi a_1 a_2 h/3 . \qquad (16.6\text{-}48)$$

In the integral on the right-hand side of Equation (16.6-20) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \qquad (16.6\text{-}49)$$

as the variables of integration. In y-space, the domain of integration is then $\{y \in \mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2,\ 0 < y_3 < 1\}$. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h , \qquad (16.6\text{-}50)$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3 , \end{aligned} \qquad (16.6\text{-}51)$$

while

$$\mathrm{d}V = a_1 a_2 h \,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3 . \qquad (16.6\text{-}52)$$

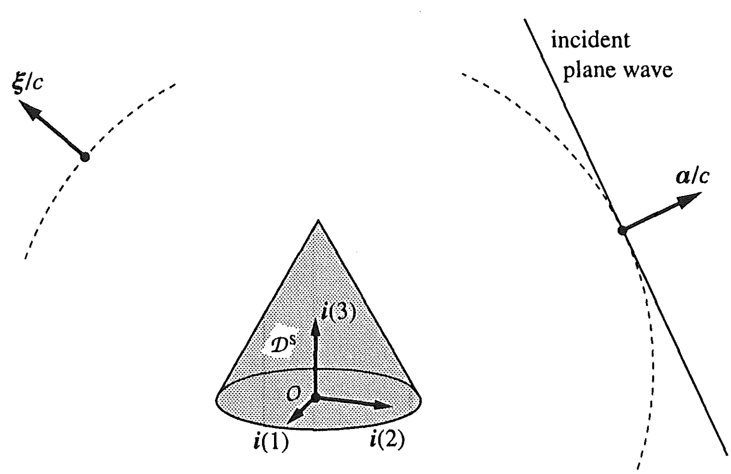


Figure 16.6-8 Scatterer in the shape of an elliptical cone of finite height.

The integration then runs as follows:

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h \int_{y_3=0}^{1} \mathrm{d}y_3 \int_{(y_1,y_2)\in\Delta^2(y_3)} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_1\,\mathrm{d}y_2\,, \tag{16.6-53}$$

where $\Delta^2(y_3) = \{(y_1,y_2)\in\mathcal{R}^2;\, 0\leqslant y_1^2 + y_2^2 < y_3^2\}$ is the circular disc of radius y_3. With a reasoning similar to that used in Equations (16.6-42)–(16.6-44), we obtain

$$\int_{(y_1,y_2)\in\Delta^2(y_3)} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_1\,\mathrm{d}y_2$$

$$= U^{-1} y_3 \int_{\phi=0}^{2\pi} \cos(\phi)\partial_t a\left[t + U y_3 \cos(\phi) + U_3 y_3\right]\mathrm{d}\phi\,, \tag{16.6-54}$$

in which

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0\,. \tag{16.6-55}$$

Furthermore,

$$\int_{y_3=0}^{1} y_3 \partial_t a\left[t + U y_3 \cos(\phi) + U_3 y_3\right]\mathrm{d}y_3$$

$$= \left[U\cos(\phi) + U_3\right]^{-1} a\left[t + U\cos(\phi) + U_3\right]$$

$$- \left[U\cos(\phi) + U_3\right]^{-2}\left\{\mathrm{I}_t a\left[t + U\cos(\phi) + U_3\right] - \mathrm{I}_t a(t)\right\}. \tag{16.6-56}$$

Collecting the results, we end up with

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} \int_{\phi=0}^{2\pi} \cos(\phi)\Big\{\left[U\cos(\phi) + U_3\right]^{-1} a\left[t + U\cos(\phi) + U_3\right]$$

$$- \left[U\cos(\phi) + U_3\right]^{-2}\left\{\mathrm{I}_t a\left[t + U\cos(\phi) + U_3\right] - \mathrm{I}_t a(t)\right\}\Big\}\,\mathrm{d}\phi\,. \tag{16.6-57}$$

Special cases occur for $U\downarrow 0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (16.6-57) by using the pertaining Taylor expansions in the right-hand side. In particular, it can be verified that for $U\downarrow 0$ and $U_3 \to 0$ the result is in accordance with Equation (16.6-19).

Tetrahedron

Let the tetrahedron be defined by (see Equation (A.9-17) and Figure 16.6-9)

$$\mathcal{D}^{\mathrm{S}} = \left\{ \boldsymbol{x}\in\mathcal{R}^3;\, \boldsymbol{x} = \sum_{I=0}^{3} \lambda(I)\boldsymbol{x}(I),\quad 0<\lambda(I)<1,\ \sum_{I=0}^{3} \lambda(I) = 1 \right\}, \tag{16.6-58}$$

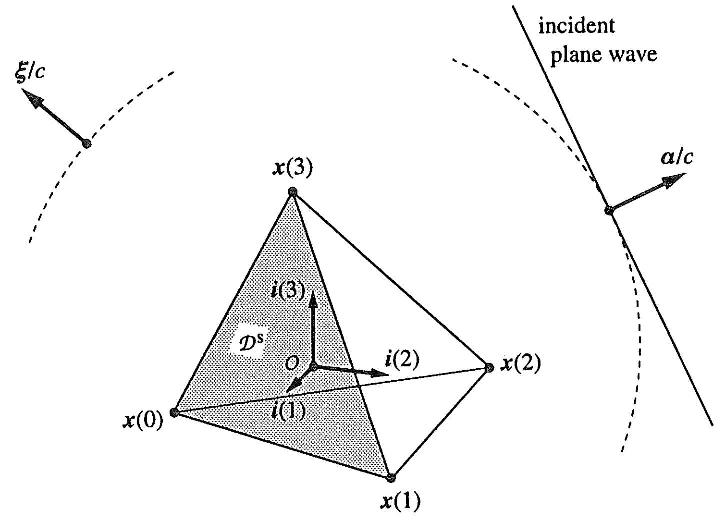


Figure 16.6-9 Scatterer in the shape of a tetrahedron (3-simplex).

in which $\{x(0),x(1),x(2),x_3(3)\}$ are the position vectors of the vertices and $\{\lambda(0), \lambda(1), \lambda(2),\lambda(3)\}$ are the barycentric coordinates. Its volume is (see Equations (A.10-29) and (A.10-33))

$$V^s = \det\,[x(1)-x(0),x(2)-x(1),x(3)-x(2)]\,/6\,. \qquad (16.6\text{-}59)$$

In the integral on the right-hand side of Equation (16.6-20) we replace $\lambda(0)$ by $1-\lambda(1)-\lambda(2)-\lambda(3)$ and introduce $\{\lambda(1),\lambda(2),\lambda(3)\}$ as the (dimensionless) variables of integration. In $\{\lambda(1),\lambda(2),\lambda(3)\}$-space the domain of integration is then $\{0<\lambda(1)<1,\ 0<\lambda(2)<1-\lambda(1),\ 0<\lambda(3)<1-\lambda(1)-\lambda(2)\}$. Then, with

$$U(I) = u_s x_s(I) \qquad \text{for } I = 0, 1, 2, 3\,, \qquad (16.6\text{-}60)$$

we have

$$\begin{aligned} u_s x_s &= \lambda(0)U(0)+\lambda(1)U(1)+\lambda(2)U(2)+\lambda(3)U(3) \\ &= [1-\lambda(1)-\lambda(2)-\lambda(3)]\ U(0)+\lambda(1)U(1)+\lambda(2)U(2)+\lambda(3)U(3) \\ &= U(0)+[U(1)-U(0)]\ \lambda(1)+[U(2)-U(0)]\ \lambda(2)+[U(3)-U(0)]\ \lambda(3)\,, \qquad (16.6\text{-}61) \end{aligned}$$

while, with the Jacobian (see Equation (A.10-31))

$$\frac{\partial(x_1,x_2,x_3)}{\partial[\lambda(1),\lambda(2),\lambda(3)]} = 6V^s, \qquad (16.6\text{-}62)$$

the elementary volume is expressed as

$$\mathrm{d}V = 6V^s\,\mathrm{d}\lambda(1)\,\mathrm{d}\lambda(2)\,\mathrm{d}\lambda(3)\,. \qquad (16.6\text{-}63)$$

