24

The electromagnetic field equations, constitutive relations and boundary conditions in the time Laplace-transform domain (complex frequency domain)

In a large number of cases met in practice, one is interested in the behaviour of causal electromagnetic fields in linear, time-invariant configurations. Mathematically, one can take advantage of this situation by carrying out a Laplace transformation with respect to time and considering the equations governing the electromagnetic field in the corresponding time Laplace-transform domain or *complex frequency domain*. In the complex frequency-domain relations, the time coordinate has been eliminated, and a field problem in space remains in which the Laplace-transform parameter s occurs as a parameter. Causality of the field is taken into account by taking $\mathrm{Re}(s) > 0$, and requiring that all causal field quantities are, in the case where the electromagnetic field is excited by sources of finite amplitude and bounded extent, analytic functions of s in the right half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane. The complex frequency-domain solution to an electromagnetic field problem exhibits itself already a number of features that are characteristic for the configuration in which the field is present. If one is, in addition, interested in the actual pulse shapes of the field, one has to carry out the inverse Laplace transformation, either by analytical or by numerical methods.

In a number of electromagnetic field and wave problems, the transform parameter s is profitably chosen to be real and positive. On the other hand, by taking $s = \mathrm{j}\omega$ where j is the imaginary unit and ω is real and positive, the complex steady-state representation of sinusoidally in time oscillating fields of *angular frequency* ω follows, the complex representation having the complex time factor $\exp(\mathrm{j}\omega t)$. For arbitrary complex values of s in the domain of analyticity, all complex frequency-domain field quantities and constitutive relaxation functions are Laplace transforms of real-valued functions of the time coordinate t. As a consequence, the complex frequency-domain field quantities and constitutive relaxation functions are real-valued for real and positive values of s. On account of Schwarz's reflection principle of complex function theory, the relevant functions then take on complex conjugate values in conjugate complex points of the s plane.

In the present chapter, the electromagnetic field equations, constitutive relations, and boundary conditions in the complex frequency domain are given and the complex frequency-domain electromagnetic vector potentials and Green's functions (point-source solutions) are introduced. The notations of Appendix B are used.

24.1 The complex frequency-domain electromagnetic field equations

We subject the electromagnetic field Equations (18.3-11) and (18.3-12) to a Laplace transformation over the interval $\mathcal{T} = \{t \in \mathcal{R};\ t > t_0\}$. For completeness, we allow a non-vanishing field to be present at $t = t_0$, although in the majority of cases we are interested in the causal field generated by sources that are switched on at the instant $t = t_0$, in which case the initial values of the field are taken to be zero. Since, with the use of the notations of Appendix B and the properties of the Laplace transformation,

$$\int_{t=t_0}^{\infty} \exp(-st)\partial_t D_k(\boldsymbol{x},t)\,\mathrm{d}t = -D_k(\boldsymbol{x},t_0)\exp(-st_0) + s\hat{D}_k(\boldsymbol{x},s) \tag{24.1-1}$$

and

$$\int_{t=t_0}^{\infty} \exp(-st)\partial_t B_j(\boldsymbol{x},t)\,\mathrm{d}t = -B_j(\boldsymbol{x},t_0)\exp(-st_0) + s\hat{B}_j(\boldsymbol{x},s)\,, \tag{24.1-2}$$

we arrive at

$$-\varepsilon_{k,m,p}\partial_m\hat{H}_p + \hat{J}_k + s\hat{D}_k = -\hat{J}_k^{\,\mathrm{ext}} + D_k(\boldsymbol{x},t_0)\exp(-st_0)\,, \tag{24.1-3}$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}_r + s\hat{B}_j = -\hat{K}_j^{\,\mathrm{ext}} + B_j(\boldsymbol{x},t_0)\exp(-st_0)\,. \tag{24.1-4}$$

From Equations (24.1-3) and (24.1-4) it follows that, in the complex frequency domain, one can take into account the influence of a non-vanishing initial electromagnetic field by properly incorporating its values in the complex frequency-domain volume densities of external electric and magnetic source currents. In the remainder of our analysis it will be tacitly understood that non-zero initial field values have been accounted for in this manner.

The application of the time Laplace transformation to Equations (18.3-13) and (18.3-14) leads to the complex frequency-domain compatibility relations

$$\partial_k(\hat{J}_k + s\hat{D}_k) = -\partial_k\hat{J}_k^{\,\mathrm{ext}} + \partial_k D_k(\boldsymbol{x},t_0)\exp(-st_0) \tag{24.1-5}$$

and

$$s\partial_j\hat{B}_j = -\partial_j\hat{K}_j^{\,\mathrm{ext}} + \partial_j B_j\,(\boldsymbol{x},t_0)\exp(-st_0)\,. \tag{24.1-6}$$

Equation (24.1-5) also follows from applying the differentiation ∂_k to Equation (24.1-3) and observing that $\partial_k\varepsilon_{k,m,p}\partial_m = 0$, while Equation (24.1-6) also follows from applying the differentiation ∂_j to Equation (24.1-4) and observing that $\partial_j\varepsilon_{j,n,r}\partial_n = 0$.

After transforming back to the time domain, the reconstructed field values are zero in the interval $t \in \mathcal{T}'$, where $\mathcal{T}' = \{t \in \mathcal{R};\ t < t_0\}$, and equal to the actual field values when $t \in \mathcal{T}$. In addition, many of the Laplace inversion algorithms, in particular the complex Bromwich inversion integral Equation (B.1-19) (of which the Fourier inversion integral is a limiting case), yield half the field values at the instant $t \in \partial\mathcal{T}$, where $\partial\mathcal{T} = \{t \in \mathcal{R};\ t = t_0\}$. Notationally, this can be expressed by employing the characteristic function $\chi_{\mathcal{T}} = \chi_{\mathcal{T}}t(t)$ of the set $\mathcal{T}$, defined as

$$\chi_{\mathcal{T}} = \{1,½,0\} \qquad \text{for } t \in \{\mathcal{T},\partial\mathcal{T},\mathcal{T}'\} . \tag{24.1-7}$$

With this notation, we have for the standard inversion applied to the Laplace transform $\hat{f}(\boldsymbol{x},s)$ of any space–time function $f = f(\boldsymbol{x},t)$ the result

$$\text{Inverse Laplace transform of } \hat{f}(\boldsymbol{x},s) = \chi_{\mathcal{T}}(t)\, f(\boldsymbol{x},t) . \tag{24.1-8}$$

Exercises

Exercise 24.1-1

(a) What volume density of external electric source current corresponds in the complex frequency domain to the initial field $D_k(\boldsymbol{x},t_0)$?
(b) What would be the corresponding volume density of external electric source current in the space–time domain?

Answers: (a) $-D_k(\boldsymbol{x},t_0)\exp(-st_0)$; (b) $-D_k(\boldsymbol{x},t_0)\delta(t - t_0)$.

Exercise 24.1-2

(a) What volume density of external magnetic source current corresponds in the complex frequency domain to the initial field $B_j(\boldsymbol{x},t_0)$?
(b) What would be the corresponding volume density of external magnetic source current in the space–time domain?

Answers: (a) $-B_j(\boldsymbol{x},t_0)\exp(-st_0)$; (b) $-B_j(\boldsymbol{x},t_0)\delta(t - t_0)$.

24.2 The complex frequency-domain electromagnetic constitutive relations; Kramers–Kronig causality relations for a medium with relaxation

In the time Laplace transformation of the constitutive relations we separately discuss: media with relaxation and instantaneously reacting media.

Medium with relaxation

The electromagnetic constitutive relations for a linear, time-invariant, locally reacting medium are given by (see Equations (18.3-9), (18.3-10) and Equations (19.3-8)–(19.3-10))

$$J_k(\boldsymbol{x},t) = \int_{t'=0}^{\infty} \kappa_{\mathrm{c};k,r}(\boldsymbol{x},t')E_r(\boldsymbol{x},t-t')\,\mathrm{d}t' , \tag{24.2-1}$$

$$D_k(x,t) = \varepsilon_0 E_k(x,t) + \varepsilon_0 \int_{t'=0}^{\infty} \kappa_{e;k,r}(x,t') E_r(x,t-t')\,dt' , \tag{24.2-2}$$

$$B_j(x,t) = \mu_0 H_j(x,t) + \mu_0 \int_{t'=0}^{\infty} \kappa_{m;j,p}(x,t') H_p(x,t-t')\,dt' . \tag{24.2-3}$$

