27

Plane electromagnetic waves in homogeneous media

In this chapter the notion of plane electromagnetic wave that has turned up in the local description of the electromagnetic field in the far-field region of the electromagnetic radiation by extended sources, is generalised to cases where the wave amplitudes are arbitrary functions of the angular wave vector, not specifically the ones of the type that occurs in the far-field approximation. The concepts of dispersion equation and wave slowness are introduced for homogeneous, arbitrarily anisotropic media, with the homogeneous, isotropic medium as a special case. For the real frequency domain the attenuation and the phase propagation of a plane wave are discussed.

27.1 Plane waves in the complex frequency domain

In the complex frequency domain *plane waves* are solutions of the complex frequency-domain source-free electromagnetic equations (see Equations (24.4-1) and (24.4-2))

$$-\varepsilon_{k,m,p}\partial_m\hat{H}_p + \hat{\eta}_{k,r}\hat{E}_r = 0\,, \tag{27.1-1}$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}_r + \hat{\zeta}_{j,p}\hat{H}_p = 0\,, \tag{27.1-2}$$

of the form

$$\{\hat{E}_r,\hat{H}_p\} = \{\hat{e}_r,\hat{h}_p\}\exp(-\hat{\gamma}_s x_s)\,, \tag{27.1-3}$$

in which the *amplitudes* (or *polarisation vectors*) $\{\hat{e}_r,\hat{h}_p\}$ and the propagation vector $\hat{\gamma}_s$ are independent of $\boldsymbol{x}$. The factor $\exp(-\hat{\gamma}_s x_s)$ is denoted as the *propagation factor.* Substitution of Equation (27.1-3) in Equations (27.1-1) and (27.1-2) leads, in view of the relation

$$\partial_m\,[\exp(-\hat{\gamma}_s x_s)] = -\hat{\gamma}_m\exp(-\hat{\gamma}_s x_s)\,, \tag{27.1-4}$$

to

$$\varepsilon_{k,m,p}\hat{\gamma}_m\hat{h}_p + \hat{\eta}_{k,r}\hat{e}_r = 0\,, \tag{27.1-5}$$

$$-\varepsilon_{j,n,r}\hat{\gamma}_n\hat{e}_r + \hat{\zeta}_{j,p}\hat{h}_p = 0\,. \tag{27.1-6}$$

Equations (27.1-5) and (27.1-6) constitute a homogeneous system of linear algebraic equations in $\hat{e}_r$ and $\hat{h}_p$, which for arbitrary values of $\hat{\gamma}_s$ has only the trivial solution $\hat{e}_r = 0$ and $\hat{h}_p = 0$. For a non-trivial solution to exist, $\hat{\gamma}_s$ must be chosen appropriately. The condition to be put on $\hat{\gamma}_s$ in this respect could be found by setting equal to zero the determinant of the system of six linear algebraic Equations (27.1-5) and (27.1-6), but a more efficient way to find the relevant relation is to eliminate one of the field quantities and to consider the then resulting system of linear, algebraic equations. For the elimination of $\hat{h}_p$ we use (see Equation (27.1-6))

$$\hat{h}_p = \hat{\zeta}^{-1}_{p,j}\varepsilon_{j,n,r}\hat{\gamma}_n\hat{e}_r \,. \tag{27.1-7}$$

Substitution of this expression in Equation (27.1-5) leads to

$$\varepsilon_{k,m,p}\hat{\gamma}_m\hat{\zeta}^{-1}_{p,j}\varepsilon_{j,n,r}\hat{\gamma}_n\hat{e}_r + \hat{\eta}_{k,r}\hat{e}_r = 0 \,. \tag{27.1-8}$$

The condition for this system of equations to have a non-trivial solution is that $\hat{\gamma}_s$ has to satisfy the determinantal equation

$$\det(\varepsilon_{k,m,p}\hat{\gamma}_m\hat{\zeta}^{-1}_{p,j}\varepsilon_{j,n,r}\hat{\gamma}_n + \hat{\eta}_{k,r}) = 0 \,. \tag{27.1-9}$$

Similarly, for the elimination of $\hat{e}_r$ we use (see Equation (27.1-5))

$$\hat{e}_r = -\hat{\eta}^{-1}_{r,k}\varepsilon_{k,m,p}\hat{\gamma}_m\hat{h}_p \,. \tag{27.1-10}$$

Substitution of this expression in Equation (27.1-6) leads to

$$\varepsilon_{j,n,r}\hat{\gamma}_n\hat{\eta}^{-1}_{r,k}\varepsilon_{k,m,p}\hat{\gamma}_m\hat{h}_p + \hat{\zeta}_{j,p}\hat{h}_p = 0 \,. \tag{27.1-11}$$

The condition for this system of equations to have a non-trivial solution is that $\hat{\gamma}_s$ has to satisfy the determinantal equation

$$\det(\varepsilon_{j,n,r}\hat{\gamma}_n\hat{\eta}^{-1}_{r,k}\varepsilon_{k,m,p}\hat{\gamma}_m + \hat{\zeta}_{j,p}) = 0 \,. \tag{27.1-12}$$

Both Equations (27.1-9) and (27.1-12) must lead to the same set of admissible values of the propagation coefficient $\hat{\gamma}_s$. From the determinantal equations (27.1-9) and (27.1-12) is clear that with any admissible value of $\hat{\gamma}_s$ also $-\hat{\gamma}_s$ is an admissible value. This property reduces the solution space in which admissible values of $\hat{\gamma}_s$ are to be sought. Equation (27.1-9) and (27.1-12) are known as the complex frequency-domain plane wave *dispersion equations* for the propagation vector.

Once a value for $\hat{\gamma}_s$ satisfying Equation (27.1-9) has been chosen, Equation (27.1-8) is used to determine the corresponding electric-field amplitude vector (or polarisation vector) $\hat{e}_r$ of the plane wave. Since the system of Equations (27.1-8) is homogeneous, one is free to choose one linear relation between the three components of $\hat{e}_r$ (*electric-field normalisation condition*). Similarly, once a value for $\hat{\gamma}_s$ satisfying Equation (27.1-12) has been chosen, Equation (27.1-11) is used to determine the corresponding magnetic-field amplitude vector (or polarisation vector) $\hat{h}_p$ of the plane wave. Since the system of Equations (27.1-11) is homogeneous, one is free to choose one linear relation between the three components of $\hat{h}_p$ (*magnetic-field normalisation condition*). For a particular value of the propagation coefficient the electric-field polarisation vector and the magnetic-field polarisation vector are not independent of each other. To exhibit their relationship, Equation (27.1-7) is rewritten as

$$\hat{h}_p = \hat{Y}_{p,r}\hat{e}_r \,, \tag{27.1-13}$$

in which

$$\hat{Y}_{p,r} = \hat{\zeta}^{-1}_{p,j}\varepsilon_{j,n,r}\hat{\gamma}_n \tag{27.1-14}$$

is the tensorial *electromagnetic plane wave admittance* of the wave, while Equation (27.1-10) is rewritten as

$$\hat{e}_r = \hat{Z}_{r,p}\hat{h}_p \, , \tag{27.1-15}$$

in which

$$\hat{Z}_{r,p} = \hat{\eta}^{-1}_{r,k}\,\varepsilon_{k,m,p}\hat{\gamma}_m \tag{27.1-16}$$

is the tensorial *electromagnetic plane wave impedance* of the wave. Substitution of Equation (27.1-15) in Equation (27.1-13) leads to

$$\hat{Y}_{p,r}\hat{Z}_{r,p'} = \delta_{p,p'} \, . \tag{27.1-17}$$

Substitution of Equation (27.1-13) in Equation (27.1-15) leads to

$$\hat{Z}_{r,p}\hat{Y}_{p,r'} = \delta_{r,r'} \, . \tag{27.1-18}$$

On account of Equations (27.1-17) and (27.1-18) the tensorial electromagnetic plane wave admittance and the tensorial electromagnetic plane wave impedance are each other's inverses.

For arbitrary values of $\hat{\gamma}_s$, satisfying Equations (27.1-9) and (27.1-12), the resulting expressions of the type of Equation (27.1-3) are denoted as *non-uniform plane waves*.

Plane wave compatibility relations

Upon contracting Equation (27.1-5) with $\hat{\gamma}_k$ and using the property that $\hat{\gamma}_k\varepsilon_{k,m,p}\hat{\gamma}_m = 0$, it follows that

$$\hat{\gamma}_k\hat{\eta}_{k,r}\hat{e}_r = 0 \, . \tag{27.1-19}$$

This equation is a *compatibility relation for the electric-field polarisation vector.*

Similarly, upon contracting Equation (27.1-6) with $\hat{\gamma}_j$ and using the property that $\hat{\gamma}_j\varepsilon_{j,n,r}\hat{\gamma}_n = 0$, it follows that

$$\hat{\gamma}_j\hat{\zeta}_{j,p}\hat{h}_p = 0 \, . \tag{27.1-20}$$

This equation is a *compatibility relation for the magnetic-field polarisation vector.*

Uniform plane waves

Uniform plane waves are a subset of the general class of non-uniform plane waves. For a *uniform plane wave* the propagation vector is of the special shape

$$\hat{\gamma}_s = \hat{\gamma}\xi_s \, , \tag{27.1-21}$$

where ξ_s is a real unit vector that specifies the *direction of propagation* of the wave. (Since ξ_s is a unit vector, we have $\xi_s\xi_s = 1$.) Now, $\hat{\gamma}$ is the (scalar) *propagation coefficient* of the uniform plane wave. Substitution of Equation (27.1-21) in Equation (27.1-9) yields

$$\det(\varepsilon_{k,m,p}\xi_m\hat{\zeta}^{-1}_{p,j}\varepsilon_{j,n,r}\xi_n\hat{\gamma}^2 + \hat{\eta}_{k,r}) = 0 \, . \tag{27.1-22}$$

Substitution of Equation (27.1-21) in Equation (27.1-12) yields

$$\det(\varepsilon_{j,n,r}\xi_n\hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\xi_m\hat{\gamma}^2 + \hat{\zeta}_{j,p}) = 0 \,. \tag{27.1-23}$$

Equations (27.1-22) and (27.1-23) are the *dispersion equations for uniform plane waves.* Causality of the wave motion entails the condition that $\exp(-\hat{\gamma}\xi_s x_s)$ should remain bounded as $|x|\to\infty$ in the half-space where $\xi_s x_s > 0$. This yields the condition $\mathrm{Re}(\hat{\gamma}) > 0$ for $\mathrm{Re}(s) > 0$. Equations (27.1-22) and (27.1-23) clearly indicate that the value of the propagation coefficient changes with the direction of propagation of the uniform plane wave, a property that is indicative for the presence of anisotropy.

The expression for the tensorial electromagnetic plane wave admittance reduces for a uniform plane wave to (see Equation (27.1-14))

$$\hat{Y}_{p,r} = \hat{\zeta}_{p,j}^{-1}\varepsilon_{j,n,r}\xi_n\hat{\gamma} \tag{27.1-24}$$

and the expression for the tensorial electromagnetic plane wave impedance reduces for a uniform plane wave to (see Equation (27.1-16))

$$\hat{Z}_{r,p} = \hat{\eta}_{r,k}^{-1}\varepsilon_{k,m,p}\xi_m\hat{\gamma} \,. \tag{27.1-25}$$

For a uniform plane wave the compatibility relation of Equation (27.1-19) for the electric-field polarisation vector leads to

$$\xi_k\hat{\eta}_{k,r}\hat{e}_r = 0 \tag{27.1-26}$$

and the compatibility relation of Equation (27.1-20) for the magnetic-field polarisation vector leads to

$$\xi_j\hat{\zeta}_{j,p}\hat{h}_p = 0 \,. \tag{27.1-27}$$

Here, the property has been used that $\hat{\gamma} \neq 0$ which property follows from the dispersion Equations (27.1-22) and (27.1-23) upon using the physical conditions that $\det(\hat{\eta}_{k,r}) \neq 0$ and $\det(\hat{\zeta}_{j,p}) \neq 0$, which conditions in their turn follow from the consideration that the complex frequency-domain constitutive relations of any medium should be uniquely invertible.