After some lengthy but elementary calculations it is found that

$$\hat{T}(u,t) = 6V^{\mathrm{s}} \left\{ \frac{1}{U(0)-U(1)} \frac{1}{U(0)-U(2)} \frac{1}{U(0)-U(3)} \mathrm{I}_t a\,[t+U(0)] \right.$$

$$+ \frac{1}{U(1)-U(0)} \frac{1}{U(1)-U(2)} \frac{1}{U(1)-U(3)} \mathrm{I}_t a\,[t+U(1)]$$

$$+ \frac{1}{U(2)-U(0)} \frac{1}{U(2)-U(1)} \frac{1}{U(2)-U(3)} \mathrm{I}_t a\,[t+U(2)]$$

$$\left. + \frac{1}{U(3)-U(0)} \frac{1}{U(3)-U(1)} \frac{1}{U(3)-U(2)} \mathrm{I}_t a\,[t+U(3)] \right\} . \qquad (16.6\text{-}64)$$

In a symmetrical fashion, this result can be written as

$$T(u,t) = 6V^{\mathrm{s}} \sum_{I=0}^{3} \frac{1}{U(I)-U(J)} \frac{1}{U(I)-U(K)} \frac{1}{U(I)-U(L)} \mathrm{I}_t a\,[t+U(I)] , \qquad (16.6\text{-}65)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$.

Special cases occur for $U(I) = U(J)$ and/or $U(I) = U(K)$ and/or $U(I) = U(L)$. The easiest way to arrive at the expressions for the relevant cases is to redo the integrations that need modifications.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the expressions for the scattered wave amplitude in the far-field region in the first-order Rayleigh–Gans–Born approximation follow, with the use of Equations (16.1-51), (16.1-52), (16.1-72)–(16.1-79) and (16.5-13)–(16.5-16) for $n = 0$.

Incident *P*-wave

For the incident plane *P*-wave given by Equations (16.1-55)–(16.1-58) the far-field scattered *P*-wave amplitude is obtained as (Figure 16.6-10)

$$\hat{v}_r^{\,\mathrm{s};P,P,\infty} = -\xi_r \xi_k \hat{\Lambda}_{k,r'}^{\mu;P}(\boldsymbol{\xi}/c_P - \boldsymbol{\alpha}^P/c_P,s) V_{r'}^P$$

$$+ (\rho c_P)^{-1} \xi_m \xi_r \xi_k \hat{\Lambda}_{k,m,p',q'}^{\chi;P}(\boldsymbol{\xi}/c_P - \boldsymbol{\alpha}^P/c_P,s) T_{p',q'}^P , \qquad (16.6\text{-}66)$$

with

$$\hat{\Lambda}_{k,r'}^{\mu;P}(\boldsymbol{u},s) = s^2 \hat{a}^P(s) \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\rho^{-1}\hat{\mu}_{k,r'}(\boldsymbol{x},s) - \delta_{k,r'}\right] \exp(s u_s x_s)\,\mathrm{d}V \qquad (16.6\text{-}67)$$

and

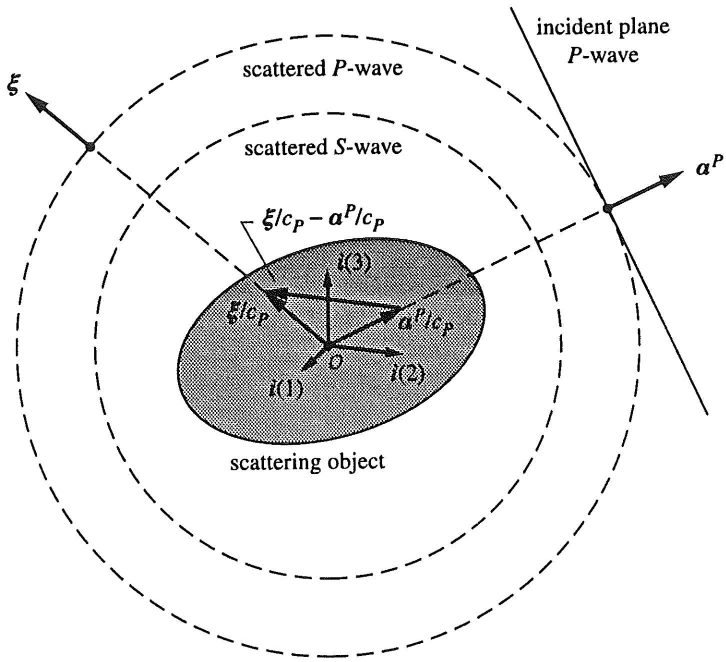


Figure 16.6-10 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident P-wave, scattered P-wave).

$$\hat{\Lambda}^{\chi;P}_{k,m,p',q'}(\boldsymbol{u},s) = s^2\hat{a}^P(s)\int_{\boldsymbol{x}\in\mathcal{D}^s}\left[C_{k,m,i,j}\hat{\chi}_{i,j,p',q'}(\boldsymbol{x},s) - \Delta^+_{k,m,p',q'}\right]\exp(su_sx_s)\,\mathrm{d}V\,, \tag{16.6-68}$$

while

$$\tau^{s;P,S,\infty}_{p,q} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{s;P,P,\infty}) + 2\mu(\xi_k v_k^{s;P,P,\infty})\xi_p\xi_q\right], \tag{16.6-69}$$

and the far-field scattered S-wave amplitudes are (Figure 16.6-11)

$$\begin{aligned}\hat{v}_r^{s;S,P,\infty} = &-(\delta_{r,k}-\xi_r\xi_k)\hat{\Lambda}^{\mu;P}_{k,r'}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^P/c_P,s)V_{r'}^P \\ &+ (\rho c_S)^{-1}(\delta_{r,k}-\xi_r\xi_k)\hat{\Lambda}^{\chi;P}_{k,m,p',q'}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^P/c_P,s)T^P_{p',q'}\,,\end{aligned} \tag{16.6-70}$$

while

$$\tau^{s;S,P,\infty}_{p,q} = -c_S^{-1}\mu(\xi_p\hat{v}_q^{s;S,P,\infty} + \xi_q\hat{v}_p^{s;S,P,\infty}). \tag{16.6-71}$$

For a *homogeneous object*, Equations (16.6-67) and (16.6-68) reduce to

$$\hat{\Lambda}^{\mu;P}_{k,r'}(\boldsymbol{u},s) = s^2\hat{a}^P(s)\left[\rho^{-1}\hat{\mu}_{k,r'}(s) - \delta_{k,r'}\right]\hat{\Upsilon}(\boldsymbol{u},s)\,, \tag{16.6-72}$$

and

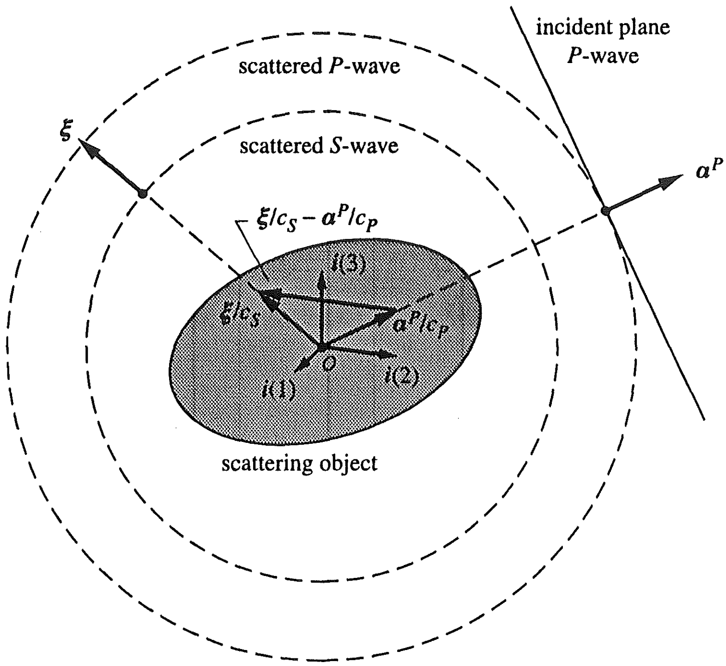


Figure 16.6-11 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident P-wave, scattered S-wave).

