19

The electromagnetic constitutive relations

The electromagnetic constitutive relations are representative for the macroscopic electromagnetic properties of passive matter. In their general form, the relations constitute an equivalent of three vectorial relations between the five vectorial quantities $\{J_k,P_k,M_j,E_r,H_p\}$. For reasons that are connected with the transfer of energy by electromagnetic fields, they express, in their standard form, $\{J_k,P_k,M_j\}$ in terms of $\{E_r,H_p\}$. Several terminological aspects of the relationship are enumerated below. In view of the assumed passivity of the medium, it is required that for any type of matter we have $\{J_k,P_k,M_j\}\to 0$ as $\{E_r,H_p\}\to 0$.

Linearity

In case the constitutive operators that express the values of $\{J_k,P_k,M_j\}$ in terms of the values of $\{E_r,H_p\}$ are linear, the medium is denoted as *linear* in its electromagnetic behaviour. If this is not the case, the medium is denoted as *non-linear* in its electromagnetic behaviour. A wide class of materials is found to behave linearly in the presence of not too strong electromagnetic fields. An exception to this are the ferromagnetic materials that already show a non-linear response to weak magnetic fields.

Time invariance

In case the constitutive operators that express the values of $\{J_k,P_k,M_j\}$ in terms of the values of $\{E_r,H_p\}$ are time invariant, the medium is denoted as *time invariant*. Otherwise, the medium is *time variant* or parametrically affected. Unless the medium parameters are affected by a time variant external mechanism (for example, acoustically, or otherwise mechanically), most materials are time invariant in their electromagnetic behaviour.

Relaxation

In case the constitutive operators express the values of $\{J_k,P_k,M_j\}$ at some instant in terms of the values of $\{E_r,H_p\}$ at that same instant only, the medium is denoted as *instantaneously* *reacting* in its electromagnetic behaviour. When, on the other hand, the values of $\{J_k, P_k, M_j\}$ are expressed in terms of the values of $\{E_r, H_p\}$ at other (usually all previous) instants, the medium is said to show electromagnetic *relaxation*. The property that only the past is involved in relaxation phenomena, is known as the *principle of causality*.

Local reactivity

In case the constitutive operators express the values of $\{J_k, P_k, M_j\}$ at some position in space in terms of the values of $\{E_r, H_p\}$ at that same position only, the medium is denoted as *locally reacting* in its electromagnetic behaviour. When, on the other hand, the values of $\{E_r, H_p\}$ elsewhere are involved in the constitutive operator, the medium is non-locally reacting in its electromagnetic behaviour. Almost all media are locally reacting in their electromagnetic behaviour. An exception to this are the warm plasmas that, through acoustic wave interaction associated with their compressibility, are often non-locally reacting.

Homogeneity

If in a certain domain in space the constitutive operators that express $\{J_k, P_k, M_j\}$ in terms of $\{E_r, H_p\}$ are shift invariant, the medium is denoted as *homogeneous*; in that domain in space where the shift invariance does not apply, the medium is denoted as *inhomogeneous*.

Isotropy

If at a point in space the constitutive operators that express $\{J_k, P_k, M_j\}$ in terms of $\{E_r, H_p\}$ are orientation invariant, the medium is denoted as *isotropic* at that point. If this property does not apply, the medium is denoted as *anisotropic*. Isotropic materials have no inner structure on a macroscopic scale; anisotropy is a structural characteristic of, for example, all crystals.

Several cases of commonly encountered electromagnetic constitutive relations will be discussed in subsequent sections.

19.1 Conductivity, permittivity and permeability of an isotropic material

For a wide class of materials, the quantities J_k and P_k only depend on E_r (and not on H_p), while the quantity M_j only depends on H_p (and not on E_r). For a medium that is, in addition, linear, time invariant, instantaneously reacting, locally reacting and isotropic in its electromagnetic behaviour, we then have

$$J_k(\boldsymbol{x},t) = \sigma(\boldsymbol{x})E_k(\boldsymbol{x},t) , \qquad (19.1\text{-}1)$$

$$P_k(\boldsymbol{x},t) = \varepsilon_0\chi_e(\boldsymbol{x})E_k(\boldsymbol{x},t) , \qquad (19.1\text{-}2)$$

$$M_j(x,t) = \chi_{\rm m}(x)H_j(x,t)\,, \tag{19.1-3}$$

where

σ = (electrical) conductivity (S/m),

$\chi_{\rm e}$ = electric susceptibility,

$\chi_{\rm m}$ = magnetic susceptibility.

(The inclusion of the factor ε_0 on the right-hand side of Equation (19.1-2) is a matter of convention to make the electric and magnetic susceptibilities dimensionless.) Next, Equation (19.1-2) is substituted in Equation (18.3-9), Equation (19.1-3) is substituted in Equation (18.3-10), and the resulting relations are written as

$$D_k(x,t) = \varepsilon(x)E_k(x,t) = \varepsilon_0\varepsilon_{\rm r}(x)E_k(x,t)\,, \tag{19.1-4}$$

$$B_j(x,t) = \mu(x)H_j(x,t) = \mu_0\mu_{\rm r}(x)H_j(x,t)\,, \tag{19.1-5}$$

where

ε = (absolute) permittivity (F/m),

$\varepsilon_{\rm r}$ = relative permittivity,

μ = (absolute) permeability (H/m),

$\mu_{\rm r}$ = relative permeability.

Comparison of the corresponding expressions shows that

$$\varepsilon = \varepsilon_0(1+\chi_{\rm e}) \quad \text{or} \quad \varepsilon_{\rm r} = 1+\chi_{\rm e}\,, \tag{19.1-6}$$

$$\mu = \mu_0(1+\chi_{\rm m}) \quad \text{or} \quad \mu_{\rm r} = 1+\chi_{\rm m}\,. \tag{19.1-7}$$

For an isotropic medium of the indicated kind, the conductivity, the electric susceptibility, the magnetic susceptibility, the permittivity and the permeability are scalars (tensors of rank zero).

In a domain where the constitutive coefficients introduced here change with position, the medium is inhomogeneous; in a domain where they are constant, the medium is homogeneous.

19.2 Conductivity, permittivity and permeability of an anisotropic material

We again consider the class of materials for which J_k and P_k only depend on E_r (and not on H_p), while the quantity M_j only depends on H_p (and not on E_r). Let the medium, in addition, be linear, time invariant, instantaneously reacting and locally reacting in its electromagnetic behaviour, and let us concentrate on the effect of anisotropy. In that case, the relations given in Equations (19.1-1)–(19.1-3) are replaced by

$$J_k(x,t) = \sigma_{k,r}(x)E_r(x,t)\,, \tag{19.2-1}$$

$$P_k(x,t) = \varepsilon_0\chi_{{\rm e};k,r}(x)E_r(x,t)\,, \tag{19.2-2}$$

$$M_j(x,t) = \chi_{{\rm m};j,p}(x)H_p(x,t)\,, \tag{19.2-3}$$

and Equations (19.1-4) and (19.1-5) by

$$D_k(x,t) = \varepsilon_{k,r}(x)E_r(x,t) = \varepsilon_0\varepsilon_{\mathrm{r};k,r}(x)E_r(x,t)\ , \tag{19.2-4}$$

$$B_j(x,t) = \mu_{j,p}(x)H_p(x,t) = \mu_0\mu_{\mathrm{r};j,p}(x)H_p(x,t)\ , \tag{19.2-5}$$

where

$$\varepsilon_{k,r} = \varepsilon_0(\delta_{k,r} + \chi_{\mathrm{e};k,r}) \qquad \text{or} \quad \varepsilon_{\mathrm{r};k,r} = \delta_{k,r} + \chi_{\mathrm{e};k,r}\ , \tag{19.2-6}$$

$$\mu_{j,p} = \mu_0(\delta_{j,p} + \chi_{\mathrm{m};j,p}) \qquad \text{or} \quad \mu_{\mathrm{r};j,p} = \delta_{j,p} + \chi_{\mathrm{m};j,p}\ . \tag{19.2-7}$$

For an anisotropic medium of the indicated kind, the conductivity, the electric susceptibility, the magnetic susceptibility, the permittivity and the permeability are tensors of rank two.

In a domain where the constitutive coefficients introduced here change with position, the medium is inhomogeneous; in a domain where they are constant, the medium is homogeneous.

19.3 Conductivity, permittivity and permeability of a material with relaxation

We again consider the class of materials for which J_k and P_k only depend on E_r (and not on H_p), while the quantity M_j only depends on H_p (and not on E_r). Let the medium, in addition, be linear, time invariant, and locally reacting in its electromagnetic behaviour, and let us concentrate on the effect of relaxation. For simplicity, we shall give the formulas for isotropic media; the extension to anisotropic media is elementary, and will be covered in an exercise. For a causal behaviour of the indicated kind, Equations (19.1-1)–(19.1-3) are replaced by

$$J_k(x,t) = \int_{t'=0}^{\infty} \kappa_{\mathrm{c}}(x,t')E_k(x,t-t')\,\mathrm{d}t'\ , \tag{19.3-1}$$

$$P_k(x,t) = \varepsilon_0 \int_{t'=0}^{\infty} \kappa_{\mathrm{e}}(x,t')E_k(x,t-t')\,\mathrm{d}t'\ , \tag{19.3-2}$$

$$M_j(x,t) = \int_{t'=0}^{\infty} \kappa_{\mathrm{m}}(x,t')H_j(x,t-t')\,\mathrm{d}t'\ , \tag{19.3-3}$$

respectively, in which

κ_{c} = conduction relaxation function (S/m•s),

κ_{e} = dielectric relaxation function (s^{-1}),

κ_{m} = magnetic relaxation function (s^{-1}).

Note that in Equations (19.3-1)–(19.3-3) the causality of the medium's response has been enforced by restricting the interval of integration of the elapse time t' to the interval $\{t'\in\mathcal{R};\ t'>0\}$, which implies that only the values of $\{E_k,H_j\}$ prior to t contribute to the values of $\{J_k,P_k,M_j\}$ at the instant t. Mathematically, one can express the same property by defining the relaxation functions on the entire interval $\{t'\in\mathcal{R}\}$ and requiring that the relaxation functions satisfy the condition

$$\{\kappa_{\mathrm{c}},\kappa_{\mathrm{e}},\kappa_{\mathrm{m}}\}(x,t') = 0 \qquad \text{for} \quad t'<0\ . \tag{19.3-4}$$

The time invariance of the medium is obvious from the property that in the right-hand sides of Equations (19.3-1)–(19.3-3) only the elapse time t' between "cause" (at the instant $t - t'$) and "effect" (at the instant t) occurs in the relaxation functions.