Mathematically, the relaxation parts of the right-hand sides of Equations (24.2-1)–(24.2-3) are *convolutions in time*. (The notion that convolutions in time can serve as the mathematical description of (mechanical) relaxation goes back to Boltzmann (see Boltzmann, 1876). From this it can be expected that the Laplace transformation possibly reveals additional properties of the relaxation functions. Carrying out the Laplace transformation of Equations (24.2-1)–(24.2-3) over the interval $t \in \mathcal{R}$ and assuming that the field quantities are of a transient nature, we obtain

$$\hat{J}_k(x,s) = \hat{\sigma}_{k,r}(x,s)\hat{E}_r(x,s) , \tag{24.2-4}$$

$$\hat{D}_k(x,s) = \hat{\varepsilon}_{k,r}(x,s)\hat{E}_r(x,s) , \tag{24.2-5}$$

$$\hat{B}_j(x,s) = \hat{\mu}_{j,p}(x,s)\hat{H}_p(x,s) , \tag{24.2-6}$$

where, in terms of the relaxation functions the complex frequency-domain conductivity, permittivity and permeability tensors are given by

$$\hat{\sigma}_{k,r} = \hat{\kappa}_{c;k,r} , \tag{24.2-7}$$

$$\hat{\varepsilon}_{k,r} = \varepsilon_0(\delta_{k,r} + \hat{\kappa}_{e;k,r}) , \tag{24.2-8}$$

$$\hat{\mu}_{j,p} = \mu_0(\delta_{j,p} + \hat{\kappa}_{m;j,p}) , \tag{24.2-9}$$

respectively, where

$$\hat{\kappa}_{c;k,r}(x,s) = \int_{t'=0}^{\infty} \exp(-st')\kappa_{c;k,r}(x,t')\,dt' , \tag{24.2-10}$$

$$\hat{\kappa}_{e;k,r}(x,s) = \int_{t'=0}^{\infty} \exp(-st')\kappa_{e;k,r}(x,t')\,dt' , \tag{24.2-11}$$

$$\hat{\kappa}_{m;j,p}(x,s) = \int_{t'=0}^{\infty} \exp(-st')\kappa_{m;j,p}(x,t')\,dt' . \tag{24.2-12}$$

Evidently, the complex frequency-domain conduction, electric and magnetic relaxation functions are the Laplace transforms of causal functions of time. Therefore, as has been shown in Section B.3, their real and imaginary parts for imaginary values of $s = j\omega$, with $\omega \in \mathcal{R}$, satisfy the Kramers–Kronig causality relations (see Equations (B.3-18) and (B.3-19)).

Consider the generic expression, omitting the different subscripts,

$$\hat{\kappa}(x,s) = \int_{t'=0}^{\infty} \exp(-st')\kappa(x,t')\,dt' \tag{24.2-13}$$

for $s = j\omega$ and write

$$\hat{\kappa}(x,j\omega) = \hat{\kappa}'(x,\omega) - j\hat{\kappa}''(x,\omega) \qquad \text{for } \omega \in \mathcal{R} . \tag{24.2-14}$$

Then, the *Kramers–Kronig causality relations* are given by

$$\kappa''(x,\omega) = -\frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\kappa'(x,\omega')}{\omega' - \omega} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R} \tag{24.2-15}$$

and

$$\kappa'(x,\omega) = \frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\kappa''(x,\omega')}{\omega' - \omega} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}. \tag{24.2-16}$$

Equations (24.2-15) and (24.2-16) imply that κ' and κ'' form pairs of Hilbert transforms. Another property of $\hat{\kappa}(x,\mathrm{j}\omega)$ is that (see Equations (B.3-6) and (B.3-7))

$$\kappa'(x,-\omega) = \kappa'(x,\omega) \qquad \text{for all } \omega \in \mathcal{R} \tag{24.2-17}$$

and

$$\kappa''(x,-\omega) = -\kappa''(x,\omega) \qquad \text{for all } \omega \in \mathcal{R}, \tag{24.2-18}$$

i.e. κ' is an even function of ω and κ'' is an odd function of ω for $\omega \in \mathcal{R}$. Using these properties in the right-hand sides of Equations (24.2-15) and (24.2-16), these relations can be rewritten as (see Equations (B.3-26) and (B.3-27))

$$\kappa''(x,\omega) = -\frac{2}{\pi} \int_{\omega'=0}^{\infty} \frac{\kappa'(x,\omega')\omega}{(\omega')^2 - \omega^2} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R} \tag{24.2-19}$$

and

$$\kappa'(x,\omega) = \frac{2}{\pi} \int_{\omega'=0}^{\infty} \frac{\kappa''(x,\omega')\omega'}{(\omega')^2 - \omega^2} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}. \tag{24.2-20}$$

In case the right-hand sides of either Equations (24.2-15) and (24.2-16) or Equations (24.2-19) and (24.2-20) have to be evaluated numerically, the Cauchy principal values of the integrals may present a difficulty. To circumvent this difficulty, we can rewrite Equations (24.2-15) and (24.2-16) as (see Equations (B.3-24) and (B.3-25))

$$\kappa''(x,\omega) = -\frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\kappa'(x,\omega') - \kappa'(x,\omega)}{\omega' - \omega} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R} \tag{24.2-21}$$

and

$$\kappa'(x,\omega) = \frac{1}{\pi} \int_{\omega'=-\infty}^{\infty} \frac{\kappa''(x,\omega') - \kappa''(x,\omega)}{\omega' - \omega} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}, \tag{24.2-22}$$

and Equations (24.2-19) and (24.2-20) as (see Equations (B.3-29) and (B.3-30))

$$\kappa''(x,\omega) = -\frac{2}{\pi} \int_{\omega'=0}^{\infty} \frac{[\kappa'(x,\omega') - \kappa'(x,\omega)]\omega}{(\omega')^2 - \omega^2} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R} \tag{24.2-23}$$

and

$$\kappa'(x,\omega) = \frac{2}{\pi} \int_{\omega'=0}^{\infty} \frac{\kappa''(x,\omega')\omega' - \kappa''(x,\omega)\omega}{(\omega')^2 - \omega^2} \mathrm{d}\omega' \qquad \text{for } \omega \in \mathcal{R}. \tag{24.2-24}$$

Equations (24.2-21)–(24.2-24) are the *Bode relations* for the relaxation functions (see Section B.3). In their right-hand sides only proper integrals occur.

Instantaneously reacting medium

For an instantaneously reacting medium, Equations (19.2-1), (19.2-4) and (19.2-5) lead, after time Laplace transformation, to

$$\hat{J}_k(\boldsymbol{x},s) = \sigma_{k,r}(\boldsymbol{x})\hat{E}_r(\boldsymbol{x},s)\,, \tag{24.2-25}$$

$$\hat{D}_k(\boldsymbol{x},s) = \varepsilon_{k,r}(\boldsymbol{x})\hat{E}_r(\boldsymbol{x},s)\,, \tag{24.2-26}$$

$$\hat{B}_j(\boldsymbol{x},s) = \mu_{j,p}(\boldsymbol{x})\hat{H}_p(\boldsymbol{x},s)\,, \tag{24.2-27}$$

respectively, in which the constitutive coefficients $\sigma_{k,r}$, $\varepsilon_{k,r}$ and $\mu_{j,p}$ are independent of s.

For the cases of the conduction relaxation function of a metal (Section 19.5), the conduction relaxation function of an electron plasma (Section 19.6) and the electric relaxation function of an isotropic dielectric (Section 19.6) the complex steady-state values for sinusoidally in time varying fields are given below.

Complex conduction relaxation function of a metal (sinusoidal oscillations)

The complex steady-state conduction relaxation function of a metal for sinusoidally, with angular frequency ω, in time varying fields follows upon substituting $s = \mathrm{j}\omega$ in the expression for $\hat{\kappa}_{\mathrm{c};k,r}(\boldsymbol{x},s)$ given in Equation (19.5-38). The result is denoted as $\hat{\sigma}_{k,r}(\boldsymbol{x},\mathrm{j}\omega)$ and is decomposed into its real and imaginary parts according to

$$\hat{\sigma}_{k,r}(\boldsymbol{x},\mathrm{j}\omega) = \sigma'_{k,r}(\boldsymbol{x},\omega) - \mathrm{j}\sigma''_{k,r}(\boldsymbol{x},\omega)\,. \tag{24.2-28}$$

To express the frequency behaviour, the relevant expression is rewritten as

$$\hat{\sigma}_{k,r}(\boldsymbol{x},\mathrm{j}\omega) = \sigma[Y_1\delta_{k,r} + \varepsilon_{k,q,r}(\omega_{\mathrm{ce},q}/\omega_{\mathrm{ce}})Y_2 + (\omega_{\mathrm{ce};k}\omega_{\mathrm{ce};r}/\omega_{\mathrm{ce}}^2)Y_3]\,, \tag{24.2-29}$$

in which

$$Y_1 = \frac{\mathrm{j}\omega/\nu_\mathrm{c} + 1}{(\mathrm{j}\omega/\nu_\mathrm{c} + 1)^2 + (\omega_{\mathrm{ce}}/\nu_\mathrm{c})^2}\,, \tag{24.2-30}$$

$$Y_2 = \frac{\omega_{\mathrm{ce}}/\nu_\mathrm{c}}{(\mathrm{j}\omega/\nu_\mathrm{c} + 1)^2 + (\omega_{\mathrm{ce}}/\nu_\mathrm{c})^2}\,, \tag{24.2-31}$$

$$Y_3 = \frac{\omega_{\mathrm{ce}}^2/\nu_\mathrm{c}^2}{(\mathrm{j}\omega/\nu_\mathrm{c} + 1)\left[(\mathrm{j}\omega/\nu_\mathrm{c} + 1)^2 + (\omega_{\mathrm{ce}}/\nu_\mathrm{c})^2\right]}\,. \tag{24.2-32}$$

Figures 24.2-1–24.2-6 show the real and imaginary parts of Y_1, Y_2 and Y_3, respectively, as a function of ω/ν_c, with $\omega_{\mathrm{ce}}/\nu_\mathrm{c}$ as a parameter. Since for conduction in a metal the effect of collisions is predominant, we have considered the range $\omega_{\mathrm{ce}}/\nu_\mathrm{c} \leqslant 1$. The coefficients