$$\hat{\Lambda}^{\chi;P}_{k,m,p',q'}(\boldsymbol{u},s) = s^2\hat{a}^P(s)\left[C_{k,m,i,j}\hat{\chi}_{i,j,p',q'}(s) - \Delta^+_{k,m,p',q'}\right]\hat{\Upsilon}(\boldsymbol{u},s)\,, \tag{16.6-73}$$

respectively, in which

$$\hat{\Upsilon}(\boldsymbol{u},s) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \exp(s u_s x_s)\,\mathrm{d}V \tag{16.6-74}$$

is the *complex frequency-domain shape factor* of the scattering object.

Incident S-wave

For the incident uniform plane S-wave given by Equations (16.1-59)–(16.1-62) the far-field scattered P-wave amplitude is obtained as (Figure 16.6-12)

$$\hat{v}_r^{s;P,S,\infty} = -\xi_r\xi_k\hat{\Lambda}^{\mu;S}_{k,r'}(\boldsymbol{\xi}/c_P - \boldsymbol{a}^S/c_S,s)V^S_{r'} + (\rho c_P)^{-1}\xi_m\xi_r\xi_k\hat{\Lambda}^{\chi;S}_{k,m,p',q'}(\boldsymbol{\xi}/c_P - \boldsymbol{a}^S/c_S,s)T^S_{p',q'}\,, \tag{16.6-75}$$

with

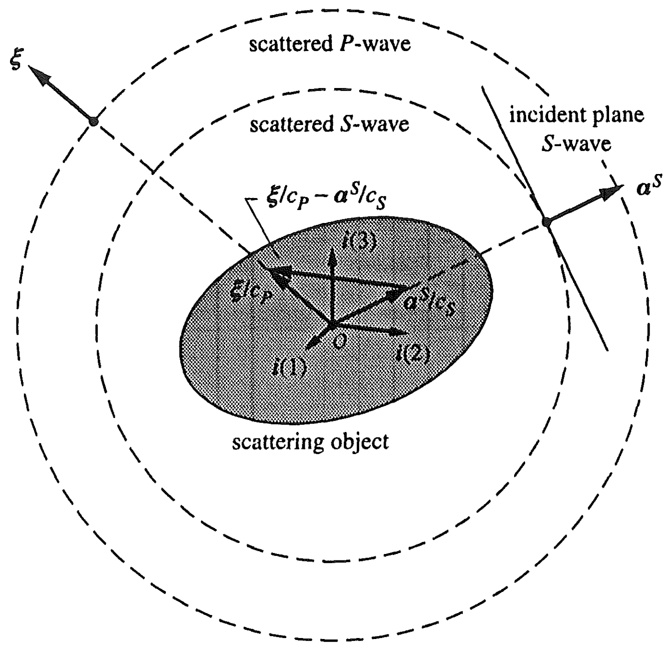


Figure 16.6-12 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident S-wave, scattered P-wave).

$$\hat{\Lambda}_{k,r'}^{\mu;S}(\boldsymbol{u},s) = s^2 \hat{a}^S(s) \int_{\boldsymbol{x}\in\mathcal{D}^s} \left[\rho^{-1}\hat{\mu}_{k,r'}(\boldsymbol{x},s) - \delta_{k,r'}\right] \exp(su_s x_s)\,\mathrm{d}V \tag{16.6-76}$$

and

$$\hat{\Lambda}_{k,m,p',q'}^{\chi;S}(\boldsymbol{u},s) = s^2 \hat{a}^S(s) \int_{\boldsymbol{x}\in\mathcal{D}^s} \left[C_{k,m,i,j}\hat{\chi}_{i,j,p',q'}(\boldsymbol{x},s) - \Delta^{+}_{k,m,p',q'}\right] \exp(su_s x_s)\,\mathrm{d}V\,, \tag{16.6-77}$$

while

$$\tau_{p,q}^{s;P,S,\infty} = -c_P^{-1}\left[\lambda\delta_{p,q}(\xi_k v_k^{s;P,S,\infty}) + 2\mu(\xi_k v_k^{s;P,S,\infty})\xi_p\xi_q\right], \tag{16.6-78}$$

and the far-field scattered S-wave amplitudes are (Figure 16.6-13)

$$\begin{aligned}\hat{v}_r^{s;S,S,\infty} &= -(\delta_{r,k} - \xi_r\xi_k)\hat{\Lambda}_{k,r'}^{\mu;S}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^S/c_S, s)V_{r'}^S \\ &\quad + (\rho c_S)^{-1}\xi_m(\delta_{r,k} - \xi_r\xi_k)\hat{\Lambda}_{k,m,p',q'}^{\chi;S}(\boldsymbol{\xi}/c_S - \boldsymbol{a}^S/c_S, s)T_{p',q'}^S\,,\end{aligned} \tag{16.6-79}$$

while

$$\tau_{p,q}^{s;S,S,\infty} = -c_S^{-1}\mu(\xi_p\hat{v}_q^{s;S,S,\infty} + \xi_q\hat{v}_p^{s;S,S,\infty}). \tag{16.6-80}$$

For a *homogeneous object*, Equations (16.6-76) and (16.6-77) reduce to

$$\hat{\Lambda}_{k,r'}^{\mu;S}(\boldsymbol{u},s) = s^2 \hat{a}^S(s)\left[\rho^{-1}\hat{\mu}_{k,r'}(s) - \delta_{k,r'}\right]\hat{\Upsilon}(\boldsymbol{u},s)\,, \tag{16.6-81}$$

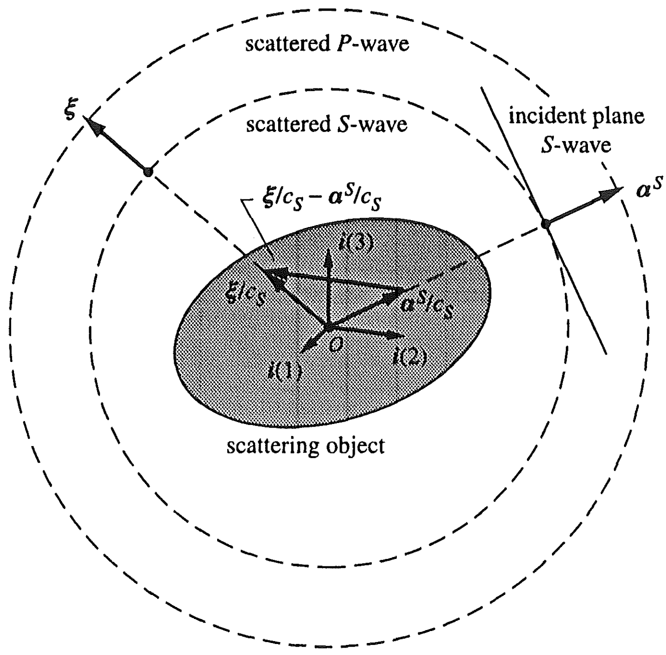


Figure 16.6-13 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation (incident S-wave, scattered S-wave).

and

$$\hat{A}^{\chi;S}_{k,m,p',q'}(\boldsymbol{u},s) = s^2\hat{a}^S(s)\left[C_{k,m,i,j}\hat{\chi}_{i,j,p',q'}(s) - \Delta^{+}_{k,m,p',q'}\right]\hat{\Upsilon}(\boldsymbol{u},s)\,, \tag{16.6-82}$$

respectively, in which

$$\hat{\Upsilon}(\boldsymbol{u},s) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \exp(su_s x_s)\,\mathrm{d}V \tag{16.6-83}$$

is the *complex frequency-domain shape factor* of the scattering object.