It follows from the microscopic theory of the electromagnetic behaviour of matter that the relaxation functions are bounded functions of their time arguments. For the limiting case of an instantaneously reacting material, the relaxation functions approach an impulse (delta-distribution) time behaviour. In this limiting case, Equations (19.3-1)–(19.3-3) reduce to Equations (19.1-1)–(19.1-3), provided that we take

$$\kappa_c(\boldsymbol{x},t') = \sigma(\boldsymbol{x})\delta(t') , \tag{19.3-5}$$

$$\kappa_e(\boldsymbol{x},t') = \chi_e(\boldsymbol{x})\delta(t') , \tag{19.3-6}$$

$$\kappa_m(\boldsymbol{x},t') = \chi_m(\boldsymbol{x})\delta(t') , \tag{19.3-7}$$

where $\delta(t')$ is the Dirac impulse distribution operative at $t' = 0$.

Some simple examples of relaxation functions, in particular those that follow from the elementary microscopic considerations within the realm of Lorentz's theory of electrons, will be discussed in Sections 19.5–19.7.

Exercises

Exercise 19.3-1

Give the constitutive relations for a time invariant, locally reacting, anisotropic medium with relaxation that is linear and causal in its electromagnetic behaviour.

Answer:

$$J_k(\boldsymbol{x},t) = \int_{t'=0}^{\infty} \kappa_{c;k,r}(\boldsymbol{x},t')E_r(\boldsymbol{x},t-t')\,dt' , \tag{19.3-8}$$

$$P_k(\boldsymbol{x},t) = \varepsilon_0 \int_{t'=0}^{\infty} \kappa_{e;k,r}(\boldsymbol{x},t')E_r(\boldsymbol{x},t-t')\,dt' , \tag{19.3-9}$$

$$M_j(\boldsymbol{x},t) = \int_{t'=0}^{\infty} \kappa_{m;j,p}(\boldsymbol{x},t')H_p(\boldsymbol{x},t-t')\,dt' . \tag{19.3-10}$$

Exercise 19.3-2

In which special case do Equations (19.3-8)–(19.3-10) reduce to Equations (19.2-1)–(19.2-3), respectively?

Answer: If

$$\kappa_{c;k,r}(\boldsymbol{x},t') = \sigma_{k,r}(\boldsymbol{x})\delta(t') , \tag{19.3-11}$$

$$\kappa_{e;k,r}(\boldsymbol{x},t') = \chi_{e;k,r}(\boldsymbol{x})\delta(t') , \tag{19.3-12}$$

$$\kappa_{m;j,p}(\boldsymbol{x},t') = \chi_{m;j,p}(\boldsymbol{x})\delta(t') . \tag{19.3-13}$$

19.4 Electric current as a flow of electrically charged particles. The conservation of electric charge

The basic concept of Lorentz's theory of electrons is that an electric current, whether it is present in an electrically conducting solid (for example, a metal), or an ionised liquid or gas, can be conceived as a flow of electrically charged particles. The concept of electric charge cannot further be reduced and is physically defined through a set of standard experiments. The electric charge q of a particle can be either positive (for example, $q > 0$ for a proton), or zero (for example, $q = 0$ for a neutron), or negative (for example, $q < 0$ for an electron). It follows from experiments that electric charge is quantised: only integer multiples of the elementary charge e occur in nature. From the experiments, the value of e is found as

$$e = 1.60217733 \times 10^{-19}\ \mathrm{C}\,. \tag{19.4-1}$$

Our analysis starts by considering a collection of identifiable particles whose geometrical dimensions are negligibly small. The collection is present in some domain $\mathcal{D}$ in three-dimensional space $\mathcal{R}^3$. Each particle carries a label by which is can be distinguished from all the other particles, and the particle with the label p occupies, at the instant t, the position $\boldsymbol{x}^{(p)}(t)$. If $\boldsymbol{x}^{(p)}$ changes with time, the (instantaneous) velocity $w_r^{(p)}$ of the particle is given by $w_r^{(p)} = \mathrm{d}_t x_r^{(p)}$, where d_t indicates the change of position with time that an observer registers when moving along with the particle. We now select a standard, shift- and time-invariant subdomain $\mathcal{D}_\varepsilon$ of $\mathcal{D}$, a so-called *representative elementary domain*, whose maximum diameter is small compared to the geometrical dimensions of the macroscopic system we are analysing and small compared to the scale on which the macroscopic quantities we are going to introduce show spatial changes, but that nevertheless contains so large a number of particles that it can be considered as an elementary part of a continuuum on the macroscopic scale (Figures 19.4-1 and 19.4-2).

The basic assumption (of statisical physics) is that appropriate spatial averages over $\mathcal{D}_\varepsilon$ of the microscopic quantities lead to the associated macroscopic quantities, the latter being assumed to vary piecewise continuously with position. (This is the so-called *continuum hypothesis*.)

Number density

Let $\boldsymbol{x}$ be the position of the (bary)centre of $\mathcal{D}_\varepsilon(\boldsymbol{x})$, and let $N_\varepsilon = N_\varepsilon(\boldsymbol{x}, t)$ be the number of particles present in $\mathcal{D}_\varepsilon(\boldsymbol{x})$. Then, the macroscopic *number density* $n = n(\boldsymbol{x},t)$ of the collection of particles attributed to the position $\boldsymbol{x}$ and taken at time t is defined as

$$n = N_\varepsilon(\boldsymbol{x},t)/V_\varepsilon\,, \tag{19.4-2}$$

where

$$V_\varepsilon = \int_{\boldsymbol{x}'\in\mathcal{D}_\varepsilon(\boldsymbol{x})} \mathrm{d}V = \int_{\boldsymbol{\xi}\in\mathcal{D}_\varepsilon(\boldsymbol{0})} \mathrm{d}V \tag{19.4-3}$$

is the volume of $\mathcal{D}_\varepsilon$. (The shift invariance of $\mathcal{D}_\varepsilon$ implies that if $\boldsymbol{x}'\in\mathcal{D}_\varepsilon(\boldsymbol{x})$, then $\boldsymbol{\xi}\in\mathcal{D}_\varepsilon(\boldsymbol{0})$, where $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\xi}$.)

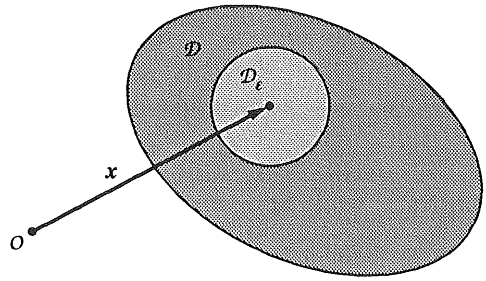

Figure 19.4-1 Domain $\mathcal{D}$ in which a collection of moving particles is present; $\mathcal{D}_\varepsilon$ is a time- and shift-invariant representative elementary domain with centre $\boldsymbol{x}$.

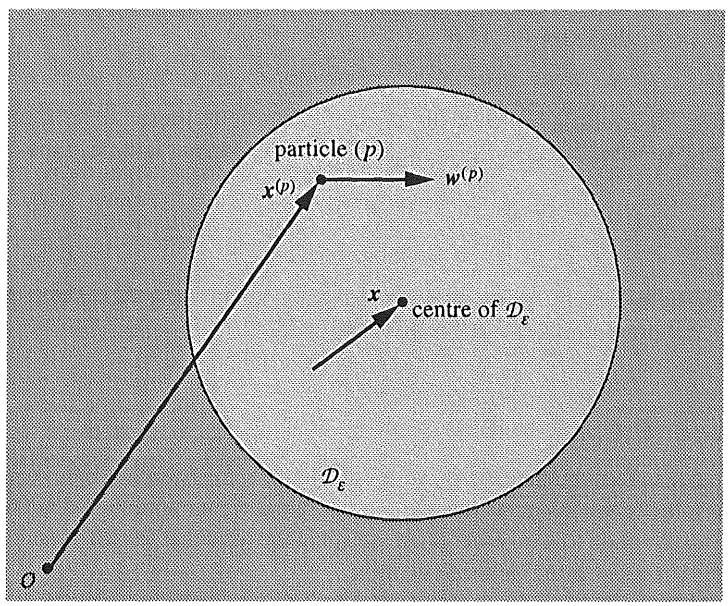

Figure 19.4-2 Representative elementary domain $\mathcal{D}_\varepsilon$ with a collection of moving particles.

Note: The position of the barycentre of $\mathcal{D}_\varepsilon(\boldsymbol{x})$ is defined as

$$\boldsymbol{x} = V_\varepsilon^{-1} \int_{\boldsymbol{x}' \in \mathcal{D}_\varepsilon(\boldsymbol{x})} \boldsymbol{x}' \, \mathrm{d}V . \qquad (19.4\text{-}4)$$

The continuum hypothesis states that $n = n(\boldsymbol{x},t)$ is a piecewise continuous function of $\boldsymbol{x}$. The total number of particles $N = N(t)$ present in some bounded domain $\mathcal{D} = \mathcal{D}(t)$ then follows as the sum of the numbers of particles present in the representative elementary subdomains that belong to $\mathcal{D}(t)$, i.e.

$$N(t) = \int_{\boldsymbol{x} \in \mathcal{D}(t)} n(\boldsymbol{x},t) \, \mathrm{d}V . \qquad (19.4\text{-}5)$$

Drift velocity

Next, the average velocity, transport velocity, or *drift velocity*, v_r of the particles is introduced as

$$v_r(x,t) = \langle w_r \rangle(x,t) = [N_\varepsilon(x,t)]^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} w_r^{(p)}(t) \,, \qquad (19.4\text{-}6)$$

where $\langle ... \rangle$ denotes the arithmetic mean of the quantity in angular brackets. It is noted that the chaotic part of the motion of the particles, which determines the thermodynamic notion of their temperature, averages out in Equation (19.4-6) and does not contribute to the right-hand side.