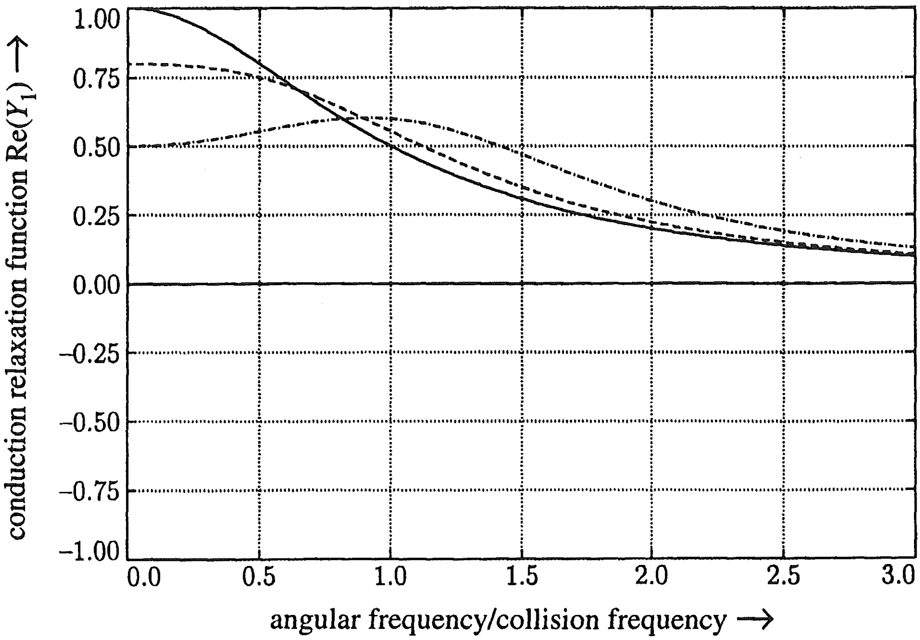

Figure 24.2-1 Complex conduction relaxation function of a metal as a function of frequency: Real part of Y_1. (——): $\omega_{ce}/\nu_c = 0$; (- - -): $\omega_{ce}/\nu_c = 0.5$; (-·-·-): $\omega_{ce}/\nu_c = 1.0$.

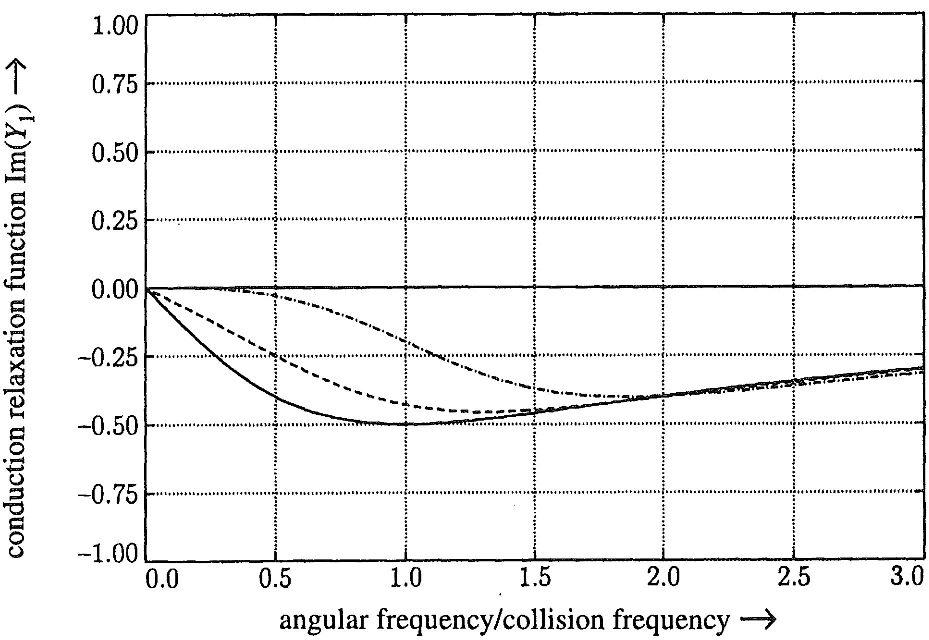

Figure 24.2-2 Complex conduction relaxation function of a metal as a function of frequency: Imaginary part of Y_1. (——): $\omega_{ce}/\nu_c = 0$; (- - -): $\omega_{ce}/\nu_c = 0.5$; (-·-·-): $\omega_{ce}/\nu_c = 1.0$.

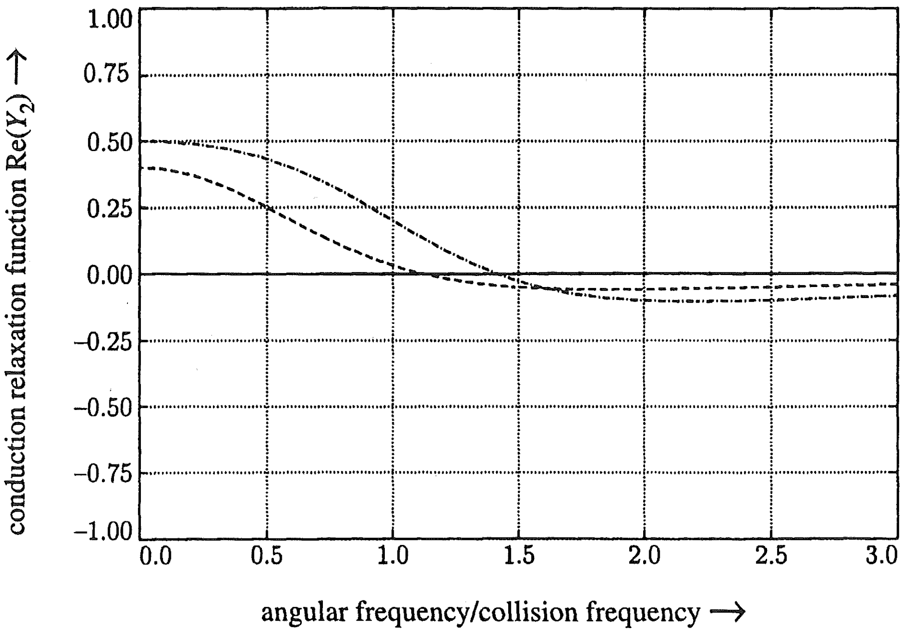

Figure 24.2-3 Complex conduction relaxation function of a metal as a function of frequency: Real part of Y_2. (——): $\omega_{ce}/\nu_c = 0$; (- - -): $\omega_{ce}/\nu_c = 0.5$; (-·-·-): $\omega_{ce}/\nu_c = 1.0$.

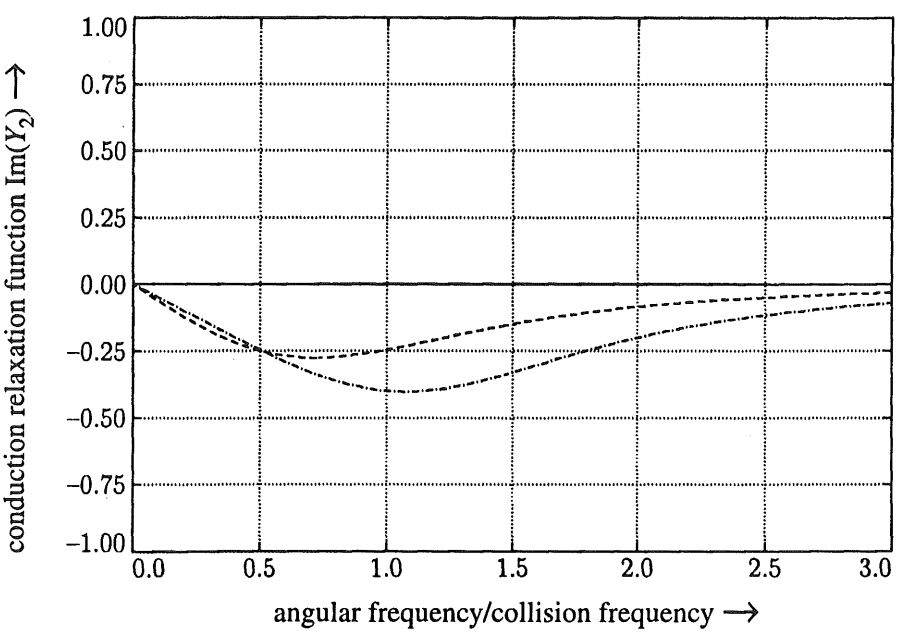

Figure 24.2-4 Complex conduction relaxation function of a metal as a function of frequency: Imaginary part of Y_2. (——): $\omega_{ce}/\nu_c = 0$; (- - -): $\omega_{ce}/\nu_c = 0.5$; (-·-·-): $\omega_{ce}/\nu_c = 1.0$.