From Equations (16.6-74) and (16.6-83) it immediately follows that for $\boldsymbol{u} = \boldsymbol{0}$, we have

$$\hat{\Upsilon}(\boldsymbol{0},s) = V^{\mathrm{s}}, \tag{16.6-84}$$

where V^{s} is the *volume of the scatterer*. Note that $\boldsymbol{u} = \boldsymbol{0}$ occurs for P-wave/P-wave scattering when $\boldsymbol{\xi} = \boldsymbol{a}^P$ and for S-wave/S-wave scattering for $\boldsymbol{\xi} = \boldsymbol{a}^S$, i.e. in both cases, for observation "behind" the scatterer or in "forward scattering". Note, also, that $\boldsymbol{u} = \boldsymbol{0}$ can never occur for P-wave/S-wave scattering or for S-wave/P-wave scattering.

Below, we shall derive for a number of canonical geometries of the scatterer, closed-form analytic expressions for the shape factor

$$\hat{\Upsilon}(\boldsymbol{u},s) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \exp(su_s x_s)\,\mathrm{d}V\,. \tag{16.6-85}$$

Ellipsoid

Let the scattering ellipsoid be defined by (see Equation (A.9-21) and Figure 16.6-14).

$$\mathcal{D}^s = \left\{ x \in \mathcal{R}^3 ;\ 0 \leqslant (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 < 1 \right\} . \tag{16.6-86}$$

Its volume is

$$V^s = (4\pi/3) a_1 a_2 a_3 . \tag{16.6-87}$$

In the integral on the right-hand side of Equation (16.6-85) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{16.6-88}$$

as the variables of integration. In y-space, the domain of integration is then the unit ball $\{y \in \mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 + y_3^2 < 1\}$. The integration over this unit ball is carried out with the aid of spherical polar coordinates $\{r,\theta,\phi\}$, with $0 \leqslant r < 1$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, about the vector $u_1 a_1 \boldsymbol{i}(1) + u_2 a_2 \boldsymbol{i}(2) + u_3 a_3 \boldsymbol{i}(3)$ as polar axis. Then

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3 = (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 a_3) y_3 = U r \cos(\theta) , \tag{16.6-89}$$

where

$$U = \left[(u_1 a_1)^2 + (u_2 a_2)^2 + (u_3 a_3)^2 \right]^{1/2} \geqslant 0 , \tag{16.6-90}$$

while

$$\mathrm{d}V = a_1 a_2 a_3 r^2 \sin(\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi . \tag{16.6-91}$$

The integration then runs as follows:

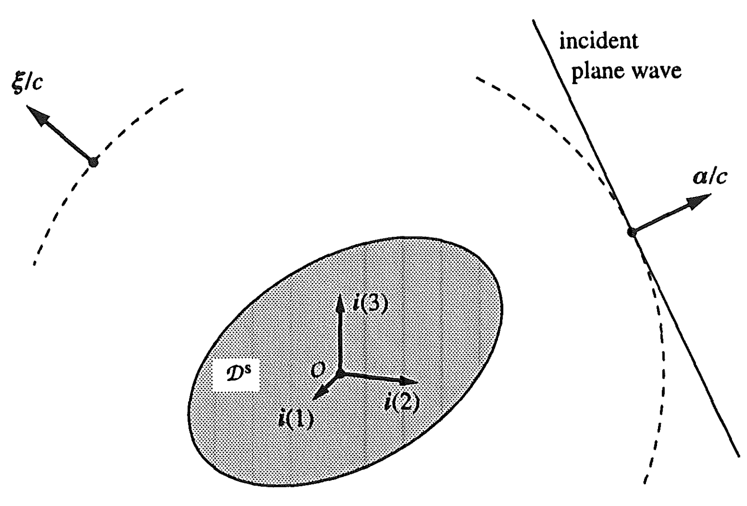


Figure 16.6-14 Scatterer in the shape of an ellipsoid.

$$\hat{\Gamma}(u,s) = a_1 a_2 a_3 \int_{r=0}^{1} r^2 \,\mathrm{d}r \int_{\theta=0}^{\pi} \sin(\theta)\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \exp\,[sUr\cos(\theta)]\,\mathrm{d}\phi$$

$$= 2\pi a_1 a_2 a_3 \int_{r=0}^{1} r^2 \,\mathrm{d}r \int_{\theta=0}^{\pi} \exp\,[sUr\cos(\theta)]\sin(\theta)\,\mathrm{d}\theta$$

$$= 2\pi a_1 a_2 a_3 (sU)^{-1} \int_{r=0}^{1} [\exp(sUr) - \exp(-sUr)]\, r\,\mathrm{d}r$$

$$= 2\pi a_1 a_2 a_3 (sU)^{-2} \left\{ \exp(sU) + \exp(-sU) - \int_{r=0}^{1} [\exp(sUr) + \exp(-sUr)]\,\mathrm{d}r \right\}$$

$$= 2\pi a_1 a_2 a_3 (sU)^{-2} \left\{ \exp(sU) + \exp(-sU) - (sU)^{-1}\,[\exp(sU) - \exp(-sU)] \right\}$$

$$= 3V^{\mathrm{s}} \frac{sU\cosh(sU) - \sinh(sU)}{(sU)^3}. \tag{16.6-92}$$

By using the Taylor expansion of the right-hand side about $U = 0$ and taking the limit $U \downarrow 0$, it can be verified that the result is in accordance with Equation (16.6-84).

Rectangular block

Let the scattering domain be the rectangular block defined by (see Equation (A.9-14) and Figure 16.6-15)

$$\mathcal{D}^{\mathrm{s}} = \left\{ x \in \mathcal{R}^3 ; -a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2,\ -a_3 < x_3 < a_3 \right\}. \tag{16.6-93}$$

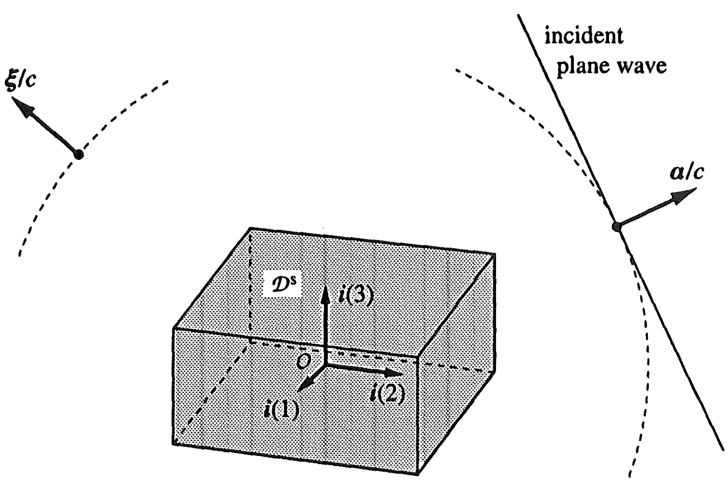


Figure 16.6-15 Scatterer in the shape of a rectangular block.