Conservation of particles

Upon following, for a short while Δt, the collection of particles present in $\mathcal{D}(t)$ on its course, a conservation law is arrived at. Let the number of particles present in $\mathcal{D}(t)$ at the instant t be

$$N(t) = \int_{x \in \mathcal{D}(t)} n(x,t) \,\mathrm{d}V \,, \qquad (19.4\text{-}7)$$

and let these particles occupy at the instant $t + \Delta t$ the domain $\mathcal{D}(t + \Delta t)$. Then, the number of particles $N(t + \Delta t)$ present in $\mathcal{D}(t + \Delta t)$ is given by

$$N(t + \Delta t) = \int_{x \in \mathcal{D}(t+\Delta t)} n(x,t + \Delta t) \,\mathrm{d}V \,, \qquad (19.4\text{-}8)$$

where $n(x,t + \Delta t)$ is the number density at the instant $t + \Delta t$. Assume that, in the meantime, particles have somehow been created at the overall rate $\mathrm{d}_t N_{\mathrm{cr}}$ or annihilated at the overall rate $\mathrm{d}_t N_{\mathrm{ann}}$. The consideration that no other processes than the ones that are mentioned are involved leads up to the first order in Δt balance equation

$$N(t + \Delta t) = N(t) + [\mathrm{d}_t N_{\mathrm{cr}} - \mathrm{d}_t N_{\mathrm{ann}}] \,\Delta t + o(\Delta t) \qquad \text{as } \Delta t \to 0 \,. \qquad (19.4\text{-}9)$$

Note: For the definition of Landau's *order symbols* o and O, see Equations (A.8-1)–(A.8-6).
Now, assuming that $n = n(x,t)$ is, throughout $\mathcal{D}(t)$, continuously differentiable with respect to t, we have the Taylor expansion

$$n(x,t + \Delta t) = n(x,t) + \partial_t n(x,t) \,\Delta t + o(\Delta t) \qquad \text{as } \Delta t \to 0 \,. \qquad (19.4\text{-}10)$$

Furthermore, the geometry of the domain $\mathcal{D}(t)$ as it changes with t entails that (see Figure 19.4-3, where $\mathcal{D}(t + \Delta t)$ is decomposed into the part that it has in common with $\mathcal{D}(t)$, the part that has been left behind and the part that has been acquired)

$$\int_{x \in \mathcal{D}(t+\Delta t)} n(x,t) \,\mathrm{d}V = \int_{x \in \mathcal{D}(t)} n(x,t) \,\mathrm{d}V + \int_{x \in \partial\mathcal{D}(t)} n(x,t) v_r(x,t) \,\Delta t \,\mathrm{d}A_r + o(\Delta t) \quad \text{as} \quad \Delta t \to 0 \,, \qquad (19.4\text{-}11)$$

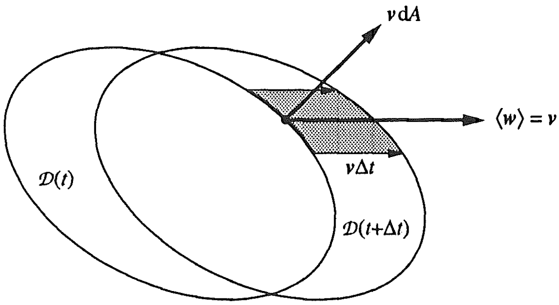

Figure 19.4-3 Conservation law for particles occupying the domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ and moving with a drift velocity $\boldsymbol{\nu}$.

where $\partial\mathcal{D}(t)$ is the boundary surface of $\mathcal{D}(t)$ and ν_r is the local drift velocity with which the particles on $\partial\mathcal{D}(t)$ move, while

$$\int_{x\in\mathcal{D}(t+\Delta t)} \partial_t n(\boldsymbol{x},t)\,\mathrm{d}V = \int_{x\in\mathcal{D}(t)} \partial_t n(\boldsymbol{x},t)\,\mathrm{d}V + o(1) \quad \text{as} \quad \Delta t \to 0\,. \tag{19.4-12}$$

Combining Equations (19.4-10), (19.4-11) and (19.4-12), the result

$$\begin{aligned}\int_{x\in\mathcal{D}(t+\Delta t)} n(\boldsymbol{x},t+\Delta t)\,\mathrm{d}V &= \int_{x\in\mathcal{D}(t+\Delta t)} n(\boldsymbol{x},t)\,\mathrm{d}V + \int_{x\in\mathcal{D}(t+\Delta t)} \partial_t n(\boldsymbol{x},t)\,\Delta t\,\mathrm{d}V + o(\Delta t)\\ &= \int_{x\in\mathcal{D}(t)} n(\boldsymbol{x},t)\,\mathrm{d}V + \int_{x\in\partial\mathcal{D}(t)} n(\boldsymbol{x},t)\nu_r(\boldsymbol{x},t)\,\Delta t\,\mathrm{d}A_r\\ &\quad + \int_{x\in\mathcal{D}(t)} \partial_t n(\boldsymbol{x},t)\,\Delta t\,\mathrm{d}V + o(\Delta t) \quad \text{as} \quad \Delta t\to 0\end{aligned} \tag{19.4-13}$$

is obtained. From Equations (19.4-13) and (19.4-7)–(19.4-9) it follows by dividing by Δt and taking the limit $\Delta t\to 0$, that

$$\int_{x\in\mathcal{D}(t)} \partial_t n(\boldsymbol{x},t)\,\mathrm{d}V + \int_{x\in\partial\mathcal{D}(t)} n(\boldsymbol{x},t)\nu_r(\boldsymbol{x},t)\,\mathrm{d}A_r = \mathrm{d}_t N_{\mathrm{cr}}(t) - \mathrm{d}_t N_{\mathrm{ann}}(t)\,. \tag{19.4-14}$$

Equation (19.4-14) is known as the *conservation law of particle flow.*

Introducing the volume densities of the rates of particle creation $\dot{n}_{\mathrm{cr}}$ and particle annihilation $\dot{n}_{\mathrm{ann}}$ similar to Equation (19.4-2), we can write

$$\mathrm{d}_t N_{\mathrm{cr}}(t) = \mathrm{d}_t \int_{x\in\mathcal{D}(t)} n_{\mathrm{cr}}(\boldsymbol{x},t)\,\mathrm{d}V = \int_{x\in\mathcal{D}(t)} \dot{n}_{\mathrm{cr}}(\boldsymbol{x},t)\,\mathrm{d}V\,, \tag{19.4-15}$$

$$\mathrm{d}_t N_{\mathrm{ann}}(t) = \mathrm{d}_t \int_{x\in\mathcal{D}(t)} n_{\mathrm{ann}}(\boldsymbol{x},t)\,\mathrm{d}V = \int_{x\in\mathcal{D}(t)} \dot{n}_{\mathrm{ann}}(\boldsymbol{x},t)\,\mathrm{d}V\,. \tag{19.4-16}$$

(The dot over a symbol is a standard notation in physics to indicate the time rate of change.) Using these expressions in Equation (19.4-14) and applying Gauss' integral theorem to the second integral on the left-hand side of Equation (19.4-14) under the assumption that nv_r is continuously differentiable throughout $\mathcal{D}(t)$, we obtain

$$\int_{x\in\mathcal{D}(t)} [\partial_t n + \partial_r(nv_r)]\,\mathrm{d}V = \int_{x\in\mathcal{D}(t)} (\dot{n}_{\mathrm{cr}} - \dot{n}_{\mathrm{ann}})\,\mathrm{d}V\,. \tag{19.4-17}$$

Since Equation (19.4-17) has to hold for any domain, and the integrands are assumed to be continuous functions of position, we arrive at (for the justification of this step, see Exercise 19.4-5)

$$\partial_t n + \partial_r(nv_r) = \dot{n}_{\mathrm{cr}} - \dot{n}_{\mathrm{ann}}\,. \tag{19.4-18}$$

Equation (19.4-18) is known as the *continuity equation of particle flow*.

Volume density of electric charge. Volume density of electric (convection) current

Next, we concentrate on the electrical properties of the particles. Consider again the representative elementary domain $\mathcal{D}_\varepsilon(x)$ and let $N_\varepsilon(x,t)$ be the number of particles present in it. In addition, let $q^{(p)}$ be the electric charge of the particle with the label p, then the *volume density of electric charge* ρ is defined as

$$\rho(x,t) = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} q^{(p)}\,, \tag{19.4-19}$$

and the *volume density of electric (convection) current* J_k as

$$J_k(x,t) = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} q^{(p)} w_k^{(p)}(t)\,. \tag{19.4-20}$$

Upon using Equation (19.4-2), Equation (19.4-19) can be rewritten as

$$\begin{aligned}\rho(x,t) &= [N_\varepsilon(x,t)/V_\varepsilon]\,[N_\varepsilon(x,t)]^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} q^{(p)} \\ &= n\langle q\rangle\,,\end{aligned} \tag{19.4-21}$$

and Equation (19.4-20) as

$$\begin{aligned}J_k(x,t) &= [N_\varepsilon(x,t)/V_\varepsilon]\,[N_\varepsilon(x,t)]^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} q^{(p)} w_k^{(p)}(t) \\ &= n\langle q w_k\rangle\,.\end{aligned} \tag{19.4-22}$$

In terms of the volume density of electric charge, the total amount $Q = Q(t)$ of electric charge of the particles in some domain $\mathcal{D}(t)$ is given by

$$Q(t) = \int_{x \in \mathcal{D}(t)} \rho(x,t)\, \mathrm{d}V . \tag{19.4-23}$$

In what follows, it will be necessary to distinguish between the different types of particles as far as their electric charges are concerned. Let the subscript B be the label that indicates the value of the electric charge of the particles of the type B. (As far as the electric properties are concerned, the subscript B is indicative of the different electrical substances out of which the collection of particles is composed.) For example, one can take

$$q_{\mathrm{B}} = \mathrm{B}e . \tag{19.4-24}$$

Let, further, the superscript p denote the label of an individual particle within the collection of a certain type. For all particles of the type B we obviously have

$$q_{\mathrm{B}}^{(p)} = q_{\mathrm{B}} . \tag{19.4-25}$$

Now let $N_{\varepsilon,\mathrm{B}} = N_{\varepsilon,\mathrm{B}}(x,t)$ denote the number of particles of type B present in the representative elementary domain $\mathcal{D}_\varepsilon(x)$. The number density of particles of type B is then given by

$$n_{\mathrm{B}}(x,t) = N_{\varepsilon,\mathrm{B}}(x,t) / V_\varepsilon , \tag{19.4-26}$$

their volume density of electric charge by

$$\begin{aligned} \rho_{\mathrm{B}}(x,t) &= V_\varepsilon^{-1} \sum_{p=1}^{N_{\varepsilon,\mathrm{B}}(x,t)} q_{\mathrm{B}}^{(p)} \\ &= [N_{\varepsilon,\mathrm{B}}(x,t) / V_\varepsilon]\, q_{\mathrm{B}} \\ &= n_{\mathrm{B}}(x,t)\, q_{\mathrm{B}} , \end{aligned} \tag{19.4-27}$$

and their volume density of electric convection current by

$$\begin{aligned} J_{\mathrm{B};k}(x,t) &= V_\varepsilon^{-1} \sum_{p=1}^{N_{\varepsilon,\mathrm{B}}(x,t)} q_{\mathrm{B}}^{(p)} w_{\mathrm{B};k}^{(p)}(t) \\ &= V_\varepsilon^{-1} q_{\mathrm{B}} \sum_{p=1}^{N_{\varepsilon,\mathrm{B}}(x,t)} w_{\mathrm{B};k}^{(p)}(t) \\ &= [N_{\varepsilon,\mathrm{B}}(x,t) / V_\varepsilon]\, [N_{\varepsilon,\mathrm{B}}(x,t)]^{-1} q_{\mathrm{B}} \sum_{p=1}^{N_{\varepsilon,\mathrm{B}}(x,t)} w_{\mathrm{B};k}^{(p)}(t) \\ &= n_{\mathrm{B}}(x,t) q_{\mathrm{B}} v_{\mathrm{B};k}(x,t) \\ &= \rho_{\mathrm{B}}(x,t) v_{\mathrm{B};k}(x,t) . \end{aligned} \tag{19.4-28}$$

Taking all types of particles together, the contributions from the different substances add up to the total volume density of electric charge.