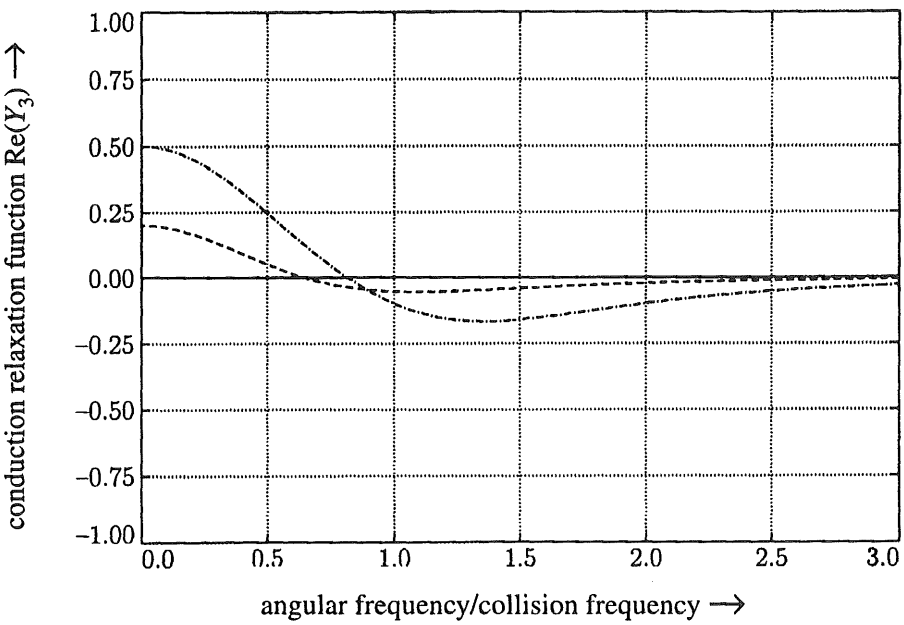

Figure 24.2-5 Complex conduction relaxation function of a metal as a function of frequency: Real part of Y_3. (——): $\omega_{ce}/\nu_c = 0$; (- - -): $\omega_{ce}/\nu_c = 0.5$; (-·-·-): $\omega_{ce}/\nu_c = 1.0$.

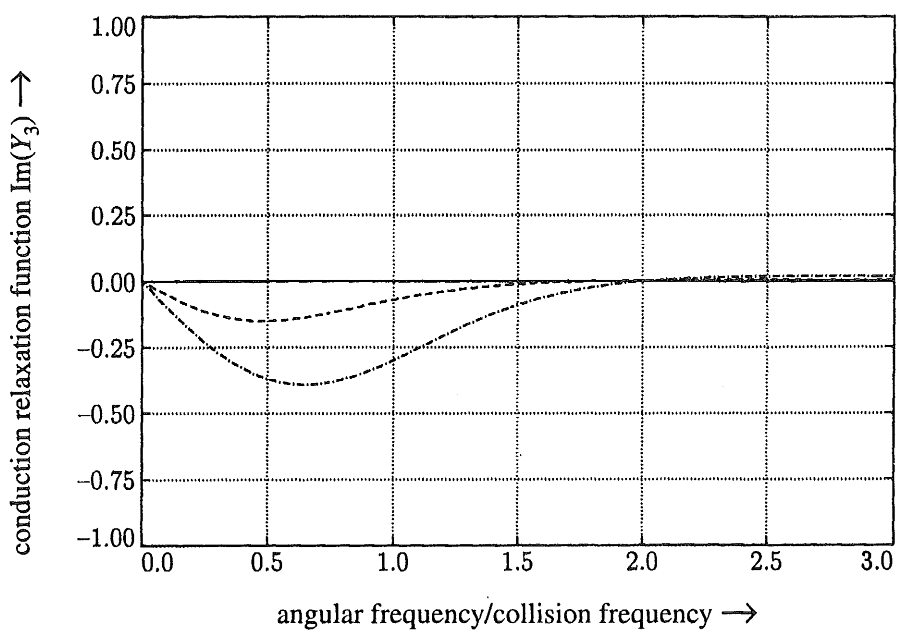

Figure 24.2-6 Complex conduction relaxation function of a metal as a function of frequency: Imaginary part of Y_3. (——): $\omega_{ce}/\nu_c = 0$; (- - -): $\omega_{ce}/\nu_c = 0.5$; (-·-·-): $\omega_{ce}/\nu_c = 1.0$.

$\omega_{\mathrm{ce},k}/\omega_{\mathrm{ce}}$ occurring in Equation (24.2-29) are the direction cosines of the orientation of the external magnetic field with respect to the chosen Cartesian reference frame.

Complex conduction relaxation function of an electron plasma (sinusoidal oscillations)

The complex conduction relaxation function of an electron plasma for sinusoidally, with angular frequency ω, in time varying fields follows upon substibuting $s = \mathrm{j}\omega$ in the expression for $\hat{\kappa}_{\mathrm{c};k,r}(\boldsymbol{x},s)$ given in Equation (19.6-4). The result is denoted as $\hat{\sigma}_{k,r}(\boldsymbol{x},\mathrm{j}\omega)$ and is decomposed into its real and imaginary parts according to

$$\hat{\sigma}_{k,r}(\boldsymbol{x},\mathrm{j}\omega) = \sigma'_{k,r}(\boldsymbol{x},\omega) - \mathrm{j}\sigma''_{k,r}(\boldsymbol{x},\omega)\,. \tag{24.2-33}$$

To express the frequency behaviour, the relevant expression is rewritten as

$$\hat{\sigma}_{k,r}(\boldsymbol{x},\mathrm{j}\omega) = \varepsilon_0\omega_{\mathrm{ce}}^{-1}\omega_{\mathrm{pe}}^2\,[X_1\delta_{k,r} + \varepsilon_{k,q,r}(\omega_{\mathrm{ce};q}/\omega_{\mathrm{ce}})X_2 + (\omega_{\mathrm{ce};k}\omega_{\mathrm{ce};r}/\omega_{\mathrm{ce}}^2)X_3]\,, \tag{24.2-34}$$

in which

$$X_1 = \frac{\mathrm{j}\omega/\omega_{\mathrm{ce}} + \nu_{\mathrm{c}}/\omega_{\mathrm{ce}}}{(\mathrm{j}\omega/\omega_{\mathrm{ce}} + \nu_{\mathrm{c}}/\omega_{\mathrm{ce}})^2 + 1}\,, \tag{24.2-35}$$

$$X_2 = \frac{1}{(\mathrm{j}\omega/\omega_{\mathrm{ce}} + \nu_{\mathrm{c}}/\omega_{\mathrm{ce}})^2 + 1}\,, \tag{24.2-36}$$

$$X_3 = \frac{1}{(\mathrm{j}\omega/\omega_{\mathrm{ce}} + \nu_{\mathrm{c}}/\omega_{\mathrm{ce}})[(\mathrm{j}\omega/\omega_{\mathrm{ce}} + \nu_{\mathrm{c}}/\omega_{\mathrm{ce}})^2 + 1]}\,. \tag{24.2-37}$$

Figures 24.2-7–24.2-12 show the real and imaginary parts of X_1, X_2 and X_3, respectively, as a function of $\omega/\omega_{\mathrm{ce}}$, with $\nu_{\mathrm{c}}/\omega_{\mathrm{ce}}$ as a parameter. Since for the plasma properties the effect of inertia (in this case of the electrons) is predominant, we have considered the range $\nu_{\mathrm{c}}/\omega_{\mathrm{ce}} \leqslant 1$. The coefficients $\omega_{\mathrm{ce};k}/\omega_{\mathrm{ce}}$ occurring in Equation (24.2-34) are the direction cosines of the orientation of the external magnetic field with respect to the chosen Cartesian reference frame.

Complex dielectric relaxation function of an isotropic dielectric (sinusoidal oscillations)

The complex susceptibility of an isotropic dielectric for sinusoidally, with angular frequency ω, in time varying fields follows upon substituting $s = \mathrm{j}\omega$ in the expression (19.7-9) for $\hat{\kappa}_{\mathrm{e}}(\boldsymbol{x},s)$. The result is denoted as $\hat{\kappa}_{\mathrm{e}}(\boldsymbol{x},\mathrm{j}\omega)$ and is decomposed into its real and imaginary parts according to

$$\hat{\kappa}_{\mathrm{e}}(\boldsymbol{x},\mathrm{j}\omega) = \kappa'_{\mathrm{e}}(\boldsymbol{x},\omega) - \mathrm{j}\kappa''_{\mathrm{e}}(\boldsymbol{x},\omega)\,. \tag{24.2-38}$$

To express the frequency behaviour, the relevant expression is rewritten as

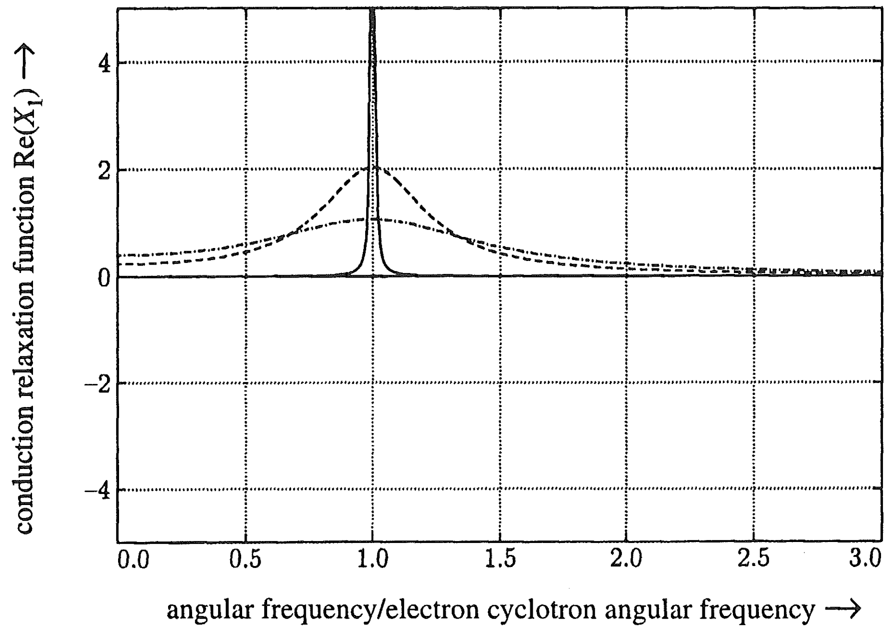

Figure 24.2-7 Complex conduction relaxation function of an electron plasma as a function of frequency: Real part of X_1. (——): $\nu_c/\omega_{ce} = 0.001$; (- - -): $\nu_c/\omega_{ce} = 0.25$; (-·-·-): $\nu_c/\omega_{ce} = 0.5$.

