28

Electromagnetic reciprocity theorems and their applications

In this chapter we discuss the basic reciprocity theorems for electromagnetic wave fields in time-invariant configurations, together with a variety of their applications. The theorems will be presented both in the time domain and in the complex frequency domain. In view of the time invariance of the configurations to be considered, there exist two versions of the theorems as far as their operations on the time coordinate are concerned, viz. a version that is denoted as the *time convolution type* and a version that is denoted as the *time correlation type.* The two versions are related via a time inversion operation. Each of the two versions has its counterpart in the complex frequency domain.

The application of the theorems to the reciprocity in *transmitting/receiving properties of electromagnetic sources and receivers (antennas),* and to the formulations of the *direct (forward) source* and *inverse source* and the *direct (forward) scattering* and *inverse scattering problems* will be discussed. Furthermore, it is indicated how the theorems lead, in a natural way, to the *integral equation formulation* of electromagnetic wave-field problems for numerical implementation. Finally, it is shown how the reciprocity theorems lead to a mathematical formulation of *Huygens' principle* and of the *Ewald–Oseen extinction theorem.*

28.1 The nature of the reciprocity theorems and the scope of their consequences

A reciprocity theorem interrelates, in a specific manner, the field or wave quantities that characterise two admissible states that could occur in one and the same time-invariant domain $\mathcal{D}\subset\mathcal{R}^3$ in space. Each of the two states can be associated with its own set of time-invariant medium parameters and its own set of source distributions. It is assumed that the media in the two states are linear in their electromagnetic behaviour, i.e. the medium parameters are independent of the values of the field or wave quantities. The domain $\mathcal{D}$ to which the reciprocity theorems apply, may be bounded or unbounded. The application to unbounded domains will always be handled as a limiting case where the boundary surface $\partial\mathcal{D}$ of $\mathcal{D}$ recedes (partially or entirely) to infinity.

From the pertaining electromagnetic field equations, first the *local form* of a reciprocity theorem will be derived, which form applies to each point of any subdomain of $\mathcal{D}$ where the electromagnetic field quantities are continuously differentiable. By integrating the local form over such subdomains and adding the results, the *global form* of the reciprocity theorem is arrived at. In it, a boundary integral over $\partial\mathcal{D}$ occurs, the integrand of which always contains the unit vector ν_m along the normal to $\partial\mathcal{D}$, oriented away from $\mathcal{D}$ (Figure 28.1-1).

The two states will be denoted by the superscripts A and B. The construction of the time-domain reciprocity theorems will be based on the electromagnetic field equations (see Equations (18.3-7) and (18.3-8), and (19.3-8)–(19.3-10))

$$-\varepsilon_{k,m,p}\partial_m H_p^{\mathrm{A}} + \partial_t \mathrm{C}_t(\varepsilon_{k,r}^{\mathrm{A}}, E_r^{\mathrm{A}}; x,t) = -J_k^{\mathrm{A}}, \tag{28.1-1}$$

$$\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{A}} + \partial_t \mathrm{C}_t(\mu_{j,p}^{\mathrm{A}}, H_p^{\mathrm{A}}; x,t) = -K_j^{\mathrm{A}}, \tag{28.1-2}$$

for state A, and

$$-\varepsilon_{k,m,p}\partial_m H_p^{\mathrm{B}} + \partial_t \mathrm{C}_t(\varepsilon_{k,r}^{\mathrm{B}}, E_r^{\mathrm{B}}; x,t) = -J_k^{\mathrm{B}}, \tag{28.1-3}$$

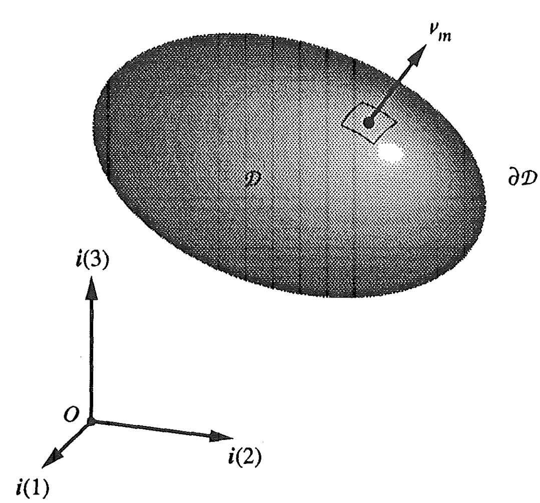
$$\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{B}} + \partial_t \mathrm{C}_t(\mu_{j,p}^{\mathrm{B}}, H_p^{\mathrm{B}}; x,t) = -K_j^{\mathrm{B}}, \tag{28.1-4}$$

for state B, where C_t denotes the time convolution operator (see Equation (B.1-11)) (Figure 28.1-2). In the incorporation of the medium properties we have refrained from introducing a separate symbol for the electric or magnetic relaxation functions and use for them the same symbol as for the instantaneous constitutive parameters. In each of the cases, the meaning will be clear from the context.

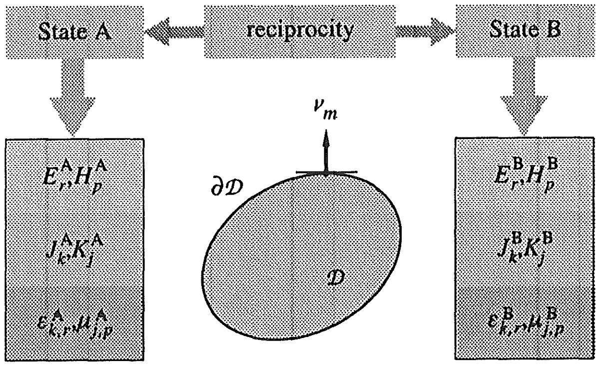
If, in $\mathcal{D}$, either surfaces of discontinuity in electromagnetic properties or electromagnetically impenetrable objects are present, Equations (28.1-1)–(28.1-4) are supplemented by boundary conditions of the type discussed in Chapter 6, both for state A and state B. These are either (see Equations (20.1-2) and (20.1-3))

$$\varepsilon_{k,m,p}\nu_m H_p^{\mathrm{A,B}} \quad \text{is continuous across any interface}\,, \tag{28.1-5}$$

and



28.1-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ and unit vector ν_m along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$, to which the reciprocity theorems apply.



28.1-2 Bounded domain $\mathcal{D}$ and States A and B to which the reciprocity theorems apply.

$$\varepsilon_{j,n,r}\nu_r E_r^{A,B} \quad \text{is continuous across any interface}\,, \tag{28.1-6}$$

where ν_m is the unit vector along the normal to the interface, or (see Equation (20.2-1))

$$\lim_{h\downarrow 0}\varepsilon_{j,n,r}\nu_n E_r^{A,B}(x+h\nu,t) = 0$$
$$\text{on the boundary of an electrically impenetrable object}\,, \tag{28.1-7}$$

where ν is the unit vector along the normal to the boundary of the object, pointing away from the object, or (see Equation (20.3-1))

$$\lim_{h\downarrow 0}\varepsilon_{k,m,p}\nu_m H_p^{A,B}(x+h\nu,t) = 0$$
$$\text{on the boundary of a magnetically impenetrable object}\,, \tag{28.1-8}$$

where ν is the unit vector along the normal to the boundary of the object, pointing away from the object.

To handle unbounded domains, we assume that outside some sphere $\mathcal{S}(O,\Delta_0)$, with its centre at the origin of the chosen reference frame and radius Δ_0, the medium is homogeneous, isotropic and lossless, with permittivity ε_0, permeability μ_0 and electromagnetic wave speed $c_0 = (\varepsilon_0\mu_0)^{-1/2}$, as well as source-free. These parameters are positive constants, but they need not have the values pertaining to a vacuum domain. In the domain outside that sphere, the so-called *embedding*, the asymptotic causal far-field representations are (see Equation (26.12-5))

$$\{E_r^{A,B}, H_p^{A,B}\} = \frac{\{E_r^{\infty;A,B}, H_p^{\infty;A,B}\}(\xi, t-|x|/c_0)}{4\pi|x|}[1+O(|x|^{-1})] \quad \text{as } |x|\to\infty\,, \tag{28.1-9}$$

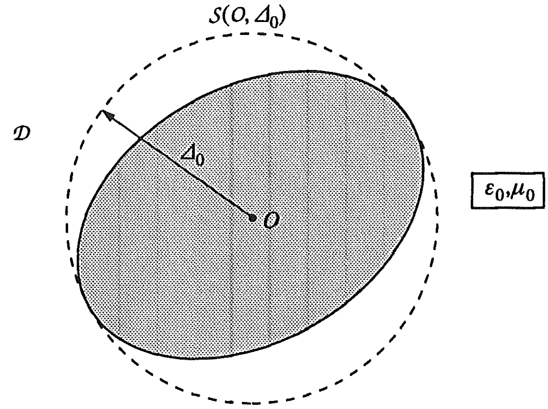
where $\{E_r^{\infty;A,B}, H_p^{\infty;A,B}\}(\xi,t)$ denote the far-field wave amplitudes, which are interrelated through (see Equations (26.12-11) and (26.12-12))

$$\varepsilon_{k,m,p}(\xi_m/c_0)H_p^{\infty;A,B} + \varepsilon_0 E_k^{\infty;A,B} = 0\,, \tag{28.1-10}$$

$$-\varepsilon_{j,n,r}(\xi_n/c_0)E_r^{\infty;A,B} + \mu_0 H_j^{\infty;A,B} = 0\,, \tag{28.1-11}$$

with

$$\xi_r E_r^{\infty;A,B} = 0\,, \tag{28.1-12}$$



28.1-3 Unbounded domain $\mathcal{D}$ for the application of a reciprocity theorem. Outside the sphere $S(O,\Delta_0)$, the medium is homogeneous, isotropic and lossless. Inside $S(O,\Delta_0)$, interfaces between different media, electrically impenetrable objects and magnetically impenetrable objects may be present.

$$\xi_p H_p^{\infty;\mathrm{A,B}} = 0\,, \tag{28.1-13}$$

(see Figure 28.1-3).

The construction of the *complex frequency-domain* reciprocity theorems will be based on the complex frequency-domain electromagnetic field equations (see Equations (24.4-1) and (24.4-2))

$$-\varepsilon_{k,m,p}\partial_m \hat{H}_p^{\mathrm{A}} + \hat{\eta}_{k,r}^{\mathrm{A}}\hat{E}_r^{\mathrm{A}} = -\hat{J}_k^{\mathrm{A}}, \tag{28.1-14}$$

$$\varepsilon_{j,n,r}\partial_n \hat{E}_r^{\mathrm{A}} + \hat{\zeta}_{j,p}^{\mathrm{A}}\hat{H}_p^{\mathrm{A}} = -\hat{K}_j^{\mathrm{A}}, \tag{28.1-15}$$

for state A, and

$$-\varepsilon_{k,m,p}\partial_m \hat{H}_p^{\mathrm{B}} + \hat{\eta}_{k,r}^{\mathrm{B}}\hat{E}_r^{\mathrm{B}} = -\hat{J}_k^{\mathrm{B}}, \tag{28.1-16}$$

$$\varepsilon_{j,n,r}\partial_n \hat{E}_r^{\mathrm{B}} + \hat{\zeta}_{j,p}^{\mathrm{B}}\hat{H}_p^{\mathrm{B}} = -\hat{K}_j^{\mathrm{B}}, \tag{28.1-17}$$

for state B. If, in $\mathcal{D}$, either surfaces of discontinuity in electromagnetic properties or electromagnetically impenetrable objects are present, Equations (28.1-14)–(28.1-17) are supplemented by boundary conditions of the type discussed in Section 24.3, both for state A and state B. These are either (see Equations (24.3-1) and (24.3-2))

$$\varepsilon_{k,m,p}\nu_m \hat{H}_p^{\mathrm{A,B}} \quad \text{is continuous across any interface}\,, \tag{28.1-18}$$

and

$$\varepsilon_{j,n,r}\nu_n \hat{E}_r^{\mathrm{A,B}} \quad \text{is continuous across any interface}\,, \tag{28.1-19}$$

where ν_r is the unit vector along the normal to the interface, or (see Equation (24.3-3))

$$\lim_{h\downarrow 0}\varepsilon_{j,n,r}\nu_n \hat{E}_r^{\mathrm{A,B}}(x + h\nu, s) = 0$$

$$\text{on the boundary of an electrically impenetrable object}\,, \tag{28.1-20}$$

where ν is the unit vector along the normal to the boundary of the object, pointing away from the object, or (see Equation (24.3-4))

$$\lim_{h\downarrow 0}\varepsilon_{k,m,p}\nu_m\hat{H}_p^{A,B}(x+h\nu,s)=0$$

$$\text{on the boundary of a magnetically impenetrable object}, \qquad (28.1\text{-}21)$$

where ν is the unit vector along the normal to the boundary of the object, pointing away from the object.

In the complex frequency domain, too, we assume, to handle unbounded domains, that outside some sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ with its centre at the origin of the chosen reference frame and radius Δ_0, the medium is homogeneous, isotropic and lossless, with permittivity ε_0, permeability μ_0 and electromagnetic wave speed $c_0=(\varepsilon_0\mu_0)^{-1/2}$, as well as source-free. In the domain outside that sphere, the *embedding*, the asymptotic causal far-field representations are (see Equation (26.11-10))

$$\{\hat{E}_r^{A,B},\hat{H}_p^{A,B}\}=\{\hat{E}_r^{\infty;A,B},\hat{H}_p^{\infty;A,B}\}(\xi,s)\,\frac{\exp(-s|x|/c_0)}{4\pi|x|}\,[1+O(|x|^{-1})]$$

$$\text{as } |x|\to\infty, \qquad (28.1\text{-}22)$$

where $\{\hat{E}_r^{\infty;A,B},\hat{H}_p^{\infty;A,B}\}(\xi,s)$ denote the far-field wave amplitudes, which are interrelated through (see Equations (26.11-13) and (26.11-14))

$$\varepsilon_{k,m,p}(\xi_m/c_0)\hat{H}_p^{\infty;A,B}+\varepsilon_0\hat{E}_k^{\infty;A,B}=0, \qquad (28.1\text{-}23)$$

$$-\varepsilon_{j,n,r}(\xi_n/c_0)\hat{E}_r^{\infty;A,B}+\mu_0\hat{H}_j^{\infty;A,B}=0, \qquad (28.1\text{-}24)$$

with

$$\xi_r\hat{E}_r^{\infty;A,B}=0, \qquad (28.1\text{-}25)$$

$$\xi_p\hat{H}_p^{\infty;A,B}=0. \qquad (28.1\text{-}26)$$

As a rule, state A will be chosen to correspond to the actual electromagnetic wave field in the configuration, or one of its constituents. This wave field will therefore satisfy the condition of causality, or, for short, will be a *causal wave field.* If state B is another physical state, for example, a state that corresponds to source distributions and/or electromagnetic medium parameters that differ from the ones in state A, state B will also be a causal wave field. If, however, state B is a computational state, i.e. a state that is representative for the manner in which the wave-field quantities in state A are computed, or a state that is representative for the manner in which the electromagnetic field measurement data pertaining to state A are processed, there is no necessity to take state B to be a causal wave field as well, and it may, for example, be taken to be an anti-causal wave field (i.e. a wave field that is time reversed with respect to a causal wave field) or no wave field at all (which happens, as an example, if one of the corresponding constitutive parameters is taken to be zero). No matter how the source distributions and the constitutive parameters are chosen, the wave-field quantities will always be assumed to satisfy the pertaining electromagnetic field equations and the pertaining boundary conditions.

To accommodate causal, anti-causal as well as non-causal states in the complex frequency-domain analysis of reciprocity, the Laplace transform with respect to time of any *transient*, not necessarily causal or anti-causal, wave function $f=f(x,t)$ will always be taken as (see Equation (B.1-5))

$$\hat{f}(x,s) = \int_{t \in \mathcal{R}} \exp(-st) f(x,t)\, \mathrm{d}t \qquad \text{for } \mathrm{Re}(s) = s_0 \,, \tag{28.1-27}$$

i.e. the support of the wave function is, in principle, taken to be the entire interval of real values of time. Whenever appropriate, the support of the wave function will be indicated explicitly. For neither causal, nor anti-causal wave fields (but of a transient nature) the right-hand side of Equation (28.1-27) should exist for some value $\mathrm{Re}(s) = s_0$ on a line parallel to the imaginary axis of the complex s plane (Figure 28.1-4) for the transformation to make any sense at all.