$$\rho = \sum_{\mathrm{B}} \rho_{\mathrm{B}} \tag{19.4-29}$$

and the total volume density of electric (convection) current

$$J_k = \sum_{\mathrm{B}} J_{\mathrm{B};k} . \tag{19.4-30}$$

Conservation of electric charge

A relationship between ρ_{B} and $J_{\mathrm{B};k}$ is established when the conservation law Equation (19.4-17) is applied to the particles of the type B. Multiplication of Equation (19.4-14) by q_{B} leads to

$$\int_{x\in\mathcal{D}(t)} \partial_t \rho_{\mathrm{B}}\, \mathrm{d}V + \int_{x\in\partial\mathcal{D}(t)} J_{\mathrm{B};k}\, \mathrm{d}A_k = \int_{x\in\mathcal{D}(t)} (\dot{\rho}_{\mathrm{B,cr}} - \dot{\rho}_{\mathrm{B,ann}})\, \mathrm{d}V \,, \qquad (19.4\text{-}31)$$

where $\dot{\rho}_{\mathrm{B,cr}}$ and $\dot{\rho}_{\mathrm{B,ann}}$ denote the volume densities of the rates at which electric charge is created and annihilated, respectively, through particles of the type B. In the same way, multiplication of Equation (19.4-18) by q_{B} leads to

$$\partial_t \rho_{\mathrm{B}} + \partial_k J_{\mathrm{B};k} = \dot{\rho}_{\mathrm{B,cr}} - \dot{\rho}_{\mathrm{B,ann}} \,. \qquad (19.4\text{-}32)$$

Now, it is an experimentally established fact that on a macroscopic scale no net electric charge can either be created or annihilated. Therefore, summing over all types of electrically charged particles, we have

$$\sum_{\mathrm{B}} \dot{\rho}_{\mathrm{B,cr}} = 0 \quad \text{and} \quad \sum_{\mathrm{B}} \dot{\rho}_{\mathrm{B,ann}} = 0 \,. \qquad (19.4\text{-}33)$$

Consequently, the summing of Equation (19.4-31) over all types of electrically charged particles yields

$$\int_{x\in\mathcal{D}(t)} \partial_t \rho\, \mathrm{d}V + \int_{x\in\partial\mathcal{D}(t)} J_k\, \mathrm{d}A_k = 0 \,, \qquad (19.4\text{-}34)$$

while the summing of Equation (19.4-32) over all types of electrically charged particles yields

$$\partial_t \rho + \partial_k J_k = 0 \,. \qquad (19.4\text{-}35)$$

Equation (19.4-34) is known as the *conservation law of electric charge*; Equation (19.4-35) is known as the *continuity equation of electric (convection) current.*

Stationary electric currents

A flow of particles is called *stationary*, or *steady*, when n, v_r, $\dot{n}_{\mathrm{cr}}$ and $\dot{n}_{\mathrm{ann}}$ are independent of time. Correspondingly, for a stationary electric current (i.e. a stationary flow of electrically charged particles), since also the electric charge of a particle is independent of time, the quantities ρ, J_k, $\dot{\rho}_{\mathrm{cr}}$ and $\dot{\rho}_{\mathrm{ann}}$ are independent of time.

Static distribution of electric charge

A distribution of particles is called *static* if no macroscopic transport of particles takes place; hence, $v_r = 0$ for a static distribution of particles. For a static distribution of electric charge (i.e. a static distribution of electrically charged particles) we correspondingly have $J_k = 0$.

Exercises

Exercise 19.4-1

In a collection of particles with number density n we consider a domain in the shape of a cube with edge length a. (a) What is the value of a if the cube is to contain, on average, a single particle? (b) What is the value of a if $n = 5.0 \times 10^{28}\ \mathrm{m}^{-3}$ (number density of atoms in silicon)?

Answers: (a) $a = n^{-1/3}$; (b) $a = 2.71 \times 10^{-10}$ m.

Exercise 19.4-2

In an ionised gas two types of electrically charged particles are present, viz. positive ions with number density n^+, drift velocity v_k^+ and electric charge q^+, and negative ions with number density n^-, drift velocity v_k^- and electric charge q^-. Give the expressions for (a) ρ, (b) J_k.

Answer: (a) $\rho = n^+q^+ + n^-q^-$; (b) $J_k = n^+q^+v_k^+ + n^-q^-v_k^-$.

Exercise 19.4-3

In a metal conductor the electric current consists of a flow of (conduction) electrons with number density n_e, electric charge $-e$, and drift velocity $v_{\mathrm{e};k}$. The host lattice is at rest and contains positively charged nuclei such that the conductor as a whole is electrically neutral. Give the expressions for (a) ρ; (b) J_k.

Answer: (a) $\rho = 0$; (b) $J_k = -n_\mathrm{e}ev_{\mathrm{e};k}$.

Exercise 19.4-4

In a semiconductor the electric current is composed of a flow of electrons with number density n_e and drift velocity $v_{\mathrm{e};k}$, and holes with number density n_h and drift velocity $v_{\mathrm{h};k}$. The electric charge of an electron is $-e$. The electric charge of a hole is e . Give the expressions for (a) ρ; (b) J_k.

Answer: (a) $\rho = -n_\mathrm{e}e + n_\mathrm{h}e$; (b) $J_k = -n_\mathrm{e}ev_{\mathrm{e};k} + n_\mathrm{h}ev_{\mathrm{h};k}$.

Exercise 19.4-5

Show that if $f = f(\boldsymbol{x},t)$ is a continuous function of $\boldsymbol{x}$ and t and

$$\int_{\boldsymbol{x}\in\mathcal{D}(t)} f(\boldsymbol{x},t)\,\mathrm{d}V = 0$$

for any $\mathcal{D}(t)$, then $f(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x}\in\mathcal{D}(t)$. (*Hint*: The proof follows by *reductio ad absurdum.* Assume that $f(\boldsymbol{x}_0,t) > 0$, then, on account of the assumed continuity, $f(\boldsymbol{x},t) > 0$ in some neighbourhood $\mathcal{B}_0$ of $\boldsymbol{x}_0$. By taking $\mathcal{D}$ to be this neighbourhood it follows that

$$\int_{\boldsymbol{x}\in\mathcal{B}_0} f(\boldsymbol{x},t)\,\mathrm{d}V > 0\,,$$

which is contrary to what is given. Repeat the same argument for $f(x_0,t) < 0$ and draw the conclusion.)

Exercise 19.4-6

Derive Equation (19.4-18) directly from Equation (19.4-17) by taking for the domain $\mathcal{D}$ the three-dimensional rectangle $\mathcal{D} = \{x' \in \mathcal{R}^3;\ x_m - \Delta x_m/2 < x'_m < x_m + \Delta x_m/2\}$. Assume that nv_r is continuously differentiable and use for nv_r everywhere on $\partial\mathcal{D}$ the first-order Taylor expansion $[nv_r](x',t) = [nv_r](x,t) + (x'_m - x_m)\partial_m[nv_r](x,t) + o(|x - x'|)$ as $|x - x'| \to 0$. Divide the resulting expression by $\Delta x_1 \Delta x_2 \Delta x_3$ and take the limit $\Delta x_1 \to 0$, $\Delta x_2 \to 0$, $\Delta x_3 \to 0$.

Exercise 19.4-7

Show that for a stationary flow of particles the conservation law of particle flow, Equation (19.4-14), reduces to

$$\int_{x \in \partial\mathcal{D}} n(x) v_r(x)\, \mathrm{d}A_r = \mathrm{d}_t N_{\mathrm{cr}}(t) - \mathrm{d}_t N_{\mathrm{ann}}(t) = \int_{x \in \mathcal{D}} [\dot{n}_{\mathrm{cr}}(x) - \dot{n}_{\mathrm{ann}}(x)]\, \mathrm{d}V . \qquad (19.4\text{-}36)$$

Exercise 19.4-8

Show that for a stationary electric current the conservation law of electric charge, Equation (19.4-34), reduces to

$$\int_{x \in \partial\mathcal{D}} J_k(x)\, \mathrm{d}A_k = 0 . \qquad (19.4\text{-}37)$$

Exercise 19.4-9

Show that for a stationary flow of particles the continuity equation of particle flow, Equation (19.4-18), reduces to

$$\partial_r(nv_r) = \dot{n}_{\mathrm{cr}} - \dot{n}_{\mathrm{ann}} . \qquad (19.4\text{-}38)$$

Exercise 19.4-10

Show that for a stationary electric current the continuity equation of electric (convection) current, Equation (19.4-35), reduces to

$$\partial_k J_k = 0 . \qquad (19.4\text{-}39)$$

Exercise 19.4-11

Prove that for any continuously differentiable function $\Phi = \Phi(x,t)$ the following properties hold:

(a) $$\partial_t \left[V_\varepsilon^{-1} \int_{x' \in \mathcal{D}_\varepsilon(x)} \Phi(x',t)\, \mathrm{d}V \right] = V_\varepsilon^{-1} \int_{x' \in \mathcal{D}_\varepsilon(x)} \partial_t \Phi(x',t)\, \mathrm{d}V ;$$

(b) $$\partial_p\left[V_\varepsilon^{-1}\int_{x'\in\mathcal{D}_\varepsilon(x)}\Phi(x',t)\,\mathrm{d}V\right]=V_\varepsilon^{-1}\int_{x'\in\mathcal{D}_\varepsilon(x)}\partial_p{}'\Phi(x',t)\,\mathrm{d}V\,.$$

Here, $\mathcal{D}_\varepsilon(x)$ is a representative elementary domain and $\partial_p{}'\Phi(x',t)$ means differentiation with respect to x_p'. (*Hint*: (a) follows by using the definition of derivative with respect to t; (b) follows by using the definition of derivative with respect to x_p, showing that

$$\partial_p\left[V_\varepsilon^{-1}\int_{x'\in\mathcal{D}_\varepsilon(x)}\Phi(x',t)\,\mathrm{d}V\right]=V_\varepsilon^{-1}\int_{x'\in\partial\mathcal{D}_\varepsilon(x)}\Phi(x',t)\,\mathrm{d}A_p\,,$$

and applying Gauss' integral theorem to the last integral.)

Exercise 19.4-12

Show, by using the method that has led from Equations (19.4-7)–(19.4-9) to Equation (19.4-14), that for any function $\Psi=\Psi(x,t)$ that is associated with the conservative flow of particles with drift velocity v_r we have

$$\mathrm{d}_t\int_{x\in\mathcal{D}(t)}\Psi(x,t)\,\mathrm{d}V=\int_{x\in\mathcal{D}(t)}\partial_t\Psi(x,t)\,\mathrm{d}V+\int_{x\in\partial\mathcal{D}(t)}\Psi(x,t)v_r(x,t)\,\mathrm{d}A_r\,. \qquad (19.4\text{-}40)$$

(This result is known as *Reynolds' transport theorem*.)