Its volume is given by

$$V^{s} = 8a_1a_2a_3 \,. \tag{16.6-94}$$

In the integral on the right-hand side of Equation (16.6-85) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{16.6-95}$$

as the variables of integration. In y-space the domain of integration is then the cube $\{y \in \mathcal{R}^3;\ -1 < y_1 < 1,\ -1 < y_2 < 1,\ -1 < y_3 < 1\}$ with edge lengths 2. With

$$U_1 = u_1a_1,\ U_2 = u_2a_2,\ U_3 = u_3a_3 \,, \tag{16.6-96}$$

furthermore, we have

$$\begin{aligned} u_s x_s &= u_1x_1 + u_2x_2 + u_3x_3 \\ &= (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3a_3)y_3 = U_1y_1 + U_2y_2 + U_3y_3 \,, \end{aligned} \tag{16.6-97}$$

while

$$\mathrm{d}V = a_1a_2a_3\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3 \,. \tag{16.6-98}$$

The integration then runs as follows:

$$\begin{aligned} \hat{\Upsilon}(u,s) &= a_1a_2a_3 \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \mathrm{d}y_2 \int_{y_1=-1}^{1} \exp\,[s(U_1y_1 + U_2y_2 + U_3y_3)]\,\mathrm{d}y_1 \\ &= a_1a_2a_3 \int_{y_3=-1}^{1} \exp(sU_3y_3)\,\mathrm{d}y_3 \int_{y_2=-1}^{1} \exp(sU_2y_2)\,\mathrm{d}y_2 \int_{y_1=-1}^{1} \exp(sU_1y_1)\,\mathrm{d}y_1 \\ &= a_1a_2a_3 \frac{\exp(sU_3) - \exp(-sU_3)}{sU_3} \frac{\exp(sU_2) - \exp(-sU_2)}{sU_2} \frac{\exp(sU_1) - \exp(-sU_1)}{sU_1} \\ &= V^{s} \frac{\sinh(sU_3)}{sU_3} \frac{\sinh(sU_2)}{sU_2} \frac{\sinh(sU_1)}{sU_1} \,. \end{aligned} \tag{16.6-99}$$

Special cases occur for either $U_1 \to 0$, $U_2 \to 0$, and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (16.6-99) by using the relevant Taylor expansions in the right-hand side. In particular, it can be verified that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$ the result is in accordance with Equation (16.6-84).

Elliptical cylinder of finite height

Let the elliptical cylinder of finite height be defined by (Figure 16.6-16)

$$\mathcal{D}^{s} = \left\{x \in \mathcal{R}^3;\ 0 \leq x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\ -h < x_3 < h\right\} . \tag{16.6-100}$$

Its volume is

$$V^{s} = 2\pi a_1a_2h \,. \tag{16.6-101}$$

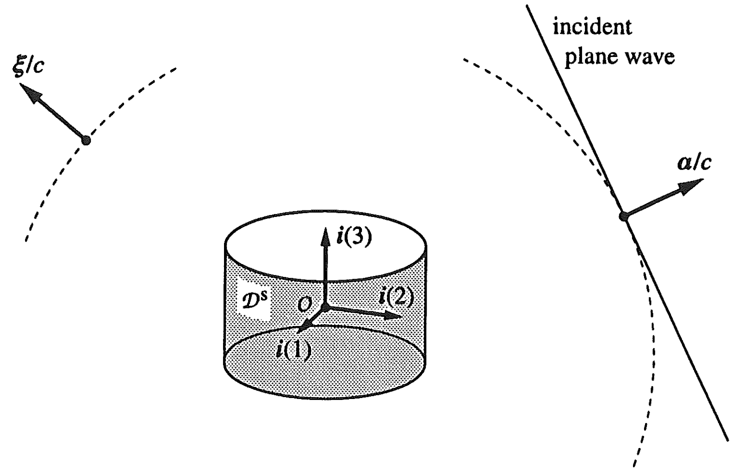


Figure 16.6-16 Scatterer in the shape of an elliptical cylinder of finite height.

In the integral on the right-hand side of Equation (16.6-85) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{16.6-102}$$

as the variables of integration. In y space, the domain of integration is then the Cartesian product of the unit disk $\Delta^2 = \{(y_1,y_2)\in\mathcal{R}^3;\ 0\leqslant y_1^2 + y_2^2 < 1\}$ and the interval $\{y_3\in\mathcal{R};\ -1 < y_3 < 1\}$ along the axis of the cylinder. Then, with

$$U_1 = u_1a_1,\ U_2 = u_2a_2,\ U_3 = u_3h\,, \tag{16.6-103}$$

we have

$$\begin{aligned} u_sx_s &= u_1x_1 + u_2x_2 + u_3x_3 \\ &= (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3h)y_3 = U_1y_1 + U_2y_2 + U_3y_3\,, \end{aligned} \tag{16.6-104}$$

while

$$\mathrm{d}V = a_1a_2h\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3\,. \tag{16.6-105}$$

The integration then runs as follows:

$$\begin{aligned} \hat{\Upsilon}(u,s) &= a_1a_2h\int_{(y_1,y_2)\in\Delta^2}\mathrm{d}y_1\,\mathrm{d}y_2\int_{y_3=-1}^{1}\exp\,[s(U_1y_1 + U_2y_2 + U_3y_3)]\,\mathrm{d}y_3 \\ &= a_1a_2h\int_{(y_1,y_2)\in\Delta^2}(sU_3)^{-1}\Big\{\exp\,[s(U_1y_1 + U_2y_2 + U_3)] \\ &\quad - \exp\,[s(U_1y_1 + U_2y_2 - U_3)]\Big\}\,\mathrm{d}y_1\,\mathrm{d}y_2\,. \end{aligned} \tag{16.6-106}$$

Next, we observe that

$$\exp\,[s(U_1y_1+U_2y_2\pm U_3)] = (s^2U_1^2+s^2U_2^2)^{-1}(\partial_{y_1}^2+\partial_{y_2}^2)\exp\,[s(U_1y_1+U_2y_2\pm U_3)] \quad \text{for } U_1^2+U_2^2\neq 0\,. \tag{16.6-107}$$

Now, applying Gauss' divergence theorem to the integration over Δ^2, we obtain

$$\begin{aligned}&\int_{(y_1,y_2)\in\Delta^2}(\partial_{y_1}^2+\partial_{y_2}^2)\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2\\&=\int_{(y_1,y_2)\in C^2}(y_1\partial_{y_1}+y_2\partial_{y_2})\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}\sigma\\&=\int_{(y_1,y_2)\in C^2}s(U_1y_1+U_2y_2)\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}\sigma,\end{aligned} \tag{16.6-108}$$

where $\mathrm{d}\sigma$ is the elementary arc length along the unit circle C^2 that forms the boundary of the unit disk Δ^2 and where we have used the property that the unit vector along the normal to C^2 pointing away from Δ^2 is given by $\boldsymbol{\nu}=y_1\boldsymbol{i}(1)+y_2\boldsymbol{i}(2)$. In the integral on the right-hand side of Equation (16.6-108) we introduce the polar coordinates $\{r,\phi\}$, with $r=1$ and $0\leqslant\phi<2\pi$, about the vector $U_1\boldsymbol{i}(1)+U_2\boldsymbol{i}(2)$ as polar axis, as the variables of integration. This yields

$$\begin{aligned}&\int_{(y_1,y_2)\in C^2}(U_1y_1+U_2y_2)\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}\sigma\\&=\int_{\phi=0}^{2\pi}U\cos(\phi)\exp[sU\cos(\phi)\pm sU_3]\,\mathrm{d}\phi=2\pi U\exp(\pm sU_3)I_1(sU)\,,\end{aligned} \tag{16.6-109}$$

where I_1 is the modified Bessel function of the first kind and order one (Abramowitz and Stegun, 1964) and

$$U=(U_1^2+U_2^2)^{1/2}\geqslant 0\,. \tag{16.6-110}$$

Collecting the results, we end up with

$$\begin{aligned}\hat{\Gamma}(\boldsymbol{u},s)&=2\pi a_1a_2hs^{-2}U^{-1}U_3^{-1}I_1(sU)\,[\exp(sU_3)-\exp(-sU_3)]\\&=2V^{\mathrm{S}}s^{-2}U^{-1}U_3^{-1}I_1(sU)\sinh(sU_3)\,.\end{aligned} \tag{16.6-111}$$

Special cases occur for $U\downarrow 0$ and/or $U_3\to 0$. The corresponding limits easily follow from Equation (16.6-111) by using the relevant Taylor expansions in the right-hand side. In particular, it can be verified that for $U\downarrow 0$ and $U_3\to 0$ the result is in accordance with Equation (16.6-84).