For *causal wave functions* with support $\mathcal{T}^+ = \{t \in \mathcal{R}; t > t_0\}$, Equation (28.1-27) yields

$$\hat{f}(x,s) = \int_{t=t_0}^{\infty} \exp(-st) f(x,t)\, \mathrm{d}t \qquad \text{for } \mathrm{Re}(s) > s_0^+ \,. \tag{28.1-28}$$

Here, the right-hand side is regular in some *right half* $\mathrm{Re}(s) > s_0^+$ of the complex s plane (Figure 28.1-5).

For *anti-causal wave functions* with support $\mathcal{T}^- = \{t \in \mathcal{R}; t < t_0\}$, Equation (28.1-27) yields

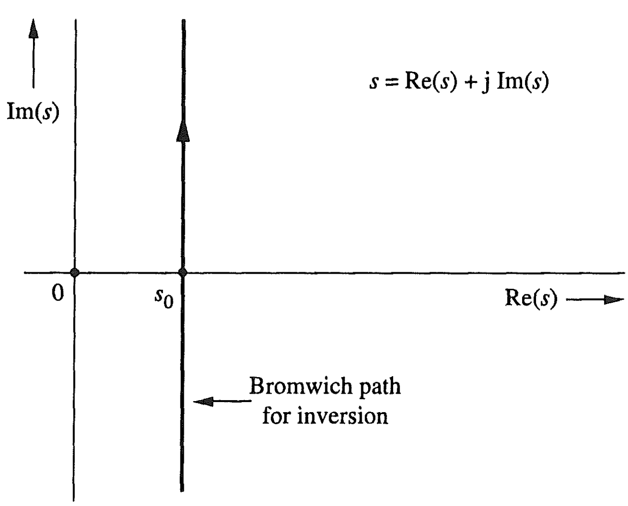
$$\hat{f}(x,s) = \int_{t=-\infty}^{t_0} \exp(-st) f(x,t)\, \mathrm{d}t \qquad \text{for } \mathrm{Re}(s) < s_0^- \,. \tag{28.1-29}$$

Here, the right-hand side is regular in some *left half* $\mathrm{Re}(s) < s_0^-$ of the complex s plane (Figure 28.1-6).

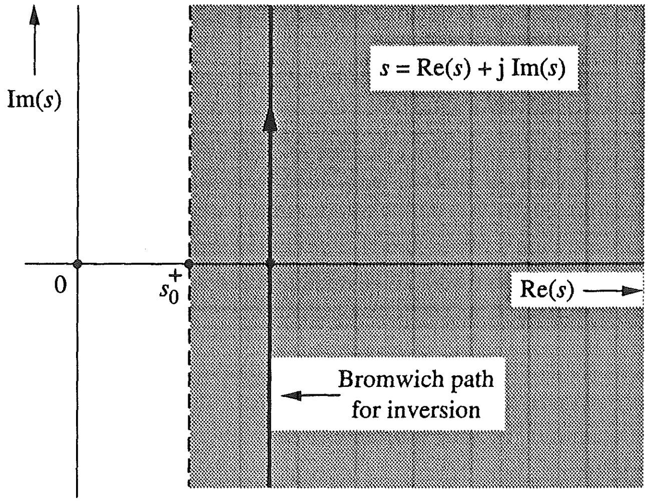
A consequence of Equation (28.1-29) is that $\hat{f}(x,-s)$ is regular in the right half $\mathrm{Re}(s) > -s_0^-$ of the complex s plane if $\hat{f}(x,s)$ is regular in the left half $\mathrm{Re}(s) < s_0^-$. This result will be needed in reciprocity theorems of the time correlation type.

For the *time convolution* $\mathrm{C}_t\,(f_1,f_2;x,t)$ of any two transient wave functions we have (see Equations (B.1-11) and (B.1-12))

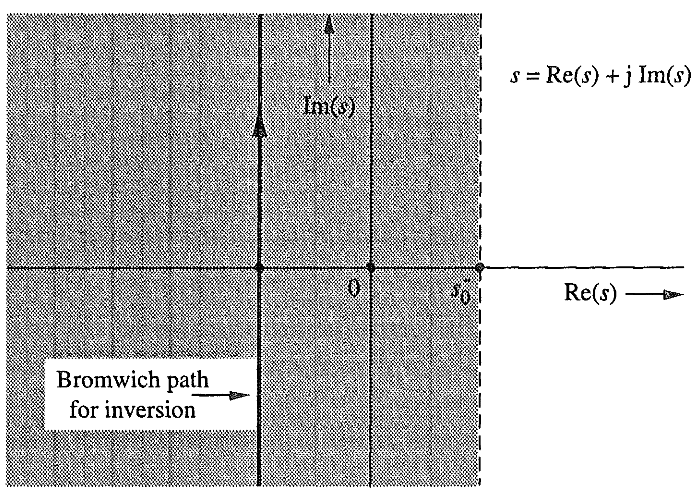
$$\hat{\mathrm{C}}_t(f_1,f_2;x,s) = \hat{f}_1(x,s)\,\hat{f}_2(x,s) \,. \tag{28.1-30}$$



28.1-4 Line $\mathrm{Re}(s) = s_0$, parallel to the imaginary axis of the complex s plane, at which the time Laplace transform of a wave field that is neither causal, nor anti-causal, but of a *transient* nature, exists.



28.1-5 Right half Re(s) > s_0^+ of the complex s plane, in which the time Laplace transform of a *causal* wave function exists.



28.1-6 Left half Re(s) < s_0^- of the complex s plane, in which the time Laplace transform of an *anti-causal* wave function exists.

This relation only holds in the common domain of regularity of $\hat{f}_1(x,s)$ and $\hat{f}_2(x,s)$. If, in particular, both $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are causal wave functions, they have a certain right half of the complex s plane as the domain of regularity in common. (Note that in this case $\hat{f}_1(x,s)$ is regular in some right half of the complex s plane, while $\hat{f}_2(x,s)$ is also regular in some right half of the complex s plane.) If, on the other hand, $f_1 = f_1(x,t)$ is a causal wave function and $f_2 = f_2(x,t)$ is an anti-causal wave function, the common domain of regularity where Equation (28.1-30) holds, is at most a strip of finite width parallel to the imaginary axis of the complex s plane. (Note that in this case $\hat{f}_1(x,s)$ is regular in some right half of the complex s plane, while $\hat{f}_2(x,s)$ is regular in some left half of the complex s plane.)

For the *time correlation* $\mathrm{R}_t(f_1,f_2;x,t)$ of any two transient wave functions we have (see Equations (B.1-14) and (B.1-15))

$$\hat{\mathrm{R}}_t(f_1,f_2;x,s) = \hat{f}_1(x,s)\hat{f}_2(x,-s)\,. \tag{28.1-31}$$

This relation only holds in the common domain of regularity of $\hat{f}_1(x,s)$ and $\hat{f}_2(x,s)$. If, in particular, both $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are causal wave functions, the common domain of regularity where Equation (28.1-31) holds, is at most a strip of finite width parallel to the imaginary axis of the complex s plane. (Note in this case that $\hat{f}_1(x,s)$ is regular in some right half of the complex s plane, and that $\hat{f}_2(x,-s)$ is regular in some left half of the complex s plane.) If, on the other hand, $f_1 = f_1(x,t)$ is a causal wave function and $f_2 = f_2(x,t)$ is an anti-causal wave function, the common domain of regularity where Equation (28.1-31) holds is some right half of the complex s plane. (Note that in this case $\hat{f}_1(x,s)$ is regular in some right half of the complex s plane, and that $\hat{f}_2(x,-s)$ is also regular in some right half of the complex s plane.)

In subsequent calculations the time correlation will, whenever appropriate, be replaced by (see Equation (B.1-18))

$$\mathrm{R}_t(f_1,f_2;x,t) = \mathrm{C}_t(f_1,\mathrm{J}_t(f_2);x,t)\,, \tag{28.1-32}$$

where J_t is the *time reversal* operator. The latter operator changes causal wave functions into anti-causal ones, and vice versa.

Exercises

Exercise 28.1-1

Of what type is the domain of regularity of the Laplace transform of the time convolution $\mathrm{C}_t(f_1,f_2;x,t)$ of two wave functions $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ that are both anti-causal?

Answer: Some left half of the complex s plane.

Exercise 28.1-2

Of what type is the domain of regularity of the Laplace transform of the time correlation $\mathrm{R}_t(f_1,f_2;x,t)$ of the anti-causal wave function $f_1 = f_1(x,t)$ and the causal wave function $f_2 = f_2(x,t)$?

Answer: Some left half of the complex s plane.

28.2 The time-domain reciprocity theorem of the time convolution type

The time-domain reciprocity theorem of the time convolution type follows upon considering the *local interaction quantity* $\varepsilon_{m,r,p}\partial_m\,[\mathrm{C}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,t)]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\begin{aligned}&\varepsilon_{m,r,p}\partial_m\left[\mathrm{C}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,t)\right]\\&= \varepsilon_{j,n,r}\partial_n\mathrm{C}_t(E_r^{\mathrm{A}},H_j^{\mathrm{B}};x,t) + \varepsilon_{k,m,p}\partial_m\mathrm{C}_t(E_k^{\mathrm{B}},H_p^{\mathrm{A}};x,t)\end{aligned}$$

$$= \mathrm{C}_t(\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{A}}, H_j^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{A}}, \varepsilon_{r,n,j}\partial_n H_j^{\mathrm{B}};x,t)$$
$$- \mathrm{C}_t(\varepsilon_{p,m,k}\partial_m E_k^{\mathrm{B}}, H_p^{\mathrm{A}};x,t) + \mathrm{C}_t(E_k^{\mathrm{B}}, \varepsilon_{k,m,p}\partial_m H_p^{\mathrm{A}};x,t)\,. \tag{28.2-1}$$

With the aid of Equations (28.1-1)–(28.1-4), the different terms on the right-hand side become

$$\mathrm{C}_t(\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{A}}, H_j^{\mathrm{B}};x,t) = -\partial_t \mathrm{C}_t(\mu_{j,p}^{\mathrm{A}}, H_p^{\mathrm{A}}, H_j^{\mathrm{B}};x,t) - \mathrm{C}_t(K_j^{\mathrm{A}}, H_j^{\mathrm{B}};x,t)\,, \tag{28.2-2}$$
$$-\mathrm{C}_t(E_r^{\mathrm{A}}, \varepsilon_{r,n,j}\partial_n H_j^{\mathrm{B}};x,t) = -\partial_t \mathrm{C}_t(E_r^{\mathrm{A}}, \varepsilon_{r,k}^{\mathrm{B}}, E_k^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{A}}, J_r^{\mathrm{B}};x,t)\,, \tag{28.2-3}$$

and

$$-\mathrm{C}_t(\varepsilon_{p,m,k}\partial_m E_k^{\mathrm{B}}, H_p^{\mathrm{A}};x,t) = \partial_t \mathrm{C}_t(\mu_{p,j}^{\mathrm{B}}, H_j^{\mathrm{B}}, H_p^{\mathrm{A}};x,t) + \mathrm{C}_t(K_p^{\mathrm{B}}, H_p^{\mathrm{A}};x,t)\,, \tag{28.2-4}$$
$$\mathrm{C}_t(E_k^{\mathrm{B}}, \varepsilon_{k,m,p}\partial_m H_p^{\mathrm{A}};x,t) = \partial_t \mathrm{C}_t(E_k^{\mathrm{B}}, \varepsilon_{k,r}^{\mathrm{A}}, E_r^{\mathrm{A}};x,t) + \mathrm{C}_t(E_k^{\mathrm{B}}, J_k^{\mathrm{A}};x,t)\,, \tag{28.2-5}$$

in which the convolution of three functions is a shorthand notation for the convolution of a function with the convolution of two other functions. (Note that in the convolution operation the order of the operators is immaterial.) Combining Equations (28.2-2)–(28.2-5) with Equation (28.2-1), it is found that

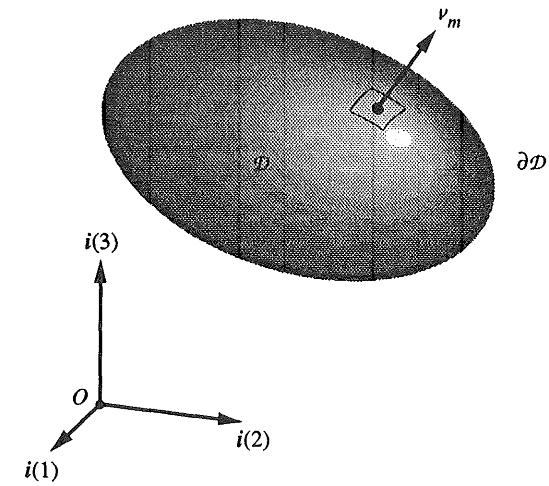
$$\varepsilon_{m,r,p}\partial_m\left[\mathrm{C}_t(E_r^{\mathrm{A}}, H_p^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{B}}, H_p^{\mathrm{A}};x,t)\right]$$
$$= -\partial_t \mathrm{C}_t(\varepsilon_{r,k}^{\mathrm{B}} - \varepsilon_{k,r}^{\mathrm{A}}, E_r^{\mathrm{A}}, E_k^{\mathrm{B}};x,t) + \partial_t \mathrm{C}_t(\mu_{p,j}^{\mathrm{B}} - \mu_{j,p}^{\mathrm{A}}, H_p^{\mathrm{A}}, H_j^{\mathrm{B}};x,t)$$
$$+ \mathrm{C}_t(J_k^{\mathrm{A}}, E_k^{\mathrm{B}};x,t) - \mathrm{C}_t(K_j^{\mathrm{A}}, H_j^{\mathrm{B}};x,t) - \mathrm{C}_t(J_r^{\mathrm{B}}, E_r^{\mathrm{A}};x,t) + \mathrm{C}_t(K_p^{\mathrm{B}}, H_p^{\mathrm{A}};x,t)\,. \tag{28.2-6}$$

Equation (28.2-6) is the *local form of electromagnetic reciprocity theorem of the time-convolution type*. The first two terms on the right-hand side are representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; these terms vanish at those locations where $\varepsilon_{r,k}^{\mathrm{B}}(x,t) = \varepsilon_{k,r}^{\mathrm{A}}(x,t)$ and $\mu_{p,j}^{\mathrm{B}}(x,t) = \mu_{j,p}^{\mathrm{A}}(x,t)$ for all $t \in \mathcal{R}$. In case the latter conditions hold, the two media are denoted as each other's *adjoint*. Note is this respect that the adjoint of a causal (anti-causal) medium is causal (anti-causal) as well. The last four terms on the right-hand side of Equation (28.2-6) are representative for the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (28.2-6) are continuous. Upon integrating Equation (28.2-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 28.2-1)

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{A}}, H_p^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{B}}, H_p^{\mathrm{A}};x,t)\right]\mathrm{d}A$$
$$= \int_{x\in\mathcal{D}} \left[-\partial_t \mathrm{C}_t(\varepsilon_{r,k}^{\mathrm{B}} - \varepsilon_{k,r}^{\mathrm{A}}, E_r^{\mathrm{A}}, E_k^{\mathrm{B}};x,t) + \partial_t \mathrm{C}_t(\mu_{p,j}^{\mathrm{B}} - \mu_{j,p}^{\mathrm{A}}, H_p^{\mathrm{A}}, H_j^{\mathrm{B}};x,t)\right]\mathrm{d}V$$
$$+ \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(J_k^{\mathrm{A}}, E_k^{\mathrm{B}};x,t) - \mathrm{C}_t(K_j^{\mathrm{A}}, H_j^{\mathrm{B}};x,t)\right.$$
$$\left. - \mathrm{C}_t(J_r^{\mathrm{B}}, E_r^{\mathrm{A}};x,t) + \mathrm{C}_t(K_p^{\mathrm{B}}, H_p^{\mathrm{A}};x,t)\right]\mathrm{d}V\,. \tag{28.2-7}$$

Equation (28.2-7) is the *global form, for the bounded domain $\mathcal{D}$, of the electromagnetic reciprocity theorem of the time convolution type.* Note that in the process of adding the



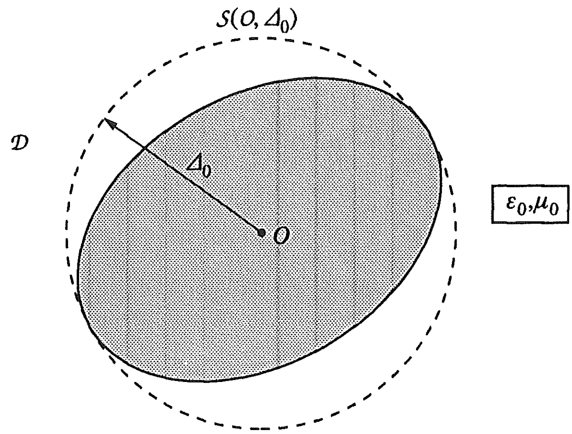
28.2-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (28.1-5) and (28.1-6)), and that the contributions from boundary surfaces of electromagnetically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (28.1-7) or Equation (28.1-8)). In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an electromagnetically impenetrable object. In the right-hand side, the first integral is representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's adjoint. The second integral on the right-hand side is representative for the action of the sources present in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem Equation (28.2-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 28.1 are made and Equation (28.2-7) is first applied to the domain interior to the sphere $\mathcal{S}(O,\Delta)$ with its centre at the origin and radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 28.2-2).