Exercise 19.4-13

Show, from the result of Exercise 19.4-12, that for any continuously differentiable function $\Psi=\Psi(x,t)$ which is associated with the conservative flow of particles with continuously differentiable drift velocity $v_r=v_r(x,t)$ we have

$$\mathrm{d}_t\int_{x\in\mathcal{D}(t)}\Psi(x,t)\,\mathrm{d}V=\int_{x\in\mathcal{D}(t)}\{\partial_t\Psi(x,t)+\partial_r\,[v_r(x,t)\Psi(x,t)]\}\,\mathrm{d}V\,.$$

(*Hint*: Apply Gauss' integral theorem to the boundary integral in Equation (19.4-40).) Upon rewriting the left-hand side as

$$\mathrm{d}_t\int_{x\in\mathcal{D}(t)}\Psi(x,t)\,\mathrm{d}V=\int_{x\in\mathcal{D}(t)}\dot{\Psi}(x,t)\,\mathrm{d}V\,, \qquad (19.4\text{-}41)$$

we also conclude that

$$\dot{\Psi}(x,t)=\partial_t\Psi(x,t)+\partial_r\,[v_r(x,t)\Psi(x,t)]\,. \qquad (19.4\text{-}42)$$

Exercise 19.4-14

In M ($M\geqslant 2$) metallic conductors that are interconnected at a "node", there is a flow of stationary currents with magnitude

$$I(m)=\int_{S(m)}J_k\,\mathrm{d}A_k$$

in the mth conductor ($m = 1,...,M$); here, $\mathcal{S}(m)$ is some cross-section of the mth conductor. Show that (a) $I(m)$ is independent of the choice of the cross-section that is used to evaluate the integral; (b) $\Sigma_{m=1}^{M} I(m) = 0$ (Kirchhoff's law). (c) Which property of J_k at the interface conductor/insulator has been used?

Answer: (c) $J_k \, \mathrm{d}A_k = 0$ at any elementary surface of the interface conductor/insulator.

Exercise 19.4-15

Consider a static distribution of particles in which recombination of the particles with other particles takes place at the rate $\dot{n}_{\mathrm{ann}} = n/\tau$ where τ denotes the "life time" of the particles. Determine $n = n(\boldsymbol{x},t)$ as a function of time in the interval $t_0 < t < \infty$. (*Hint*: Use Equation (19.4-18).)

Answer: $n(\boldsymbol{x},t) = n(\boldsymbol{x},t_0) \exp[-(t - t_0)/\tau]$.

Exercise 19.4-16

Into a piece of matter that is present in a domain $\mathcal{D}$ in space we inject instantaneously N_0 particles at $t = t_0$. Due to recombination with other particles present in $\mathcal{D}$ the injected particles have a life time τ (see Exercise 19.4-15). There is no flow of particles across the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. Let $N = N(t)$ denote the total number of particles of the injected type, present in $\mathcal{D}$ at the instant t. Determine (a) the differential equation that N must satisfy; (b) the solution of this equation. (*Hint*: Use Equation (19.4-14).)

Answer:

(a) $\partial_t N = N_0 \delta(t - t_0) - N/\tau$; (b) $N = 0$ for $t < t_0$, $N = N_0 \exp[-(t - t_0)/\tau]$ for $t > t_0$.

Exercise 19.4-17

Show that for a stationary electric convection current the continuity equation of electric current, Equation (19.4-35) reduces to

$$\partial_k J_k = 0 . \qquad (19.4\text{-}38)$$

19.5 The conduction relaxation function of a metal

In a metal, conduction of electricity takes place through the transport of the conduction electrons present in it. We assume that the metal is macroscopically neutral and that the positive nuclei of its host lattice are fixed in space. The conduction electrons are set into motion by a macroscopic electromagnetic field that exerts a force on them, and the resulting drift current is taken to be identical to the induced electric current introduced into Maxwell's equations in matter. (Another mechanism that could set the electrons into motion is diffusion, where a gradient in the number density of particles causes them to flow. The corresponding diffusion current is negligible in a metal, but is of major importance in semiconductors.) During their motion, the electrons collide with the nuclei in the host lattice. The macroscopic effect of these collisions is taken into account by assuming that on the average the collisions take place at the rate of ν_c per time (ν_c = *collision frequency* (s^{-1})), and that at each collision the electron transfers its momentum completely to the host lattice (which is then set into a slow mechanical motion). The equation of motion of the conduction electron with label (p) that is present in the representative elementary domain $\mathcal{D}_\varepsilon(\boldsymbol{x})$ is under these assumptions given by

$$m_e \mathrm{d}_t w_k^{(p)} + \dot{p}_{c;k}^{(p)} = F_k(\boldsymbol{x}^{(p)}, t) \,, \tag{19.5-1}$$

where m_e is the electron mass, $w_k^{(p)} = w_k^{(p)}(t)$ is the instantaneous velocity of the electron, $\boldsymbol{x}^{(p)} = \boldsymbol{x}^{(p)}(t)$ is its position, $\dot{p}_{c;k}^{(p)}$ is the time rate of change of the electron's momentum due to the collisions, and $F_k = F_k(\boldsymbol{x}^{(p)}, t)$ is given by (see Equation (18.1-1))

$$F_k = -eE_k - e\varepsilon_{k,m,j} w_m \mu_0 H_j \,. \tag{19.5-2}$$

Equation (19.5-1) is now averaged over the representative elementary domain $\mathcal{D}_\varepsilon(\boldsymbol{x})$. To rewrite the first term on the left-hand side of the averaged equation we need the theorem

$$\mathrm{D}_t[n_e(\boldsymbol{x},t) v_k(\boldsymbol{x},t)] = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(\boldsymbol{x},t)} \mathrm{d}_t w_k^{(p)}(t) \,, \tag{19.5-3}$$

where v_k is the drift velocity and

$$\mathrm{D}_t = v_r \partial_r + \partial_t \tag{19.5-4}$$

denotes co-moving differentiation with respect to time. To prove Equation (19.5-3) it is observed that

$$n_e(\boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t) v_k(\boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t) = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t)} w_k^{(p)}(t + \Delta t) \tag{19.5-5}$$

and

$$n_e(\boldsymbol{x},t) v_k(\boldsymbol{x},t) = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(\boldsymbol{x},t)} w_k^{(p)}(t) \,. \tag{19.5-6}$$

If, now, the representative elementary domain $\mathcal{D}_\varepsilon(\boldsymbol{x} + \Delta\boldsymbol{x})$ results from $\mathcal{D}_\varepsilon(\boldsymbol{x})$ under the translation

$$\Delta\boldsymbol{x} = \boldsymbol{v}\Delta t \,, \tag{19.5-7}$$

i.e. a displacement with the local drift velocity, we have

$$N_\varepsilon(\boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t) = N_\varepsilon(\boldsymbol{x}, t) \,, \tag{19.5-8}$$

since under such a displacement the particles in $\mathcal{D}_\varepsilon(\boldsymbol{x})$ are followed on their (collective) macroscopic course and the flow of conduction electrons is conservative. Substituting the Taylor expansions

$$w_k^{(p)}(t + \Delta t) = w_k^{(p)}(t) + \Delta t \, \mathrm{d}_t w_k^{(p)}(t) + o(\Delta t) \qquad \text{as } \Delta t \to 0 \,, \tag{19.5-9}$$

and

$$\begin{aligned} &n_e(\boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t) v_k(\boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t) \\ &= n_e(\boldsymbol{x},t) v_k(\boldsymbol{x},t) + \Delta t \{ v_r \partial_r [n_e(\boldsymbol{x},t) v_k(\boldsymbol{x},t)] \\ &\quad + \partial_t [n_e(\boldsymbol{x},t) v_k(\boldsymbol{x},t)] \} + o(\Delta t) \qquad \text{as} \quad \Delta t \to 0 \end{aligned} \tag{19.5-10}$$

in Equation (19.5-5), subtracting Equation (19.5-6) from the result, dividing the resulting equation by Δt, and taking the limit $\Delta t \rightarrow 0$, we end up with

$$v_r \partial_r [n_e(x,t) v_k(x,t)] + \partial_t [n_e(x,t) v_k(x,t)] = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} \mathrm{d}_t w_k^{(p)}(t) , \qquad (19.5\text{-}11)$$

which is Equation (19.5-3).

The second term in the equation that results after averaging Equation (19.5-1) is written as

$$V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} \dot{p}_{\mathrm{c};k}^{(p)} = n_e m_e \nu_c v_k , \qquad (19.5\text{-}12)$$

where n_e is the number density of the electrons, ν_c is their collision frequency and $m_e v_k$ is the volume averaged momentum of the electrons. Identifying the volume density of material electric current (see Equation (18.3-4)) with the volume density of electric conduction current (see Equation (19.4-28)), leads to

$$J_k^{\mathrm{ind}} = -n_e e v_k = J_k . \qquad (19.5\text{-}13)$$

The averaging of Equation (19.5-1) then leads to the following differential equation for the volume density of electron conduction current:

$$\mathrm{D}_t J_k + \nu_c J_k = (n_e e^2 / m_e) E_k - \varepsilon_{k,m,j} J_m (e \mu_0 H_j / m_e) , \qquad (19.5\text{-}14)$$

where D_t is given by Equation (19.5-4).

The term $v_r \partial_r$ in D_t makes the differential equation non-linear (note that $v_r = -J_r / n_e e$). In most practical cases, the drift velocity is so low that $|v_r \partial_r J_k| \ll \partial_t J_k$. In view of this, we take

$$\mathrm{D}_t \approx \partial_t \qquad (19.5\text{-}15)$$

in Equation (19.5-14). Under this assumption, Equation (19.5-14) reduces for stationary currents ($\partial_t J_k = 0$) and in the absence of an external magnetic field (i.e. $H_j = 0$) to Ohm's law

$$J_k = \sigma E_k , \qquad (19.5\text{-}16)$$

provided that we approximate n_e by its static background value and take

$$\sigma = n_e e^2 / m_e \nu_c \qquad (19.5\text{-}17)$$

as the *conductivity* for stationary currents. Furthermore,

$$\omega_{\mathrm{ce};j} = e \mu_0 H_j / m_e \qquad (19.5\text{-}18)$$

is the vectorial *electron cyclotron angular frequency* of an electron gyrating around the magnetic field. In this respect it is observed that the magnetic field associated, according to Maxwell's equations, with a time-varying electric field yields a contribution to the force on a point charge that is $w_m w_m / c_0^2$ times the value of the electric force; this contribution can therefore be neglected in Equation (19.5-14) and, in the approximation in which we work, H_j can be identified with the external magnetic field (if present). Taking for H_j in Equation (19.5-18) a static external magnetic field, the coefficients in the linearised form of the differential equation

Equation (19.5-14) become time independent. In its standard form the differential equation then becomes

$$\partial_t J_k + \nu_c J_k = \nu_c \sigma E_k - \varepsilon_{k,m,j} J_m \omega_{ce;j} \,. \tag{19.5-19}$$

Equation (19.5-19) is already a constitutive relation that expresses the relationship between J_k and E_r, be it that this constitutive relation is not in the standard form. From it we conclude that in a metal the phenomenon of (electron) conduction is a local effect and that in its electric conduction behaviour the metal is isotropic in the absence of an external magnetic field, while it is anisotropic in the presence of an external magnetic field. To arrive at the corresponding constitutive relation in the standard form, Equation (19.5-19) as a differential equation in time has to be solved. This is most easily done with the aid of a Laplace transformation with respect to time. Carrying out this transformation, Equation (19.5-19) changes into

$$(s + \nu_c)\hat{J}_k = \nu_c \sigma \hat{E}_k - \varepsilon_{k,m,j} \hat{J}_m \omega_{ce;j} \,. \tag{19.5-20}$$

Several cases will be considered separately below.