Isotropic media

For an *isotropic medium* we have

$$\hat{\eta}_{k,r} = \hat{\eta}\delta_{k,r} \tag{27.1-28}$$

and

$$\hat{\zeta}_{j,p} = \hat{\zeta}\delta_{j,p} \,. \tag{27.1-29}$$

With these relations, Equations (27.1-5) and (27.1-6) change into

$$\varepsilon_{k,m,p}\hat{\gamma}_m\hat{h}_p + \hat{\eta}\hat{e}_k = 0 \,, \tag{27.1-30}$$

$$-\varepsilon_{j,n,r}\hat{\gamma}_n\hat{e}_r + \hat{\zeta}\hat{h}_j = 0 \,, \tag{27.1-31}$$

while the compatibilty relations Equations (27.1-19) and (27.1-20) change into

$$\hat{\gamma}_k\hat{e}_k = 0 \,, \tag{27.1-32}$$

$$\hat{\gamma}_j \hat{h}_j = 0 \,, \tag{27.1-33}$$

since $\hat{\eta} \neq 0$ and $\hat{\zeta} \neq 0$. (Equations (27.1-32) and (27.1-33) also follow directly from Equation (27.1-30) upon contraction with $\hat{\gamma}_k$ and Equation (27.1-31) upon contraction with $\hat{\gamma}_j$ respectively.) Substituting the expression

$$\hat{h}_p = \hat{\zeta}^{-1} \varepsilon_{p,n,r} \hat{\gamma}_n \hat{e}_r \tag{27.1-34}$$

that follows from Equation (27.1-31), in Equation (27.1-30), and using the property $\varepsilon_{k,m,p}\varepsilon_{p,n,r} = \delta_{k,n}\delta_{m,r} - \delta_{k,r}\delta_{m,n}$ and the compatibility relation of Equation (27.1-32), we end up with

$$(-\hat{\zeta}^{-1} \hat{\gamma}_m \hat{\gamma}_m + \hat{\eta}) \hat{e}_k = 0 \,. \tag{27.1-35}$$

Similarly, substition of the expression $\hat{e}_r = -\hat{\eta}^{-1} \varepsilon_{r,m,p} \hat{\gamma}_m \hat{h}_p$ that follows from Equation (27.1-30), in Equation (27.1-31), and using the property $\varepsilon_{j,n,r}\varepsilon_{r,m,p} = \delta_{j,m}\delta_{n,p} - \delta_{j,p}\delta_{n,m}$ and the compatibility relation of Equation (27.1-33), we end up with

$$(-\hat{\eta}^{-1} \hat{\gamma}_n \hat{\gamma}_n + \hat{\zeta}) \hat{h}_j = 0 \,. \tag{27.1-36}$$

Both Equations (27.1-35) and (27.1-36) lead to the dispersion relation for non-uniform plane waves in an isotropic medium

$$\hat{\gamma}_s \hat{\gamma}_s = \hat{\eta} \hat{\zeta} \,. \tag{27.1-37}$$

For a uniform plane wave (for which Equation (27.1-21) holds), the dispersion relation in an isotropic medium follows from Equation (27.1-37) as

$$\hat{\gamma}^2 = \hat{\eta} \hat{\zeta} \,. \tag{27.1-38}$$

The solution of the latter equation is given by

$$\hat{\gamma} = (\hat{\eta}\hat{\zeta})^{1/2} \qquad \text{with } \mathrm{Re}(\ldots)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 \,. \tag{27.1-39}$$

Equation (27.1-39) clearly shows that for isotropic media the value of the propagation coefficient is independent of the direction of propagation of the uniform plane wave, a property that is indicative for the isotropy of the medium and, hence, for the absence of anisotropy.

For uniform plane waves in isotropic media, the tensorial electromagnetic plane wave admittance attains the form (see Equation (27.1-14))

$$\hat{Y}_{p,r} = \hat{Y} \varepsilon_{p,n,r} \xi_n \,, \tag{27.1-40}$$

in which

$$\hat{Y} = \hat{\gamma}/\hat{\zeta} = (\hat{\eta}/\hat{\zeta})^{1/2} \qquad \text{with } \mathrm{Re}(\ldots)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 \tag{27.1-41}$$

is the scalar electromagnetic plane wave admittance, and the tensorial electromagnetic plane wave impedance attains the form (see Equation (27.1-16))

$$\hat{Z}_{r,p} = \hat{Z} \varepsilon_{r,m,p} \xi_m \,, \tag{27.1-42}$$

in which

$$\hat{Z} = \hat{\gamma}/\hat{\eta} = (\hat{\zeta}/\hat{\eta})^{1/2} \qquad \text{with } \mathrm{Re}(\ldots)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0 \tag{27.1-43}$$

is the scalar electromagnetic plane wave impedance. Accordingly, Equations (27.1-30) and (27.1-31) can now be written as

$$\varepsilon_{k,m,p}\xi_m\hat{h}_p + \hat{Y}\hat{e}_k = 0 \,, \tag{27.1-44}$$

$$-\varepsilon_{j,n,r}\xi_n\hat{e}_r + \hat{Z}\hat{h}_j = 0 \,. \tag{27.1-45}$$

Furthermore, the compatibility relations now reduce to (see Equation (27.1-32))

$$\xi_k\hat{e}_k = 0 \tag{27.1-46}$$

and (see Equation (27.1-33))

$$\xi_j\hat{h}_j = 0 \,. \tag{27.1-47}$$

From Equations (27.1-46) and (27.1-47) it is clear that in an isotropic medium a uniform electromagnetic plane wave is *transverse* with respect to its direction of propagation, both in its electric and its magnetic field strengths.

Exercises

Exercise 27.1-1

Show by contracting Equation (27.1-8) with $\hat{\eta}^{-1}_{r',k}$ that the dispersion relation for plane electromagnetic waves in an anisotropic medium can also be written as

$$\det(\hat{\eta}^{-1}_{r',k}\varepsilon_{k,m,p}\gamma_m\hat{\zeta}^{-1}_{p,j}\varepsilon_{j,n,r}\hat{\gamma}_n + \delta_{r',r}) = 0 \,. \tag{27.1-48}$$

Exercise 27.1-2

Show by contracting Equation (27.1-11) with $\hat{\zeta}^{-1}_{p',j}$ that the dispersion relation for plane electromagnetic waves in an anisotropic medium can also be written as

$$\det(\hat{\zeta}^{-1}_{p',j}\varepsilon_{j,n,r}\gamma_n\hat{\eta}^{-1}_{r,k}\varepsilon_{k,m,p}\hat{\gamma}_m + \delta_{p',p}) = 0 \,. \tag{27.1-49}$$

Exercise 27.1-3

(a) Show that for an isotropic medium the dispersion relation for uniform plane waves follows from both Equations (27.1-9) and (27.1-12) as (see also Exercises 27.1 and 27.2)

$$\det(\varepsilon_{k,m,p}\xi_m\varepsilon_{p,n,r}\xi_n\hat{\gamma}^2 + \hat{\eta}\hat{\zeta}\hat{\delta}_{k,r}) = 0 \,. \tag{27.1-50}$$

(b) Show, by using the relation $\varepsilon_{k,m,p}\varepsilon_{p,n,r} = \delta_{k,n}\delta_{m,r} - \delta_{k,r}\delta_{m,n}$, that this determinantal equation can be written as

$$\det\left[\xi_k\xi_r\gamma^2 - (\hat{\gamma}^2 - \hat{\eta}\hat{\zeta})\delta_{k,r}\right] = 0 \,, \tag{27.1-51}$$

or

$$\begin{vmatrix} \xi_1^2\gamma^2 - (\hat{\gamma}^2 - \hat{\eta}\hat{\zeta}) & \xi_1\xi_2\gamma^2 & \xi_1\xi_3\gamma^2 \\ \xi_2\xi_1\gamma^2 & \xi_2^2\gamma^2 - (\hat{\gamma}^2 - \hat{\eta}\hat{\zeta}) & \xi_2\xi_3\gamma^2 \\ \xi_3\xi_1\gamma^2 & \xi_3\xi_2\gamma^2 & \xi_3^2\gamma^2 - (\hat{\gamma}^2 - \hat{\eta}\hat{\zeta}) \end{vmatrix} = 0 \,. \tag{27.1-52}$$

(c) Show that this determinantal equation leads to

$$(\gamma^2 - \hat{\eta}\hat{\zeta})^2 \hat{\eta}\hat{\zeta} = 0\,, \tag{27.1-53}$$

or, since $\hat{\eta}\hat{\zeta} \neq 0$,

$$(\gamma^2 - \hat{\eta}\hat{\zeta})^2 = 0\,. \tag{27.1-54}$$

(This result shows that the $\gamma^2 = \hat{\eta}\hat{\zeta}$ is a double root of the dispersion equation for uniform plane electromagnetic waves in isotropic media, which fact is indicative for the degeneracy of the plane wave motion in the sense that two waves with different polarisation vectors can propagate with the same propagation coefficient.)

Exercise 27.1-4

What is the expression for the scalar electromagnetic plane wave admittance in a homogeneous, isotropic medium with conductivity σ, permittivity ε, magnetic loss coefficient Γ and permeability μ?

Answer:

$$\hat{Y} = [(\sigma + s\varepsilon)/(\Gamma + s\mu)]^{1/2} \qquad \text{with } \mathrm{Re}(...)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0\,. \tag{27.1-55}$$

Exercise 27.1-5

What is the expression for the scalar electromagnetic plane wave impedance in a homogeneous, isotropic medium with conductivity σ, permittivity ε, magnetic loss coefficient Γ and permeability μ?

Answer:

$$\hat{Z} = [(\Gamma + s\mu)/(\sigma + s\varepsilon)]^{1/2} \qquad \text{with } \mathrm{Re}(...)^{1/2} > 0 \quad \text{for } \mathrm{Re}(s) > 0\,. \tag{27.1-56}$$

Exercise 27.1-6

What is the expression for the scalar electromagnetic plane wave admittance in a homogeneous, isotropic medium with permittivity ε and permeability μ?

Answer:

$$\hat{Y} = (\varepsilon/\mu)^{1/2} \qquad \text{with } (...)^{1/2} > 0\,. \tag{27.1-57}$$

Exercise 27.1-7

What is the expression for the scalar electromagnetic plane wave impedance in a homogeneous, isotropic medium with permittivity ε and permeability μ?

Answer:

$$\hat{Z} = (\mu/\varepsilon)^{1/2} \qquad \text{with } (...)^{1/2} > 0\,. \tag{27.1-58}$$

Exercise 27.1-8

Construct the one-dimensional wave solutions of the source-free complex frequency-domain electromagnetic field equations in a homogeneous, isotropic medium with transverse admittance of the medium $\hat{\eta}$ and longitudinal impedance of the medium $\hat{\zeta}$ by taking the Euclidean reference frame such that the propagation takes place along the x_3-direction. What is the propagation factor for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3? (c) Express, for the two cases, the non-vanishing components of $\hat{H}_p$ in terms of the non-vanishing components of $\hat{E}_r$.

Answers:

(a) propagation factor $\exp(-\hat{\gamma}x_3)$, $\hat{H}_2 = (\hat{\zeta}/\hat{\eta})^{½}\hat{E}_1$, $\hat{H}_1 = -(\hat{\zeta}/\hat{\eta})^{½}\hat{E}_2$,

(b) propagation factor $\exp(\hat{\gamma}x_3)$, $\hat{H}_2 = -(\hat{\zeta}/\hat{\eta})^{½}\hat{E}_1$, $\hat{H}_1 = (\hat{\zeta}/\hat{\eta})^{½}\hat{E}_2$.

Here, $\hat{\gamma} = (\hat{\zeta}\hat{\eta})^{½}$ and $\mathrm{Re}(...)^{½} > 0$ for $\mathrm{Re}(s) > 0$. (Note that for each of the two cases, two waves with linearly independent polarisations propagate in the same direction with the same propagation coefficient. This is consistent with the result of Exercise 27.1-3.)