Whether or not the surface integral contribution over $\mathcal{S}(O,\Delta)$ does vanish as $\Delta\to\infty$, depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representation of Equations (28.1-9)–(28.1-13) apply for sufficiently large values of Δ. Then, the time convolutions occurring in the integrands of Equation (28.2-7)



28.2-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the medium is homogeneous, isotropic and lossless.

are also causal in time, and at any finite value of t, Δ can be chosen so large that on $\mathcal{S}(O,\Delta)$ the integrand vanishes. In this case, the contribution from $\mathcal{S}(O,\Delta)$ vanishes. If, however, at least one of the two states is chosen to be non-causal (which, for example, can apply to the case where one of the two states is a computational one), the time convolutions occurring in Equation (28.2-7) are non-causal as well and the contribution from $\mathcal{S}(O,\Delta)$ does not vanish, no matter how large a value of Δ is chosen. As outside the sphere $\mathcal{S}(O,\Delta_0)$ that is used to define the embedding (see Section 28.1) the media are each other's adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(O,\Delta)$ is, however, independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 28.2-2).

The time-domain reciprocity theorem of the time convolution type is mainly used for investigating the transmission/reception reciprocity properties of electromagnetic systems and devices (for example, antennas) (see Sections 28.6 and 28.7) and for the modelling of direct (forward) source problems (see Section 28.8) and direct (forward) scattering problems (see Section 28.9). References to the earlier literature on the subject can be found in a paper by De Hoop (1987).

Exercises

Exercise 28.2-1

To what form reduce the contrast-in-media terms in the reciprocity theorems Equations (28.2-6) and (28.2-7) if the media in states A and B are both instantaneously reacting with permittivities $\varepsilon^{\mathrm{A}}_{k,r}$ and $\varepsilon^{\mathrm{B}}_{r,k}$, and permeabilities $\mu^{\mathrm{A}}_{j,p}$ and $\mu^{\mathrm{B}}_{p,j}$?

Answer:

$$\mathrm{C}_t(\varepsilon_{r,k}^{\mathrm{B}} - \varepsilon_{k,r}^{\mathrm{A}}, E_r^{\mathrm{A}}, E_k^{\mathrm{B}}; x,t) = \left[\varepsilon_{r,k}^{\mathrm{B}}(x) - \varepsilon_{k,r}^{\mathrm{A}}(x)\right] \mathrm{C}_t(E_r^{\mathrm{A}}, E_k^{\mathrm{B}}; x,t)$$

and

$$\mathrm{C}_t(\mu_{p,j}^{\mathrm{B}} - \mu_{j,p}^{\mathrm{A}}, H_p^{\mathrm{A}}, H_j^{\mathrm{B}}; x,t) = \left[\mu_{p,j}^{\mathrm{B}}(x) - \mu_{j,p}^{\mathrm{A}}(x)\right] \mathrm{C}_t(H_p^{\mathrm{A}}, H_j^{\mathrm{B}}; x,t) .$$

Exercise 28.2-2

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 28.2-3.

The reciprocity theorem Equation (28.2-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, neither in state A nor in state B, and the medium in $\mathcal{D}$ in state B is in its electromagnetic properties adjoint to the one in state A. Prove that

$$\varepsilon_{m,r,p} \int_{x \in \mathcal{S}_1} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{A}}, H_p^{\mathrm{B}}; x,t) - \mathrm{C}_t(E_r^{\mathrm{B}}, H_p^{\mathrm{A}}; x,t)\right] \mathrm{d}A$$

$$= \varepsilon_{m,r,p} \int_{x \in \mathcal{S}_2} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{A}}, H_p^{\mathrm{B}}; x,t) - \mathrm{C}_t(E_r^{\mathrm{B}}, H_p^{\mathrm{A}}; x,t)\right] \mathrm{d}A , \tag{28.2-8}$$

i.e. that the surface integral is an invariant.

28.3 The time-domain reciprocity theorem of the time correlation type

The time-domain reciprocity theorem of the time correlation type follows upon considering the *local interaction quantity* $\varepsilon_{m,r,p}\partial_m\,[\mathrm{R}_t(E_r^{\mathrm{A}}, H_p^{\mathrm{B}}; x,t) + \mathrm{R}_t(E_r^{\mathrm{B}}, H_p^{\mathrm{A}}; x,-t)]$. On account of Equations (B.1-14) and (B.1-18) and the symmetry of the convolution operator, this quantity can be rewritten as $\varepsilon_{m,r,p}\partial_m\,[\mathrm{C}_t(E_r^{\mathrm{A}}, \mathrm{J}_t(H_p^{\mathrm{B}}); x,t) + \mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{B}}), H_p^{\mathrm{A}}; x,t)]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\varepsilon_{m,r,p}\partial_m \left[\mathrm{C}_t(E_r^{\mathrm{A}}, \mathrm{J}_t(H_p^{\mathrm{B}}); x,t) + \mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{B}}), H_p^{\mathrm{A}}; x,t)\right]$$

$$= \varepsilon_{j,n,r}\partial_n \mathrm{C}_t(E_r^{\mathrm{A}}, \mathrm{J}_t(H_j^{\mathrm{B}}); x,t) - \varepsilon_{k,m,p}\partial_m \mathrm{C}_t(\mathrm{J}_t(E_k^{\mathrm{B}}), H_p^{\mathrm{A}}; x,t)$$

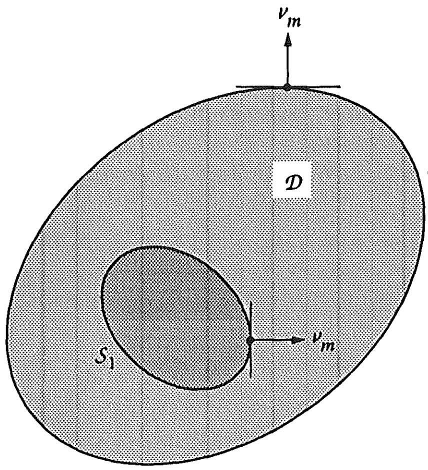
$$= \mathrm{C}_t(\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{A}}, \mathrm{J}_t(H_j^{\mathrm{B}}); x,t) - \mathrm{C}_t(E_r^{\mathrm{A}}, \varepsilon_{r,n,j}\partial_n \mathrm{J}_t(H_j^{\mathrm{B}}); x,t)$$

$$+ \mathrm{C}_t(\varepsilon_{p,m,k}\partial_m \mathrm{J}_t(E_k^{\mathrm{B}}), H_p^{\mathrm{A}}; x,t) - \mathrm{C}_t(\mathrm{J}_t(E_k^{\mathrm{B}}), \varepsilon_{k,m,p}\partial_m H_p^{\mathrm{A}}; x,t) . \tag{28.3-1}$$

With the aid of Equations (28.1-1)–(28.1-4) and the rule $\mathrm{J}_t(\partial_t f) = -\partial_t(\mathrm{J}_t(f))$, the different terms on the right-hand side become

$$\mathrm{C}_t(\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{A}}, \mathrm{J}_t(H_j^{\mathrm{B}}); x,t)$$

$$= -\partial_t \mathrm{C}_t(\mu_{j,p}^{\mathrm{A}} H_p^{\mathrm{A}}, \mathrm{J}_t(H_j^{\mathrm{B}}); x,t) - \mathrm{C}_t(K_j^{\mathrm{A}}, \mathrm{J}_t(H_j^{\mathrm{B}}); x,t) , \tag{28.3-2}$$



28.2-3 Domain $\mathcal{D}$, bounded internally by the closed surface S_1 and externally by the closed surface S_2.

$$-C_t(E_r^A, \varepsilon_{r,n,j}\partial_n J_t(H_j^B); x,t)$$
$$= \partial_t C_t(E_r^A, J_t(\varepsilon_{r,k}^B), J_t(E_k^B); x,t) - C_t(E_r^A, J_t(J_r^B); x,t)\,, \tag{28.3-3}$$

and

$$C_t(\varepsilon_{p,m,k}\partial_m J_t(E_k^B), H_p^A; x,t)$$
$$= \partial_t C_t(J_t(\mu_{p,j}^B), J_t(H_j^B), H_p^A); x,t) - C_t(J_t(K_p^B), H_p^A; x,t)\,, \tag{28.3-4}$$
$$-C_t(J_t(E_k^B), \varepsilon_{k,m,p}\partial_m H_p^A; x,t)$$
$$= -\partial_t C_t(J_t(E_k^B), \varepsilon_{k,r}^A, E_r^A; x,t) - C_t(J_t(E_k^B), J_k^A; x,t)\,, \tag{28.3-5}$$

in which the convolution of three functions is a shorthand notation for the convolution of a function with the convolution of two other functions. (Note that in the convolution operation the order of the operators is immaterial.) Combining Equations (28.3-2)–(28.3-5) with Equation (28.3-1), it is found that

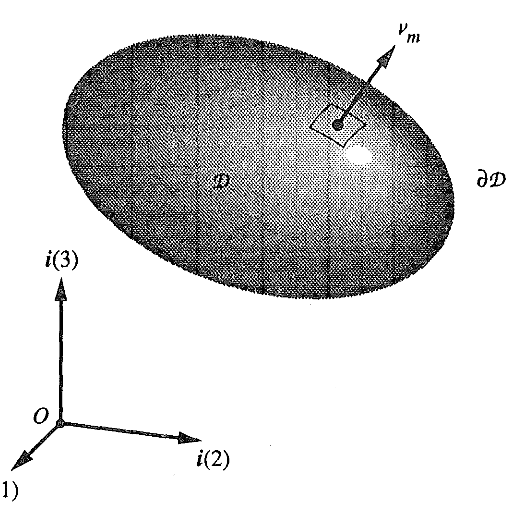
$$\varepsilon_{m,r,p}\partial_m\left[C_t(E_r^A, J_t(H_p^B); x,t) + C_t(J_t(E_r^B), H_p^A; x,t)\right]$$
$$= \partial_t C_t(J_t(\varepsilon_{r,k}^B) - \varepsilon_{k,r}^A, E_r^A, J_t(E_k^B); x,t) + \partial_t C_t(J_t(\mu_{p,j}^B) - \mu_{j,p}^A, H_p^A, J_t(H_j^B); x,t)$$
$$- C_t(J_k^A, J_t(E_k^B); x,t) - C_t(K_j^A, J_t(H_j^B); x,t)$$
$$- C_t(J_t(J_r^B), E_r^A; x,t) - C_t(J_t(K_p^B), H_p^A; x,t)\,. \tag{28.3-6}$$

Equation (28.3-6) is the *local form of electromagnetic reciprocity theorem of the time correlation type*. The first two terms on the right-hand side are representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; these terms vanish at those locations where $J_t(\varepsilon_{r,k}^B)(x,t) = \varepsilon_{k,r}^A(x,t)$ and $J_t(\mu_{p,j}^B)(x,t) = \mu_{j,p}^A(x,t)$ for all $t \in \mathcal{R}$. In case the latter conditions hold, the two media are denoted as each other's *time-reverse adjoint.* Note is this respect that the time-reverse adjoint of a causal (anti-causal) medium is an anti-causal (causal) medium. The last four terms on the right-hand side of Equation (28.3-6) are representative for the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (28.3-6) are continuous. Upon integrating Equation (28.3-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 28.3-1)

$$
\begin{aligned}
&\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{R}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t)+\mathrm{R}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,-t)\right]\mathrm{d}A\\
&=\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{A}},\mathrm{J}_t(H_p^{\mathrm{B}});x,t)+\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{B}}),H_p^{\mathrm{A}};x,t)\right]\mathrm{d}A\\
&=\int_{x\in\mathcal{D}}\left[\partial_t\mathrm{C}_t(\mathrm{J}_t(\varepsilon_{r,k}^{\mathrm{B}})-\varepsilon_{k,r}^{\mathrm{A}},E_r^{\mathrm{A}},\mathrm{J}_t(E_k^{\mathrm{B}});x,t)+\partial_t\mathrm{C}_t(\mathrm{J}_t(\mu_{p,j}^{\mathrm{B}})-\mu_{j,p}^{\mathrm{A}},H_p^{\mathrm{A}},\mathrm{J}_t(H_j^{\mathrm{B}});x,t)\right]\mathrm{d}V\\
&\quad-\int_{x\in\mathcal{D}}\left[\mathrm{C}_t(J_k^{\mathrm{A}},\mathrm{J}_t(E_k^{\mathrm{B}});x,t)+\mathrm{C}_t(K_j^{\mathrm{A}},\mathrm{J}_t(H_j^{\mathrm{B}});x,t)\right.\\
&\quad\left.+\mathrm{C}_t(\mathrm{J}_t(J_r^{\mathrm{B}}),E_r^{\mathrm{A}};x,t)+\mathrm{C}_t(\mathrm{J}_t(K_p^{\mathrm{B}}),H_p^{\mathrm{A}};x,t)\right]\mathrm{d}V\,.
\end{aligned}
\qquad (28.3\text{-}7)
$$

Equation (28.3-7) is the *global form, for the bounded domain $\mathcal{D}$, of the electromagnetic reciprocity theorem of the time correlation type.* Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (28.1-5) and



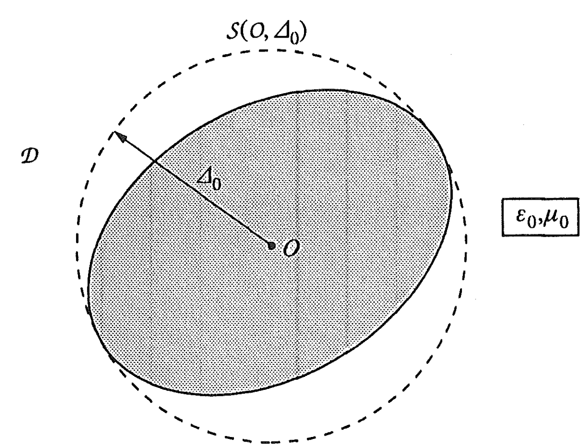
28.3-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

(28.1-6)), and that the contributions from boundary surfaces of electromagnetically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (28.1-7) or Equation (28.1-8)). In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an electromagnetically impenetrable object. In the right-hand side, the first integral is representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's time-reverse adjoint. The second integral on the right-hand side is representative for the action of the sources present in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In a number of cases the reciprocity theorem Equation (28.3-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 28.1 are made and Equation (28.3-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin and of radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 28.3-2).

Since outside the sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ that is used to define the embedding (see Section 28.1) the media are each other's time-reverse adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(\mathcal{O},\Delta)$ is, in any case, independent of the value of Δ for $\Delta > \Delta_0$ (see Exercise 28.3-2). Whether or not this contribution vanishes as $\Delta\to\infty$, depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the time correlations occurring in the integrands of Equation (28.3-7) are neither causal nor anti-causal, and the contribution of $\mathcal{S}(\mathcal{O},\Delta)$ is a non-vanishing function that is independent of the value of Δ. If, however, state A is chosen to be causal and state B is chosen to be anti-causal (which, for



28.3-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(\mathcal{O},\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(\mathcal{O},\Delta_0)$ is the sphere outside which the medium is homogeneous, isotropic and lossless.

example, can apply to the case where state B is a computational one), the correlations occurring in Equation (28.3-7) are causal as well and the contribution from $\mathcal{S}(O,\Delta)$ vanishes for sufficiently large values of Δ.