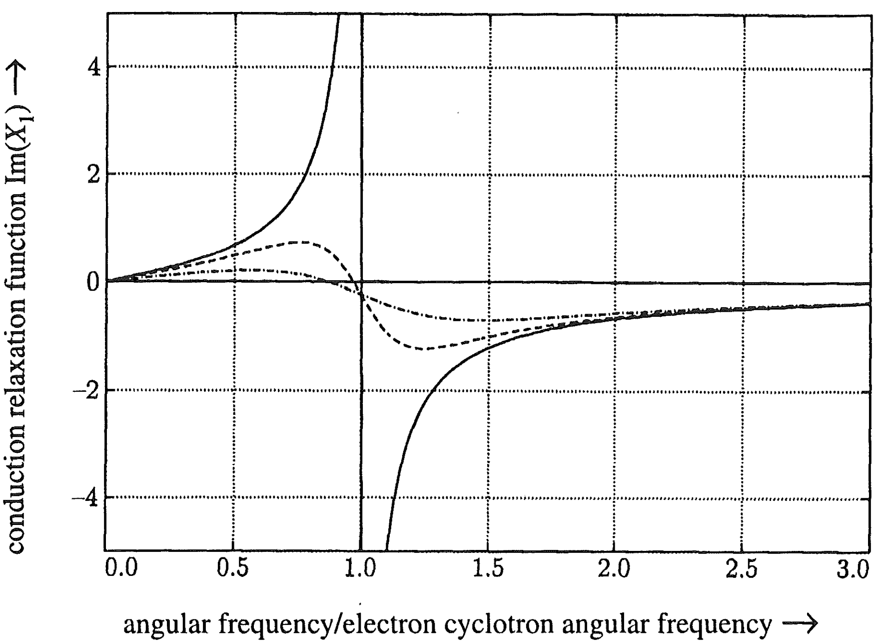

Figure 24.2-8 Complex conduction relaxation function of an electron plasma as a function of frequency: Imaginary part of X_1. (——): $\nu_c/\omega_{ce} = 0.001$; (- - -): $\nu_c/\omega_{ce} = 0.25$; (-·-·-): $\nu_c/\omega_{ce} = 0.5$.

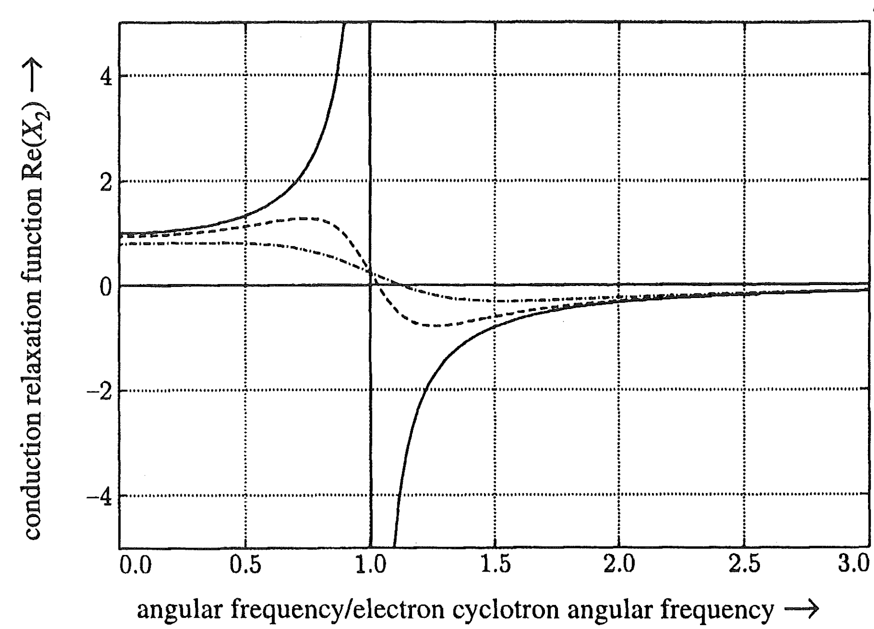

Figure 24.2-9 Complex conduction relaxation function of an electron plasma as a function of frequency: Real part of X_2. (——): $\nu_c/\omega_{ce} = 0.001$; (- - -): $\nu_c/\omega_{ce} = 0.25$; (-·-·-): $\nu_c/\omega_{ce} = 0.5$.

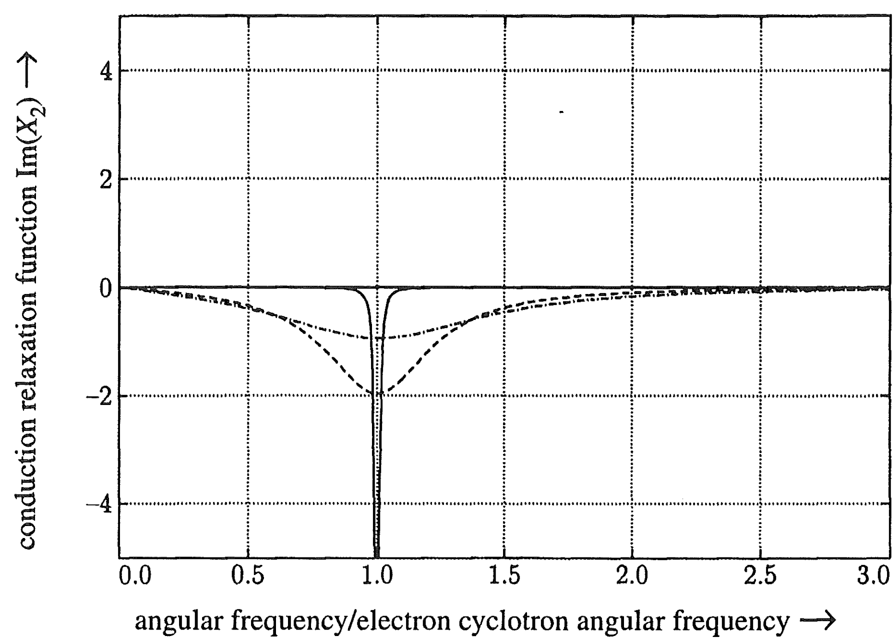

Figure 24.2-10 Complex conduction relaxation function of an electron plasma as a function of frequency: Imaginary part of X_2. (——): $\nu_c/\omega_{ce} = 0.001$; (- - -): $\nu_c/\omega_{ce} = 0.25$; (-·-·-): $\nu_c/\omega_{ce} = 0.5$.

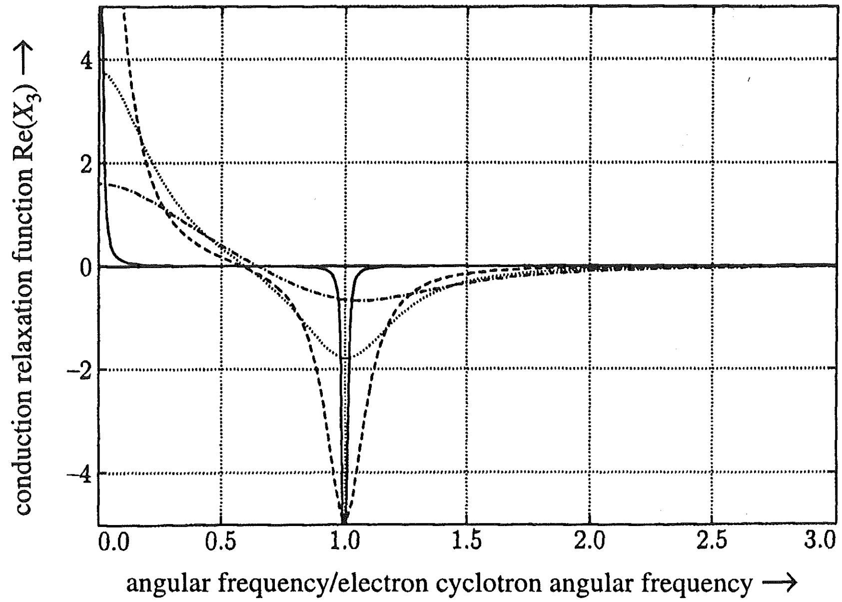

Figure 24.2-11 Complex conduction relaxation function of an electron plasma as a function of frequency: Real part of X_3. (—): $\nu_c/\omega_{ce} = 0.001$; (- - -): $\nu_c/\omega_{ce} = 0.25$; (-·-·-): $\nu_c/\omega_{ce} = 0.5$.