27.2 Plane waves in lossless media; the slowness surface

In a lossless medium the complex frequency-domain transverse admittance per length $\hat{\eta}_{k,r}$ and the longitudinal impedance per length $\hat{\zeta}_{j,p}$ reduce to

$$\hat{\eta}_{k,r} = s\varepsilon_{k,r} \tag{27.2-1}$$

and

$$\hat{\zeta}_{j,p} = s\mu_{j,p}\,, \tag{27.2-2}$$

respectively, in which $\varepsilon_{k,r}$ and $\mu_{j,p}$ are independent of s. Under these circumstances the complex propagation vector $\hat{\gamma}_s$ is written as

$$\hat{\gamma}_s = s\Lambda_s\,, \tag{27.2-3}$$

in which Λ_s is the *slowness vector*. Substitution of Equations (27.2-1)–(27.2-3) in the dispersion Equations (27.1-9) and (27.1-12) leads to

$$\det(\varepsilon_{k,m,p}\Lambda_m\mu^{-1}_{p,j}\varepsilon_{j,n,r}\Lambda_n + \varepsilon_{k,r}) = 0 \tag{27.2-4}$$

and

$$\det(\varepsilon_{j,n,r}\Lambda_n\varepsilon^{-1}_{r,k}\varepsilon_{k,m,p}\Lambda_p + \mu_{j,p}) = 0\,, \tag{27.2-5}$$

respectively, which are the equations to be satisfied by the slowness vector. Note that, although Equations (27.2-4) and (27.2-5) are independent of s and $\varepsilon_{k,r}$ and $\mu_{j,p}$ (and, hence, $\varepsilon^{-1}_{r,k}$ and $\mu^{-1}_{j,p}$) are real-valued, Λ_s can still be complex-valued. Such complex values of Λ_s correspond to *non-uniform plane waves*.

Uniform plane waves

For a uniform plane wave in a lossless medium the propagation vector is written as

$$\hat{\gamma}_s = s\Lambda\xi_s\,, \tag{27.2-6}$$

in which ξ_s is the (real) unit vector in the direction of propagation of the plane wave (note that $\xi_s\xi_s = 1$) and Λ is the *scalar slowness*. Substitution of the corresponding expression

$$\Lambda_s = \Lambda\xi_s \tag{27.2-7}$$

in Equations (27.2-4) and (27.2-5) yields

$$\det(\varepsilon_{k,m,p}\xi_m\mu_{p,j}^{-1}\varepsilon_{j,n,r}\xi_n\Lambda^2 + \varepsilon_{k,r}) = 0 \tag{27.2-8}$$

and

$$\det(\varepsilon_{j,n,r}\xi_n\varepsilon_{r,k}^{-1}\varepsilon_{k,m,p}\varepsilon_p\Lambda^2 + \mu_{j,p}) = 0\,, \tag{27.2-9}$$

respectively, which equations lead to the same admissible values of Λ^2.

In the three-dimensional Euclidean *slowness space* where $\Lambda_s = \Lambda\xi_s$ is the position vector, Equations (27.2-8) and (27.2-9) define a surface that is known as the *slowness surface*. For the class of lossless media, the slowness surface characterises geometrically the propagation properties of uniform plane waves. In particular, the shape of the slowness surface is indicative for the presence of anisotropy (and the nature of it) in the electromagnetic properties of the medium. As Equations (27.2-8) and (27.2-9) show, the slowness surface for an anisotropic, lossless medium is, in general, a three-sheeted surface with point symmetry about $\Lambda = 0$.

Isotropic media

For an isotropic, lossless medium we have

$$\varepsilon_{k,r} = \varepsilon\delta_{k,r} \tag{27.2-10}$$

and

$$\mu_{j,p} = \mu\delta_{j,p}\,. \tag{27.2-11}$$

For this case, the equation for the complex slowness of a non-uniform plane wave follows from Equation (27.1-37) as

$$\Lambda_s\Lambda_s = \varepsilon\mu\,, \tag{27.2-12}$$

while through substitution of Equation (27.2-6) in Equation (27.1-37) the equation for the slowness of a uniform plane wave follows as

$$\Lambda^2 = \varepsilon\mu\,. \tag{27.2-13}$$

The solution of the latter equation is given by

$$\Lambda = (\varepsilon\mu)^{1/2} \qquad \text{with } (\ldots)^{1/2} > 0\,, \tag{27.2-14}$$

or

$$\Lambda = 1/c\,, \tag{27.2-15}$$

where

$$c = (\varepsilon\mu)^{-1/2} \qquad \text{with } (\ldots)^{-1/2} > 0 \tag{27.2-16}$$

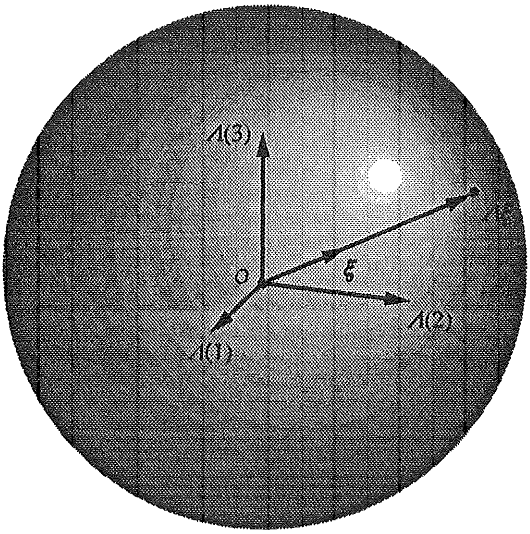


Figure 27.2-1 Slowness surface (sphere) for uniform plane waves in an isotropic, lossless medium.

is the electromagnetic wave speed. As Equations (27.2-13) and (27.2-15) show, the slowness surface for an isotropic, lossless medium is a *sphere* with radius $1/c$ (Figure 27.2-1).

Exercises

Exercise 27.2-1

Let $\{O; \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ be an orthogonal Cartesian reference frame in three-dimensional Euclidean slowness space. Use Equation (27.2-13) to construct the equation of the spherical slowness surface.

Answer:

$$\mathcal{A}_1^2 + \mathcal{A}_2^2 + \mathcal{A}_3^2 = \varepsilon\mu = 1/c^2 . \qquad (27.2\text{-}17)$$

27.3 Plane waves in the real frequency domain; attenuation vector and phase vector

In the signal processing of electromagnetic wave phenomena extensive use is made of the highly efficient Fast-Fourier-Transform (FFT-)algorithms that apply to the imaginary values $s = j\omega$ (j = imaginary unit, ω = (real) angular frequency) of the complex frequency s to transform wave-field quantities from the time domain to the complex frequency domain, and vice versa. As a consequence, the corresponding imaginary values of s are of particular interest. Now, for imaginary values of s, the condition of causality can no longer be easily invoked on the frequency-domain wave quantities. To control the causality one must always consider the imaginary values of s as the limiting ones upon approaching, in the complex s plane, the imaginary axis via the right half $\mathrm{Re}(s) > 0$ of the complex s plane. For $s = \mathrm{j}\omega$ it is customary to decompose the complex propagation vector $\hat{\gamma}_s = \hat{\gamma}_s(\mathrm{j}\omega)$ into its real and imaginary parts according to

$$\hat{\gamma}_s(\mathrm{j}\omega) = \alpha_s(\omega) + \mathrm{j}\beta_s(\omega)\,, \tag{27.3-1}$$

where α_s is the attenuation vector (SI unit: neper/metre (Np/m)), and β_s is the phase vector (SI unit: radian/metre (rad/m)). In view of the property

$$|\exp(-\hat{\gamma}_s x_s)| = \exp\,(-\alpha_s x_s)\,, \tag{27.3-2}$$

which holds because $|\exp(-\mathrm{j}\beta_s x_s)| = 1$, the family of planes $\{\alpha \in \mathcal{R}^3, x \in \mathcal{R}^3; \alpha_s x_s = \text{constant}\}$ defines a set of *planes of equal amplitude*, while in view of the property

$$\arg\,[\exp(-\hat{\gamma}_s x_s)] = -\beta_s x_s\,, \tag{27.3-3}$$

which holds because $\arg[\exp(-\alpha_s x_s)] = 0$, the family of planes $\{\beta \in \mathcal{R}^3, x \in \mathcal{R}^3; \beta_s x_s = \text{constant}\}$ defines a set of *planes of equal phase*. These two properties elucidate the term "plane wave" for complex frequency-domain solutions of the electromagnetic field equations of the type given in Equation (27.1-3).

Uniform plane waves

For a uniform plane wave propagating in the direction of the unit vector ξ_s we have (see Equations (27.1-21) and (27.3-1))

$$\alpha_s = \alpha\xi_s \tag{27.3-4}$$

and

$$\beta_s = \beta\xi_s\,, \tag{27.3-5}$$

where

α = (scalar) attenuation coefficient (Np/m),

β = (scalar) phase coefficient (rad/m),

and

$$\hat{\gamma}(\mathrm{j}\omega) = \alpha(\omega) + \mathrm{j}\beta(\omega)\,. \tag{27.3-6}$$

For uniform plane waves the set of planes of equal amplitude coincides with the set of planes of equal phase (Figure 27.3-1).

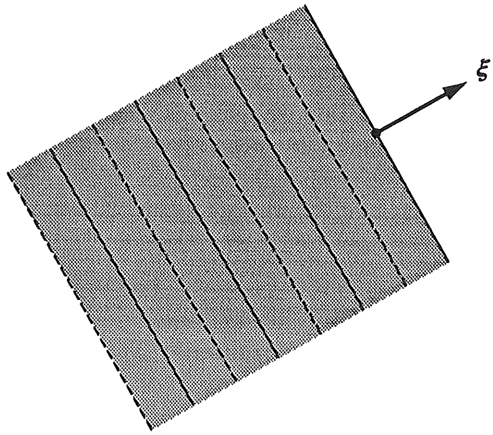


Figure 27.3-1 Planes of equal amplitude (——) and planes of equal phase (- - -) of a uniform plane wave in the real frequency domain.

Propagation in an isotropic medium with conductive electric and linear hysteresis magnetic losses

As a first example we shall discuss the propagation of plane waves in an isotropic medium with conductive electric and linear hysteresis magnetic losses. For such a medium, we have (see Equations (26.1-7) and (26.1-8))

$$\hat{\eta}(\mathrm{j}\omega) = \sigma + \mathrm{j}\omega\varepsilon \tag{27.3-7}$$

and

$$\hat{\zeta}(\mathrm{j}\omega) = \Gamma + \mathrm{j}\omega\mu \,. \tag{27.3-8}$$

Substitution of Equation (27.3-1) in the corresponding dispersion equation (see Equation (27.1-37)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = (\sigma + \mathrm{j}\omega\varepsilon)(\Gamma + \mathrm{j}\omega\mu) \,. \tag{27.3-9}$$

Separation of Equation (27.3-9) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = \sigma\Gamma - \omega^2\varepsilon\mu \tag{27.3-10}$$

and

$$2\alpha_s\beta_s = \omega(\sigma\mu + \Gamma\varepsilon) \,. \tag{27.3-11}$$

From Equation (27.3-10) it is clear that for $\sigma\Gamma < \omega^2\varepsilon\mu$ we have $\alpha_s\alpha_s < \beta_s\beta_s$, i.e. phase propagation is the predominant phenomenon, while for $\sigma\Gamma > \omega^2\varepsilon\mu$ we have $\alpha_s\alpha_s > \beta_s\beta_s$, i.e. attenuation is the predominant phenomenon. Note that the condition about which of the two phenomena is predominant, is frequency dependent. Furthermore, Equation (27.3-11) indicates that $\alpha_s\beta_s \neq 0$ for non-zero frequency, since the right-hand side differs from zero for non-zero frequency. Hence, for the type of medium under consideration, the set of planes of equal

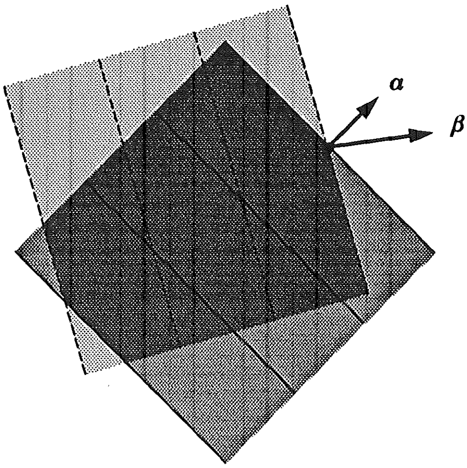


Figure 27.3-2 Planes of equal amplitude (—) and planes of equal phase (- - -) of a non-uniform plane wave in a medium with conductive electric and linear hysteresis magnetic losses in the real frequency domain.

amplitude is, for non-uniform plane waves, inclined with respect to the set of planes of equal phase (Figure 27.3-2).