The time-domain reciprocity theorem of the time correlation type is mainly used in the modelling of inverse source problems (see Section 28.10) and inverse scattering problems (see Section 28.11). References to the earlier literature on the subject can be found in a paper by De Hoop (1987).

Exercises

Exercise 28.3-1

To what form reduce the contrast-in-media terms in the reciprocity theorems Equations (28.3-6) and (28.3-7) if the media in states A and B are both instantaneously reacting?

Answer:

$$\partial_t \mathrm{C}_t(\mathrm{J}_t(\varepsilon^{\mathrm{B}}_{r,k}) - \varepsilon^{\mathrm{A}}_{k,r}, E^{\mathrm{A}}_r, \mathrm{J}_t(E^{\mathrm{B}}_k); x,t) = \left[\varepsilon^{\mathrm{B}}_{r,k}(x) - \varepsilon^{\mathrm{A}}_{k,r}(x)\right] \partial_t \mathrm{C}_t(E^{\mathrm{A}}_r, \mathrm{J}_t(E^{\mathrm{B}}_k); x,t)$$

and

$$\partial_t \mathrm{C}_t(\mathrm{J}_t(\mu^{\mathrm{B}}_{p,j}) - \mu^{\mathrm{A}}_{j,p}, H^{\mathrm{A}}_p, \mathrm{J}_t(H^{\mathrm{B}}_p); x,t) = \left[\mu^{\mathrm{B}}_{p,j}(x) - \mu^{\mathrm{A}}_{j,p}(x)\right] \partial_t \mathrm{C}_t(H^{\mathrm{A}}_p, \mathrm{J}_t(H^{\mathrm{B}}_j); x,t) .$$

Exercise 28.3-2

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 28.3-3.

The reciprocity theorem Equation (28.3-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, in either state A or state B, and the medium in $\mathcal{D}$ in state B is in its electromagnetic properties the time-reverse adjoint to the one in state A. Prove that

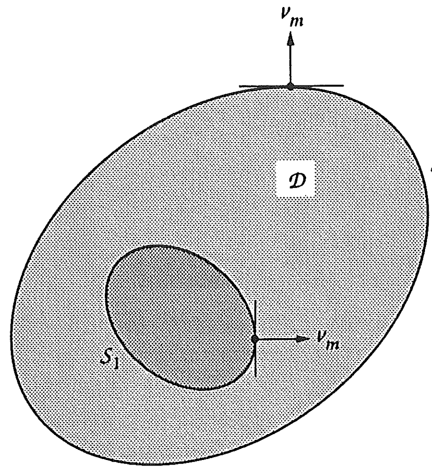
$$\varepsilon_{m,r,p} \int_{x \in \mathcal{S}_1} \nu_m \left[\mathrm{C}_t(E^{\mathrm{A}}_r, \mathrm{J}_t(H^{\mathrm{B}}_p); x,t) + \mathrm{C}_t(\mathrm{J}_t(E^{\mathrm{B}}_r), H^{\mathrm{A}}_p; x,t)\right] \mathrm{d}A$$

$$= \varepsilon_{m,r,p} \int_{x \in \mathcal{S}_2} \nu_m \left[\mathrm{C}_t(E^{\mathrm{A}}_r, \mathrm{J}_t(H^{\mathrm{B}}_p); x,t) + \mathrm{C}_t(\mathrm{J}_t(E^{\mathrm{B}}_r), H^{\mathrm{A}}_p; x,t)\right] \mathrm{d}A , \qquad (28.3\text{-}8)$$

i.e. the surface integral is an invariant.

28.4 The complex frequency-domain reciprocity theorem of the time convolution type

The complex frequency-domain reciprocity theorem of the time- convolution type follows upon considering the *local interaction quantity* $\varepsilon_{m,r,p}\partial_m\,[\hat{E}^{\mathrm{A}}_r(x,s)\hat{H}^{\mathrm{B}}_p(x,s) - \hat{E}^{\mathrm{B}}_r(x,s)\hat{H}^{\mathrm{A}}_p(x,s)]$. Using



28.3-3 Domain $\mathcal{D}$, bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$
\begin{aligned}
&\varepsilon_{m,r,p}\partial_m\left[\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{H}_p^{\mathrm{B}}(\boldsymbol{x},s) - \hat{E}_r^{\mathrm{B}}(\boldsymbol{x},s)\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s)\right]\\
&= \varepsilon_{j,n,r}\partial_n\left[\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s)\right] + \varepsilon_{k,m,p}\partial_m\left[\hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s)\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s)\right]\\
&= \left[\varepsilon_{j,n,r}\partial_n\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s) - \hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\left[\varepsilon_{r,n,j}\partial_n\hat{H}_k^{\mathrm{B}}(\boldsymbol{x},s)\right]\\
&\quad - \left[\varepsilon_{p,m,k}\partial_m\hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s)\right]\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s) + \hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s)\left[\varepsilon_{k,m,p}\partial_m\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s)\right]. \qquad (28.4\text{-}1)
\end{aligned}
$$

With the aid of Equations (28.1-14)–(28.1-17), the different terms on the right-hand side become

$$
\left[\varepsilon_{j,n,r}\partial_n\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\right]\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s) = -\hat{\zeta}_{j,p}^{\mathrm{A}}(\boldsymbol{x},s)\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s)\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s) - \hat{K}_j^{\mathrm{A}}(\boldsymbol{x},s)\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s)\,, \qquad (28.4\text{-}2)
$$

$$
-\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\left[\varepsilon_{r,n,j}\partial_n\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s)\right] = -\hat{\eta}_{r,k}^{\mathrm{B}}(\boldsymbol{x},s)\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s) - \hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{J}_r^{\mathrm{B}}(\boldsymbol{x},s)\,, \qquad (28.4\text{-}3)
$$

and

$$
-\left[\varepsilon_{p,m,k}\partial_m\hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s)\right]\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s) = \hat{\zeta}_{p,j}^{\mathrm{B}}(\boldsymbol{x},s)\hat{H}_j^{\mathrm{B}}(\boldsymbol{x},s)\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s) + \hat{K}_p^{\mathrm{B}}(\boldsymbol{x},s)\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s)\,, \qquad (28.4\text{-}4)
$$

$$
\hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s)\left[\varepsilon_{k,m,p}\partial_m\hat{H}_p^{\mathrm{A}}(\boldsymbol{x},s)\right] = \hat{\eta}_{k,r}^{\mathrm{A}}(\boldsymbol{x},s)\hat{E}_r^{\mathrm{A}}(\boldsymbol{x},s)\hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s) + \hat{E}_k^{\mathrm{B}}(\boldsymbol{x},s)\hat{J}_k^{\mathrm{A}}(\boldsymbol{x},s)\,. \qquad (28.4\text{-}5)
$$

Combining Equations (28.4-2)–(28.4-5) with Equation (28.4-1), it is found that

$$\varepsilon_{m,r,p}\partial_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s) - \hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]$$

$$= -\left[\hat{\eta}_{r,k}^{\mathrm{B}}(x,s) - \hat{\eta}_{k,r}^{\mathrm{A}}(x,s)\right]\hat{E}_r^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s)$$

$$+\left[\hat{\zeta}_{p,j}^{\mathrm{B}}(x,s) - \hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)\right]\hat{H}_p^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)$$

$$+\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s) - \hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)$$

$$-\hat{J}_r^{\mathrm{B}}(x,s)\hat{E}_r^{\mathrm{A}}(x,s) + \hat{K}_p^{\mathrm{B}}(x,s)\hat{H}_j^{\mathrm{A}}(x,s)\,. \tag{28.4-6}$$

Equation (28.4-6) is the *local form of the complex frequency-domain counterpart of the electromagnetic reciprocity theorem of the time convolution type.* The first two terms on the right-hand side are representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; these terms vanish at those locations where $\hat{\eta}_{r,k}^{\mathrm{B}}(x,s) = \hat{\eta}_{k,r}^{\mathrm{A}}(x,s)$ and $\hat{\zeta}_{p,j}^{\mathrm{B}}(x,s) = \hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)$ for all s in the domain in the complex s plane where Equation (28.4-6) holds. In case the latter conditions hold, the two media are denoted as each other's *adjoint.* The last four terms on the right-hand side of Equation (28.4-6) are representative for the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (28.4-6) are continuous. Upon integrating Equation (28.4-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 28.4-1)

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s) - \hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}A$$

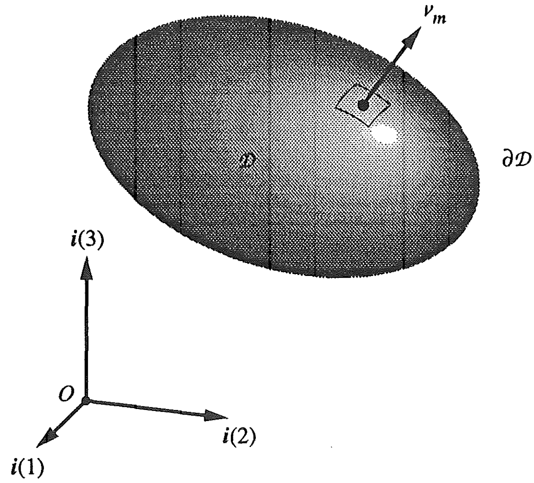
$$= \int_{x\in\mathcal{D}} \left\{-\left[\hat{\eta}_{r,k}^{\mathrm{B}}(x,s) - \hat{\eta}_{k,r}^{\mathrm{A}}(x,s)\right]\hat{E}_r^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s)\right.$$

$$\left.+\left[\hat{\zeta}_{p,j}^{\mathrm{B}}(x,s) - \hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)\right]\hat{H}_p^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)\right\}\mathrm{d}V$$

$$+\int_{x\in\mathcal{D}} \left[\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s) - \hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)\right.$$

$$\left.-\hat{J}_r^{\mathrm{B}}(x,s)\hat{E}_r^{\mathrm{A}}(x,s) + \hat{K}_p^{\mathrm{B}}(x,s)\hat{H}_j^{\mathrm{A}}(x,s)\right]\mathrm{d}V\,. \tag{28.4-7}$$

Equation (28.4-7) is the *global form,* for the bounded domain $\mathcal{D}$, of the *complex frequency-domain counterpart of the electromagnetic reciprocity theorem of the time convolution type.* Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (28.1-18) and (28.1-19)), and that the contributions from boundary surfaces of electromagnetically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equation (28.1-20) or Equation (28.1-21)).



28.4-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

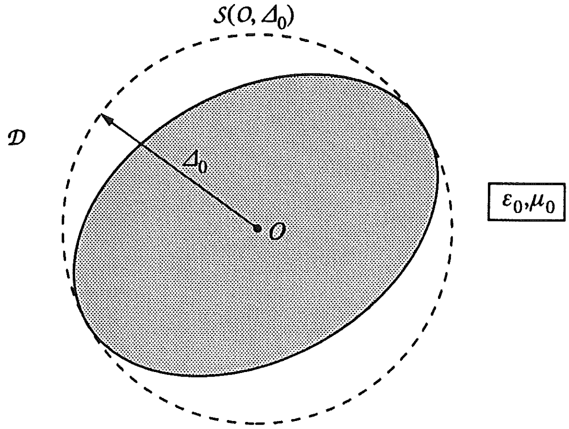
In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an electromagnetically impenetrable object. In the right-hand side, the first integral is representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's adjoint. The second integral on the right-hand side is representative for the action of the sources in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem Equation (28.4-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 28.1 are made and Equation (28.4-7) is first applied to the domain interior to the sphere $\mathcal{S}(O,\Delta)$ with its centre at the origin and of radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 28.4-2).

Whether or not the surface integral contribution over $\mathcal{S}(O,\Delta)$ does vanish as $\Delta\to\infty$, depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (28.1-22)–(28.1-26) apply for sufficiently large values of Δ. Then the contribution from $\mathcal{S}(O,\Delta)$ vanishes in the limit $\Delta\to\infty$. If, however, at least one of the two states is chosen to be non-causal (which, for example, can apply to the case where one of the two states is a computational one), the contribution from $\mathcal{S}(O,\Delta)$ does not vanish, no matter how large Δ is chosen. However, since outside the sphere $\mathcal{S}(O,\Delta_0)$ that is used to define the embedding (see Section 28.1) the media are each other's adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(O,\Delta)$ is independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 28.4-4).

The complex frequency-domain reciprocity theorem of the time convolution type is mainly used for investigating the transmission/reception reciprocity properties of electromagnetic



28.4-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the medium is homogeneous, isotropic and lossless.

systems and devices (for example, antennas) (see Sections 28.6 and 28.7) and for the modelling of direct (forward) source problems (see Section 28.8) and direct (forward) scattering problems (see Section 28.9).

Exercises

Exercise 28.4-1

Show, by taking the Laplace transform with respect to time, that Equation (28.4-6) follows from Equation (28.2-6).

Exercise 28.4-2

Show, by taking the Laplace transform with respect to time, that Equation (28.4-7) follows from Equation (28.2-7).

Exercise 28.4-3

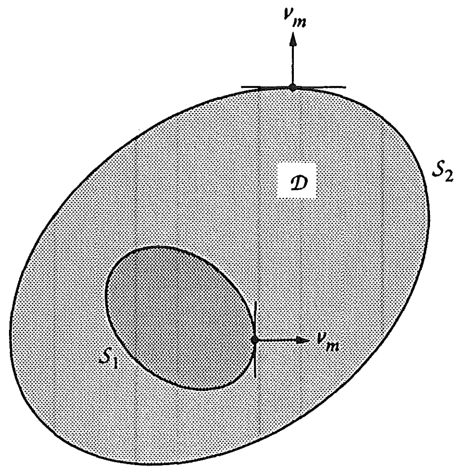
To what form reduce the contrast-in-media terms in the reciprocity theorems Equations (28.4-6) and (28.4-7) if the media in states A and B are both instantaneously reacting?

Answers:

$$\left[\hat{\eta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},s) - \hat{\eta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)\right]\hat{E}^{\mathrm{A}}_{r}(\boldsymbol{x},s)\hat{E}^{\mathrm{B}}_{k}(\boldsymbol{x},s) = s\left[\varepsilon^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \varepsilon^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right]\hat{E}^{\mathrm{A}}_{r}(\boldsymbol{x},s)\hat{E}^{\mathrm{B}}_{k}(\boldsymbol{x},s)$$

and

$$\left[\hat{\zeta}^{\mathrm{B}}_{p,j}(\boldsymbol{x},s) - \hat{\zeta}^{\mathrm{A}}_{j,p}(\boldsymbol{x},s)\right]\hat{H}^{\mathrm{A}}_{p}(\boldsymbol{x},s)\hat{H}^{\mathrm{B}}_{j}(\boldsymbol{x},s) = s\left[\mu^{\mathrm{B}}_{p,j}(\boldsymbol{x}) - \mu^{\mathrm{A}}_{j,p}(\boldsymbol{x})\right]\hat{H}^{\mathrm{A}}_{p}(\boldsymbol{x},s)\hat{H}^{\mathrm{B}}_{j}(\boldsymbol{x},s)\,.$$



28.4-3 Domain $\mathcal{D}$, bounded internally by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$.

Exercise 28.4-4

Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 28.4-3.

The reciprocity theorem Equation (28.4-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present, in either state A or state B, and the medium in $\mathcal{D}$ in state B is in its electromagnetic properties adjoint to the one in state A. Prove that

$$\varepsilon_{m,r,p}\int_{x\in\mathcal{S}_1}\nu_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s)-\hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}A$$

$$=\varepsilon_{m,r,p}\int_{x\in\mathcal{S}_2}\nu_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s)-\hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}A\,, \qquad (28.4\text{-}8)$$

i.e. the surface integral is an invariant.