No external magnetic field present; isotropic conductor

First, the case is investigated where no external magnetic field is present. Then, $\omega_{ce;j} = 0$ and Equation (19.5-19) reduces to

$$\partial_t J_k + \nu_c J_k = \nu_c \sigma E_k \,. \tag{19.5-21}$$

Correspondingly, Equation (19.5-20) reduces to

$$(s + \nu_c)\hat{J}_k = \nu_c \sigma \hat{E}_k \,. \tag{19.5-22}$$

The solution to Equation (19.5-22) is obtained as

$$\hat{J}_k = \hat{\kappa}_c \hat{E}_k \,, \tag{19.5-23}$$

with

$$\hat{\kappa}_c = \frac{\nu_c \sigma}{s + \nu_c} \,. \tag{19.5-24}$$

The time-domain equivalent of Equation (19.5-23) is the time convolution

$$J_k(\boldsymbol{x},t) = \int_{t'=0}^{\infty} \kappa_c(\boldsymbol{x},t) E_k(\boldsymbol{x},t - t') \,\mathrm{d}t' \,, \tag{19.5-25}$$

in which the conduction relaxation function follows from Equation (19.5-24) as

$$\kappa_c = \nu_c \sigma \exp(-\nu_c t) H(t) \,, \tag{19.5-26}$$

in which $H(t)$ is the Heaviside unit step function.

No external magnetic field present; isotropic superconductor

In certain materials the collision frequency ν_c drops to zero value upon cooling the material below a certain critical temperature T_c. Such a material is denoted as a *superconductor*. In this case Equation (19.5-21) is, in view of Equation (19.5-17), replaced by

$$\partial_t J_k = (n_e e^2 / m_e) E_k \,. \tag{19.5-27}$$

From this equation the constitutive relation in its standard form follows as

$$J_k(x,t) = \frac{n_e e^2}{m_e} \int_{t''=-\infty}^{t} E_k(x,t'')\,\mathrm{d}t'' = \frac{n_e e^2}{m_e} \int_{t'=0}^{\infty} E_k(x,t-t')\,\mathrm{d}t' \,. \tag{19.5-28}$$

Equations (19.5-27) and (19.5-28) are as the *first London* equations. In fact, Equations (19.5-27) and (19.5-28) are the constitutive relations for an isotropic collisionless plasma (see Section 19.6). Note that Equation (19.5-28) also results from Equations (19.5-25) and (19.5-26) upon taking the limit $\nu_c \downarrow 0$.

External magnetic field present; anisotropic conductor

In the presence of an external magnetic field, the vectorial Equation (19.5-20) must be solved in its full form, i.e. $\hat{J}_k$ must be expressed in terms of $\hat{E}_r$. To achieve this, we first apply to Equation (19.5-20) the operator $\omega_{ce;k}$. This leads to

$$(s+\nu_c)\omega_{ce;k}\hat{J}_k = \nu_c \sigma \omega_{ce;k}\hat{E}_k \,, \tag{19.5-29}$$

where the property has been used that

$$\omega_{ce;k}\varepsilon_{k,m,j}\hat{J}_m\omega_{ce;j} = 0 \,. \tag{19.5-30}$$

Secondly, the operator $\varepsilon_{r,q,k}\omega_{ce;q}$ is applied to Equation (19.5-20). This leads to

$$\begin{aligned}&(s+\nu_c)\varepsilon_{r,q,k}\omega_{ce;q}\hat{J}_k \\ &= \nu_c\sigma\varepsilon_{r,q,k}\omega_{ce;q}\hat{E}_k - \varepsilon_{r,q,k}\omega_{ce;q}\varepsilon_{k,m,j}\hat{J}_m\omega_{ce;j} \,.\end{aligned} \tag{19.5-31}$$

However, (see Equation (A.7-51))

$$\varepsilon_{r,q,k}\varepsilon_{k,m,j} = \delta_{r,m}\delta_{q,j} - \delta_{r,j}\delta_{q,m} \,. \tag{19.5-32}$$

Using Equation (19.5-32) in the last term on the right-hand side of Equation (19.5-31), we obtain

$$\begin{aligned}&(s+\nu_c)\varepsilon_{r,q,k}\omega_{ce;q}\hat{J}_k \\ &= \nu_c\sigma\varepsilon_{r,q,k}\omega_{ce;q}\hat{E}_k - \omega_{ce;q}\omega_{ce;q}\hat{J}_r + \omega_{ce;r}\omega_{ce;q}\hat{J}_q \,.\end{aligned} \tag{19.5-33}$$

Now, using Equation (19.5-29) in the last term on the right-hand side, Equation (19.5-33) becomes

$$\begin{aligned}&(s+\nu_c)\varepsilon_{r,q,k}\omega_{ce;q}\hat{J}_k \\ &= \nu_c\sigma\varepsilon_{r,q,k}\omega_{ce;q}\hat{E}_k - \omega_{ce}^2\hat{J}_r + (s+\nu_c)^{-1}\nu_c\sigma\omega_{ce;r}\omega_{ce;k}\hat{E}_k \,,\end{aligned} \tag{19.5-34}$$

in which

$$\omega_{\rm ce}^2 = \omega_{{\rm ce};q}\omega_{{\rm ce};q} \,. \tag{19.5-35}$$

Upon multiplying Equation (19.5-20) by $s + \nu_{\rm c}$ and using Equation (19.5-34), we end up with

$$\left[(s+\nu_{\rm c})^2 + \omega_{\rm ce}^2\right]\hat{J}_k$$
$$= (s+\nu_{\rm c})\nu_{\rm c}\sigma\hat{E}_k + \nu_{\rm c}\sigma\varepsilon_{k,q,r}\omega_{{\rm ce};q}\hat{E}_r + (s+\nu_{\rm c})^{-1}\nu_{\rm c}\sigma\omega_{{\rm ce};k}\omega_{{\rm ce};r}\hat{E}_r \,, \tag{19.5-36}$$

where some subscripts have appropriately been changed.

From Equation (19.5-36) the complex frequency-domain tensorial conduction relaxation function $\hat{\kappa}_{{\rm c};k,r} = \hat{\kappa}_{{\rm c};k,r}(\boldsymbol{x},s)$, introduced through

$$\hat{J}_k = \hat{\kappa}_{{\rm c};k,r}\hat{E}_r \,, \tag{19.5-37}$$

follows as

$$\hat{\kappa}_{{\rm c};k,r} = \frac{\nu_{\rm c}\sigma}{\left[(s+\nu_{\rm c})^2 + \omega_{\rm ce}^2\right]}\left[(s+\nu_{\rm c})\delta_{k,r} + \varepsilon_{k,q,r}\omega_{{\rm ce};q} + (s+\nu_{\rm c})^{-1}\omega_{{\rm ce};k}\omega_{{\rm ce};r}\right]. \tag{19.5-38}$$

With the aid of the Laplace-transform pairs

$$\frac{(s+\nu_{\rm s})}{(s+\nu_{\rm c})^2 + \omega_{\rm ce}^2} \leftrightarrow \exp(-\nu_{\rm c}t)\cos(\omega_{\rm ce}t)H(t) \,, \tag{19.5-39}$$

$$\frac{1}{(s+\nu_{\rm c})^2 + \omega_{\rm ce}^2} \leftrightarrow \omega_{\rm ce}^{-1}\exp(-\nu_{\rm c}t)\sin(\omega_{\rm ce}t)H(t) \,, \tag{19.5-40}$$

and

$$\frac{1}{(s+\nu_{\rm c})}\,\frac{1}{(s+\nu_{\rm c})^2 + \omega_{\rm ce}^2} \leftrightarrow \omega_{\rm ce}^{-2}\exp(-\nu_{\rm c}t)[1-\cos(\omega_{\rm ce}t)]H(t) \,, \tag{19.5-41}$$

in which $H(t)$ denotes the Heaviside unit step function ($H(t) = 0$ if $t < 0$, $H(0) = ½$, $H(t) = 1$ if $t > 0$), the time-domain counterpart of Equation (19.5-37) is obtained as

$$\kappa_{{\rm c};k,r} = \nu_{\rm c}\sigma\exp(-\nu_{\rm c}t)\left\{\cos(\omega_{\rm ce}t)\delta_{k,r} + \varepsilon_{k,q,r}(\omega_{{\rm ce};q}/\omega_{\rm ce})\sin(\omega_{\rm ce}t)\right.$$
$$\left. + (\omega_{{\rm ce};k}\omega_{{\rm ce};r}/\omega_{\rm ce}^2)\,[1-\cos(\omega_{\rm ce}t)]\right\}H(t) \,. \tag{19.5-42}$$

The time-domain constitutive relation finally follows as the convolution

$$J_k(\boldsymbol{x},t) = \int_{t'=0}^{\infty}\kappa_{{\rm c};k,r}(\boldsymbol{x},t')E_r(\boldsymbol{x},t-t')\,{\rm d}t' \,. \tag{19.5-43}$$

External magnetic field present: anisotropic superconductor

For the case of a superconductor in the presence of an external magnetic field, Equation (19.5-20) reduces to

$$s\hat{J}_k = (n_{\rm e}e^2/m_{\rm e})\hat{E}_k - \varepsilon_{k,m,j}\hat{J}_m\omega_{{\rm ce};j} \,. \tag{19.5-44}$$

To express $\hat{J}_k$ in terms of $\hat{E}_r$, the steps leading to Equations (19.5-37) and (19.5-38) can be repeated. The result also follows from Equation (19.5-38) upon replacing $\nu_c\sigma$ by $n_e e^2/m_e$ and further substituting $\nu_c = 0$. For this case, the result

$$\hat{\kappa}_{c;k,r} = \frac{n_e e^2/m_e}{s^2+\omega_{ce}^2}\left[s\delta_{k,r} + \varepsilon_{k,q,r}\omega_{ce;q} + s^{-1}\omega_{ce;k}\omega_{ce;r}\right] \tag{19.5-45}$$

is then obtained. Using Equations (19.5-39)–(19.5-41) for $\nu_c = 0$, the time-domain counterpart of Equation (19.5-45) then follows as

$$\kappa_{c;k,r} = \frac{n_e e^2}{m_e}\left\{\cos(\omega_{ce}t)\delta_{k,r} + \varepsilon_{k,q,r}(\omega_{ce;q}/\omega_{ce})\sin(\omega_{ce}t)\right.$$
$$\left. + (\omega_{ce;k}\omega_{ce;r}/\omega_{ce}^2)\,[1-\cos(\omega_{ce}t)]\right\} H(t)\,, \tag{19.5-46}$$

which is, for the case of an anisotropic superconductor, to be used in the convolution in Equation (19.5-43). Equations (19.5-46) and (19.5-43) form, in fact, the constitutive relation for an anisotropic collisionless plasma (see Section 19.6)