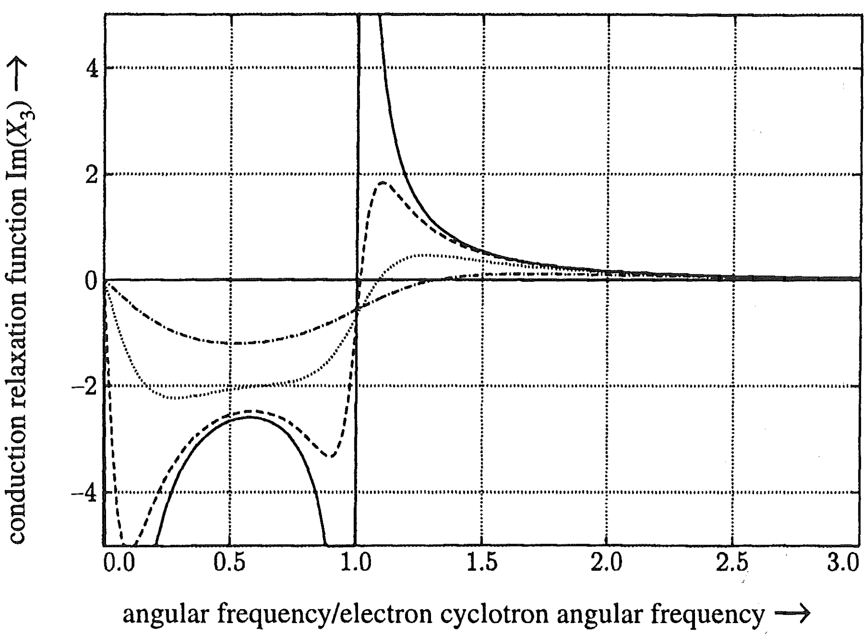

Figure 24.2-12 Complex conduction relaxation function of an electron plasma as a function of frequency: Imaginary part of X_3. (—): $\nu_c/\omega_{ce} = 0.001$; (- - -): $\nu_c/\omega_{ce} = 0.25$; (-·-·-): $\nu_c/\omega_{ce} = 0.5$.

$$\hat{\chi}_e(\boldsymbol{x},j\omega) = Z\,, \tag{24.2-39}$$

in which

$$Z = \frac{\omega_p^2/\omega_0^2}{-\omega^2/\omega_0^2 + j(\omega/\omega_0)(\Gamma/\omega_0) + 1 - \omega_p^2/3\omega_0^2}\,. \tag{24.2-40}$$

Figures 24.2-13–24.2-16 show the real and imaginary parts of Z as a function of ω/ω_0, with Γ/ω_0 and ω_p/ω_0 as parameters. Since in a solid (in our case a dielectric) the mechanical restoring force is predominant, we have considered the range $\Gamma/\omega_0 \leqslant 1$ and $\omega_p/\omega_0 \leqslant 1$.

Exercises

Exercise 24.2-1

Give the complex frequency-domain electromagnetic constitutive relations for an isotropic medium with relaxation.

Answer:

$$\hat{J}_k(\boldsymbol{x},s) = \hat{\sigma}(\boldsymbol{x},s)\hat{E}_k(\boldsymbol{x},s)\,, \tag{24.2-41}$$

$$\hat{D}_k(\boldsymbol{x},s) = \hat{\varepsilon}(\boldsymbol{x},s)\hat{E}_k(\boldsymbol{x},s)\,, \tag{24.2-42}$$

$$\hat{B}_j(\boldsymbol{x},s) = \hat{\mu}(\boldsymbol{x},s)\hat{H}_j(\boldsymbol{x},s)\,, \tag{24.2-43}$$

in which

$$\hat{\sigma} = \hat{\kappa}_c\,, \tag{24.2-44}$$

$$\hat{\varepsilon} = \varepsilon_0(1 + \hat{\kappa}_e)\,, \tag{24.2-45}$$

$$\hat{\mu} = \mu_0(1 + \hat{\kappa}_m)\,, \tag{24.2-46}$$

with

$$\hat{\kappa}_{c;e;m}(\boldsymbol{x},s) = \int_{t'=0}^{\infty} \exp(-st')\kappa_{c;e;m}(\boldsymbol{x},t')\,dt'\,. \tag{24.2-47}$$

Exercise 24.2-2

Give the complex frequency-domain electromagnetic constitutive relations for an instantaneously reacting isotropic medium.

Answer:

$$\hat{J}_k(\boldsymbol{x},s) = \sigma(\boldsymbol{x})\hat{E}_k(\boldsymbol{x},s)\,, \tag{24.2-48}$$

$$\hat{D}_k(\boldsymbol{x},s) = \varepsilon(\boldsymbol{x})\hat{E}_k(\boldsymbol{x},s)\,, \tag{24.2-49}$$

$$\hat{B}_j(\boldsymbol{x},s) = \mu(\boldsymbol{x})\hat{H}_j(\boldsymbol{x},s)\,, \tag{24.2-50}$$

in which

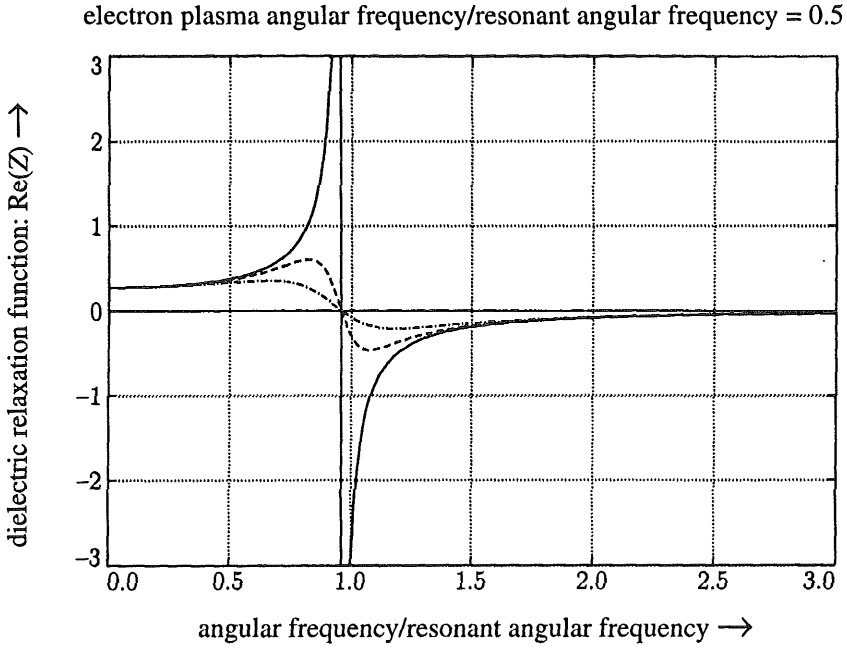

Figure 24.2-13 Complex dielectric relaxation function as a function of frequency: Real part of Z. (—): $\Gamma/\omega_0 = 0.001$; (- - -): $\Gamma/\omega_0 = 0.25$; (-·-·-): $\Gamma/\omega_0 = 0.5$.

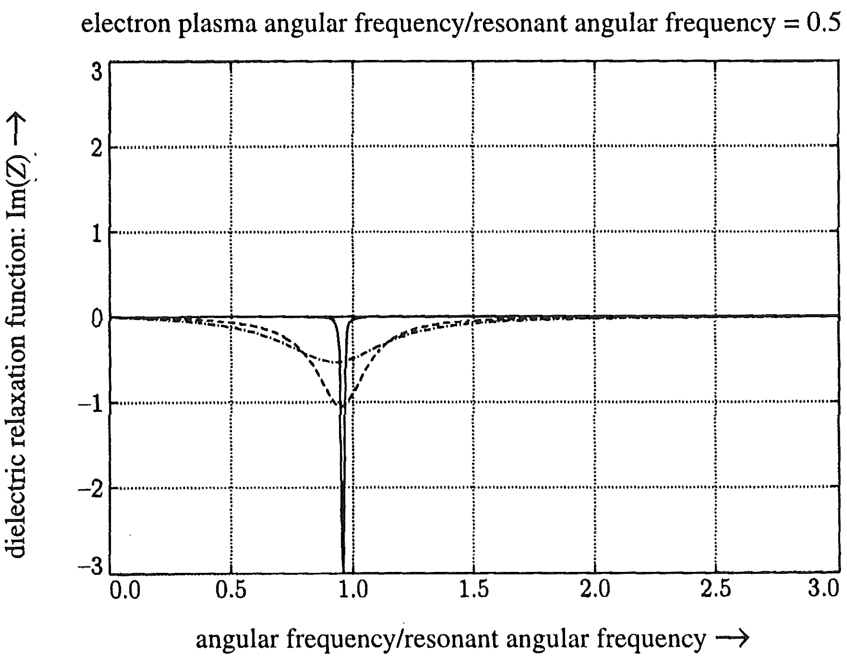

Figure 24.2-14 Complex dielectric relaxation function as a function of frequency: Imaginary part of Z. (—): $\Gamma/\omega_0 = 0.001$; (- - -): $\Gamma/\omega_0 = 0.25$; (-·-·-): $\Gamma/\omega_0 = 0.5$.