28.5 The complex frequency-domain reciprocity theorem of the time correlation type

The complex frequency-domain reciprocity theorem of the time correlation type follows upon considering the *local interaction quantity* $\varepsilon_{m,r,p}\partial_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,-s)+\hat{E}_r^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\right]$. Using standard rules for spatial differentiation and adjusting the subscripts to later convenience, we obtain

$$\varepsilon_{m,r,p}\partial_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,-s)+\hat{E}_r^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\right]$$
$$=\varepsilon_{j,n,r}\partial_n\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)\right]-\varepsilon_{k,m,p}\partial_m\left[\hat{E}_k^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\right]$$
$$=\left[\varepsilon_{j,n,r}\partial_n\hat{E}_r^{\mathrm{A}}(x,s)\right]\hat{H}_j^{\mathrm{B}}(x,-s)-\hat{E}_r^{\mathrm{A}}(x,s)\left[\varepsilon_{r,n,j}\partial_n\hat{H}_j^{\mathrm{B}}(x,-s)\right]$$
$$+\left[\varepsilon_{p,m,k}\partial_m\hat{E}_k^{\mathrm{B}}(x,-s)\right]\hat{H}_p^{\mathrm{A}}(x,s)-\hat{E}_k^{\mathrm{B}}(x,-s)\left[\varepsilon_{k,m,p}\partial_m\hat{H}_p^{\mathrm{A}}(x,s)\right]. \tag{28.5-1}$$

With the aid of Equations (28.1-14)–(28.1-17), the different terms on the right-hand side become

$$\left[\varepsilon_{j,n,r}\partial_n\hat{E}_r^{\mathrm{A}}(x,s)\right]\hat{H}_j^{\mathrm{B}}(x,-s)$$
$$=-\hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)-\hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)\,, \tag{28.5-2}$$
$$-\hat{E}_r^{\mathrm{A}}(x,s)\left[\varepsilon_{r,n,j}\partial_n\hat{H}_j^{\mathrm{B}}(x,-s)\right]$$
$$=-\hat{\eta}_{r,k}^{\mathrm{B}}(x,-s)\hat{E}_r^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,-s)-\hat{E}_r^{\mathrm{A}}(x,s)\hat{J}_r^{\mathrm{B}}(x,-s)\,, \tag{28.5-3}$$

and

$$\left[\varepsilon_{p,m,k}\partial_m\hat{E}_k^{\mathrm{B}}(x,-s)\right]\hat{H}_p^{\mathrm{A}}(x,s)$$
$$=-\hat{\zeta}_{p,j}^{\mathrm{B}}(x,-s)\hat{H}_j^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)-\hat{K}_p^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\,, \tag{28.5-4}$$
$$-\hat{E}_k^{\mathrm{B}}(x,-s)\left[\varepsilon_{k,m,p}\partial_m\hat{H}_p^{\mathrm{A}}(x,s)\right]$$
$$=-\hat{\eta}_{k,r}^{\mathrm{A}}(x,s)\hat{E}_r^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,-s)-\hat{E}_k^{\mathrm{B}}(x,-s)\hat{J}_k^{\mathrm{A}}(x,s)\,. \tag{28.5-5}$$

Combining Equations (28.5-2)–(28.5-5) with Equation (28.5-1), it is found that

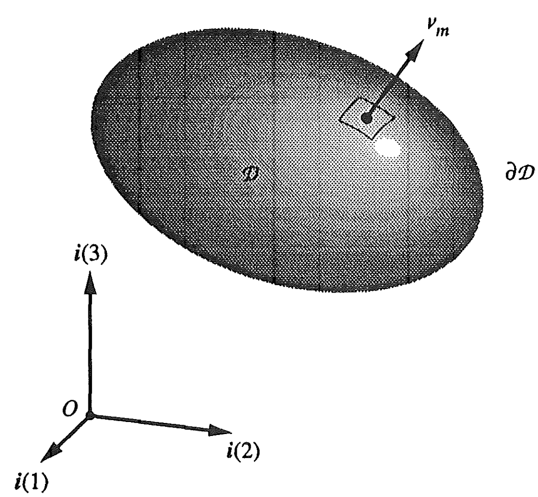
$$\varepsilon_{m,r,p}\partial_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,-s)+\hat{E}_r^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\right]$$
$$=-\left[\hat{\eta}_{r,k}^{\mathrm{B}}(x,-s)+\hat{\eta}_{k,r}^{\mathrm{A}}(x,s)\right]\hat{E}_r^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,-s)$$
$$-\left[\hat{\zeta}_{p,j}^{\mathrm{B}}(x,-s)+\hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)\right]\hat{H}_p^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)$$
$$-\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,-s)-\hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)$$
$$-\hat{J}_r^{\mathrm{B}}(x,-s)\hat{E}_r^{\mathrm{A}}(x,s)-\hat{K}_p^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\,. \tag{28.5-6}$$

Equation (28.5-6) is the *local form of the complex frequency-domain counterpart of the electromagnetic reciprocity theorem of the time correlation type.* The first two terms on the right-hand side are representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; these terms vanish at those locations where $\hat{\eta}_{r,k}^{\mathrm{B}}(x,-s)=-\hat{\eta}_{k,r}^{\mathrm{A}}(x,s)$ and $\hat{\zeta}_{p,j}^{\mathrm{B}}(x,-s)=-\hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)$ for all s in the domain in the complex s plane where Equation (28.5-6) holds. In case the latter conditions hold, the two media are denoted as each other's *time-reverse adjoint*. The last four terms on the right-hand side of Equation (28.5-6) are representative for the action of the sources in the two states; these terms vanish at those locations where no sources are present.

To arrive at the global form of the reciprocity theorem for some bounded domain $\mathcal{D}$, it is assumed that $\mathcal{D}$ is the union of a finite number of subdomains in each of which the terms occurring in Equation (28.5-6) are continuous. Upon integrating Equation (28.5-6) over each of these subdomains, applying Gauss' divergence theorem (Equation (A.12-1)) to the resulting left-hand side, and adding the results, we arrive at (Figure 28.5-1)

$$\begin{aligned}
&\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}} \nu_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,-s)+\hat{E}_r^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}A\\
&=\int_{x\in\mathcal{D}}\Big\{\left[-\hat{\eta}_{r,k}^{\mathrm{B}}(x,-s)-\hat{\eta}_{k,r}^{\mathrm{A}}(x,s)\right]\hat{E}_r^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,-s)\\
&\quad+\left[-\hat{\zeta}_{p,j}^{\mathrm{B}}(x,-s)-\hat{\zeta}_{j,p}^{\mathrm{A}}(x,s)\right]\hat{H}_p^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)\Big\}\,\mathrm{d}V\\
&\quad-\int_{x\in\mathcal{D}}\Big[\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,-s)+\hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,-s)\\
&\quad+\hat{J}_r^{\mathrm{B}}(x,-s)\hat{E}_r^{\mathrm{A}}(x,s)+\hat{K}_p^{\mathrm{B}}(x,-s)\hat{H}_p^{\mathrm{A}}(x,s)\Big]\,\mathrm{d}V\,. \qquad (28.5\text{-}7)
\end{aligned}$$

Equation (28.5-7) is the *global form*, for the bounded domain $\mathcal{D}$, of the *complex frequency-domain counterpart of the electromagnetic reciprocity theorem of the time correlation type*. Note that in the process of adding the contributions from the subdomains of $\mathcal{D}$, the contributions from common interfaces have cancelled in view of the boundary conditions of the continuity type (Equations (28.1-18) and (28.1-19)), and that the contributions from boundary surfaces of



28.5-1 Bounded domain $\mathcal{D}$ with boundary surface $\partial\mathcal{D}$ to which the reciprocity theorems apply.

electromagnetically impenetrable parts of the configuration have vanished in view of the pertaining boundary conditions of the explicit type (Equations (28.1-20) or (28.1-21)). In the left-hand side, therefore, only a contribution from the outer boundary $\partial\mathcal{D}$ of $\mathcal{D}$ remains insofar as parts of this boundary do not coincide with the boundary surface of an electromagnetically impenetrable object. In the right-hand side, the first integral is representative for the differences (contrasts) in the electromagnetic properties of the media present in the two states; this term vanishes if the media in the two states are, throughout $\mathcal{D}$, each other's time-reverse adjoint. The second integral on the right-hand side is representative for the action of the sources in $\mathcal{D}$ in the two states; this term vanishes if no sources are present in $\mathcal{D}$.

The limiting case of an unbounded domain

In quite a number of cases the reciprocity theorem Equation (28.5-7) will be applied to an unbounded domain. To handle such cases, the embedding provisions of Section 28.1 are made and Equation (28.5-7) is first applied to the domain interior to the sphere $\mathcal{S}(\mathcal{O},\Delta)$ with its centre at the origin and of radius Δ, after which the limit $\Delta\to\infty$ is taken (Figure 28.5-2).

Whether or not the surface integral contribution over $\mathcal{S}(\mathcal{O},\Delta)$ does vanish as $\Delta\to\infty$, depends on the nature of the time behaviour of the wave fields in the two states. In case the wave fields in state A and state B are both causal in time (which is the case if both states apply to physical wave fields), the far-field representations of Equations (28.1-22)–(28.1-26) apply for sufficiently large values of Δ. Then, since outside the sphere $\mathcal{S}(\mathcal{O},\Delta_0)$ that is used to define the embedding (see Section 28.1) the media are each other's time-reverse adjoint and no sources are present, the surface integral contribution from $\mathcal{S}(\mathcal{O},\Delta)$ is, in any case, independent of the value of Δ as long as $\Delta > \Delta_0$ (see Exercise 28.5-4). If, however, state A is chosen to be causal and state B is chosen to be anti-causal (which, for example, can apply to the case where state B is a computational one), the contribution from $\mathcal{S}(\mathcal{O},\Delta)$ vanishes for sufficiently large values of Δ.

The complex frequency-domain reciprocity theorem of the time correlation type is mainly used in the modelling of inverse source problems (see Section 28.10) and inverse scattering problems (see Section 28.11).

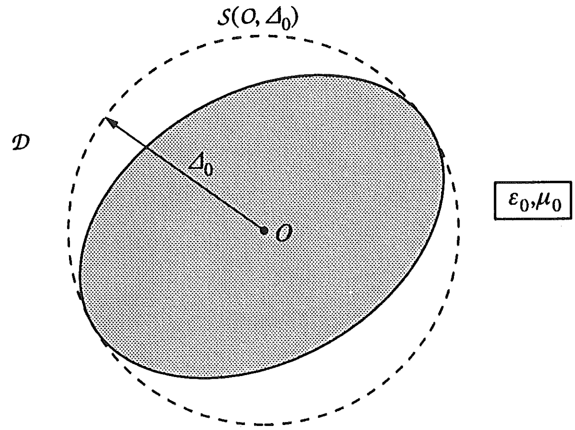
Exercises

Exercise 28.5-1

Show, by taking the Laplace transform with respect to time, that Equation (28.5-6) follows from Equation (28.3-6).

Exercise 28.5-2

Show, by taking the Laplace transform with respect to time, that Equation (28.5-7) follows from Equation (28.3-7).



28.5-2 Unbounded domain $\mathcal{D}$ to which the reciprocity theorems apply. $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity; $\mathcal{S}(O,\Delta_0)$ is the sphere outside which the medium is homogeneous, isotropic and lossless.

Exercise 28.5-3

To what form reduce the contrast-in-media terms in the reciprocity theorems Equations (28.5-6) and (28.5-7) if the media in states A and B are both instantaneously reacting?

Answer:

$$\left[-\hat{\eta}^{\mathrm{B}}_{r,k}(\boldsymbol{x},-s) - \hat{\eta}^{\mathrm{A}}_{k,r}(\boldsymbol{x},s)\right]\hat{E}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{E}^{\mathrm{B}}_k(\boldsymbol{x},-s) = s\left[\varepsilon^{\mathrm{B}}_{r,k}(\boldsymbol{x}) - \varepsilon^{\mathrm{A}}_{k,r}(\boldsymbol{x})\right]\hat{E}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{E}^{\mathrm{B}}_k(\boldsymbol{x},-s)$$

and

$$\left[-\hat{\zeta}^{\mathrm{B}}_{p,j}(\boldsymbol{x},-s) - \hat{\zeta}^{\mathrm{A}}_{j,p}(\boldsymbol{x},s)\right]\hat{H}^{\mathrm{A}}_p(\boldsymbol{x},s)\hat{H}^{\mathrm{B}}_j(\boldsymbol{x},-s) = s\left[\mu^{\mathrm{B}}_{p,j}(\boldsymbol{x}) - \mu^{\mathrm{A}}_{j,p}(\boldsymbol{x})\right]\hat{H}^{\mathrm{A}}_p(\boldsymbol{x},s)\hat{H}^{\mathrm{B}}_j(\boldsymbol{x},-s)\,.$$

Exercise 28.5-4

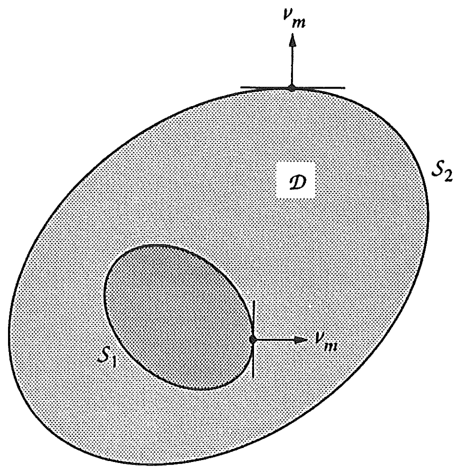
Let $\mathcal{D}$ be the bounded domain that is internally bounded by the closed surface $\mathcal{S}_1$ and externally by the closed surface $\mathcal{S}_2$. The unit vectors along the normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen as shown in Figure 28.5-3.

The reciprocity theorem Equation (28.5-7) is applied to the domain $\mathcal{D}$. In $\mathcal{D}$, no sources are present either in state A or in state B, and the medium in $\mathcal{D}$ in state B is in its electromagnetic properties the time-reverse adjoint to the one in state A. Prove that

$$\varepsilon_{m,r,p}\int_{\boldsymbol{x}\in\mathcal{S}_1}\nu_m\left[\hat{E}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{H}^{\mathrm{B}}_p(\boldsymbol{x},-s) + \hat{E}^{\mathrm{B}}_r(\boldsymbol{x},-s)\hat{H}^{\mathrm{A}}_p(\boldsymbol{x},s)\right]\mathrm{d}A$$

$$= \varepsilon_{m,r,p}\int_{\boldsymbol{x}\in\mathcal{S}_2}\nu_m\left[\hat{E}^{\mathrm{A}}_r(\boldsymbol{x},s)\hat{H}^{\mathrm{B}}_p(\boldsymbol{x},-s) + \hat{E}^{\mathrm{B}}_r(\boldsymbol{x},-s)\hat{H}^{\mathrm{A}}_p(\boldsymbol{x},s)\right]\mathrm{d}A\,, \tag{28.5-8}$$

i.e. the surface integral is an invariant.



28.5-3 Domain $\mathcal{D}$, bounded internally by the closed surface S_1 and externally by the closed surface S_2.

28.6 Transmission/reception reciprocity properties of a pair of electromagnetic antennas

Any electromagnetic system or device, when in operation, emits electromagnetic radiation into its environment and, reciprocally, any electromagnetic system or device is susceptible to an electromagnetic field that is present in its environment. In a number of cases, this phenomenon is an unwanted effect because it can lead to an undesired interaction between systems and devices that are not designed to do so. Quantitatively, the potentiality of a system or device to emit electromagnetic radiation and its ability to pick up an electromagnetic field from its environment are interrelated via reciprocity. These aspects will be further discussed in Chapter 30 in the realm of the ElectroMagnetic Compatibility of electromagnetic systems and devices.

A device or system that is specifically designed to emit electromagnetic radiation into its environment or to pick up an electromagnetic field from its environment, is commonly denoted as an (electromagnetic) *antenna*. An active antenna (that is usually electronically activated via a low-frequency termination) is denoted as a *transmitting antenna*; a passive antenna (that is usually connected to a passive load via a low-frequency termination) is denoted as a *receiving antenna*. Here, too, the potentiality of an electromagnetic device or system to act as a transmitting antenna and the potentiality of such a system to act as a receiving antenna are quantitatively interrelated via reciprocity. This aspect will be further disussed in the present section.

Transmitting antennas can be divided into two types, viz. the ones whose action can be computationally modelled by prescribed values of (volume or surface distributed) *electric current densities*, and the ones whose action can be computationally modelled by prescribed values of the (volume or surface distributed) *magnetic current densities*. Examples of electric current transmitting antennas are the wire electric dipoles as used in radio broadcasting in the medium-wave and short-wave frequency bands. Examples of magnetic current antennas are the wire loop magnetic dipoles as used in mobile communication systems. Also the receiving antennas are divided into two types, viz. the ones whose action can computationally be modelled by their sensitivity to the *electric field strength* (throughout their volume or at their surface), and the ones whose action can computationally be modelled by their sensitivity to the *magnetic field strength* (throughout their volume or at their surface). Examples of electric field sensitive receiving antennas are the electric dipole antennas as used in the reception of television broadcast signals. Examples of magnetic field sensitive receiving antennas are the coil-wound ferrite rod antennas as used in the reception of medium-wave radio broadcast signals.