Elliptical cone of finite height

Let the elliptical cone of finite height be defined by (Figure 16.6-17)

$$\mathcal{D}^{\mathrm{S}}=\left\{\boldsymbol{x}\in\mathcal{R}^3;\ 0\leqslant x_1^2/a_1^2+x_2^2/a_2^2<x_3^2/h^2,\ 0<x_3<h\right\}. \tag{16.6-112}$$

Its volume is

$$V^{\mathrm{S}}=\pi a_1a_2h/3\,. \tag{16.6-113}$$

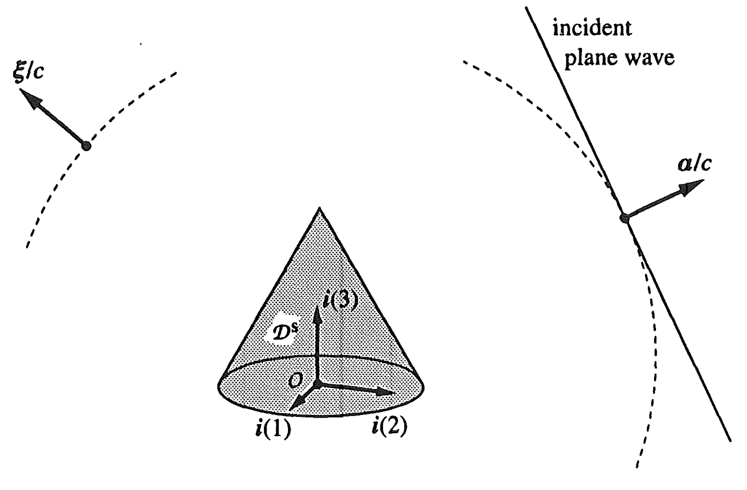


Figure 16.6-17 Scatterer in the shape of an elliptical cone of finite height.

In the integral on the right-hand side of Equation (16.6-85) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{16.6-114}$$

as the variables of integration. In y space, the domain of integration is then $\{y \in \mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2,\ 0 < y_3 < 1\}$. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \tag{16.6-115}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3\,, \end{aligned} \tag{16.6-116}$$

while

$$dV = a_1 a_2 h\, dy_1\, dy_2\, dy_3\,. \tag{16.6-117}$$

The integration then runs as follows:

$$\hat{\Upsilon}(u,s) = a_1 a_2 h \int_{y_3=0}^{1} dy_3 \int_{(y_1,y_2)\in\Delta^2(y_3)} \exp\,[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\, dy_1\, dy_2\,, \tag{16.6-118}$$

where $\Delta^2(y_3) = \{(y_1,y_2)\in\mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2\}$ is the circular disc of radius y_3. With a reasoning similar to that as used in Equations (16.6-107)–(16.6-109), we obtain

$$\begin{aligned} &\int_{(y_1,y_2)\in\Delta^2(y_3)} \exp[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\, dy_1\, dy_2 \\ &= (sU)^{-1} y_3 \int_{\phi=0}^{2\pi} \cos(\phi) \exp[s(U y_3 \cos(\phi) + U_3 y_3)]\, d\phi\,, \end{aligned} \tag{16.6-119}$$

in which

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0 . \tag{16.6-120}$$

Furthermore,

$$\int_{y_3=0}^{1} y_3 \exp[s(U y_3 \cos(\phi) + U_3 y_3)]\, \mathrm{d}y_3$$

$$= [s(U\cos(\phi) + U_3)]^{-1} \left\{ \exp[s(U\cos(\phi) + U_3)] - \int_{y_3=0}^{1} \exp[s(U y_3 \cos(\phi) + U_3 y_3)]\, \mathrm{d}y_3 \right\}$$

$$= [s(U\cos(\phi) + U_3)]^{-1} \exp[s(U\cos(\phi) + U_3)]$$

$$- [s(U\cos(\phi) + U_3)]^{-2} \left\{ \exp[s(U y_3 \cos(\phi) + U_3)] - 1 \right\} . \tag{16.6-121}$$

Collecting the results, we end up with

$$\hat{\Upsilon}(u,s) = 6V^{\mathrm{s}}(sU)^{-1} \int_{\phi=0}^{2\pi} \cos(\phi)$$

$$\times \frac{1}{2\pi} \left\{ \frac{\exp[s(U\cos(\phi) + U_3)]}{s(U\cos(\phi) + U_3)} - \frac{\exp[s(U\cos(\phi) + U_3)] - 1}{s^2(U\cos(\phi) + U_3)^2} \right\} \mathrm{d}\phi . \tag{16.6-122}$$

Special cases occur for $U \downarrow 0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (16.6-122) by using the relevant Taylor expansions in the right-hand side. In particular, it can be verified that for $U \downarrow 0$ and $U_3 \to 0$ the result is in accordance with Equation (16.6-84).

Tetrahedron

Let the tetrahedron be defined by (see Equation (A.9-17) and Figure 16.6-18)

$$\mathcal{D}^{\mathrm{s}} = \left\{ x \in \mathcal{R}^3 ;\ x = \sum_{I=0}^{3} \lambda(I) x(I),\ 0 < \lambda(I) < 1,\ \sum_{I=0}^{3} \lambda(I) = 1 \right\} , \tag{16.6-123}$$

in which $\{x(0),x(1),x(2),x(3)\}$ are the position vectors of the vertices and $\{\lambda(0),\ \lambda(1),\ \lambda(2),\lambda(3)\}$ are the barycentric coordinates. Its volume is given by (see Equations (A.10-29) and (A.10-33))

$$V^{\mathrm{s}} = \det[x(1) - x(0), x(2) - x(1), x(3) - x(2)] / 6 . \tag{16.6-124}$$

In the integral on the right-hand side of Equation (16.6-85) we replace $\lambda(0)$ by $1 - \lambda(1) - \lambda(2) - \lambda(3)$ and introduce $\{\lambda(1),\lambda(2),\lambda(3)\}$ as the (dimensionless) variables of integration. In $\{\lambda(1),\lambda(2),\lambda(3)\}$-space the domain of integration is then $\{0<\lambda(1)<1,\ 0<\lambda(2)<1-\lambda(1),\ 0<\lambda(3)<1-\lambda(1)-\lambda(2)\}$. Then, with

$$U(I) = u_s x_s(I) \qquad \text{for } I = 0, 1, 2, 3 , \tag{16.6-125}$$

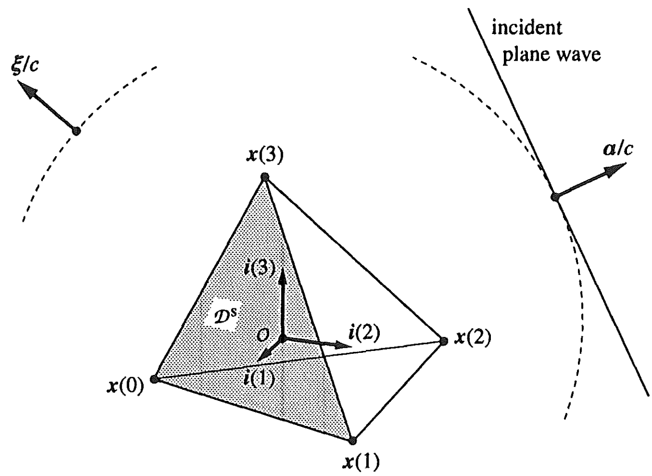


Figure 16.6-18. Scatterer in the shape of a tetrahedron (3-simplex).