For a uniform plane wave in the type of medium under consideration, substitution of Equations (27.3-4) and (27.3-5) in Equations (27.3-10) and (27.3-11) leads to

$$\alpha^2 - \beta^2 = \sigma\Gamma - \omega^2\varepsilon\mu \tag{27.3-12}$$

and

$$2\alpha\beta = \omega(\sigma\mu + \Gamma\varepsilon)\,, \tag{27.3-13}$$

where the property $\xi_s\xi_s = 1$ has been used. Since α must be non-negative in view of the condition of causality, Equation (27.3-13) indicates that $\beta \gtrless 0$ for $\omega \gtrless 0$. After some algebraic manipulations we obtain (see Exercise 27.3-1)

$$\alpha = \left[\frac{1}{2}\left(\sigma\Gamma - \omega^2\varepsilon\mu + \left\{(\sigma\Gamma)^2 + \omega^2\left[(\sigma\mu)^2 + (\Gamma\varepsilon)^2\right] + \omega^4(\varepsilon\mu)^2\right\}^{1/2}\right)\right]^{1/2} \tag{27.3-14}$$

and

$$\beta = \pm\left[\frac{1}{2}\left(-\sigma\Gamma + \omega^2\varepsilon\mu + \left\{(\sigma\Gamma)^2 + \omega^2\left[(\sigma\mu)^2 + (\Gamma\varepsilon)^2\right] + \omega^4(\varepsilon\mu)^2\right\}^{1/2}\right)\right]^{1/2}$$

$$\text{for } \omega \gtrless 0\,, \tag{27.3-15}$$

in which all square root expressions are non-negative. For very low and very high frequencies these results yield the asymptotic representations

$$\alpha \sim (\sigma\Gamma)^{1/2} \quad \text{and} \quad \beta \sim \omega\frac{\sigma\mu + \Gamma\varepsilon}{2(\sigma\Gamma)^{1/2}} \qquad \text{as } |\omega| \to 0 \tag{27.3-16}$$

and

$$\alpha \sim \frac{\sigma\mu + \Gamma\varepsilon}{2(\varepsilon\mu)^{1/2}} \quad \text{and} \quad \beta \sim \omega(\varepsilon\mu)^{1/2} \qquad \text{as } |\omega| \to \infty \tag{27.3-17}$$

respectively, where it has been assumed that $\sigma\Gamma \neq 0$. To put the results in a normalised form, we introduce the critical angular frequency of the conductive electric losses ω_c and the critical angular frequency of the linear hysteresis magnetic losses ω_h via

$$\omega_c = \sigma/\varepsilon \tag{27.3-18}$$

and

$$\omega_h = \Gamma/\mu\,, \tag{27.3-19}$$

and write

$$\alpha = (\sigma\Gamma)^{1/2}\,\bar{\alpha}\,, \tag{27.3-20}$$

$$\beta = (\sigma\Gamma)^{1/2}\,\bar{\beta} \tag{27.3-21}$$

and

$$\bar{\omega} = \omega/(\omega_c\omega_h)^{1/2}\,, \tag{27.3-22}$$

where $\bar{\alpha}$ is the normalised attenuation coefficient, $\bar{\beta}$ is the normalised phase coefficient, and $\bar{\omega}$ is the normalised angular frequency. In terms of these quantities, Equations (27.3-12) and (27.3-13) become

$$\bar{\alpha}^2 - \bar{\beta}^2 = 1 - \bar{\omega}^2 \tag{27.3-23}$$

and

$$2\bar{\alpha}\bar{\beta} = \bar{\omega}\left[\left(\frac{\omega_c}{\omega_h}\right)^{1/2} + \left(\frac{\omega_h}{\omega_c}\right)^{1/2}\right], \tag{27.3-24}$$

with the solution (see Equations (27.3-14) and (27.3-15))

$$\bar{\alpha} = \left[\frac{1}{2}\left(1 - \bar{\omega}^2 + \left\{1 + \bar{\omega}^2\,(\omega_c/\omega_h + \omega_h/\omega_c) + \bar{\omega}^4\right\}^{1/2}\right)\right]^{1/2}, \tag{27.3-25}$$

and

$$\bar{\beta} = \pm\left[\frac{1}{2}\left(-1 + \bar{\omega}^2 + \left\{1 + \bar{\omega}^2\,(\omega_c/\omega_h + \omega_h/\omega_c) + \bar{\omega}^4\right\}^{1/2}\right)\right]^{1/2} \quad \text{for } \bar{\omega} \gtrless 0\,. \tag{27.3-26}$$

The asymptotic representations Equation (27.3-16) and (27.3-17) become in their normalised form

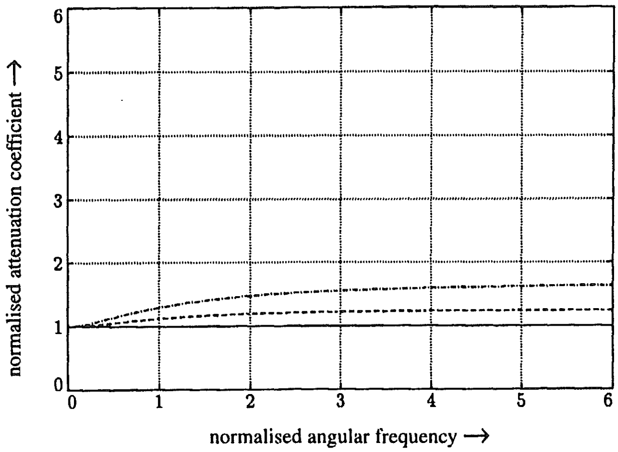
$$\bar{\alpha} \sim 1 \qquad \text{and} \quad \bar{\beta} \sim \bar{\omega}\,\frac{(\omega_c/\omega_h)^{1/2} + (\omega_h/\omega_c)^{1/2}}{2} \qquad \text{as } |\bar{\omega}| \to 0 \tag{27.3-27}$$

and

$$\bar{\alpha} \sim \frac{1}{2}\left[(\omega_c/\omega_h)^{1/2} + (\omega_h/\omega_c)^{1/2}\right] \qquad \text{and } \bar{\beta} \sim \bar{\omega} \qquad \text{as } |\bar{\omega}| \to \infty\,. \tag{27.3-28}$$

Figure 27.3-3 shows $\bar{\alpha}$ and $\bar{\beta}$ as a function of $\bar{\omega}$ with ω_c/ω_h as a parameter.

27-3-3(a)



27-3-3(b)

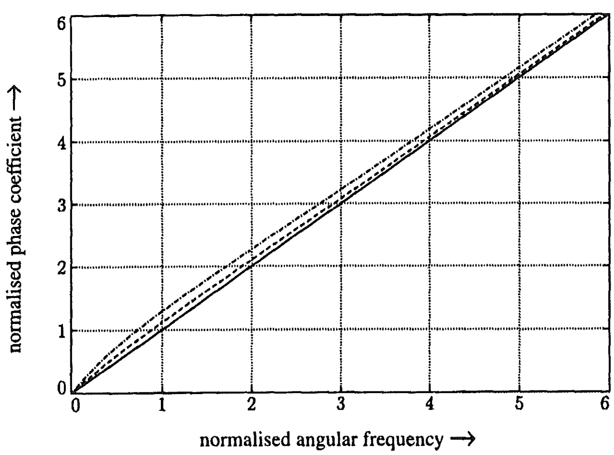


Figure 27.3-3 Normalised attenuation coefficient $\bar{\alpha} = \alpha/(\sigma\Gamma)^{1/2}$ and normalised phase coefficient $\bar{\beta} = \beta/(\sigma\Gamma)^{1/2}$ as a function of normalised angular frequency $\bar{\omega} = \omega/(\omega_c\omega_h)^{1/2}$ with ω_c/ω_h as a parameter for a uniform plane wave in an isotropic medium with conductive electric and linear hysteresis magnetic losses ($\omega_c = \sigma/\varepsilon$, $\omega_h = \Gamma/\mu$). (——): $\omega_c/\omega_h = 1$; (- - -): $\omega_c/\omega_h = 4$; (-·-·-): $\omega_c/\omega_h = 9$.

Propagation in an isotropic medium with conductive electric losses only

As a second example we shall discuss the propagation of plane waves in an isotropic medium with conductive electric losses only. For such a medium, we have (see Equations (24.4-7) and (24.4-8))

$$\hat{\eta}(\mathrm{j}\omega) = \sigma + \mathrm{j}\omega\varepsilon \qquad (27.3\text{-}29)$$

and

$$\hat{\zeta}(\mathrm{j}\omega) = \mathrm{j}\omega\mu \,. \tag{27.3-30}$$

Substitution of Equation (27.3-1) in the corresponding dispersion equation (see Equation (27.1-37)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = (\sigma + \mathrm{j}\omega\varepsilon)\mathrm{j}\omega\mu \,. \tag{27.3-31}$$

Separation of Equation (27.3-31) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2\varepsilon\mu \tag{27.3-32}$$

and

$$2\alpha_s\beta_s = \omega\sigma\mu \,. \tag{27.3-33}$$

From Equation (27.3-32) it is clear that we have $\alpha_s\alpha_s < \beta_s\beta_s$ i.e. phase propagation is, for all frequencies, the predominant phenomenon. Furthermore, Equation (27.3-33) indicates that $\alpha_s\beta_s \neq 0$ for non-zero frequency, since the right-hand side differs from zero for non-zero frequency. Hence, for the type of medium under consideration, the set of planes of equal amplitude is, for non-uniform plane waves, inclined with respect to the set of planes of equal phase (Figure 27.3-4).

For a uniform plane wave in the type of medium under consideration, substitution of Equations (27.3-4) and (27.3-5) in Equations (27.3-32) and (27.3-33) leads to

$$\alpha^2 - \beta^2 = -\omega^2\varepsilon\mu \tag{27.3-34}$$

and

$$2\alpha\beta = \omega\sigma\mu \,, \tag{27.3-35}$$

where the property $\xi_s\xi_s = 1$ has been used. Since α must be non-negative in view of the condition of causality, Equation (27.3-35) indicates that $\beta \gtrless 0$ for $\omega \gtrless 0$. After some algebraic manipulations we obtain (see Exercise 27.3-5)

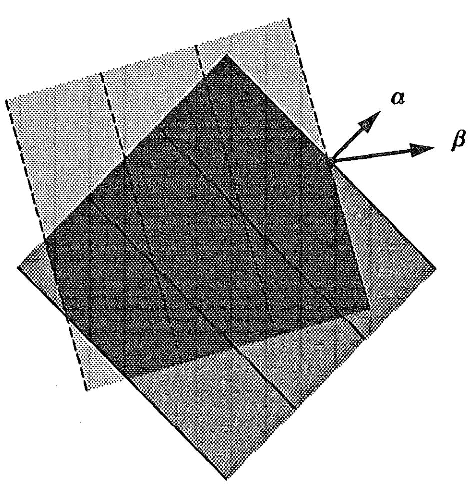


Figure 27.3-4 Planes of equal amplitude (—) and planes of equal phase (- - -) of a non-uniform plane wave in a medium with conductive electric losses only in the real frequency domain.