To analyse the reciprocity properties of the different antennas in their transmitting and receiving situations, we consider the fundamental configuration of two antennas that are surrounded by an arbitrarily inhomogeneous and anisotropic medium. The configuration occupies the entire three-dimensional space $\mathcal{R}^3$. Antenna A occupies the bounded domain Ant_A with boundary surface $\partial\mathrm{Ant}_\mathrm{A}$, and unit vector along the normal ν_m oriented away from Ant_A. Antenna B occupies the bounded domain Ant_B with boundary surface $\partial\mathrm{Ant}_\mathrm{B}$, and unit vector along the normal ν_m oriented away from Ant_B. The domain exterior to $\mathrm{Ant}_\mathrm{A}\cup\partial\mathrm{Ant}_\mathrm{A}$ is denoted by Ant_A'; the domain exterior to $\mathrm{Ant}_\mathrm{B}\cup\partial\mathrm{Ant}_\mathrm{B}$ is denoted by Ant_B'. The domains Ant_A and Ant_B are disjoint (Figure 28.6-1).

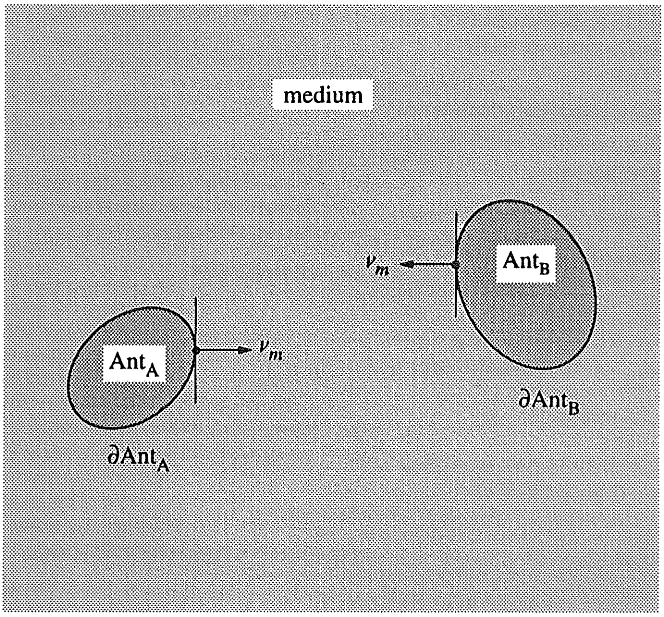
As to the boundary conditions across interfaces between parts of the surrounding medium with different electromagnetic properties and the boundary conditions at the boundary surfaces of electromagnetically impenetrable objects, the provisions necessary for the global reciprocity theorems to hold are made. The standard limiting procedure of Section 28.1 for the handling of an unbounded domain applies. Since transmission and reception are both causal phenomena, the transmission/reception reciprocity properties are based on the reciprocity theorems (28.2-7) and (28.4-7) of the time convolution type, in which theorem causality is preserved.

Volume action antennas

A *volume action antenna* is characterised by the property that in the transmitting mode its action can be accounted for by prescribed values of the volume source densities of electric current and magnetic current, whose common support is the domain occupied by that antenna, while in the receiving mode it is sensitive to the electric field strength and/or the magnetic field strength over the domain it occupies. To investigate the transmission/reception reciprocity properties of a pair of such antennas, we take state A to be the causal electromagnetic state for which the volume source densities have the support Ant_A (i.e. in state A, Antenna A is the transmitting antenna and Antenna B is the receiving antenna). Furthermore, we take state B to be the causal electromagnetic state for which the volume source densities have the support Ant_B (i.e. in state B, Antenna B is the transmitting antenna and Antenna A is the receiving antenna). Application of the global time-domain reciprocity theorem of the time convolution type, Equation (28.2-7), to the entire configuration yields, assuming the embedding medium to be self-adjoint,

$$\int_{x\in\mathrm{Ant}_\mathrm{A}} \left[\mathrm{C}_t(J_k^\mathrm{A},E_k^\mathrm{B};x,t) - \mathrm{C}_t(K_j^\mathrm{A},H_j^\mathrm{B};x,t)\right]\mathrm{d}V$$

$$= \int_{x\in\mathrm{Ant}_\mathrm{B}} \left[\mathrm{C}_t(J_r^\mathrm{B},E_r^\mathrm{A};x,t) - \mathrm{C}_t(K_p^\mathrm{B},H_p^\mathrm{A};x,t)\right]\mathrm{d}V\,. \qquad (28.6\text{-}1)$$



28.6-1 Configuration for the transmission/reception reciprocity properties of a pair of electromagnetic antennas Ant_A and Ant_B.

The complex frequency-domain counterpart of Equation (28.6-1) follows from Equation (28.4-7) as

$$\int_{x \in \mathrm{Ant}_\mathrm{A}} \left[\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s) - \hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)\right]\mathrm{d}V$$

$$= \int_{x \in \mathrm{Ant}_\mathrm{B}} \left[\hat{J}_r^{\mathrm{B}}(x,s)\hat{E}_r^{\mathrm{A}}(x,s) - \hat{K}_p^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}V\,. \qquad (28.6\text{-}2)$$

In Equations (28.6-1) and (28.6-2), the terms containing the volume source densities of electric current are representative for the action of the antenna as a (volume distributed) *electric current transmitting antenna*, while the terms containing the volume densities of magnetic current are representative for the action of the antenna as a (volume distributed) *magnetic current transmitting antenna*. Furthermore, the terms containing the electric field strength quantify the action of the antenna as a (volume distributed) *electric field receiving antenna*, while the terms containing the magnetic field strength quantify the action of the antenna as a (volume distributed) *magnetic field receiving antenna*. From Equations (28.6-1) and (28.6-2), and hence from the principle of reciprocity, it is concluded that a spatially distributed electric current antenna is only sensitive to the electric field (and insensitive to the magnetic field), while a spatially distributed magnetic current antenna is only sensitive to the magnetic field (and insensitive to the electric field). The reciprocity relations imply that the different sensitivities are related (viz. through Equations (28.6-1) and (28.6-2)).

Surface action antennas

A *surface action antenna* is characterised by the property that in its transmitting mode its action can be accounted for by prescribed values of the tangential components of the electric field strength and the magnetic field strength at its boundary surface, while in its receiving mode it is sensitive to the tangential components of the electric field strength and/or the magnetic field strength at that surface. This description of the action of the antenna is employed when the description of its action by volume sources is either inapplicable or irrelevant. (Most applications of this type of antenna are found in microwave engineering, where surface action antennas are commonly denoted as *"aperture antennas"*.) To investigate the reciprocity properties of a pair of such antennas, we take state A to be the causal electromagnetic state for which the prescribed surface source densities have the support $\partial\mathrm{Ant}_{\mathrm{A}}$ (i.e. in state A, Antenna A is the transmitting antenna and Antenna B is the receiving antenna). Furthermore, we take state B to be the causal electromagnetic state for which the prescribed surface source densities have the support $\partial\mathrm{Ant}_{\mathrm{B}}$ (i.e. in state B, Antenna B is the transmitting antenna and Antenna A is the receiving antenna). Application of the global time-domain reciprocity theorem of the time-convolution type, Equation (28.2-7), to the entire domain $\mathcal{R}^3 \cap \mathrm{Ant}'_{\mathrm{A}} \cap \mathrm{Ant}'_{\mathrm{B}}$ exterior to the antennas yields, assuming the embedding medium to be self-adjoint,

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}_{\mathrm{A}}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t) - \mathrm{C}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,t)\right]\mathrm{d}A$$

$$= \varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}_{\mathrm{B}}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,t) - \mathrm{C}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t)\right]\mathrm{d}A\,. \qquad (28.6\text{-}3)$$

The complex frequency-domain counterpart of Equation (28.6-3) follows from Equation (28.4-7) as

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}_{\mathrm{A}}} \nu_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s) - \hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}A$$

$$= \varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}_{\mathrm{B}}} \nu_m\left[\hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s) - \hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s)\right]\mathrm{d}A\,. \qquad (28.6\text{-}4)$$

In Equations (28.6-3) and (28.6-4), the terms containing the source densities of electric surface current (i.e. $\varepsilon_{r,m,p}\nu_m H_p$) are representative for the action of the antenna as a (surface distributed) *electric current transmitting antenna*, while the terms containing the source densities of magnetic surface current (i.e. $-\varepsilon_{j,n,r}\nu_n E_r$) are representative for the action of the antenna as a (surface distributed) *magnetic current transmitting antenna*. Furthermore, the terms containing the tangential electric field strength quantify the sensitivity of the antenna as a (surface distributed) *electric field receiving antenna*, while the terms containing the tangential magnetic field strength quantify the sensitivity of the antenna as a (surface distributed) *magnetic field receiving antenna*. From Equations (28.6-3) and (28.6-4) it is concluded that a surface distributed electric current antenna only senses the electric field, while a surface distributed magnetic current antenna only senses the magnetic field. The reciprocity relations imply that the different sensitivities are related (viz. through Equations (28.6-3) and (28.6-4)).

Exercises

Exercise 28.6-1

Use Equation (28.2-7) to derive the time-domain transmission/reception reciprocity theorem for a pair of antennas A and B if Antenna A is a volume action antenna and Antenna B is a surface action antenna. (Note the orientation of the unit vector ν_m along the normal on $\partial\mathrm{Ant}_\mathrm{B}$.)

Answer:

$$\int_{x\in\mathrm{Ant}_\mathrm{A}} \left[\mathrm{C}_t(J_k^{\mathrm{A}},E_k^{\mathrm{B}};x,t) - \mathrm{C}_t(K_j^{\mathrm{A}},H_j^{\mathrm{B}};x,t)\right] \mathrm{d}V$$

$$= \varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}_\mathrm{B}} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,t) - \mathrm{C}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t)\right] \mathrm{d}A \,. \tag{28.6-5}$$

Exercise 28.6-2

Use Equation (28.4-7) to derive the complex frequency-domain transmission/reception reciprocity theorem for a pair of antennas A and B if Antenna A is a volume action antenna and Antenna B is a surface action antenna. (Note the orientation of the unit vector ν_m along the normal on $\partial\mathrm{Ant}_\mathrm{B}$.)

Answer:

$$\int_{x\in\mathrm{Ant}_\mathrm{A}} \left[\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s) - \hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)\right] \mathrm{d}V$$

$$= \varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}_\mathrm{B}} \nu_m \left[\hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s) - \hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s)\right] \mathrm{d}A \,. \tag{28.6-6}$$

Exercise 28.6-3

If in the interior of Ant_A and Ant_B the electromagnetic field quantities would be set equal to zero and these wave-field quantities would on $\partial\mathrm{Ant}_\mathrm{A}$ and $\partial\mathrm{Ant}_\mathrm{B}$ jump to their respective boundary values, the jumps would, on account of Equations (28.1-1)–(28.1-4), give rise to surface electric current densities with equivalent volume densities

$$J_k^{\mathrm{A,B}} = \varepsilon_{k,m,p}\nu_m H_p^{\mathrm{A,B}}\delta_{\partial\mathrm{Ant}_{\mathrm{A,B}}}(x)$$

and surface magnetic current densities with equivalent volume densities

$$K_j^{\mathrm{A,B}} = -\varepsilon_{j,n,r}\nu_n E_r^{\mathrm{A,B}}\delta_{\partial\mathrm{Ant}_{\mathrm{A,B}}}(x) \,,$$

where $\delta_{\mathcal{S}}(x)$ is the surface Dirac delta distribution operative on the surface $\mathcal{S}$. Show, by taking the time convolution of the inner products of $J_k^{\mathrm{A,B}}$ with $E_k^{\mathrm{B,A}}$ and of $K_j^{\mathrm{A,B}}$ with $H_j^{\mathrm{B,A}}$, that in this physical picture Equation (28.6-3) is compatible with Equation (28.6-1).

Exercise 28.6-4

Show, in a manner similar to Exercise 28.6-3, that Equation (28.6-4) is compatible with Equation (28.6-2).

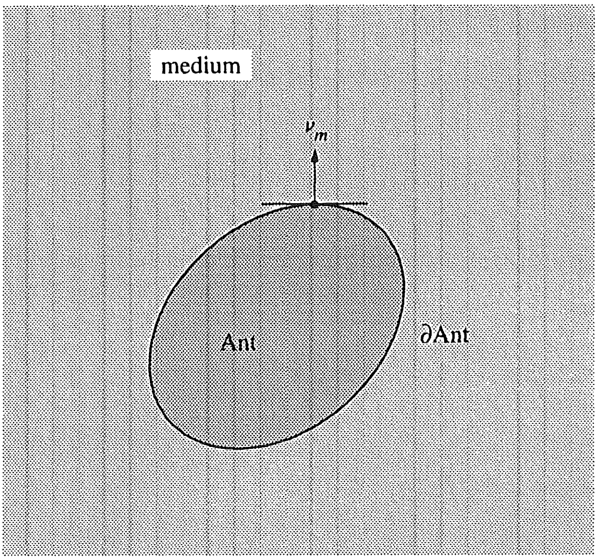
28.7 Transmission/reception reciprocity properties of a single electromagnetic antenna

To analyse the transmission/reception reciprocity properties of a single antenna, we consider the fundamental configuration of a single antenna surrounded by an arbitrarily inhomogeneous and anisotropic medium. The configuration occupies the entire three-dimensional space $\mathcal{R}^3$. The antenna occupies the bounded domain Ant, with boundary surface ∂Ant and unit vector along the normal ν_m oriented away from Ant (Figure 28.7-1).

The domain exterior to Ant$\cup\partial$Ant is denoted by Ant$'$. As to the boundary conditions across interfaces between parts of the configuration with different electromagnetic medium properties and the boundary conditions at the boundary surfaces of electromagnetically impenetrable objects, the provisions necessary for the global reciprocity theorems to hold are made. The standard limiting procedure of Section 28.1 for the handling of an unbounded domain applies. Since transmission and reception are both causal phenomena, the transmission/reception reciprocity properties are based on the reciprocity theorems (28.2-7) and (28.4-7) of the time convolution type in which theorems causality is preserved.