Stationary electric current. Hall effect

For stationary electric currents, we have $\partial_t J_k$ and Equation (19.5-19) reduces to

$$E_k = \sigma^{-1}J_k + (\nu_c\sigma)^{-1}\varepsilon_{k,m,j}J_m\omega_{ce;j}\,. \tag{19.5-47}$$

Using Equation (19.5-17) for the value of σ and Equation (19.5-18) for the value of $\omega_{ce;j}$, this relation can be rewritten as

$$E_k = \sigma^{-1}J_k - R_H\varepsilon_{k,m,j}J_m\mu_0 H_j\,, \tag{19.5-48}$$

in which

$$R_H = -\frac{1}{n_e e} \tag{19.5-49}$$

is the *Hall coefficient*. As Equation (19.5-48) shows, the presence of an external magnetic field introduces anisotropy in the conduction properties of a material for stationary electric currents; this effect is known as the *Hall effect*. In particular, the effect that a given electric current density yields a contribution to the electric field that is perpendicular to the current density (the second term on the right-hand side of Equation (19.5-48)) is used to measure the value of the magnetic field strength by causing a known stationary electric current to flow through a piece of metal (or semiconductor) and measuring the resulting electric field strength, assuming that the quantities specifying the properties of the piece of matter used are known. Conversely, the second term on the right-hand side of Equation (19.5-48) is used to determine the number density and the nature of the moving carriers of electric charge in metals or semiconductors, assuming that the external magnetic field is known and the electric field strength perpendicular to the electric current density is measured.

Exercises

Exercise 19.5-1

Take, in Equation (19.5-25), $E_k = 0$ when $-\infty < t < t_0$ and $E_k = E_k^{(0)}$ when $t_0 < t < \infty$, where $E_k^{(0)}$ is time independent. Show that in the limit $t \to \infty$, upon using Equation (19.5-26), Equation (19.5-25) yields $J_k \to \sigma E_k$. (*Hint*: Observe that

$$\int_{t'=0}^{t-t_0} \kappa_c(x,t')\,\mathrm{d}t' \to \int_{t'=0}^{\infty} \kappa_c(x,t')\,\mathrm{d}t' \quad \text{as} \quad t \to \infty,$$

and evaluate the resulting integral.)

19.6 The conduction relaxation function of an electron plasma

A *plasma* is a gas in which an ionisation of the atoms has taken place. In an electron plasma the negative ions are electrons. In the gaseous state considered, the positive ions are not fixed in a lattice (as is the case in a solid conductor), but under most conditions their drift velocity is negligibly small compared to the drift velocity of the electrons. This is due to the property that thermodynamic equilibrium of the plasma is established through the collisions between positive ions and electrons, and the fact that even the lightest positive ion (viz. a proton) has a mass that is 1836 times as large as the electron mass. In this degree of accuracy, therefore only the contribution from the electrons to the volume density of material electric current has to be taken into account. The analysis of Section 19.5 can be repeated. Using Equation (19.5-17), Equation (19.5-19) is, for the present application, rewritten as

$$\partial_t J_k + \nu_c J_k = \varepsilon_0 \omega_{pe}^2 E_k - \varepsilon_{k,m,j} J_m \omega_{ce;j} \tag{19.6-1}$$

and Equation (19.5-20) as

$$(s + \nu_c)\hat{J}_k = \varepsilon_0 \omega_{pe}^2 \hat{E}_k - \varepsilon_{k,m,j} \hat{J}_m \omega_{ce;j}\,, \tag{19.6-2}$$

in which

$$\omega_{pe} = \left[\frac{n_e e^2}{\varepsilon_0 m_e}\right]^{1/2} \tag{19.6-3}$$

is the *electron plasma angular frequency*. The reason for introducing the electron plasma angular frequency in stead of the static conductivity into the expression for the conduction relaxation function is that under usual circumstances in a plasma the inertia of the electrons predominantly influences its conductive behaviour, while in a metal under usual circumstances the effect of collisions is predominant. Proceeding as in Section 19.5, the complex frequency-domain conduction relaxation function is then found as (see Equation (19.5-38))

$$\hat{\kappa}_{c;k,r} = \varepsilon_0 \omega_{pe}^2 \frac{1}{(s+\nu_c)^2 + \omega_{ce}^2}\left[(s+\nu_c)\delta_{k,r} + \varepsilon_{k,q,r}\omega_{ce;q} + (s+\nu_c)^{-1}\omega_{ce;k}\omega_{ce;r}\right], \tag{19.6-4}$$

with the corresponding constitutive relation

$$\hat{J}_k = \hat{\kappa}_{c;k,r}\hat{E}_r \tag{19.6-5}$$

and the time-domain conduction relaxation function as (see Equation (19.5-42))

$$\kappa_{c;k,r} = \varepsilon_0\omega_{pe}^2 \exp(-\nu_c t) \{\cos(\omega_{ce}t)\delta_{k,r} + \varepsilon_{k,q,r}(\omega_{ce;q}/\omega_{ce}) \sin(\omega_{ce}t) + (\omega_{ce;k}\omega_{ce;r}/\omega_{ce}^2 [1 - \cos(\omega_{ce}t)]\} H(t) , \tag{19.6-6}$$

with the corresponding constitutive relation

$$J_k(x,t) = \int_{t'=0}^{\infty} \kappa_{c;k,r}(x,t')E_k(x,t-t')\,dt' . \tag{19.6-7}$$

An important application of the constitutive behaviour of an electron plasma is found in the theory of *ionospheric radio wave propagation*. The ionosphere is an ionised part of the atmosphere surrounding the Earth, with an altitude in between 50 km and 250 km above the Earth's surface. The ionisation takes place under the action of the ultraviolet radiation emitted by the Sun. Therefore, the presence and absence of the ionosphere follows the rhythm of day and night, which explains the difference between day and night in radiowave propagation (Ratcliffe 1959, Budden 1961).

In the theory of electromagnetic wave propagation through the ionosphere it is customary to stress the dielectric properties rather than the conduction properties. In accordance with this, a dielectric relaxation function is introduced such that (see Equations (19.3-8) and (19.3-9))

$$J_k = \varepsilon_0\partial_t P_k , \tag{19.6-8}$$

with

$$P_k(x,t) = \varepsilon_0 \int_{t'=0}^{\infty} \kappa_{e;k,r}(x,t')E_k(x,t-t')\,dt' , \tag{19.6-9}$$

with the complex frequency-domain counterparts

$$\hat{J}_k = \varepsilon_0 s\hat{P}_k \tag{19.6-10}$$

and

$$\hat{P}_k = \varepsilon_0\hat{\kappa}_{e;k,r}\hat{E}_k . \tag{19.6-11}$$

A comparison of Equations (19.6-4)–(19.6-7) with Equations (19.6-8)–(19.6-11) shows that

$$\hat{\kappa}_{e;k,r} = s^{-1}\omega_{pe}^2 \frac{1}{(s+\nu_c)^2 + \omega_{ce}^2}\left[(s+\nu_c)\delta_{k,r} + \varepsilon_{k,q,r}\omega_{ce;q} + (s+\nu_c)^{-1}\omega_{ce;k}\omega_{ce;r}\right], \tag{19.6-12}$$

with the time-domain counterpart

$$\kappa_{e;k,r} = \mathrm{I}_t\Big[\omega_{pe}^2 \exp(-\nu_c t) \{\cos(\omega_{ce}t)\delta_{k,r} + \varepsilon_{k,q,r}(\omega_{ce;q}/\omega_{ce}) \sin(\omega_{ce}t) + (\omega_{ce;k}\omega_{ce;r}/\omega_{ce}^2) [1 - \cos(\omega_{ce}t)]\} H(t)\Big] . \tag{19.6-13}$$

In case the collision effects can be neglected ($\nu_c = 0$), the plasma becomes a *collisionless* plasma.

In the presence of an external magnetic field the plasma is anisotropic in its electromagnetic behaviour; in the absence of an external magnetic field the plasma is isotropic in its electromagnetic behaviour.

Exercises

Exercise 19.6-1

Determine the value of the electron plasma frequency $f_{\rm pe} = \omega_{\rm pe}/2\pi$, for a plasma with number density of electrons $n_{\rm e} = 1 \times 10^9$ m^{-3} ($m_{\rm e} = 9.10938 \times 10^{-31}$ kg).

Answer: $f_{\rm pe} = 2.83930 \times 10^5$ Hz.

Exercise 19.6-2

Determine the value of the electron cyclotron frequency $f_{\rm ce} = \omega_{\rm ce}/2\pi$, if the external magnetic field has the value $\mu_0 H = B$ with $B = 1$ T ($m_{\rm e} = 9.10938 \times 10^{-31}$ kg).

Answer: $f_{\rm ce} = 2.7992 \times 10^{10}$ Hz.

Exercise 19.6-3

Determine the value of the electron cyclotron frequency $f_{\rm ce} = \omega_{\rm ce}/2\pi$, if the external magnetic field has the value $\mu_0 H = 0.5 \times 10^{-4}$ T (the Earth's magnetic field) ($m_{\rm e} = 9.10938 \times 10^{-31}$ kg).

Answer: $f_{\rm ce} = 1.39962 \times 10^6$ Hz.

Exercise 19.6-4

Determine the number density of electrons in the ionosphere (a plasma layer in the Earth's atmosphere) if its electron plasma frequency $f_{\rm pe} = \omega_{\rm pe}/2\pi$ has the value $f_{\rm ce} = 4.5 \times 10^6$ Hz ($m_{\rm e} = 9.10938 \times 10^{-31}$ kg).

Answer: $n_{\rm e} = 2.5119 \times 10^{11}$ m^{-3}.

Exercise 19.6-5

Give the complex frequency-domain dielectric relaxation function of a collisionless isotropic electron plasma (no external magnetic field). (*Hint*: Substitute $\nu_{\rm c} = 0$ and $\omega_{{\rm ce};q} = 0$ in Equation (19.6-12).)

Answer:

$$\hat{\kappa}_{{\rm e};k,r}(x,s) = \omega_{\rm pe}^2(x) s^{-2} \delta_{k,r} \, . \qquad (19.6\text{-}14)$$

Exercise 19.6-6

Give the time-domain dielectric relaxation function of a collisionless isotropic electron plasma (no external magnetic field). (*Hint*: Use the result of Exercise 19.6-5.)