$$\alpha = \left[\frac{1}{2} \left\{ -\omega^2 \varepsilon\mu + \left[(\omega\sigma\mu)^2 + \omega^4 (\varepsilon\mu)^2 \right]^{1/2} \right\} \right]^{1/2} \tag{27.3-36}$$

and

$$\beta = \pm \left[\frac{1}{2} \left\{ \omega^2 \varepsilon\mu + \left[(\omega\sigma\mu)^2 + \omega^4 (\varepsilon\mu)^2 \right]^{1/2} \right\} \right]^{1/2} \qquad \text{for } \omega \gtrless 0 , \tag{27.3-37}$$

in which all square root expressions are non-negative. For very low and very high frequencies these results yield the asymptotic representations

$$\alpha \sim (|\omega| \mu\sigma/2)^{1/2} \quad \text{and} \quad \beta \sim \pm(|\omega| \mu\sigma/2)^{1/2} \qquad \text{as } \omega \to \pm 0 \tag{27.3-38}$$

and

$$\alpha \sim (\sigma/2)(\mu/\varepsilon)^{1/2} \quad \text{and} \quad \beta \sim \omega(\varepsilon\mu)^{1/2} \qquad \text{as } |\omega| \to \infty , \tag{27.3-39}$$

respectively, where it has been assumed that $\sigma \neq 0$. To put the results in a normalised form, we introduce the critical angular frequency of the conductive electric losses ω_c via

$$\omega_c = \sigma/\varepsilon \tag{27.3-40}$$

and write

$$\alpha = \sigma(\mu/\varepsilon)^{1/2} \bar{\alpha} , \tag{27.3-41}$$

$$\beta = \sigma(\mu/\varepsilon)^{1/2} \bar{\beta} \tag{27.3-42}$$

and

$$\bar{\omega} = \omega/\omega_c , \tag{27.3-43}$$

where $\bar{\alpha}$ is the normalised attenuation coefficient, $\bar{\beta}$ is the normalised phase coefficient, and $\bar{\omega}$ is the normalised angular frequency. In terms of these quantities, Equations (27.3-34) and (27.3-35) become

$$\bar{\alpha}^2 - \bar{\beta}^2 = -\bar{\omega}^2 \tag{27.3-44}$$

and

$$2\bar{\alpha}\bar{\beta} = \bar{\omega} , \tag{27.3-45}$$

with the solution (see Equations (27.3-36) and (27.3-37))

$$\bar{\alpha} = \left[\frac{1}{2} \left(-\bar{\omega}^2 + \{ \bar{\omega}^2 + \bar{\omega}^4 \}^{1/2} \right) \right]^{1/2} \tag{27.3-46}$$

and

$$\bar{\beta} = \pm\bar{\omega} \left[\frac{1}{2} \left(1 + \{ \bar{\omega}^{-2} + 1 \}^{1/2} \right) \right]^{1/2} . \tag{27.3-47}$$

The asymptotic representations Equation (27.3-38) and (27.3-39) become in their normalised form

$$\bar{\alpha} \sim (|\bar{\omega}|/2)^{1/2} \qquad \text{and} \quad \bar{\beta} \sim \pm(|\bar{\omega}|/2)^{1/2} \qquad \text{as } |\bar{\omega}| \to \pm 0 \tag{27.3-48}$$

and

$$\bar{\alpha} \sim \tfrac{1}{2} \quad \text{and } \bar{\beta} \sim \bar{\omega} \quad \text{as } |\bar{\omega}| \rightarrow \infty . \tag{27.3-49}$$

Figure 27.3-5 shows $\bar{\alpha}$ and $\bar{\beta}$ as a function of $\bar{\omega}$.

Propagation in a lossless isotropic medium

As a third example we shall discuss the propagation of plane waves in a lossless isotropic medium. For such a medium, we have (see Equations (24.4-7) and (24.4-8))

$$\hat{\eta}(\mathrm{j}\omega) = \mathrm{j}\omega\varepsilon \tag{27.3-50}$$

and

$$\hat{\zeta}(\mathrm{j}\omega) = \mathrm{j}\omega\mu . \tag{27.3-51}$$

Substitution of Equation (27.3-1) in the corresponding dispersion equation (see Equation (27.1-37)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = -\omega^2 \varepsilon\mu . \tag{27.3-52}$$

Separation of Equation (27.3-52) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2 \varepsilon\mu \tag{27.3-53}$$

and

$$2\alpha_s\beta_s = 0 . \tag{27.3-54}$$

From Equation (27.3-53) it is clear that we have $\alpha_s\alpha_s < \beta_s\beta_s$, i.e. phase propagation is, for all frequencies, the predominant phenomenon. Furthermore, Equation (27.3-54) indicates that $\alpha_s\beta_s = 0$ for all frequencies. Hence, for the type of medium under consideration, the set of planes of equal amplitude is, for non-uniform plane waves, perpendicular to the set of planes of equal phase (Figure 27.3-6).

For a uniform plane wave in the type of medium under consideration, substitution of Equations (27.3-4) and (27.3-5) in Equations (27.3-53) and (27.3-54) leads to

$$\alpha^2 - \beta^2 = -\omega^2 \varepsilon\mu \tag{27.3-55}$$

and

$$2\alpha\beta = 0 , \tag{27.3-56}$$

where the property $\xi_s\xi_s = 1$ has been used. These equations have as their only solution

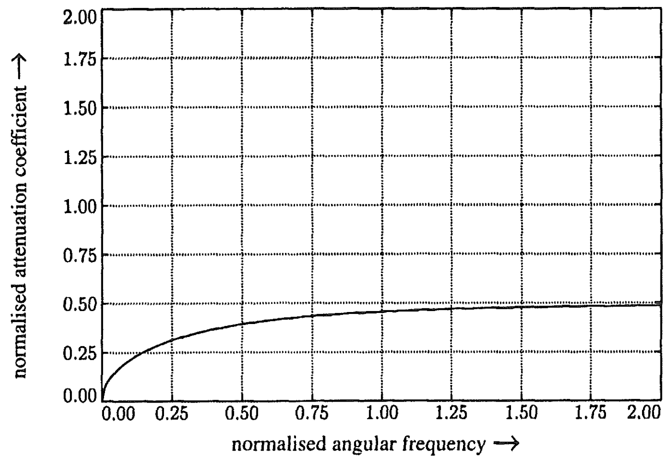
$$\alpha = 0 \tag{27.3-57}$$

and

$$\beta = \omega(\varepsilon\mu)^{1/2} , \tag{27.3-58}$$

in which the square root expression is positive. Hence, in the lossless medium there is no attenuation and the phase coefficient varies linearly with the angular frequency.

27-3-5(a)



27-3-5(b)

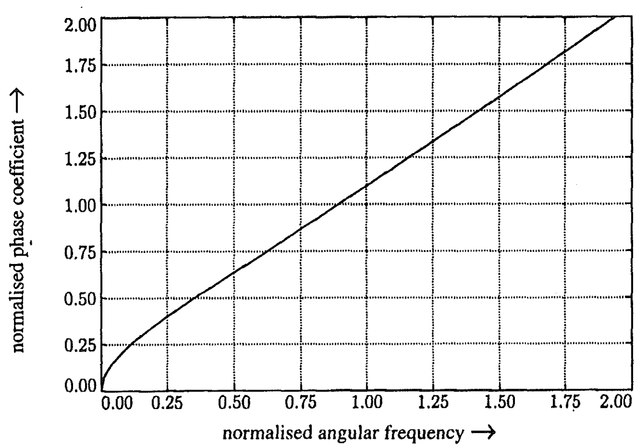


Figure 27.3-5 Normalised attenuation coefficient $\overline{\alpha} = \alpha/\sigma(\mu/\varepsilon)^{1/2}$ and normalised phase coefficient $\overline{\beta} = \beta/\sigma(\mu/\varepsilon)^{1/2}$ as a function of normalised angular frequency $\overline{\omega} = \omega/\omega_c$ for a uniform plane wave in an isotropic medium with conductive electric losses only ($\omega_c = \sigma/\varepsilon$).

Propagation in a collisionless electron plasma in a vacuum background

As a fourth example we shall discuss the propagation of plane waves in a collisionless electron plasma in a vacuum background. For such a medium, we have (see Equations (24.4-13) and (24.4-14))

$$\hat{\eta}(\mathrm{j}\omega) = \mathrm{j}\omega\varepsilon_0\left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) \tag{27.3-59}$$

and

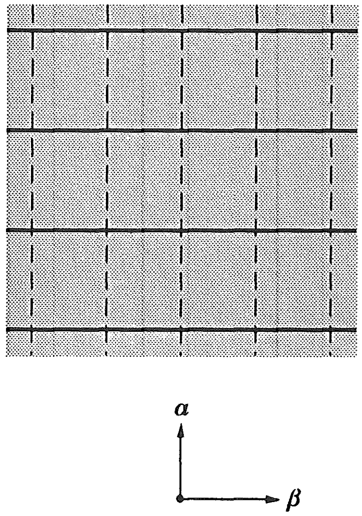


Figure 27.3-6 Planes of equal amplitude (—) and planes of equal phase (- - -) of a non-uniform plane wave in a lossless medium in the real frequency domain.

$$\hat{\zeta}(j\omega) = j\omega\mu_0 \,. \tag{27.3-60}$$

Substitution of Equation (27.3-1) in the corresponding dispersion equation (see Equation (27.1-37)) yields

$$(\alpha_s + j\beta_s)(\alpha_s + j\beta_s) = -\omega^2 \varepsilon_0 \mu_0 \left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) . \tag{27.3-61}$$

Separation of Equation (27.3-61) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2 \varepsilon_0 \mu_0 \left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) \tag{27.3-62}$$

and

$$2\alpha_s\beta_s = 0 \,. \tag{27.3-63}$$

From Equation (27.3-62) it is clear that we have $\alpha_s\alpha_s > \beta_s\beta_s$ for $|\omega| < \omega_{\mathrm{pe}}$, i.e. attenuation is, for all frequencies below the electron plasma frequency, the predominant phenomenon, while $\alpha_s\alpha_s < \beta_s\beta_s$ for $|\omega| > \omega_{\mathrm{pe}}$, i.e. phase propagation is, for all frequencies above the electron plasma frequency, the predominant phenomenon. Furthermore, Equation (27.3-63) indicates that $\alpha_s\beta_s = 0$ for all frequencies. Hence, for the type of medium under consideration, the set of planes of equal amplitude is, for non-uniform plane waves, perpendicular to the set of planes of equal phase (Figure 27.3-7).

For a uniform plane wave in the type of medium under consideration, substitution of Equations (27.3-4) and (27.3-5) in Equations (27.3-62) and (27.3-63) leads to

$$\alpha^2 - \beta^2 = -\omega^2 \varepsilon_0 \mu_0 \left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) \tag{27.3-64}$$

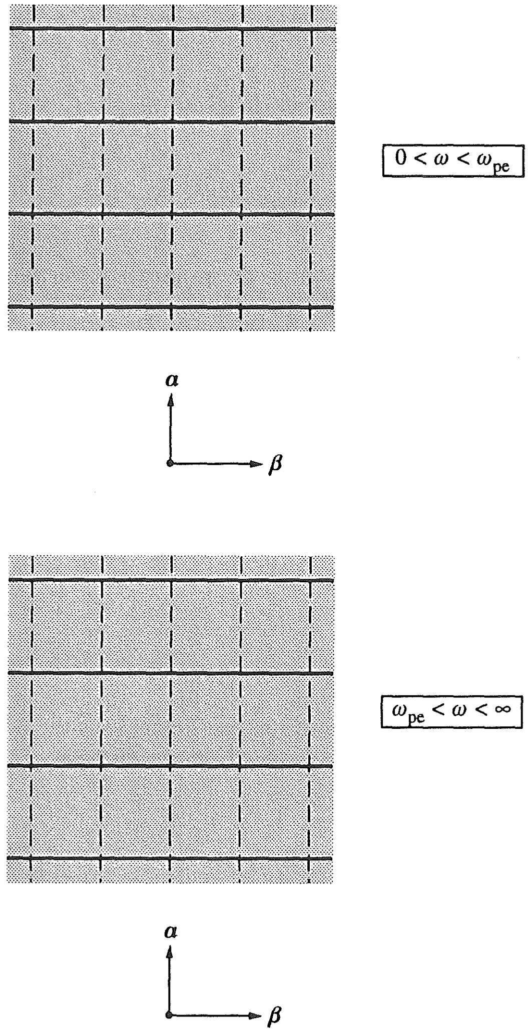


Figure 27.3-7 Planes of equal amplitude (—) and planes of equal phase (- - -) of a non-uniform plane wave in a collisionless electron plasma in a vacuum background in the real frequency domain.