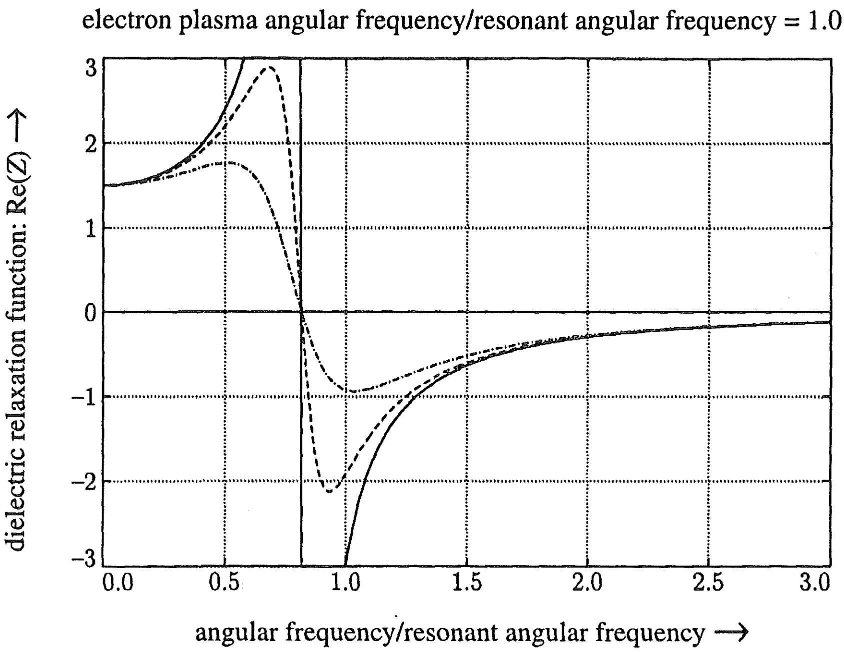

Figure 24.2-15 Complex dielectric relaxation function as a function of frequency: Real part of Z. (—): $\Gamma/\omega_0 = 0.001$; (- - -): $\Gamma/\omega_0 = 0.25$; (-·-·-): $\Gamma/\omega_0 = 0.5$.

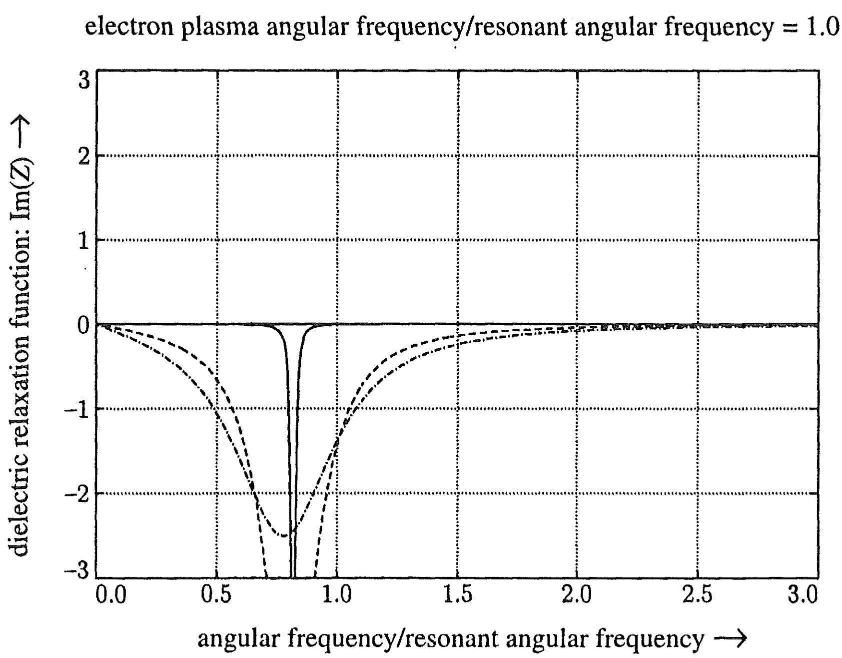

Figure 24.2-16 Complex dielectric relaxation function as a function of frequency: Imaginary part of Z. (—): $\Gamma/\omega_0 = 0.001$; (- - -): $\Gamma/\omega_0 = 0.25$; (-·-·-): $\Gamma/\omega_0 = 0.5$.

$$\varepsilon = \varepsilon_0(1 + \chi_e)\,, \tag{24.2-51}$$

$$\mu = \mu_0(1 + \chi_m)\,. \tag{24.2-52}$$

Exercise 24.2-3

Give the expression for the complex conduction relaxation function of a metal (sinusoidal oscillations) in the absence of an external magnetic field. (*Hint*: Substitute $\omega_{ce;q} = 0$ in Equations (24.2-29)–(24.2-32).)

Answer:

$$\hat{\sigma}(\boldsymbol{x},j\omega) = \sigma'(\boldsymbol{x},\omega) - j\sigma''(\boldsymbol{x},\omega)\,, \tag{24.2-53}$$

with

$$\hat{\sigma} = \frac{\sigma}{j\omega/\nu_c + 1} = \frac{\sigma(1 - j\omega/\nu_c)}{1 + \omega^2/\nu_c^2}\,, \tag{24.2-54}$$

and hence

$$\sigma' = \frac{\sigma}{1 + \omega^2/\nu_c^2}\,, \tag{24.2-55}$$

$$\sigma'' = \frac{\sigma\omega/\nu_c}{1 + \omega^2/\nu_c^2}\,. \tag{24.2-56}$$

Here, $\sigma = n_e e^2/m_e\nu_c$ (see Equation (19.5-17)).

Exercise 24.2-4

Give the expression for the complex frequency-domain conductivity of a superconductor (sinusoidal oscillations) in the absence of an external magnetic field. (*Hint*: Take the limit $\nu_c \to 0$ in the result of Exercise 24.2-3.)

Answer:

$$\hat{\sigma}(\boldsymbol{x},j\omega) = \sigma'(\boldsymbol{x},\omega) - j\sigma''(\boldsymbol{x},\omega)\,, \tag{24.2-57}$$

with

$$\sigma' = 0\,, \tag{24.2-58}$$

$$\sigma'' = \frac{n_e e^2}{\omega m_e}\,. \tag{24.2-59}$$

(Note that this conductivity is imaginary, but not (real and) infinite.)

Exercise 24.2-5

Give the expression for the complex frequency-domain conduction relaxation function of a electron plasma (sinusoidal oscillations) in the absence of an external magnetic field. (*Hint*: Substitute $\omega_{ce;q} = 0$ in Equations (24.2-34)–(24.2-37).)

Answer:

$$\hat{\sigma}(\boldsymbol{x},\mathrm{j}\omega) = \sigma'(\boldsymbol{x},\omega) - \mathrm{j}\sigma''(\boldsymbol{x},\omega)\,, \tag{24.2-60}$$

with

$$\hat{\sigma} = \frac{\varepsilon_0\omega_{\mathrm{pe}}^2}{\mathrm{j}\omega + \nu_{\mathrm{c}}} = \varepsilon_0\omega_{\mathrm{pe}}^2\frac{\nu_{\mathrm{c}} - \mathrm{j}\omega}{\omega^2 + \nu_{\mathrm{c}}^2}\,, \tag{24.2-61}$$

and hence

$$\sigma' = \frac{\varepsilon_0\omega_{\mathrm{pe}}^2\nu_{\mathrm{c}}}{\omega^2 + \nu_{\mathrm{c}}^2}\,, \tag{24.2-62}$$

$$\sigma'' = \frac{\varepsilon_0\omega_{\mathrm{pe}}^2\omega}{\omega^2 + \nu_{\mathrm{c}}^2}\,. \tag{24.2-63}$$

Here, $\omega_{\mathrm{pe}}^2 = n_{\mathrm{e}}e^2/m_{\mathrm{e}}\varepsilon_0$ (see Equation (19.5-17)).

24.3 The complex frequency-domain boundary conditions

The boundary conditions that have been discussed in Chapter 20 apply to time-invariant boundaries. As a consequence of this, they can directly be transferred to the complex frequency domain. At the interface of two different media we therefore have (see Equations (20.1-2) and (20.1-3))

$$\varepsilon_{k,m,p}\nu_m\hat{H}_p \quad \text{continuous across source-free interface,} \tag{24.3-1}$$

and

$$\varepsilon_{j,n,r}\nu_n\hat{E}_r \quad \text{continuous across source-free interface,} \tag{24.3-2}$$

while from Equation (20.2-1) we have

$$\lim_{h\downarrow 0}\varepsilon_{j,n,r}\nu_n\hat{E}_r(\boldsymbol{x} + h\boldsymbol{\nu},s) = 0$$
$$\text{for } \boldsymbol{x} \in \{\text{boundary of an electrically impenetrable object}\}, \tag{24.3-3}$$

and from Equation (20.3-1)

$$\lim_{h\downarrow 0}\varepsilon_{k,m,p}\nu_m\hat{H}_p(\boldsymbol{x} + h\boldsymbol{\nu},s) = 0$$
$$\text{for } \boldsymbol{x} \in \{\text{boundary of a magnetically impenetrable object}\}. \tag{24.3-4}$$

It is clear that these boundary conditions would also follow if the procedure of Chapter 20 were applied to the complex frequency-domain electromagnetic field Equations (24.1-3) and (24.1-4).