Answer:

$$\hat{\kappa}_{{\rm e};k,r}(x,t) = \omega_{\rm pe}^2(x) t H(t) \delta_{k,r} \, . \qquad (19.6\text{-}15)$$

19.7 The dielectric relaxation function of an isotropic dielectric

A dielectric material is conceived to consist of a collection of neutral atoms. Each atom is modelled as a (light) movable cloud of negative electric charge that is elastically bound to a (heavier) fixed nucleus of equal and opposite positive electric charge. The movable negatively charged cloud is set into motion by a macroscopic electromagnetic field; the motion of the positively charged nucleus is neglected. The drift velocity v_m of the negatively charged cloud is assumed to be so small that, on the assumption that no magnetostatic external field is present, the influence of the magnetic force can be neglected, the latter being of the order of $w_m w_m / c_0^2$ times the electric force. Consider the atoms of a particular substance. Let the mass of their cloud of movable negative electric charge be m, and let their atomic number be Z; then, $m = Zm_e$, where m_e is the electron mass, $-Ze$ is the amount of the negative electric charge of the cloud, and Ze is the amount of the positive electric charge of the nucleus. The atoms of the substance under consideration that are present in the representative elementary domain $\mathcal{D}_\varepsilon(x)$ are labelled with the superscript (p). Let $u_k^{(p)} = u_k^{(p)}(t)$ denote the displacement of the barycentre of the negatively charged cloud with respect to the positively charged nucleus of the atom with label (p). Then, the equation of motion of the negatively charged cloud of this atom is given by

$$m\mathrm{d}_t^2 u_k^{(p)} + \dot{p}_{\mathrm{c};k}^{(p)} + m\omega_0^2 u_k^{(p)} = F_k(x^{(p)},t)\,, \tag{19.7-1}$$

where $\dot{p}_{\mathrm{c};k}^{(p)}$ is the time rate of change of the electron cloud's momentum due to friction with neighbouring atoms, $m\omega_0^2$ is the elastic restoring force, ω_0 is the resonant angular frequency of the mechanical system, and F_k is the electric force acting on the movable electric charge distribution. As to this force, we can, in condensed matter, not put this equal to qE_k, where $q = -Ze$ and E_k is the macroscopic electric field, since we must first remove a part of the macroscopic continuum in order to make room for our atom to move in the remaining vacuum. Taking, for an isotropic dielectric, the part to be removed to be a ball that is uniformly polarised (i.e. a ball in which the electric polarisation has the local value we want to determine from the microscopic theory) with electric polarisation P_k, we have (see Exercise 19.7-1)

$$F_k = q(E_k + P_k/3\varepsilon_0)\,. \tag{19.7-2}$$

In accordance with the concepts of Section 19.4 and with Equation (18.3-5), we put

$$P_k = nq\langle u_k\rangle\,, \tag{19.7-3}$$

where n is the number density of the atoms of the substance under investigation and $\langle u_k\rangle$ is the average displacement in $\mathcal{D}_\varepsilon(x)$. Averaging Equation (19.7-1) over $\mathcal{D}_\varepsilon(x)$ and using the theorem (see the derivation of Equation (19.5-3))

$$\mathrm{D}_t[n(x,t)\langle u_k\rangle(x,t)] = V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} \mathrm{d}_t u_k^{(p)}(t)\,,$$

we obtain the following relation between P_k and E_k

$$\mathrm{D}_t^2 P_k + \Gamma \mathrm{D}_t P_k + (\omega_0^2 - \omega_\mathrm{p}^2/3)P_k = \varepsilon_0 \omega_\mathrm{p}^2 E_k\,, \tag{19.7-4}$$

where

$$\omega_\mathrm{p} = (nq^2/\varepsilon_0 m)^{1/2} \tag{19.7-5}$$

is the *plasma angular frequency* of the movable electric charge distribution and Γ is a phenomenological damping coefficient that results from

$$V_\varepsilon^{-1} \sum_{p=1}^{N_\varepsilon(x,t)} \dot{p}_{c;k}^{(p)} = m\Gamma \mathrm{D}_t[n\langle u_k\rangle] \,. \qquad (19.7\text{-}6)$$

From Equation (19.7-4) we must express P_k in terms of E_k. To this end we use the approximation $\mathrm{D}_t \approx \partial_t$ (low-velocity linearisation), take the Laplace transform of Equation (19.7-4) with respect to time and obtain

$$(s^2 + \Gamma s + \omega_0^2 - \omega_\mathrm{p}^2/3)\hat{P}_k = \varepsilon_0 \omega_\mathrm{p}^2 \hat{E}_k \,. \qquad (19.7\text{-}7)$$

Using Equation (19.7-7), the contribution of the atoms of the type under investigation to the s-domain dielectric relaxation function $\hat{\kappa}_\mathrm{e}$, introduced through

$$\hat{P}_k = \varepsilon_0 \hat{\kappa}_\mathrm{e} \hat{E}_k \,, \qquad (19.7\text{-}8)$$

follows as

$$\hat{\kappa}_\mathrm{e} = \frac{\omega_\mathrm{p}^2}{(s + \Gamma/2)^2 + \Omega^2} \,, \qquad (19.7\text{-}9)$$

where

$$\Omega = (\omega_0^2 - \omega_\mathrm{p}^2/3 - \Gamma^2/4)^{1/2} \qquad (19.7\text{-}10)$$

is the natural angular frequency of the oscillations of the movable electric charge. The time-domain expression corresponding to Equation (19.7-8) is

$$P_k(x,t) = \varepsilon_0 \int_{t'=0}^{\infty} \kappa_\mathrm{e}(x,t') E_k(x,t-t') \,\mathrm{d}t' \,, \qquad (19.7\text{-}11)$$

in which the time-domain dielectric relaxation function is, from Equation (19.7-9), found as

$$\kappa_\mathrm{e}(x,t') = (\omega_\mathrm{p}/\Omega) \exp[-(\Gamma/2)t'] \sin(\Omega t') H(t') \,. \qquad (19.7\text{-}12)$$

The behaviour of the type of Equations (19.7-11) and (19.7-12) is typical for a so-called *Lorentzian absorption line*, for which $\omega_\mathrm{p}^2/3 + \Gamma^2/4 < \omega_0^2$ (as in all practical cases in condensed matter).

In case the dielectric consists of atoms of different substances, each substance contributes to P_k in an additive way, and the dielectric relaxation function consists of the sum of the dielectric relaxation functions of the constituting substances. In optical spectroscopy, the occurrence of (Lorentzian) absorption lines is used to analyse the composition of substances and determine which chemical elements are present in the piece of matter under investigation.

Exercises

Exercise 19.7-1

A ball of radius a is uniformly polarised with electric polarisation P_k. The corresponding electrostatic field must satisfy the electrostatic field equations $\varepsilon_{j,n,r}\partial_n E_r = 0$, $\partial_k D_k = 0$ and $D_k = \varepsilon_0 E_k + P_k$. Show that the field defined by

$$E_k = -P_k/3\varepsilon_0 \text{ for } 0 < |x| < a, \quad E_k = (3p_m x_m x_k - x_m x_m p_k)/4\pi\varepsilon_0 |x|^5 \text{ for } a < |x| < \infty,$$

where $p_k = (4\pi a^3/3)P_k$ is the electric moment of the ball, satisfies all conditions, including the one that the field must go to zero at infinity. (*Hint:* Note that the equation for E_r is satisfied if we take $E_r = -\partial_r V$, where V is the electrostatic potential. Show that then $\partial_k \partial_k V = 0$, which equation has the admissible solution $V = p_m x_m / 4\pi\varepsilon_0 a^3$ for $0 \leqslant |x| < a$ and $V = p_m x_m / 4\pi\varepsilon_0 |x|^3$ for $a < |x| < \infty$. Verify that the electrostatic potential (and hence the tangential component of the electric field strength) and the normal component of the electric flux density are continuous across the boundary $|x| = a$ of the ball.)

Exercise 19.7-2

By equating the term $Zm_e\omega_0^2 u_k$ in Equation (19.7-1) to the electrostatic force that the positively charged nucleus exerts on the movable cloud of negatively charged electrons in a ball of radius a, the value of ω_0^2 can be expressed in terms of atomic quantities. Use the result of Exercise 18.4-1 to obtain this expression.

Answer: $\omega_0^2 = Ze^2/4\pi\varepsilon_0 a^3 m_e$.

19.8 SI units of the quantities associated with the electromagnetic constitutive behaviour of matter

Table 19.8-1 lists the SI units of the quantities introduced in this chapter.

Table 19.8-1 Quantities related to the flow of electrically charged particles and the electromagnetic constitutive behaviour of matter, and their units in the International System of Units (SI)

Quantity		Unit	
Name	Symbol	Name	Symbol
Total number	N		
Volume	V	metre^3	m^3
Number density	n	metre^{-3}	m^{-3}
Drift velocity	v_r	metre/second	m/s
Rate of creation	$\dot{N}_{cr}$	second^{-1}	s^{-1}
Rate of annihilation	$\dot{N}_{ann}$	second^{-1}	s^{-1}
Volume density of rate of creation	$\dot{n}_{cr}$	metre^{-3}.second^{-1}	m^{-3}.s^{-1}
Volume density of rate of annihilation	$\dot{n}_{ann}$	metre^{-3}.second^{-1}	m^{-3}.s^{-1}
Area	A	metre^2	m^2
Volume density of electric charge	ρ	coulomb/metre^3	C/m^3
Volume density of electric current	J_k	ampere/metre^2	A/m^2
Volume density of rate of creation of electric charge	$\dot{\rho}_{cr}$	coulomb/(metre^3.second)	C/m^3.s

Table 19.8-1 (continued) Quantities related to the flow of electrically charged particles and the electromagnetic constitutive behaviour of matter, and their units in the International System of Units (SI)

Quantity		Unit	
Name	Symbol	Name	Symbol
Volume density of rate of annihilation of electric charge	$\dot{\rho}_{\rm ann}$	coulomb/(metre^3.second)	C/m^3.s
Conductivity	$\sigma_{k,r}$	siemens/metre	S/m
Absolute permittivity	$\varepsilon_{k,r}$	farad/metre	F/m
Electric susceptibility	$\chi_{{\rm e};k,r}$		
Relative permittivity	$\varepsilon_{{\rm r};k,r}$		
Absolute permeability	$\mu_{j,p}$	henry/metre	H/m
Magnetic susceptibility	$\chi_{{\rm m};j,p}$		
Relative permeability	$\mu_{{\rm r};j,p}$		
Conduction relaxation function	$\kappa_{{\rm c};k,r}$	siemens/metre.second	S/m.s
Dielectric relaxation function	$\kappa_{{\rm e};k,r}$	second^{-1}	s^{-1}
Magnetic relaxation function	$\kappa_{{\rm m};j,p}$	second^{-1}	s^{-1}
Collision frequency	$\nu_{\rm c}$	second^{-1}	s^{-1}
Electron plasma angular frequency	$\omega_{\rm pe}$	radian/second	rad/s
Electron cyclotron angular frequency	$\omega_{{\rm ce};q,p}$	radian/second	rad/s
Hall coefficient	$R_{\rm H}$	metre^3/coulomb	m^3/C

References

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