and

$$2\alpha\beta = 0\,, \tag{27.3-65}$$

where the property $\xi_s\xi_s = 1$ has been used. Since α must be non-negative in view of the condition of causality, the solution of these equations is given by

$$\alpha = \begin{cases} |\omega|(\varepsilon_0\mu_0)^{1/2}\left(\dfrac{\omega_{\mathrm{pe}}^2}{\omega^2} - 1\right)^{1/2} & \text{for } 0 \leqslant |\omega| \leqslant \omega_{\mathrm{pe}}\,, \\[2ex] 0 & \text{for } \omega_{\mathrm{pe}} \leqslant |\omega| < \infty \end{cases} \tag{27.3-66}$$

and

$$\beta = \begin{cases} 0 & \text{for } 0 \leqslant |\omega| \leqslant \omega_{pe}, \\ \omega(\varepsilon_0\mu_0)^{1/2}\left(1 - \dfrac{\omega_{pe}^2}{\omega^2}\right)^{1/2} & \text{for } \omega_{pe} \leqslant |\omega| < \infty, \end{cases} \tag{27.3-67}$$

in which all square root expressions are non-negative. For very low and very high frequencies these results yield the asymptotic representations

$$\alpha \sim \omega_{pe}(\varepsilon_0\mu_0)^{1/2} \quad \text{and} \quad \beta = 0 \quad \text{as } |\omega| \to 0 \tag{27.3-68}$$

and

$$\alpha = 0 \quad \text{and} \quad \beta \sim \omega(\varepsilon_0\mu_0)^{1/2} \quad \text{as } |\omega| \to \infty, \tag{27.3-69}$$

respectively. To put the results in a normalised form, we write

$$\alpha = \omega_{pe}(\varepsilon_0\mu_0)^{1/2}\bar{\alpha}, \tag{27.3-70}$$

$$\beta = \omega_{pe}(\varepsilon_0\mu_0)^{1/2}\bar{\beta} \tag{27.3-71}$$

and

$$\bar{\omega} = \omega/\omega_{pe}, \tag{27.3-72}$$

where $\bar{\alpha}$ is the normalised attenuation coefficient, $\bar{\beta}$ is the normalised phase coefficient, and $\bar{\omega}$ is the normalised angular frequency. In terms of these quantities, Equations (27.3-64) and (27.3-65) become

$$\bar{\alpha}^2 - \bar{\beta}^2 = -(\bar{\omega}^2 - 1) \tag{27.3-73}$$

and

$$2\bar{\alpha}\bar{\beta} = 0, \tag{27.3-74}$$

with the solution (see Equations (27.3-66) and (27.3-67))

$$\bar{\alpha} = \begin{cases} (1 - \bar{\omega}^2)^{1/2} & \text{for } 0 \leqslant |\bar{\omega}| \leqslant 1, \\ 0 & \text{for } 1 \leqslant |\bar{\omega}| < \infty \end{cases} \tag{27.3-75}$$

and

$$\bar{\beta} = \begin{cases} 0 & \text{for } 0 \leqslant |\bar{\omega}| \leqslant 1, \\ \bar{\omega}(1 - \bar{\omega}^{-2})^{1/2} & \text{for } 1 \leqslant |\bar{\omega}| \leqslant \infty. \end{cases} \tag{27.3-76}$$

The asymptotic representations Equation (27.3-68) and (27.3-69) become in their normalised form

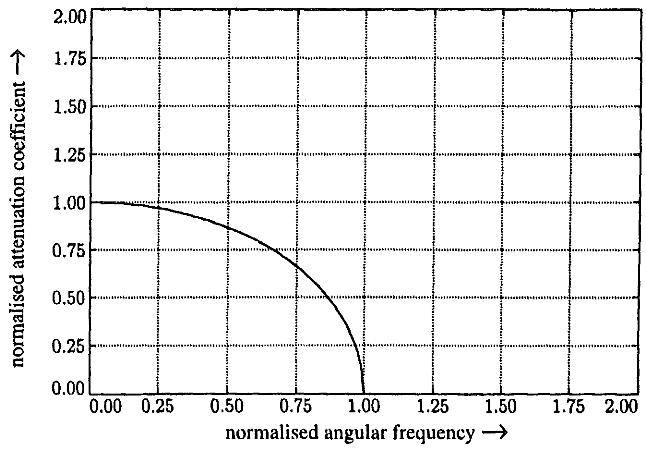
$$\bar{\alpha} \sim 1 \quad \text{and} \quad \bar{\beta} = 0 \quad \text{as } |\bar{\omega}| \to 0 \tag{27.3-77}$$

and

$$\bar{\alpha} = 0 \quad \text{and} \quad \bar{\beta} \sim \bar{\omega} \quad \text{as } |\bar{\omega}| \to \infty. \tag{27.3-78}$$

Figure 27.3-8 shows $\bar{\alpha}$ and $\bar{\beta}$ as a function of $\bar{\omega}$.

27-3-8(a)



27-3-8(b)

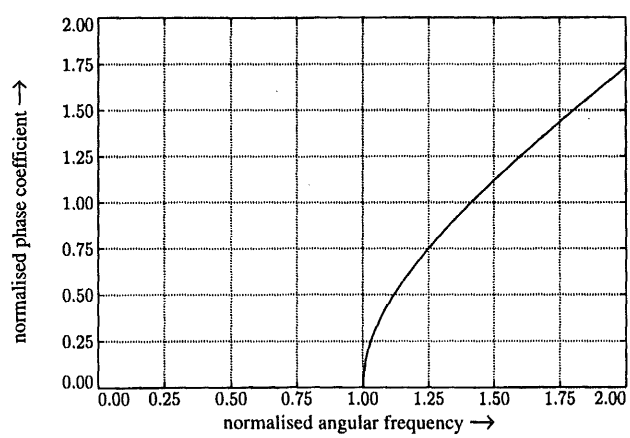


Figure 27.3-8 Normalised attenuation coefficient $\bar{\alpha} = \alpha/\omega_{pe}(\varepsilon_0\mu_0)^{1/2}$ and normalised phase coefficient $\bar{\beta} = \beta/\omega_{pe}(\varepsilon_0\mu_0)^{1/2}$ as a function of normalised angular frequency $\bar{\omega} = \omega/\omega_{pe}$ for a uniform plane wave in a collisionless electron plasma in a vacuum background (ω_{pe} = electron plasma angular frequency).

Propagation in a dielectric with a Lorentzian absorption line

As a fifth example we shall discuss the propagation of plane waves in a dielectric medium with a Lorentzian absorption line. For such a medium, we have (see Equation (24.4-40))

$$\hat{\eta}(\mathrm{j}\omega) = \mathrm{j}\omega\varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2 - \mathrm{j}\omega\Gamma - \omega_0^2 + \omega_p^2/3}\right) \tag{27.3-79}$$

and

$$\hat{\zeta}(\mathrm{j}\omega) = \mathrm{j}\omega\mu_0\,. \tag{27.3-80}$$

Substitution of Equation (27.3-1) in the corresponding dispersion equation (see Equation (27.1-37)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = -\omega^2 \varepsilon_0 \mu_0 \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2 - \mathrm{j}\omega\Gamma - \omega_0^2 + \omega_\mathrm{p}^2/3} \right). \tag{27.3-81}$$

Separation of Equation (27.3-81) into its real and imaginary parts leads to

$$\alpha_s \alpha_s - \beta_s \beta_s = -\omega^2 \varepsilon_0 \mu_0 \left(1 - \frac{\omega_\mathrm{p}^2 (\omega^2 - \omega_0^2 + \omega_\mathrm{p}^2/3)}{(\omega^2 - \omega_0^2 + \omega_\mathrm{p}^2/3)^2 + \omega^2 \Gamma^2} \right) \tag{27.3-82}$$

and

$$2\alpha_s \beta_s = \omega^2 \varepsilon_0 \mu_0 \frac{\omega_\mathrm{p}^2 \omega \Gamma}{(\omega^2 - \omega_0^2 + \omega_\mathrm{p}^2/3)^2 + \omega^2 \Gamma^2}. \tag{27.3-83}$$

For all cases met in practice, $\omega_\mathrm{p} \ll \omega_0$ and $\Gamma/2 < (\omega_0^2 - \omega_\mathrm{p}^2/3)^{1/2}$. From Equation (27.3-82) it then follows that except for values of the angular frequency ω in the neighbourhood of the angular resonant frequency ω_0, we have $\alpha_s\alpha_s < \beta_s\beta_s$, i.e. for all frequencies away from the resonant frequency of the atomic system, phase propagation is the predominant phenomenon. Furthermore, Equation (27.3-83) indicates that, for the type of medium under consideration, the set of planes of equal amplitude is, for non-uniform plane waves, inclined with respect to the set of planes of equal phase (Figure 27.3-9).

For a uniform plane wave in the type of medium under consideration, substitution of Equations (27.3-4) and (27.3-5) in Equations (27.3-82) and (27.3-83) leads to

$$\alpha^2 - \beta^2 = -\omega^2 \varepsilon_0 \mu_0 \left(1 - \frac{\omega_\mathrm{p}^2 (\omega^2 - \omega_0^2 + \omega_\mathrm{p}^2/3)}{(\omega^2 - \omega_0^2 + \omega_\mathrm{p}^2/3)^2 + \omega^2 \Gamma^2} \right) \tag{27.3-84}$$

and

$$2\alpha\beta = \omega^2 \varepsilon_0 \mu_0 \frac{\omega_\mathrm{p}^2 \omega \Gamma}{(\omega^2 - \omega_0^2 + \omega_\mathrm{p}^2/3)^2 + \omega^2 \Gamma^2}, \tag{27.3-85}$$

where the property $\xi_s \xi_s = 1$ has been used. From these equations, α and β can be solved. The expressions are rather cumbersome and are not reproduced here. In a normalised form, graphs of α and β as a function of angular frequency will be given. To this end, we write

$$\alpha = \omega_0 (\varepsilon_0 \mu_0)^{1/2} \bar{\alpha}, \tag{27.3-86}$$

$$\beta = \omega_0 (\varepsilon_0 \mu_0)^{1/2} \bar{\beta}, \tag{27.3-87}$$

and

$$\bar{\omega} = \omega/\omega_0, \tag{27.3-88}$$

where $\bar{\alpha}$ is the normalised attenuation coefficient, $\bar{\beta}$ is the normalised phase coefficient, and $\bar{\omega}$ is the normalised angular frequency, whereas

$$\bar{\Gamma} = \Gamma/\omega_0 \tag{27.3-89}$$

will occur as a normalised damping coefficient and

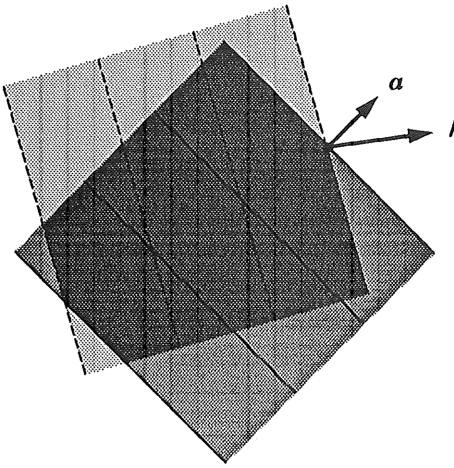


Figure 27.3-9 Planes of equal amplitude (—) and planes of equal phase (- - -) of a non-uniform plane wave in a dielectric with a Lorentzian absorption line in the real frequency domain.

$$\bar{\omega}_p = \omega_p/\omega_0 \,. \tag{27.3-90}$$

as a normalised angular plasma frequency. In terms of these quantities, Equations (27.3-84) and (27.3-85) become

$$\bar{\alpha}^2 - \bar{\beta}^2 = -\bar{\omega}^2\left(1 - \frac{\bar{\omega}_p^2(\bar{\omega}^2 - 1 + \bar{\omega}_p^2/3)}{(\bar{\omega}^2 - 1 + \bar{\omega}_p^2/3)^2 + \bar{\omega}^2\bar{\Gamma}^2}\right) \tag{27.3-91}$$

and

$$2\bar{\alpha}\bar{\beta} = \bar{\omega}^2\left(\frac{\bar{\omega}_p^2\bar{\omega}\bar{\Gamma}}{(\bar{\omega}^2 - 1 + \bar{\omega}_p^2/3)^2 + \bar{\omega}^2\bar{\Gamma}^2}\right). \tag{27.3-92}$$

Figure 27.3-10 shows $\bar{\alpha}$ and $\bar{\beta}$ as a function of $\bar{\omega}$.