we have

$$\begin{aligned}u_s x_s &= \lambda(0)U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3)\\ &= [1 - \lambda(1) - \lambda(2) - \lambda(3)]\, U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3)\\ &= U(0) + [U(1) - U(0)]\, \lambda(1) + [U(2) - U(0)]\, \lambda(2) + [U(3) - U(0)]\, \lambda(3)\,, \end{aligned} \tag{16.6-126}$$

while, with the Jacobian (see Equation (A.10-31))

$$\frac{\partial(x_1,x_2,x_3)}{\partial\,[\lambda(1),\lambda(2),\lambda(3)]} = 6V^{\rm s}, \tag{16.6-127}$$

the elementary volume is expressed as

$$\mathrm{d}V = 6V^{\rm s}\, \mathrm{d}\lambda(1)\, \mathrm{d}\lambda(2)\, \mathrm{d}\lambda(3)\,. \tag{16.6-128}$$

After some lengthy but elementary calculations it is found that

$$\begin{aligned}\hat{\Upsilon}(u,s) = 6V^{\rm s}s^{-3} \Bigg\{ &\frac{1}{U(0)-U(1)}\,\frac{1}{U(0)-U(2)}\,\frac{1}{U(0)-U(3)}\exp[sU(0)]\\ &+\frac{1}{U(1)-U(0)}\,\frac{1}{U(1)-U(2)}\,\frac{1}{U(1)-U(3)}\exp[sU(1)]\\ &+\frac{1}{U(2)-U(0)}\,\frac{1}{U(2)-U(1)}\,\frac{1}{U(2)-U(3)}\exp[sU(2)]\\ &+\frac{1}{U(3)-U(0)}\,\frac{1}{U(3)-U(1)}\,\frac{1}{U(3)-U(2)}\exp[sU(3)]\Bigg\}\,. \end{aligned} \tag{16.6-129}$$

In a symmetrical fashion, this result can be written as

$$\hat{Y}(\boldsymbol{u},s) = 6V^{\mathrm{S}}s^{-3}\sum_{I=0}^{3}\frac{1}{U(I)-U(J)}\,\frac{1}{U(I)-U(K)}\,\frac{1}{U(I)-U(L)}\exp[sU(I)]\,, \qquad (16.6\text{-}130)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$.

Special cases occur for $U(I) = U(J)$ and/or $U(I) = U(K)$ and/or $U(I) = U(L)$. The easiest way to arrive at the expressions for the relevant cases is to redo the integrations that need modifications.

Note: Since the first-order Rayleigh–Gans–Born approximation is additive in the domains occupied by the scatterers, the scattering by an arbitrary union of canonical scatterers follows by superposition. In particular, the result for the tetrahedron is the building block for scatterers in the shape of an arbitrary polyhedron.

The first-order Rayleigh–Gans–Born scattering finds numerous applications both in the forward (direct) and the inverse scattering theory. References to the earlier literature can be found in Quak (1989).

Exercises

Exercise 16.6-1

Show that Equation (16.6-92) follows from the time Laplace transform of Equation (16.6-27).

Exercise 16.6-2

Show that Equation (16.6-99) follows from the time Laplace transform of Equation (16.6-34).

Exercise 16.6-3

Show that Equation (16.6-111) follows from the time Laplace transform of Equation (16.6-46).

Exercise 16.6-4

Show that Equation (16.6-122) follows from the time Laplace transform of Equation (16.6-57).

Exercise 16.6-5

Show that Equation (16.6-130) follows from the time Laplace transform of Equation (16.6-65).

Exercise 16.6-6

Show that for $U\downarrow 0$, Equation (16.6-27) becomes Equation (16.6-19). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 16.6-7

Show that for $U_3 \to 0$, Equation (16.6-34) becomes

$$\Upsilon(\boldsymbol{u},t) = 2a_1a_2a_3(U_1U_2)^{-1}\,[a(t+U_1+U_2) - a(t+U_1-U_2) - a(t-U_1+U_2) + a(t-U_1-U_2)]\,. \tag{16.6-131}$$

(In this case, $\boldsymbol{u}$ is parallel to the x_1,x_2 plane.)

Exercise 16.6-8

Show that for $U_2 \to 0$ and $U_3 \to 0$, Equation (16.6-34) becomes

$$\hat{\Upsilon}(\boldsymbol{u},t) = 4a_1a_2a_3U_1^{-1}\,[\partial_t a(t+U_1) - \partial_t a(t-U_1)]\,. \tag{16.6-132}$$

(In this case, $\boldsymbol{u}$ is parallel to the x_1 axis.)

Exercise 16.6-9

Show that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$, Equation (16.6-34) becomes (16.6-19). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 16.6-10

Show that for $U \downarrow 0$, Equation (16.6-46) becomes

$$\hat{\Upsilon}(\boldsymbol{u},t) = \pi a_1a_2hU_3^{-1}\,[\partial_t a(t+U_3) - \partial_t a(t-U_3)]\,. \tag{16.6-133}$$

(In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 16.6-11

Show that for $U_3 \to 0$, Equation (16.6-46) becomes

$$\hat{\Upsilon}(\boldsymbol{u},t) = 2a_1a_2hU^{-1}\int_{\phi=0}^{2\pi} \cos(\phi)\partial_t a\,[t+U\cos(\phi)]\,\mathrm{d}\phi\,. \tag{16.6-134}$$

(In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cylinder.)

Exercise 16.6-12

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (16.6-46) becomes Equation (16.6-19). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 16.6-13

Show that for $U \downarrow 0$, Equation (16.6-57) becomes

$$\Upsilon(\boldsymbol{u},t) = \pi a_1 a_2 h \{U_3^{-1} [\partial_t a(t+U_3) - 2U_3^{-2} a(t+U_3)]$$

$$+ 2U_3^{-3} [\mathrm{I}_t a(t+U_3) - \mathrm{I}_t a(t)]\} . \qquad (16.6\text{-}135)$$

(In this case, $\boldsymbol{u}$ is parallel to the axis of the cone.)

Exercise 16.6-14

Show that for $U_3 \to 0$, Equation (16.6-57) becomes

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} \int_{\phi=0}^{2\pi} \{[U\cos(\phi)]^{-1} a(t+U\cos(\phi))$$

$$- [U\cos(\phi)]^{-2} [\mathrm{I}_t a(t+U\cos(\phi)) - \mathrm{I}_t a(t)]\} \cos(\phi)\, \mathrm{d}\phi . \qquad (16.6\text{-}136)$$

(In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cone.)