Volume action antenna

If Ant is a *volume action antenna*, its action in the transmitting mode is accounted for by prescribed values of the volume source densities of electric and magnetic current, whose support is the domain occupied by the antenna, while in the receiving mode it is sensitive to the electric and magnetic field strengths over the domain it occupies. To investigate the transmission/



28.7-1 Configuration for the transmission/reception reciprocity properties of a single electromagnetic antenna Ant.

reception reciprocity properties of a single antenna of this kind, state A is taken to be the causal state associated with the wave field $\{E_r^{\mathrm{T}},H_p^{\mathrm{T}}\}$ generated by the prescribed volume source densities $\{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\}$ whose support is Ant. This state is denoted as the *transmitting state* and will be denoted by the superscript T. Next, state B is taken to be the causal state associated with the wave field that is generated by unspecified sources located in the domain Ant′ exterior to the antenna. In the surrounding medium these sources would generate an *incident wave field* $\{E_r^{\mathrm{i}},H_p^{\mathrm{i}}\}$ if the antenna were not activated. The total wave field $\{E_r^{\mathrm{R}},H_p^{\mathrm{R}}\}$ in the presence of the antenna is then the superposition of the incident wave field and the *scattered wave field* $\{E_r^{\mathrm{s}},H_p^{\mathrm{s}}\}$, i.e.

$$\{E_r^{\mathrm{R}},H_p^{\mathrm{R}}\} = \{E_r^{\mathrm{i}} + E_r^{\mathrm{s}},H_p^{\mathrm{i}} + H_p^{\mathrm{s}}\} . \tag{28.7-1}$$

The relevant state is denoted as the *receiving state* and will be denoted by the superscript R. Note that in the receiving state the domain Ant occupied by the antenna is source-free and that the scattered wave field in this state is source-free in the domain Ant′ exterior to the domain occupied by the antenna. Application of the time-domain reciprocity theorem of the time-convolution type, Equation (28.2-7), to the transmitted and the scattered wave fields and to the domain Ant′ exterior to the transducer yields, assuming the medium to be self-adjoint,

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{T}},H_p^{\mathrm{s}};x,t) - \mathrm{C}_t(E_r^{\mathrm{s}},H_p^{\mathrm{T}};x,t)\right]\mathrm{d}A = 0 . \tag{28.7-2}$$

Here, we have used the property that the total wave field in the transmitting state and the scattered wave field in the receiving state are both source-free in the domain Ant′ exterior to the transducer and causally related to the action of (primary or secondary) source distributions with the domain Ant occupied by the antenna as their supports, on account of which both the volume integral over the domain exterior to the antenna and the surface integral over the outer boundary of the domain of application of Equation (28.2-7) vanish. Using Equation (28.7-1), it follows from Equation (28.7-2) that

$$\begin{aligned}&\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{T}},H_p^{\mathrm{R}};x,t) - \mathrm{C}_t(E_r^{\mathrm{R}},H_p^{\mathrm{T}};x,t)\right]\mathrm{d}A\\ &= \varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}} \nu_m\left[\mathrm{C}_t(E_p^{\mathrm{T}},H_r^{\mathrm{i}};x,t) - \mathrm{C}_t(E_p^{\mathrm{i}},H_r^{\mathrm{T}};x,t)\right]\mathrm{d}A .\end{aligned} \tag{28.7-3}$$

Next, Equation (28.2-7) is applied to the total wave fields in the transmitting and the receiving states and to the domain Ant occupied by the antenna. This yields, again assuming the medium to be self-adjoint,

$$\begin{aligned}&\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}} \nu_m\left[\mathrm{C}_t(E_p^{\mathrm{T}},H_r^{\mathrm{R}};x,t) - \mathrm{C}_t(E_p^{\mathrm{R}},H_r^{\mathrm{T}};x,t)\right]\mathrm{d}A\\ &= \int_{x\in\mathrm{Ant}} \left[\mathrm{C}_t(J_k^{\mathrm{T}},E_k^{\mathrm{R}};x,t) - \mathrm{C}_t(K_j^{\mathrm{T}},H_j^{\mathrm{R}};x,t)\right]\mathrm{d}V .\end{aligned} \tag{28.7-4}$$

Combining Equations (28.7-3) and (28.7-4), and using the continuity of the tangential components of the electric and the magnetic field strengths across ∂Ant in both states, we arrive at

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{T}},H_p^{\mathrm{i}};x,t)-\mathrm{C}_t(E_r^{\mathrm{i}},H_p^{\mathrm{T}};x,t)\right]\mathrm{d}A$$

$$=\int_{x\in\mathrm{Ant}}\left[\mathrm{C}_t(J_k^{\mathrm{T}},E_k^{\mathrm{R}};x,t)-\mathrm{C}_t(K_j^{\mathrm{T}},H_j^{\mathrm{R}};x,t)\right]\mathrm{d}V\,. \tag{28.7-5}$$

The complex frequency-domain counterpart of Equation (28.7-5) follows, in a similar manner, from Equation (28.4-7) as

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathrm{Ant}} \nu_m\left[\hat{E}_r^{\mathrm{T}}(x,s)\hat{H}_p^{\mathrm{i}}(x,s)-\hat{E}_r^{\mathrm{i}}(x,s)\hat{H}_p^{\mathrm{T}}(x,s)\right]\mathrm{d}A$$

$$=\int_{x\in\mathrm{Ant}}\left[\hat{J}_k^{\mathrm{T}}(x,s)\hat{E}_k^{\mathrm{R}}(x,s)-\hat{K}_j^{\mathrm{T}}(x,s)\hat{H}_j^{\mathrm{R}}(x,s)\right]\mathrm{d}V\,. \tag{28.7-6}$$

In view of what has been found in Section 28.6, the right-hand sides of Equations (28.7-5) and (28.7-6) are representative for the sensitivity of the antenna to a received electromagnetic field generated elsewhere in the domain exterior to the antenna. The left-hand sides express that the antenna can, in the receiving state, be conceived as to be excited, across its boundary surface, by the incident wave field. Equations (28.7-5) and (28.7-6) relate these two aspects quantitatively.

Surface action antenna

If Ant is a *surface action antenna*, its action in the transmitting mode is accounted for by prescribed values of the tangential components of the electric and the magnetic field strength at its boundary surface, while in the receiving mode it is sensitive to the tangential components of the electric field strength and the magnetic field strength at that surface. This description of the action of the antenna is employed for the typical "aperture antennas" used in radar applications. To investigate the transmission/reception reciprocity properties of a single antenna of this kind, state A is, as above, taken to be the causal state associated with the wave field $\{E_r^{\mathrm{T}},H_p^{\mathrm{T}}\}$ generated by the prescribed surface source densities of electric current (i.e. $\varepsilon_{r,m,p}\nu_m H_p^{\mathrm{T}}$) and of magnetic current (i.e. $-\varepsilon_{j,n,r}\nu_n E_r^{\mathrm{T}}$), whose support is ∂Ant. This state is denoted as the *transmitting state* and will be denoted by the superscript T. Next, state B is taken to be the causal state associated with the wave field that is generated by unspecified sources located in the domain Ant′ exterior to the antenna. In the surrounding medium these sources would generate an *incident wave field* $\{E_r^{\mathrm{i}},H_p^{\mathrm{i}}\}$ if the antenna were not activated. The total wave field $\{E_r^{\mathrm{R}},H_p^{\mathrm{R}}\}$ in the presence of the antenna is again the superposition of the incident wave field and the *scattered wave field* $\{E_r^{\mathrm{s}},H_p^{\mathrm{s}}\}$, i.e.

$$\{E_r^{\mathrm{R}},H_p^{\mathrm{R}}\}=\{E_r^{\mathrm{i}}+E_r^{\mathrm{s}},H_p^{\mathrm{i}}+H_p^{\mathrm{s}}\}\,. \tag{28.7-7}$$

The relevant state is denoted as the *receiving state* and will be denoted by the superscript R. Note that in the receiving state the scattered wave field is source-free in the domain Ant′ exterior to the domain occupied by the antenna. Application of the time-domain reciprocity theorem of the time convolution type, Equation (28.2-7), to the transmitted and the scattered wave fields and to the domain Ant′ exterior to the antenna yields, assuming the medium to be self-adjoint,

$$\varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{T}}, H_p^{\mathrm{s}}; x,t) - \mathrm{C}_t(E_r^{\mathrm{s}}, H_p^{\mathrm{T}}; x,t) \right] \mathrm{d}A = 0 \,. \tag{28.7-8}$$

Next, using Equation (28.7-7), it follows that

$$\varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{T}}, H_p^{\mathrm{R}}; x,t) - \mathrm{C}_t(E_r^{\mathrm{R}}, H_p^{\mathrm{T}}; x,t) \right] \mathrm{d}A$$

$$= \varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{T}}, H_p^{\mathrm{i}}; x,t) - \mathrm{C}_t(E_r^{\mathrm{i}}, H_p^{\mathrm{T}}; x,t) \right] \mathrm{d}A \,. \tag{28.7-9}$$

The complex frequency-domain counterpart of Equation (28.7-9) follows, in a similar manner, from Equation (28.4-7) as

$$\varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}} \nu_m \left[\hat{E}_r^{\mathrm{T}}(x,s) \hat{H}_p^{\mathrm{R}}(x,s) - \hat{E}_r^{\mathrm{R}}(x,s) \hat{H}_p^{\mathrm{T}}(x,s) \right] \mathrm{d}A$$

$$= \varepsilon_{m,r,p} \int_{x\in\partial\mathrm{Ant}} \nu_m \left[\hat{E}_r^{\mathrm{T}}(x,s) \hat{H}_p^{\mathrm{i}}(x,s) - \hat{E}_r^{\mathrm{i}}(x,s) \hat{H}_p^{\mathrm{T}}(x,s) \right] \mathrm{d}A \,. \tag{28.7-10}$$

In view of what has been found in Section 28.6, the left-hand sides of Equations (28.7-9) and (28.7-10) are representative for the sensitivity of the antenna to a received electromagnetic field generated elsewhere in the domain exterior to the antenna. The right-hand sides express that the antenna can, in the receiving state, be conceived as to be excited, across its boundary surface, by the incident wave field. Equations (28.7-9) and (28.7-10) relate these two aspects quantitatively.

28.8 The direct (forward) source problem. Point-source solutions and Green's functions

In the direct (or forward) source problem we want to express the electromagnetic field quantities in a configuration with given electromagnetic properties in terms of the source distributions that generate the wave field. The domain in which expressions for the generated electromagnetic wave field $\{E_r^{\mathrm{T}}, H_p^{\mathrm{T}}\} = \{E_r^{\mathrm{T}}, H_p^{\mathrm{T}}\}(x,t)$ are to be found is in general the entire three-dimensional space $\mathcal{R}^3$. The standard provisions of Section 28.1 for the handling of an unbounded domain are made. Since $\{E_r^{\mathrm{T}}, H_p^{\mathrm{T}}\}$ is a physical wave field, it satisfies the condition of causality. The source distributions $\{J_k^{\mathrm{T}}, K_j^{\mathrm{T}}\} = \{J_k^{\mathrm{T}}, K_j^{\mathrm{T}}\}(x,t)$ that generate the wave field, have the bounded support $\mathcal{D}^T$ (Figure 28.8-1).

The electromagnetic properties of the medium present in the configuration are characterised by the relaxation functions $\{\varepsilon_{k,r}, \mu_{j,p}\} = \{\varepsilon_{k,r}, \mu_{j,p}\}(x,t)$, which are causal functions of time. The case of an instantaneously reacting medium easily follows from the more general case of a medium with relaxation.

Time-domain analysis

For the time-domain analysis of the problem the global reciprocity theorem of the time-convolution type, Equation (28.2-7), is taken as the point of departure. In it, state A is taken to be the generated electromagnetic wave field under consideration, i.e.

$$\{E_r^{\mathrm{A}}, H_p^{\mathrm{A}}\} = \{E_r^{\mathrm{T}}, H_p^{\mathrm{T}}\}(\boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{R}^3, \tag{28.8-1}$$

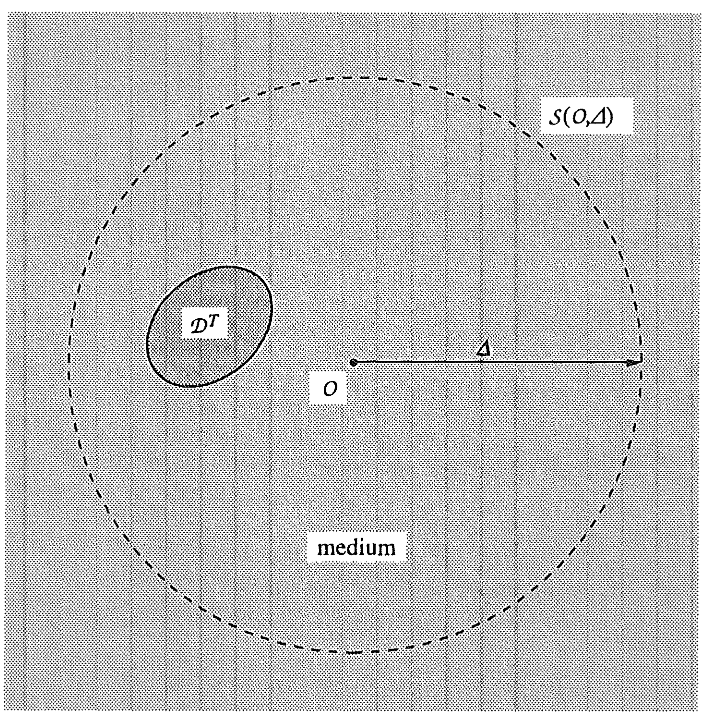
$$\{J_k^{\mathrm{A}}, K_j^{\mathrm{A}}\} = \{J_k^{\mathrm{T}}, K_j^{\mathrm{T}}\}(\boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^T, \tag{28.8-2}$$

and

$$\{\varepsilon_{k,r}^{\mathrm{A}}, \mu_{j,p}^{\mathrm{A}}\} = \{\varepsilon_{k,r}, \mu_{j,p}\}(\boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{R}^3 . \tag{28.8-3}$$

Next, state B is chosen such that the application of Equation (28.2-7) to the domain $\mathcal{D}$ leads to the values of $\{E_r^{\mathrm{T}}, H_p^{\mathrm{T}}\}$ at some arbitrary point $\boldsymbol{x}' \in \mathcal{R}^3$. Inspection of the right-hand side of Equation (28.2-7) reveals that this is accomplished if we take for the source distributions of state B a point source of electric current at $\boldsymbol{x}'$ in case we want an expression for the electric field strength at $\boldsymbol{x}'$ and a point source of magnetic current at $\boldsymbol{x}'$ in case we want an expression for the magnetic field strength at $\boldsymbol{x}'$, while the medium in state B must be taken to be adjoint to the one in state A, i.e.

$$\{\varepsilon_{r,k}^{\mathrm{B}}, \mu_{p,j}^{\mathrm{B}}\} = \{\varepsilon_{k,r}, \mu_{j,p}\}(\boldsymbol{x}, t) \qquad \text{for } \boldsymbol{x} \in \mathcal{R}^3 . \tag{28.8-4}$$



28.8-1 Configuration for the direct (forward) source problem. $\mathcal{D}^T$ is the bounded support of the source distributions. The configuration occupies the entire $\mathcal{R}^3$; $\mathcal{S}(O,\Delta)$ is the bounding sphere that recedes to infinity.

Furthermore, if in the configuration electromagnetically impenetrable objects are present, the electromagnetic field in state B must satisfy on the boundaries of these electromagnetically impenetrable objects the same boundary conditions as in state A, while the electromagnetic field in state B must be causally related to the action of its (point) sources. The two choices for the source distributions will be discussed separately below.

First, we choose

$$J_r^{\mathrm{B}} = a_r \delta(x - x', t) \quad \text{and} \quad K_p^{\mathrm{B}} = 0 , \tag{28.8-5}$$

where $\delta(x - x', t)$ represents the four-dimensional unit impulse (Dirac distribution) operative at the point $x = x'$ and at the instant $t = 0$, while a_r is an arbitrary constant vector. The electromagnetic field causally radiated by this source is denoted as

$$\{E_k^{\mathrm{B}}, H_j^{\mathrm{B}}\} = \{E_k^{J;\mathrm{B}}, H_j^{J;\mathrm{B}}\}(x,x',t) , \tag{28.8-6}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (28.2-7) is applied to $\mathcal{R}^3$. The standard provisions of Section 28.1 for the handling of an unbounded domain yield

$$\begin{aligned} &\varepsilon_{m,r,p} \int_{x \in \mathcal{S}(O,\Delta)} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{A}}, H_p^{\mathrm{B}}; x,t) - \mathrm{C}_t(E_r^{\mathrm{B}}, H_p^{\mathrm{A}}; x,t) \right] \mathrm{d}A \\ &= \varepsilon_{m,r,p} \int_{x \in \mathcal{S}(O,\Delta)} \nu_m \left[\mathrm{C}_t(E_r^{\mathrm{T}}, H_p^{J;\mathrm{B}}; x,x',t) - \mathrm{C}_t(E_r^{J;\mathrm{B}}, H_p^{\mathrm{T}}; x,x',t) \right] \mathrm{d}A \to 0 \\ &\qquad \text{as } \Delta \to \infty . \end{aligned} \tag{28.8-7}$$

Furthermore, in view of Equation (28.8-5) and the properties of $\delta(x - x', t)$,

$$\begin{aligned} &\int_{x \in \mathcal{R}^3} \left[\mathrm{C}_t(J_r^{\mathrm{B}}, E_r^{\mathrm{A}}; x,t) - \mathrm{C}_t(K_p^{\mathrm{B}}, H_p^{\mathrm{A}}; x,t) \right] \mathrm{d}V \\ &= \int_{x \in \mathcal{R}^3} \mathrm{C}_t(a_r \delta(x - x', t), E_r^{\mathrm{T}}; x,t) \, \mathrm{d}V = a_r E_r^{\mathrm{T}}(x', t) . \end{aligned} \tag{28.8-8}$$