Propagation in an isotropic metal conductor

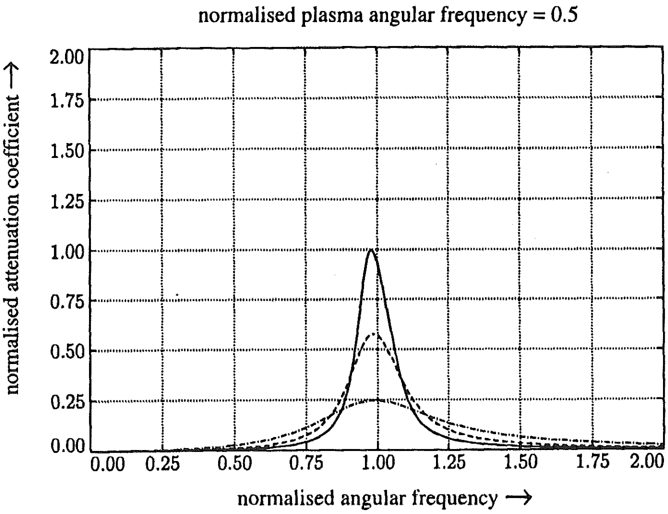
As a sixth example we shall discuss the propagation of plane waves in an isotropic metal conductor. For such a medium, we have (see Equations (24.4-13) and (24.4-14))

$$\hat{\eta}(\mathrm{j}\omega) = \frac{\varepsilon_0\omega_{pe}^2}{\mathrm{j}\omega + \nu_c} + \mathrm{j}\omega\varepsilon_0 \tag{27.3-93}$$

and

$$\hat{\zeta}(\mathrm{j}\omega) = \mathrm{j}\omega\mu_0 \,. \tag{27.3-94}$$

27-3-10(a)



27-3-10(b)

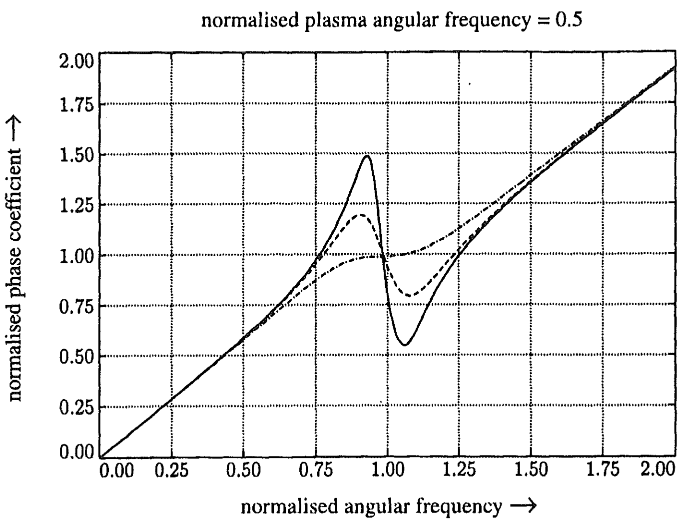


Figure 27.3-10 Normalised attenuation coefficient $\bar{\alpha} = \alpha/\omega_0(\varepsilon_0\mu_0)^{1/2}$ and normalised phase coefficient $\bar{\beta} = \beta/\omega_0(\varepsilon_0\mu_0)^{1/2}$ as a function of normalised angular frequency $\bar{\omega} = \omega/\omega_0$ for a uniform plane wave in a dielectric with a Lorentzian absorption line with normalized damping coefficient $\bar{\Gamma} = \Gamma/\omega_0$ and normalised plasma angular frequency $\bar{\omega}_p = \omega_p/\omega_0$. (——): $\omega_p/\omega_0 = 0.5$, $\Gamma/\omega_0 = 0.1$; (- - -): $\omega_p/\omega_0 = 0.5$, $\Gamma/\omega_0 = 0.2$; (-·-·-): $\omega_p/\omega_0 = 0.5$, $\Gamma/\omega_0 = 0.5$.

Substitution of Equation (27.3-1) in the corresponding dispersion equation (see Equation (27.1-37)) yields

$$(\alpha_s + \mathrm{j}\beta_s)(\alpha_s + \mathrm{j}\beta_s) = -\omega^2\varepsilon_0\mu_0\left(1 + \frac{\omega_{pe}^2}{-\omega^2 + \mathrm{j}\omega\nu_c}\right). \tag{27.3-95}$$

Separation of Equation (27.3-95) into its real and imaginary parts leads to

$$\alpha_s\alpha_s - \beta_s\beta_s = -\omega^2\varepsilon_0\mu_0\left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2 + \nu_{\mathrm{c}}^2}\right) \tag{27.3-96}$$

and

$$2\alpha\beta = \omega^2\varepsilon_0\mu_0\frac{\nu_{\mathrm{c}}\omega_{\mathrm{pe}}^2}{\omega(\omega^2 + \nu_{\mathrm{c}}^2)}\,. \tag{27.3-97}$$

Now, for a good conductor certainly $\nu_c < \omega_{\mathrm{pe}}$ and usually $\nu_c \ll \omega_{\mathrm{pe}}$. Under these conditions it follows from Equation (27.3-96) that for $\omega^2 < \omega_{\mathrm{pe}}^2 - \nu_{\mathrm{c}}^2$ we have $\alpha_s\alpha_s > \beta_s\beta_s$, i.e. attenuation is the predominant phenomenon, while for $\omega^2 > \omega_{\mathrm{pe}}^2 - \nu_{\mathrm{c}}^2$ we have $\alpha_s\alpha_s < \beta_s\beta_s$, i.e. phase propagation is the predominant phenomenon. Furthermore, Equation (27.3-97) indicates that $\alpha_s\beta_s \neq 0$ for non-zero frequency. Hence, for the type of medium under consideration, the set of planes of equal amplitude is, for non-uniform plane waves, inclined with respect to the set of planes of equal phase (Figure 27.3-11).

For a uniform plane wave in the type of medium under consideration, substitution of Equations (27.3-4) and (27.3-5) in Equations (27.3-96) and (27.3-97) leads to

$$\alpha^2 - \beta^2 = -\omega^2\varepsilon_0\mu_0\left(1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2 + \nu_{\mathrm{c}}^2}\right) \tag{27.3-98}$$

and

$$2\alpha\beta = \omega^2\varepsilon_0\mu_0\frac{\nu_{\mathrm{c}}\omega_{\mathrm{pe}}^2}{\omega(\omega^2 + \nu_{\mathrm{c}}^2)}\,, \tag{27.3-99}$$

where the property $\xi_s\xi_s = 1$ has been used. Since α must be non-negative in view of the condition of causality, Equation (27.3-99) indicates that $\beta \gtrless 0$ for $\omega \gtrless 0$. From these equations, α and β can be solved. The expressions are rather cumbersome and are not reproduced here. In a normalised form, graphs of α and β as a function of angular frequency will be given. To this end, we write

$$\alpha = \nu_{\mathrm{c}}(\varepsilon_0\mu_0)^{1/2}\bar{\alpha}\,, \tag{27.3-100}$$

$$\beta = \nu_{\mathrm{c}}(\varepsilon_0\mu_0)^{1/2}\bar{\beta}\,, \tag{27.3-101}$$

and

$$\bar{\omega} = \omega/\nu_{\mathrm{c}}\,, \tag{27.3-102}$$

where $\bar{\alpha}$ is the normalised attenuation coefficient, $\bar{\beta}$ is the normalised phase coefficient, and $\bar{\omega}$ is the normalised angular frequency, whereas

$$\bar{\omega}_{\mathrm{pe}} = \omega_{\mathrm{pe}}/\nu_{\mathrm{c}} \tag{27.3-103}$$

will occur as a normalised angular plasma frequency. In terms of these quantities, Equations (27.3-98) and (27.3-99) become

$$\bar{\alpha}^2 - \bar{\beta}^2 = -\bar{\omega}^2\left(1 - \frac{\bar{\omega}_{\mathrm{pe}}^2}{\bar{\omega}^2 + 1}\right) \tag{27.3-104}$$

and

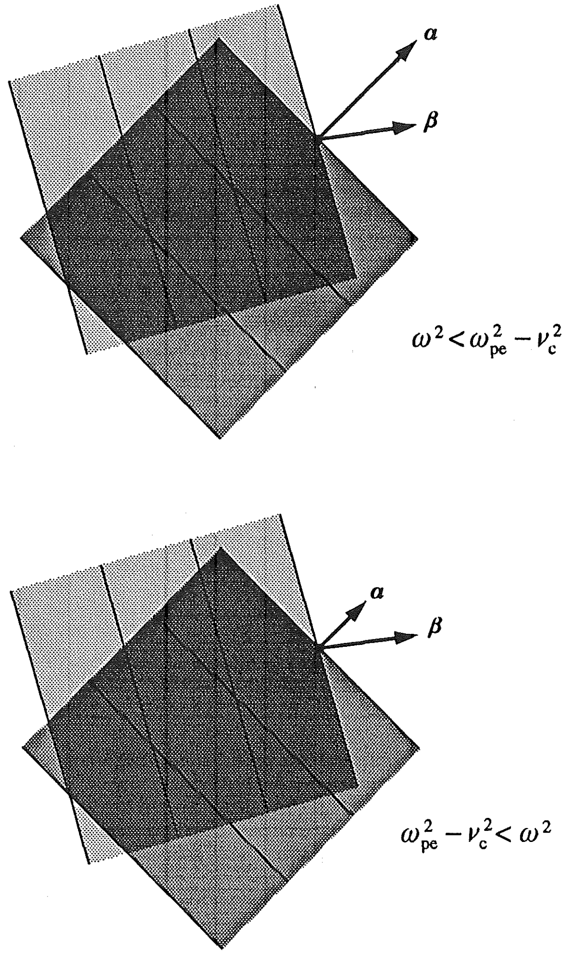


Figure 27.3-11 Planes of equal amplitude (——) and planes of equal phase (- - -) of a non-uniform plane wave in a metal conductor in the real frequency domain.

$$2\bar{\alpha}\bar{\beta} = \bar{\omega}^2 \frac{\bar{\omega}_{pe}^2}{\bar{\omega}(\bar{\omega}^2 + 1)} . \qquad (27.3\text{-}105)$$

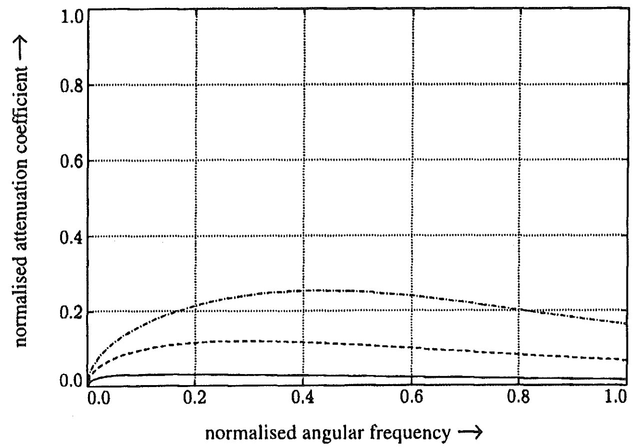
Figure 27.3-12 shows $\bar{\alpha}$ and $\bar{\beta}$ as a function of $\bar{\omega}$.

Exercises

Exercise 27.3-1

Derive Equations (27.3-14) and (27.3-15) from Equations (27.3-12) and (27.3-13). (*Hint*: Derive expressions for α^2 and β^2 from Equation (27.3-12) and the sum of the squared versions of Equations (27.3-12) and (27.3-13).)

27-3-12(a)



27-3-12(b)

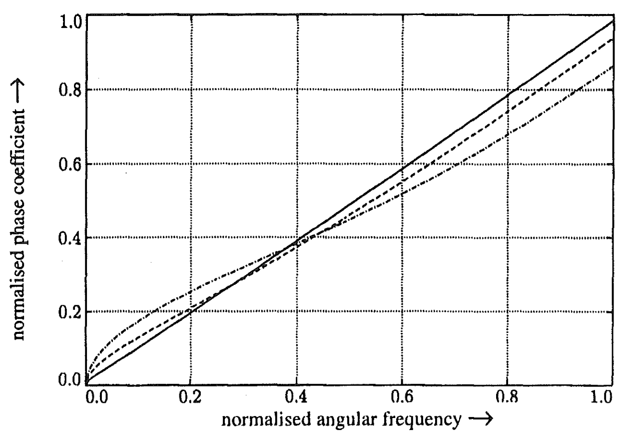


Figure 27.3-12 Normalised attenuation coefficient $\overline{\alpha} = \alpha/\nu_c(\varepsilon_0\mu_0)^{1/2}$ and normalised phase coefficient $\overline{\beta} = \beta/\nu_c(\varepsilon_0\mu_0)^{1/2}$ as a function of normalised angular frequency $\overline{\omega} = \omega/\nu_c$ for a uniform plane wave in a metal conductor with normalised plasma angular frequency $\overline{\omega}_{pe} = \omega_{pe}/\nu_c$. (——): $\omega_{pe}/\nu_c = 0.25$; (- - -): $\omega_{pe}/\nu_c = 0.50$; (-·-·-): $\omega_{pe}/\nu_c = 0.75$.