Equations (24.3-1)–(24.3-4) are the fundamental boundary conditions that follow from the basic electromagnetic field equations. As has been shown in Chapter 20, also the electromagnetic compatibility relations lead to a set of (auxiliary) boundary conditions. The complex frequency-domain counterparts of the latter follow from Equations (20.1-4) and (20.1-5) as

$$\nu_k(\hat{J}_k + s\hat{D}_k) \quad \text{continuous across source-free interface} \tag{24.3-5}$$

and

$$\nu_k \hat{B}_k \quad \text{continuous across source-free interface,} \tag{24.3-6}$$

while from Equation (20.2-2) we have

$$\lim_{h \downarrow 0} \nu_j(x) \hat{B}_j(x + h\nu, s) = 0$$
$$\text{for } x \in \{\text{boundary of an electrically impenetrable object}\}, \tag{24.3-7}$$

and from Equation (20.3-2)

$$\lim_{h \downarrow 0} \nu_k(x) \left[\hat{J}_k(x + h\nu, s) + s\hat{D}_k(x + h\nu, s) \right] = 0$$
$$\text{for } x \in \{\text{boundary of a magnetically impenetrable object}\}. \tag{24.3-8}$$

Exercises

Exercise 24.3-1

Apply the procedure of Chapter 20 to the complex frequency-domain electromagnetic field equations (24.1-3) and (24.1-4) to arrive at the fundamental complex frequency-domain boundary conditions Equations (24.3-1) and (24.3-2).

Exercise 24.3-2

Apply the procedure of Chapter 20 to the complex frequency-domain compatibility relations (24.1-5) and (24.1-6) to arrive at the auxiliary complex frequency-domain boundary conditions Equations (24.3-5) and (24.3-6).

24.4 The complex frequency-domain coupled electromagnetic wave equations

In the majority of our calculations we shall substitute Equations (24.2-4)–(24.2-6) in the complex frequency-domain electromagnetic field Equations (24.1-3) and (24.1-4) and thus obtain a system of differential equations in space, in which the number of unknowns is equal to the number of equations and in which s occurs as a parameter. The relevant equations are written as

$$-\varepsilon_{k,m,p} \partial_m \hat{H}_p + \hat{\eta}_{k,r} \hat{E}_r = -\hat{J}_k^{\text{ext}} + D_k(x, t_0) \exp(-st_0), \tag{24.4-1}$$

$$\varepsilon_{j,n,r} \partial_n \hat{E}_r + \hat{\zeta}_{j,p} \hat{H}_p = -\hat{K}_j^{\text{ext}} + B_j(x, t_0) \exp(-st_0), \tag{24.4-2}$$

in which

$$\hat{\eta}_{k,r} = \hat{\sigma}_{k,r} + s\hat{\varepsilon}_{k,r} \tag{24.4-3}$$

is the *transverse admittance per length of the medium*, and

$$\hat{\zeta}_{j,p} = s\hat{\mu}_{j,p} \tag{24.4-4}$$

is the *longitudinal impedance per length of the medium.* (The terminology is borrowed from the one that is conventional in one-dimensional transmission-line theory (see Exercise 24.4-1).)

For an anisotropic medium, $\hat{\eta}_{k,r}$ and $\hat{\zeta}_{j,p}$ are tensors of rank two. For an isotropic medium we have

$$\hat{\eta}_{k,r} = \hat{\eta}\delta_{k,r} \tag{24.4-5}$$

and

$$\hat{\zeta}_{j,p} = \hat{\zeta}\delta_{j,p} , \tag{24.4-6}$$

in which

$$\hat{\eta} = \hat{\sigma} + s\hat{\varepsilon} \tag{24.4-7}$$

and

$$\hat{\zeta} = s\hat{\mu} \tag{24.4-8}$$

are scalars.

In view of the causal behaviour of a passive medium, $\hat{\eta}_{k,r} = \hat{\eta}_{k,r}(\boldsymbol{x},s)$ and $\hat{\zeta}_{j,p} = \hat{\zeta}_{j,p}(\boldsymbol{x},s)$ are analytic functions of the complex frequency s that are regular in the right-half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane. However, the inverse constitutive relations, which involve the inverse $\hat{\eta}^{-1}_{r,k}$ to $\hat{\eta}_{k,r}$ defined through

$$\hat{\eta}_{k,r}\hat{\eta}^{-1}_{r,k'} = \delta_{k,k'} \tag{24.4-9}$$

and the inverse $\hat{\zeta}^{-1}_{p,j}$ to $\hat{\zeta}_{j,p}$ defined through

$$\hat{\zeta}_{j,p}\hat{\zeta}^{-1}_{p,j'} = \delta_{j,j'} , \tag{24.4-10}$$

are also causal. Hence, also $\hat{\eta}^{-1}_{r,k} = \hat{\eta}^{-1}_{r,k}(\boldsymbol{x},s)$ and $\hat{\zeta}^{-1}_{p,j} = \hat{\zeta}^{-1}_{p,j}(\boldsymbol{x},s)$ must be analytic functions of the complex frequency s that are regular in the right half $\{\mathrm{Re}(s) > 0\}$ of the complex s plane. As Equations (24.4-9) and (24.4-10) show, this implies that $\det(\hat{\eta}_{k,r})$ and $\det(\hat{\zeta}_{j,p})$ can have neither singularities nor zeros in the right half of the complex s plane. For an isotropic medium, Equations (24.4-9) and (24.4-10) imply that $\hat{\eta}$ and $\hat{\zeta}$ are free from zeros in the right half of the complex s plane.

Note that the inclusion of extra factors of s in the complex frequency-domain medium parameters do not invalidate these properties.

Exercises

Exercise 24.4-1

The complex frequency-domain properties of a two-wire (single transmission channel), one-dimensional transmission line operating in its dominant mode are described by a transverse admittance per length $\hat{Y}_{\mathrm{T}}$ and a longitudinal impedance per length $\hat{Z}_{\mathrm{L}}$. Let $\hat{V} = \hat{V}(z,s)$ be the voltage across the line and $\hat{I} = \hat{I}(z,s)$ the electric current in the wires, where z is the coordinate along the line. Show that $\partial_z\hat{I} + \hat{Y}_{\mathrm{T}}\hat{V} = \hat{I}^{\,\mathrm{ext}}$ and $\partial_z\hat{V} + \hat{Z}_{\mathrm{L}}\hat{I} = \hat{V}^{\,\mathrm{ext}}$, where $\hat{I}^{\,\mathrm{ext}}$ and $\hat{V}^{\,\mathrm{ext}}$ are the source electric current and the source voltage, respectively, that are per unit length induced by external fields. (In the equations, the initial values of the electric current and the voltage have been set equal to zero.) For sufficiently slowly varying fields (quasi-static fields), we have $\hat{Y}_{\mathrm{T}} = G + sC$ and $\hat{Z}_{\mathrm{L}} = R + sL$, where G, C, R and L are the conductance, capacitance, resistance and inductance per length of the transmission line.

Exercise 24.4-2

Give the expression for the complex frequency-domain transverse admittance per length $\hat{\eta}$ and the longitudinal impedance per length $\hat{\zeta}$ of a metal conductor in the absence of an external magnetic field. (*Hint*: Use Equations (19.5-24), (24.4-3) and (24.4-4).)

Answer:

$$\hat{\eta} = \frac{\sigma}{s/\nu_{\mathrm{c}} + 1} + s\varepsilon_0 \tag{24.4-11}$$

$$\hat{\zeta} = s\mu_0 \, . \tag{24.4-12}$$

Here, $\sigma = n_{\mathrm{e}}e^2/m_{\mathrm{e}}\nu_{\mathrm{c}}$.

Exercise 24.4-3

Give the expression for the complex frequency-domain transverse admittance per length $\hat{\eta}$ and the longitudinal impedance per length $\hat{\zeta}$ of an electron plasma in the absence of an external magnetic field. (*Hint*: Use Equations (19.6-4), (24.4-3) and (24.4-4).)

Answer:

$$\hat{\eta} = s\varepsilon_0 \left[1 + \frac{\omega_{\mathrm{pe}}^2}{s(s + \nu_{\mathrm{c}})} \right], \tag{24.4-13}$$

$$\hat{\zeta} = s\mu_0 \, . \tag{24.4-14}$$

Here, $\omega_{\mathrm{pe}}^2 = n_{\mathrm{e}}e^2/m_{\mathrm{e}}\varepsilon_0$.

Exercise 24.4-4

Give the expression for the complex frequency-domain transverse admittance per length $\hat{\eta}$ and the longitudinal impedance per length $\hat{\zeta}$ of an isotropic dielectric with a Lorentzian absorption line. (*Hint*: Use Equations (19.7-8)–(19.7-9), (24.4-3) and (24.4-4).)

Answer:

$$\hat{\eta} = s\varepsilon_0 \left[1 + \frac{\omega_{\mathrm{p}}^2}{(s + \Gamma/2)^2 + \Omega^2} \right], \tag{24.4-15}$$

$$\hat{\zeta} = s\mu_0 \, . \tag{24.4-16}$$

Here, $\Omega = (\omega_0^2 - \omega_{\mathrm{p}}^2 - \Gamma^2/4)^{1/2}$ is the natural angular frequency of the oscillations of the movable electric charge, ω_{p} is the plasma angular frequency of the movable electric charge, ω_0

is the resonant angular frequency of the mechanical model of the atom and Γ is a phenomenological damping coefficient.

References

Boltzmann, L., 1876, Zur Theorie der elastischen Nachwirkung, *Annalen der Physik und Chemie, Ergaenzungsband*, 7, 624–654.