Exercise 16.6-15

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (16.6-57) becomes Equation (16.6-19). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 16.6-16

Show that for $U(J) \to U(I)$, Equation (16.6-65) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}} \Bigg[\left\{ \frac{1}{U(I)-U(K)} \frac{1}{U(I)-U(L)} \right\} a[t+U(I)]$$

$$- \left\{ \frac{1}{[U(I)-U(K)]^2} \frac{1}{U(I)-U(L)} + \frac{1}{U(I)-U(K)} \frac{1}{[U(I)-U(L)]^2} \right\} \mathrm{I}_t a[t+U(I)]$$

$$+ \left\{ \frac{1}{[U(K)-U(I)]^2} \frac{1}{U(K)-U(L)} \mathrm{I}_t a[t+U(K)] \right\}$$

$$+ \left\{ \frac{1}{[U(L)-U(I)]^2} \frac{1}{U(L)-U(K)} \mathrm{I}_t a[t+U(L)] \right\} \Bigg], \qquad (16.6\text{-}137)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$.)

Exercise 16.6-17

Show that for $U(J) \to U(I)$ and $U(L) \to U(K)$, Equation (16.6-65) becomes

$$\mathit{\Upsilon}(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left(\frac{1}{[U(I)-U(K)]^2}\{a\,[t+U(I)]+a\,[t+U(K)]\}\right.$$

$$\left.-\frac{2}{[U(I)-U(K)]^3}\{\mathrm{I}_t a\,[t+U(I)]-\mathrm{I}_t a\,[t+U(K)]\}\right), \tag{16.6-138}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$, as well as perpendicular to the edge connecting the vertex $\boldsymbol{x}(K)$ with the vertex $\boldsymbol{x}(L)$.)

Exercise 16.6-18

Show that for $U(J)\to U(I)$ and $U(K)\to U(I)$, Equation (16.6-65) becomes

$$\mathit{\Upsilon}(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left(\frac{1}{U(I)-U(L)}\partial_t a\,[t+U(I)] - \frac{1}{[U(I)-U(L)]^2}a\,[t+U(I)]\right.$$

$$\left.+\frac{1}{[U(I)-U(L)]^3}\left\{\mathrm{I}_t a\,[t+U(I)]-\mathrm{I}_t a\,[t+U(L)]\right\}\right), \tag{16.6-139}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the plane containing the triangle of which $\boldsymbol{x}(I)$, $\boldsymbol{x}(J)$ and $\boldsymbol{x}(K)$ are the vertices.)

Exercise 16.6-19

Show that for $\boldsymbol{u} = \boldsymbol{0}$, Equation (16.6-65) becomes Equation (16.6-19).

Exercise 16.6-20

Show that for $U\downarrow 0$, Equation (16.6-92) becomes Equation (16.6-84).

Exercise 16.6-21

Show that for $U_3\to 0$, Equation (16.6-99) becomes

$$\hat{\mathit{\Upsilon}}(\boldsymbol{u},s) = V^{\mathrm{s}}\,\frac{\sinh(sU_2)}{sU_2}\,\frac{\sinh(sU_1)}{sU_1} \tag{16.6-140}$$

and show that the result follows from the time Laplace transform of Equation (16.6-131). (In this case, $\boldsymbol{u}$ is parallel to the x_1,x_2 plane.)

Exercise 16.6-22

Show that for $U_2\to 0$ and $U_3\to 0$, Equation (16.6-99) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = V^{\mathrm{S}} \frac{\sinh(sU_1)}{sU_1} \tag{16.6-141}$$

and show that the result follows from the time Laplace transform of Equation (16.6-132). (In this case, $\boldsymbol{u}$ is parallel to the x_1 axis.)

Exercise 16.6-23

Show that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$, Equation (16.6-99) becomes Equation (16.6-84). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 16.6-24

Show that for $U \downarrow 0$, Equation (16.6-111) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 2\pi a_1 a_2 h s^{-1} U_3^{-1} \sinh(sU_3) \tag{16.6-142}$$

and show that the result follows the time Laplace transform of Equation (16.6-133). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 16.6-25

Show that for $U_3 \to 0$, Equation (16.6-111) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 4\pi a_1 a_2 h s^{-1} U^{-1} I_1(sU) \tag{16.6-143}$$

and show that the result follows from the time Laplace transform of Equation (16.6-134). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 16.6-26

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (16.6-111) becomes Equation (16.6-84). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 16.6-27

Show that for $U \downarrow 0$, Equation (16.6-122) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = \pi a_1 a_2 h s^{-2} \left\{ \left[sU_3^{-1} - 2U_3^{-2} + 2s^{-1}U_3^{-3} \right] \exp(sU_3) - 2s^{-1}U_3^{-3} \right\} \tag{16.6-144}$$

and show that this result follows from the time Laplace transform of Equation (16.6-135). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cone.)

Exercise 16.6-28

Show that for $U_3 \to 0$, Equation (16.6-122) becomes

$$\hat{r}(u,s) = a_1 a_2 h s^{-2} U^{-1} \int_{\phi=0}^{2\pi} \{[U\cos(\phi)]^{-1} \exp[sU\cos(\phi)]$$

$$- s^{-1}[U\cos(\phi)]^{-2}[\exp(sU\cos(\phi)) - 1]\} \cos(\phi)\, d\phi \qquad (16.6\text{-}145)$$

and show that this result follows from the time Laplace transform of Equation (16.6-136). (In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cone.)

Exercise 16.6-29

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (16.6-122) becomes Equation (16.6-84). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 16.6-30

Show that for $U(J) \to U(I)$, Equation (16.6-130) becomes

$$\hat{r}(u,s) = 6V^{s} s^{-2} \Bigg[\left\{ \frac{1}{U(I) - U(K)} \frac{1}{U(I) - U(L)} \exp[sU(I)] \right\}$$

$$- \left\{ \frac{1}{[U(I) - U(K)]^2} \frac{1}{U(I) - U(L)} + \frac{1}{U(I) - U(K)} \frac{1}{[U(I) - U(L)]^2} \right\} s^{-1} \exp[sU(I)]$$

$$+ \left\{ \frac{1}{[U(K) - U(L)]^2} \frac{1}{U(K) - U(L)} s^{-1} \exp[sU(K)] \right\}$$

$$+ \left\{ \frac{1}{[U(L) - U(I)]^2} \frac{1}{U(L) - U(K)} s^{-1} \exp[sU(L)] \right\} \Bigg], \qquad (16.6\text{-}146)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$, and show that this result follows from the time Laplace transform of Equation (16.6-137). (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$.)

Exercise 16.6-31

Show that for $U(J) \to U(I)$ and $U(L) \to U(K)$, Equation (16.6-130) becomes

$$\hat{r}(u,s) = 6V^{s} s^{-2} \Bigg(\frac{1}{[U(I) - U(K)]^2} \{\exp[sU(I)] + \exp[sU(K)]\}$$

$$- \frac{2s^{-1}}{[U(I) - U(K)]^3} \{\exp[sU(I)] - \exp[sU(L)]\} \Bigg), \qquad (16.6\text{-}147)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$, and show that this result follows from the time Laplace transform of Equation (16.6-138). (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $x(I)$ with the vertex $x(J)$ as well as perpendicular to the edge connecting the vertex $x(K)$ with the vertex $x(L)$.)

Exercise 16.6-32

Show that for $U(J) \to U(I)$ and $U(K) \to U(I)$, Equation (16.6-130) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}} s^{-2} \left(\frac{s}{U(I) - U(K)} \exp[sU(I)] - \frac{1}{[U(I) - U(K)]^2} \exp[sU(I)] \right.$$
$$\left. + \frac{s^{-1}}{[U(I) - U(K)]^3} \{\exp[sU(I)] - \exp[sU(L)]\} \right), \qquad (16.6\text{-}148)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$, and show that this result is the time Laplace transform of Equation (16.6-139). (In this case, $\boldsymbol{u}$ is perpendicular to the plane containing the triangle of which $x(I)$, $x(J)$ and $x(K)$ are the vertices.)

Exercise 16.6-33

Show that for $\boldsymbol{u} = \boldsymbol{0}$, Equation (16.6-130) becomes Equation (16.6-84).

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