Exercise 27.3-2

Derive the asymptotic representations given in Equations (27.3-16) and (27.3-17) from Equations (27.3-14) and (27.3-15) and check that the result is in accordance with Equations (27.3-12) and (27.3-13).

Exercise 27.3-3

Derive Equations (27.3-25) and (27.3-26) from Equations (27.3-14) and (27.3-15) by using Equations (27.3-20)–(27.3-22).

Exercise 27.3-4

Derive Equations (27.3-27) and (27.3-28) from Equations (27.3-16) and (27.3-17) by using Equations (27.3-20)–(27.3-22).

Exercise 27.3-5

Derive Equations (27.3-36) and (27.3-37) from Equations (27.3-34) and (27.3-35). (*Hint*: Derive expressions for α^2 and β^2 from Equation (27.3-34) and the sum of the squared versions of Equations (27.3-34) and (27.3-35).)

Exercise 27.3-6

Derive the asymptotic representations given in Equations (27.3-38) and (27.3-39) from Equations (27.3-36) and (27.3-37) and check that the result is in accordance with Equations (27.3-34) and (27.3-35).

Exercise 27.3-7

Derive Equations (27.3-46) and (27.3-47) from Equations (27.3-36) and (27.3-37) by using Equations (27.3-40)–(27.3-42).

Exercise 27.3-8

Derive Equations (27.3-48) and (27.3-49) from Equations (27.3-38) and (27.3-39) by using Equations (27.3-40)–(27.3-42).

27.4 Time-domain uniform plane waves in an isotropic, lossless medium

Time-domain *uniform plane waves* in an isotropic, lossless medium are solutions of the source-free electromagnetic field equations (see Equations (18.3-11) amd (18.3-12), and Equations (19.1-4) and (19.1-5))

$$-\varepsilon_{k,m,p}\partial_m H_p + \varepsilon\partial_t E_k = 0\,, \tag{27.4-1}$$

$$\varepsilon_{j,n,r}\partial_n E_r + \mu\partial_t H_j = 0\,, \tag{27.4-2}$$

of the form

$$\{E_k, H_j\} = \{e_k, h_j\}(t - \Lambda\xi_s x_s)\,, \tag{27.4-3}$$

in which ξ_s is the *unit vector in the direction of propagation* of the wave, Λ is its *slowness*, and $e_k(t)$ and $h_j(t)$ are the *time-domain polarisation vectors* (*"signatures"*) of the electric field strength E_k and the magnetic field strength H_j, respectively. Substitution of Equation (27.4-3) in Equations (27.4-1) and (27.4-2) leads, in view of the relation

$$\partial_m\{e_k, h_j\}(t - \Lambda\xi_s x_s) = -\Lambda\xi_m\partial_t\{e_k, h_j\}(t - \Lambda\xi_s x_s)\,, \tag{27.4-4}$$

to

$$\Lambda\varepsilon_{k,m,p}\xi_m\partial_t h_p + \varepsilon\partial_t e_k = 0\,, \tag{27.4-5}$$

$$-\Lambda\varepsilon_{j,n,r}\xi_n\partial_t e_r + \mu\partial_t h_j = 0\,. \tag{27.4-6}$$

Since Equations (27.4-5) and (27.4-6) have to be satisfied for all values of t at each position x of the source-free domain under consideration, all components of $\partial_t e_k$ and $\partial_t h_j$ must have a common pulse shape. In view of this, and of the condition of causality by which e_k and h_j have the value zero prior to some instant in the finite past, all components of e_k and h_j must have a common pulse shape. In view of this, Equation (27.4-3) is replaced by

$$\{E_k,H_j\} = \{e_k,h_j\}a(t-\Lambda\xi_s x_s)\,, \tag{27.4-7}$$

where $\{e_k,h_j\}$ are now the time-independent polarisation vectors of $\{E_k,H_j\}$, and $a(t)$ is the somehow normalised pulse shape of the wave motion. With Equation (27.4-7), Equations (27.4-5) and (27.4-6) reduce to

$$\Lambda\varepsilon_{k,m,p}\xi_m h_p + \varepsilon e_k = 0\,, \tag{27.4-8}$$

$$-\Lambda\varepsilon_{j,n,r}\xi_n e_r + \mu h_j = 0\,. \tag{27.4-9}$$

Contraction of Equation (27.4-8) with ξ_k leads, in view of the relation $\xi_k\varepsilon_{k,m,p}\xi_m = 0$, to the compatibility relation

$$\xi_k e_k = 0\,; \tag{27.4-10}$$

contraction of Equation (27.4-9) with ξ_j leads, in view of the relation $\xi_j\varepsilon_{j,n,r}\xi_n = 0$, to the compatibility relation

$$\xi_j h_j = 0\,. \tag{27.4-11}$$

Substituting the expression

$$h_p = \mu^{-1}\Lambda\varepsilon_{p,n,r}\xi_n e_r \tag{27.4-12}$$

that follows from Equation (27.4-9), in Equation (27.4-10) and using the property $\varepsilon_{k,m,p}\varepsilon_{p,n,r} = \delta_{k,n}\delta_{m,r} - \delta_{k,r}\delta_{m,n}$ together with the compatibility relation of Equation (27.4-10), we end up with

$$(\Lambda^2 - \varepsilon\mu)e_k = 0\,. \tag{27.4-13}$$

Similarly, substituting the expression

$$e_r = -\varepsilon^{-1}\varepsilon_{r,m,p}\xi_m h_p \tag{27.4-14}$$

that follows from Equation (27.4-8), in Equation (27.4-9) and using the property $\varepsilon_{j,n,r}\varepsilon_{r,m,p} = \delta_{j,m}\delta_{n,p} - \delta_{j,p}\delta_{n,m}$ together with the compatibility relation of Equation (27.4-11), we end up with

$$(\Lambda^2 - \varepsilon\mu)h_j = 0\,. \tag{27.4-15}$$

Both, Equations (27.4-13) and (27.4-15), lead to the equation that the slowness must satisfy, viz.

$$\Lambda^2 = \varepsilon\mu\,, \tag{27.4-16}$$

with the solution

$$\Lambda = (\varepsilon\mu)^{1/2} \qquad \text{with } (...)^{1/2} > 0\,, \tag{27.4-17}$$

or

$$\Lambda = 1/c\,, \tag{27.4-18}$$

where

$$c = (\varepsilon\mu)^{-1/2} \qquad \text{with } (...)^{-1/2} > 0 \tag{27.4-19}$$

is the electromagnetic wave speed for a wave propagating in the direction of ξ_s, i.e. for a wave whose spatial argument $\Lambda\xi_s x_s$ increases along ξ_s. For the value of Λ as given by Equation (27.4-17), Equations (27.4-8) and (27.4-9) can be rewritten as

$$\varepsilon_{k,m,p}\xi_m h_p + Y e_k = 0\,, \tag{27.4-20}$$

$$-\varepsilon_{j,n,r}\xi_n e_r + Z h_j = 0\,, \tag{27.4-21}$$

in which

$$Y = (\varepsilon/\mu)^{1/2} \qquad \text{with } (...)^{1/2} > 0 \tag{27.4-22}$$

is the *scalar plane wave admittance of the wave* and

$$Z = (\mu/\varepsilon)^{1/2} \qquad \text{with } (...)^{1/2} > 0 \tag{27.4-23}$$

is the *scalar plane wave impedance* of the wave.

From the compatibility relations of Equations (27.4-10) and (27.4-11) it is clear that in an isotropic lossless medium the electric field strength and the magnetic field strength of a uniform plane wave are *transverse* with respect to the direction propagation.

For the Poynting vector (see Equation (21.3-8))

$$S_s = \varepsilon_{s,k,j} E_k H_j \tag{27.4-24}$$

of the uniform plane wave, two alternative expressions can be obtained from Equations (27.4-7), (27.4-20) and (27.4-21). First, upon contracting Equation (27.4-20) with $\varepsilon_{s,k,j}h_j$ and using the identity $\varepsilon_{s,k,j}\varepsilon_{k,m,p} = -\delta_{s,m}\delta_{j,p} + \delta_{s,p}\delta_{j,m}$ together with the compatibility relation $\xi_m h_m = 0$, we obtain

$$S_s = c\varepsilon E_k E_k \xi_s\,. \tag{27.4-25}$$

Secondly, upon contracting Equation (27.4-21) with $\varepsilon_{s,k,j}e_k$ and using the identity $\varepsilon_{s,k,j}\varepsilon_{j,n,r} = \delta_{s,n}\delta_{k,r} - \delta_{s,r}\delta_{k,n}$ together with the compatibility relation $\xi_n e_n = 0$, we obtain

$$S_s = c\mu H_j H_j \xi_s\,. \tag{27.4-26}$$

Upon comparing Equation (27.4-25) with Equation (27.4-26), we conclude that

$$\tfrac{1}{2}\,\varepsilon E_k E_k = \tfrac{1}{2}\,\mu H_j H_j\,, \tag{27.4-27}$$

i.e. the volume density of electric field energy is equal to the volume density of magnetic field energy in the uniform plane wave. Using this in Equation (27.4-24), we can write

$$S_s = c\left[\tfrac{1}{2}\,\varepsilon E_k E_k + \tfrac{1}{2}\,\mu H_j H_j\right]\xi_s\,. \tag{27.4-28}$$

This result leads to the picture that for a uniform plane electromagnetic wave in an isotropic, lossless medium, the Poynting vector carries the sum of the volume densities of electric field energy and magnetic field energy, with the electromagnetic wave speed, in its direction of propagation.

Exercises

Exercise 27.4-1

Construct the one-dimensional wave solutions of the source-free time-domain electromagnetic field equations in a homogeneous, isotropic and lossless medium with permittivity ε and permeability μ by taking a Cartesian reference frame such that the propagation takes place along the x_3-direction. Assume that only E_1 differs from zero and write down the expression for E_1 for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3. Express, for the two cases, the non-vanishing components of H_j in terms of E_1.

Answer:

(a) $E_1 = e_1 a(t - x_3/c),\ H_2 = (\varepsilon/\mu)^{1/2} E_1$;

(b) $E_1 = e_1 a(t + x_3/c),\ H_2 = -(\varepsilon/\mu)^{1/2} E_1$.

Here, $c = (\varepsilon\mu)^{-1/2}$.

Exercise 27.4-2

Construct the one-dimensional wave solutions of the source-free time-domain electromagnetic field equations in a homogeneous, isotropic and lossless medium with permittivity ε and permeability μ by taking a Cartesian reference frame such that the propagation takes place along the x_3-direction. Assume that only E_2 differs from zero and write down the expression for E_2 for (a) propagation in the direction of increasing x_3, (b) propagation in the direction of decreasing x_3. Express, for the two cases, the non-vanishing components of H_j in terms of E_2.

Answer:

(a) $E_2 = e_2 a(t - x_3/c),\ H_1 = -(\varepsilon/\mu)^{1/2} E_2$;

(b) $E_2 = e_2 a(t + x_3/c),\ H_1 = (\varepsilon/\mu)^{1/2} E_2$.

Here, $c = (\varepsilon\mu)^{-1/2}$.

Exercise 27.4-3

Determine the value of (a) the plane wave impedance Z_0, (b) the plane wave admittance Y_0 *in vacuo*.

Answer

(a) $Z_0 = 376.73031$ ohm; (b) $Y_0 = 2.6544187 \times 10^{-3}$ siemens.

Exercise 27.4-4

For a uniform plane wave, periodicity in time entails periodicity in space. Let T denote the time period of the wave and let $f = 1/T$ be its (fundamental) frequency. Show, with the aid of Equation (27.4-7) that the spatial period λ in the direction of propagation of the wave is related to T or f via

$$\lambda = cT = c/f \,. \qquad (27.4\text{-}29)$$

(The quantity λ is denoted as the *wavelength* of the time-periodic plane wave.)

Exercise 27.4-5

Determine the wavelength of a time-periodic plane electromagnetic wave in vacuo ($c_0 \approx 3 \times 10^8$ m/s) if this wave has a frequency of (a) $f = 1$ Hz, (b) $f = 1$ kHz, (c) $f = 1$ MHz, (d) $f = 1$ GHz, and (e) $f = 1$ THz.

Answers: (a) $\lambda = 300$ Mm, (b) $\lambda = 300$ km, (c) $\lambda = 300$ m, (d) $\lambda = 300$ mm, (e) $\lambda = 300$ μm.