Since, further, the sources have the support $\mathcal{D}^T$,

$$\begin{aligned} &\int_{x \in \mathcal{R}^3} \left[\mathrm{C}_t(J_k^{\mathrm{A}}, E_k^{\mathrm{B}}; x,t) - \mathrm{C}_t(K_j^{\mathrm{A}}, H_j^{\mathrm{B}}; x,t) \right] \mathrm{d}V \\ &= \int_{x \in \mathcal{D}^T} \left[\mathrm{C}_t(J_k^{\mathrm{T}}, E_k^{J;\mathrm{B}}; x,x',t) - \mathrm{C}_t(K_j^{\mathrm{T}}, H_j^{J;\mathrm{B}}; x,x',t) \right] \mathrm{d}V . \end{aligned} \tag{28.8-9}$$

Collecting the results, we arrive at

$$a_r E_r^{\mathrm{T}}(x', t) = \int_{x \in \mathcal{D}^T} \left[\mathrm{C}_t(E_k^{J;\mathrm{B}}, J_k^{\mathrm{T}}; x,x',t) - \mathrm{C}_t(H_j^{J;\mathrm{B}}, K_k^{\mathrm{T}}; x,x',t) \right] \mathrm{d}V \quad \text{for } x' \in \mathcal{R}^3 , \tag{28.8-10}$$

where, in the right-hand side, we have used the symmetry of the convolution in its functional arguments. From Equation (28.8-10) a representation for $E_r^{\mathrm{T}}(x', t)$ is obtained by taking into account that $E_k^{J;\mathrm{B}}$ and $H_j^{J;\mathrm{B}}$ are linearly related to a_r. The latter relationship is expressed by

$$\{E_k^{J;\mathrm{B}}, H_j^{J;\mathrm{B}}\}(x,x',t) = \{G_{k,r}^{EJ;\mathrm{B}}, G_{j,r}^{HJ;\mathrm{B}}\}(x,x',t) a_r . \tag{28.8-11}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 28.8-1 and 28.8-3)

$$\{G_{k,r}^{EJ;\mathrm{B}},G_{j,r}^{HJ;\mathrm{B}}\}(x,x',t)=\{G_{r,k}^{EJ},-G_{r,j}^{EK}\}(x',x,t) \tag{28.8-12}$$

hold, Equation (28.8-10) leads, with Equations (28.8-11) and (28.8-12), and invoking the condition that the resulting equation has to hold for arbitrary values of a_r, to the final result

$$E_r^{\mathrm{T}}(x',t)=\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(G_{r,k}^{EJ},J_k^{\mathrm{T}};x',x,t)+\mathrm{C}_t(G_{r,j}^{EK},K_j^{\mathrm{T}};x',x,t)\right]\mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 . \tag{28.8-13}$$

Equation (28.8-13) expresses the electric field strength E_r^{T} of the generated electromagnetic field at x' as the superposition of the contributions from the elementary distributed sources $J_k^{\mathrm{T}}\,\mathrm{d}V$ and $K_j^{\mathrm{T}}\,\mathrm{d}V$ at x. The intervening kernel functions are the *electric-field/electric-current Green's function* $G_{r,k}^{EJ}=G_{r,k}^{EJ}(x',x,t)$ and the *electric-field/magnetic-current Green's function* $G_{r,j}^{EK}=G_{r,j}^{EK}(x',x,t)$. These Green's functions are the electric field strength at x', radiated in the actual medium with constitutive parameters $\{\varepsilon_{k,r},\mu_{j,p}\}=\{\varepsilon_{k,r},\mu_{j,p}\}(x,t)$, by a point source of electric current at x and a point source of magnetic current at x, respectively.

Secondly, we choose

$$J_r^{\mathrm{B}}=0 \quad \text{and} \quad K_p^{\mathrm{B}}=b_p\delta(x-x',t)\,, \tag{28.8-14}$$

where b_p is an arbitrary constant vector. The electromagnetic field causally radiated by this source is denoted as

$$\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\}=\{E_k^{K;\mathrm{B}},H_j^{K;\mathrm{B}}\}(x,x',t)\,, \tag{28.8-15}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (28.2-7) is applied to $\mathcal{R}^3$. The standard provisions of Section 28.1 for the handling of an unbounded domain yield

$$\begin{aligned}&\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{A}},H_p^{\mathrm{B}};x,t)-\mathrm{C}_t(E_r^{\mathrm{B}},H_p^{\mathrm{A}};x,t)\right]\mathrm{d}A\\&=\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{T}},H_p^{K;\mathrm{B}};x,x',t)-\mathrm{C}_t(E_r^{K;\mathrm{B}},H_p^{\mathrm{T}};x,x',t)\right]\mathrm{d}A\to 0\\&\qquad\text{as } \Delta\to\infty .\end{aligned} \tag{28.8-16}$$

Furthermore, in view of Equation (28.8-14) and the properties of $\delta(x-x',t)$,

$$\begin{aligned}&\int_{x\in\mathcal{R}^3}\left[\mathrm{C}_t(J_r^{\mathrm{B}},E_r^{\mathrm{A}};x,t)-\mathrm{C}_t(K_p^{\mathrm{B}},H_p^{\mathrm{A}};x,t)\right]\mathrm{d}V\\&=-\int_{x\in\mathcal{R}^3}\mathrm{C}_t(b_p\delta(x-x',t),H_p^{\mathrm{T}};x,t)\,\mathrm{d}V=-b_pH_p^{\mathrm{T}}(x',t)\,.\end{aligned} \tag{28.8-17}$$

Since, further, the sources have the support $\mathcal{D}^T$,

$$\begin{aligned}&\int_{x\in\mathcal{R}^3}\left[\mathrm{C}_t(J_k^{\mathrm{A}},E_k^{\mathrm{B}};x,t)-\mathrm{C}_t(K_j^{\mathrm{A}},H_j^{\mathrm{B}};x,t)\right]\mathrm{d}V\\&=\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(J_k^{\mathrm{T}},E_k^{K;\mathrm{B}};x,x',t)-\mathrm{C}_t(K_j^{\mathrm{T}},H_j^{K;\mathrm{B}};x,x',t)\right]\mathrm{d}V\,.\end{aligned} \tag{28.8-18}$$

Collecting the results, we arrive at

$$b_p H_p^{\mathrm{T}}(x',t) = \int_{x \in \mathcal{D}^T} \left[-\mathrm{C}_t(E_k^{K;\mathrm{B}}, J_k^{\mathrm{T}}; x, x', t) + \mathrm{C}_t(H_j^{K;\mathrm{B}}, K_j^{\mathrm{T}}; x, x', t) \right] \mathrm{d}V$$

$$\text{for } x' \in \mathcal{R}^3 , \tag{28.8-19}$$

where, in the right-hand side, we have used the symmetry of the convolution in its functional arguments. From Equation (28.8-19) a representation for $H_p^{\mathrm{T}}(x',t)$ is obtained by taking into account that $E_k^{K;\mathrm{B}}$ and $H_j^{K;\mathrm{B}}$ are linearly related to b_p. The latter relationship is expressed by

$$\{E_k^{K;\mathrm{B}}, H_j^{K;\mathrm{B}}\}(x,x',t) = \{G_{k,p}^{EK;\mathrm{B}}, G_{j,p}^{HK;\mathrm{B}}\}(x,x',t)\, b_p . \tag{28.8-20}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 28.8-2 and 28.8-4)

$$\{G_{k,p}^{EK;\mathrm{B}}, G_{j,p}^{HK;\mathrm{B}}\}(x,x',t) = \{-G_{p,k}^{HJ}, G_{p,j}^{HK}\}(x',x,t) \tag{28.8-21}$$

hold, Equation (28.8-19) leads with Equations (28.8-20) and (28.8-21), and invoking the condition the resulting equation has to hold for arbitrary values of b_p, to the final result

$$H_p^{\mathrm{T}}(x',t) = \int_{x \in \mathcal{D}^T} \left[\mathrm{C}_t(G_{p,k}^{HJ}, J_k^{\mathrm{T}}; x', x, t) + \mathrm{C}_t(G_{p,j}^{HK}, K_j^{\mathrm{T}}; x', x, t) \right] \mathrm{d}V \qquad \text{for } x' \in \mathcal{R}^3 . \tag{28.8-22}$$

Equation (28.8-22) expresses the magnetic field strength H_p^{T} of the generated electromagnetic field at x' as the superposition of the contributions from the elementary distributed sources $J_k^{\mathrm{T}}\,\mathrm{d}V$ and $K_j^{\mathrm{T}}\,\mathrm{d}V$ at x. The intervening kernel functions are $G_{p,k}^{HJ} = G_{p,k}^{HJ}(x',x,t)$, the *magnetic-field/electric-current source Green's function and* $G_{p,j}^{HK} = G_{p,j}^{HK}(x',x,t)$, the *magnetic-field/magnetic-current source Green's function.* These Green's functions are the magnetic field strength at x', radiated in the actual medium with constitutive parameters $\{\varepsilon_{k,r}, \mu_{j,p}\} = \{\varepsilon_{k,r}, \mu_{j,p}\}(x,t)$, by a point source of electric current at x and a point source of magnetic current at x, respectively.

Complex frequency-domain analysis

For the complex frequency-domain analysis of the problem the complex frequency-domain global reciprocity theorem of the time convolution type, Equation (28.4-7), is taken as the point of departure. In it, state A is taken to be the generated electromagnetic field under consideration, i.e.

$$\{\hat{E}_r^{\mathrm{A}}, \hat{H}_p^{\mathrm{A}}\} = \{\hat{E}_r^{\mathrm{T}}, \hat{H}_p^{\mathrm{T}}\}(x,s) \qquad \text{for } x \in \mathcal{R}^3 , \tag{28.8-23}$$

$$\{\hat{J}_k^{\mathrm{A}}, \hat{K}_j^{\mathrm{A}}\} = \{\hat{J}_k^{\mathrm{T}}, \hat{K}_j^{\mathrm{T}}\}(x,s) \qquad \text{for } x \in \mathcal{D}^T , \tag{28.8-24}$$

and

$$\{\hat{\eta}_{k,r}^{\mathrm{A}}, \hat{\zeta}_{j,p}^{\mathrm{A}}\} = \{\hat{\eta}_{k,r}, \hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x \in \mathcal{R}^3 . \tag{28.8-25}$$

Next, state B is chosen such that the application of Equation (28.4-7) to the entire $\mathcal{R}^3$ leads to the values of $\{\hat{E}_r^{\mathrm{T}}, \hat{H}_p^{\mathrm{T}}\}$ at some arbitrary point $x' \in \mathcal{R}^3$. Inspection of the right-hand side of Equation (28.4-7) reveals that this is accomplished if we take for the source distributions of state B a point source of electric current at x' in case we want an expression for the electric field strength at x' and a point source of magnetic current at x' in case we want an expression for the magnetic field strength at x', while the medium in state B must be taken to be adjoint to the one in state A, i.e.

$$\{\hat{\eta}^{\rm B}_{r,k},\hat{\zeta}^{\rm B}_{p,j}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.8-26}$$

Furthermore, if in the configuration electromagnetically impenetrable objects are present, the electromagnetic field in state B must satisfy on the boundaries of these objects the same boundary conditions as in state A, while the electromagnetic field in state B must be causally related to the action of its (point) sources. The two choices for the source distributions will be discussed separately below.

First, we choose

$$\hat{J}^{\rm B}_r = \hat{a}_r(s)\delta(x-x') \quad \text{and} \quad \hat{K}^{\rm B}_p = 0 , \tag{28.8-27}$$

where $\delta(x-x')$ represents the three-dimensional unit impulse (Dirac distribution) operative at the point $x = x'$, while $\hat{a}_r = \hat{a}_r(s)$ is an arbitrary vector function of s. The electromagnetic field causally radiated by this source is denoted as

$$\{\hat{E}^{\rm B}_k,\hat{H}^{\rm B}_j\} = \{\hat{E}^{J;{\rm B}}_k,\hat{H}^{J;{\rm B}}_j\}(x,x',s) , \tag{28.8-28}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (28.4-7) is applied to $\mathcal{R}^3$. The standard provisions of Section 28.1 for the handling of an unbounded domain yield

$$\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[\hat{E}^{\rm A}_r(x,s)\hat{H}^{\rm B}_p(x,s) - \hat{E}^{\rm B}_r(x,s)\hat{H}^{\rm A}_p(x,s)\right]{\rm d}A$$

$$= \varepsilon_{m,r,p}\int_{x\in\mathcal{S}(O,\Delta)} \nu_m\left[\hat{E}^{\rm T}_r(x,s)\hat{H}^{J;{\rm B}}_p(x,x',s) - \hat{E}^{J;{\rm B}}_r(x,x',s)\hat{H}^{\rm T}_p(x,s)\right]{\rm d}A\to 0$$

$$\text{as } \Delta\to\infty . \tag{28.8-29}$$

Furthermore, in view of Equation (28.8-27) and the properties of $\delta(x-x')$,

$$\int_{x\in\mathcal{R}^3}\left[\hat{J}^{\rm B}_r(x,s)\hat{E}^{\rm A}_r(x,s) - \hat{K}^{\rm B}_p(x,s)\hat{H}^{\rm A}_p(x,s)\right]{\rm d}V$$

$$= \int_{x\in\mathcal{R}^3} \hat{a}_r(s)\delta(x-x')\hat{E}^{\rm T}_r(x,s)\,{\rm d}V = \hat{a}_r(s)\hat{E}^{\rm T}_r(x',s) . \tag{28.8-30}$$

Since, further, the sources have the support $\mathcal{D}^T$,

$$\int_{x\in\mathcal{R}^3}\left[\hat{J}^{\rm A}_k(x,s)\hat{E}^{\rm B}_k(x,s) - \hat{K}^{\rm A}_j(x,s)\hat{H}^{\rm B}_j(x,s)\right]{\rm d}V$$

$$= \int_{x\in\mathcal{D}^T}\left[\hat{J}^{\rm T}_k(x,s)\hat{E}^{J;{\rm B}}_k(x,x',s) - \hat{K}^{\rm T}_j(x,s)\hat{H}^{J;{\rm B}}_j(x,x',s)\right]{\rm d}V . \tag{28.8-31}$$

Collecting the results, we arrive at

$$\hat{a}_r(s)\hat{E}_r^{\mathrm{T}}(x',s) = \int_{x\in\mathcal{D}^T} \left[\hat{E}_k^{J;\mathrm{B}}(x,x',s)\hat{J}_k^{\mathrm{T}}(x,s) - \hat{H}_j^{J;\mathrm{B}}(x,x',s)\hat{K}_j^{\mathrm{T}}(x,s)\right] \mathrm{d}V$$
$$\text{for } x'\in\mathcal{R}^3 \,. \tag{28.8-32}$$

From Equation (28.8-32) a representation for $\hat{E}_r^{\mathrm{T}}(x',s)$ is obtained by taking into account that $\hat{E}_k^{J;\mathrm{B}}$ and $\hat{H}_j^{J;\mathrm{B}}$ are linearly related to $\hat{a}_r(s)$. The latter relationship is expressed by

$$\{\hat{E}_k^{J;\mathrm{B}},\hat{H}_j^{J;\mathrm{B}}\}(x,x',s) = \{\hat{G}_{k,r}^{EJ;\mathrm{B}},\hat{G}_{j,r}^{HJ;\mathrm{B}}\}(x,x',s)\hat{a}_r(s) \,. \tag{28.8-33}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 28.8-5 and 28.8-7)

$$\{\hat{G}_{k,r}^{EJ;\mathrm{B}},\hat{G}_{j,r}^{HJ;\mathrm{B}}\}(x,x',s) = \{\hat{G}_{r,k}^{EJ},-\hat{G}_{r,j}^{EK}\}(x',x,s) \tag{28.8-34}$$

hold, Equation (28.8-32) leads, with Equations (28.8-33) and (28.8-34), and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{a}_r(s)$, to the final result

$$\hat{E}_r^{\mathrm{T}}(x',s) = \int_{x\in\mathcal{D}^T} \left[\hat{G}_{r,k}^{EJ}(x',x,s)\hat{J}_k^{\mathrm{T}}(x,s) + \hat{G}_{r,j}^{EK}(x',x,s)\hat{K}_j^{\mathrm{T}}(x',s)\right] \mathrm{d}V \quad \text{for } x'\in\mathcal{R}^3 \,. \tag{28.8-35}$$

Equation (28.8-35) expresses the electric field strength $\hat{E}_r^{\mathrm{T}}$ of the generated electromagnetic field at x' as the superposition of the contributions from the elementary distributed sources $\hat{J}_k^{\mathrm{T}}\,\mathrm{d}V$ and $\hat{K}_j^{\mathrm{T}}\,\mathrm{d}V$ at x. The intervening kernel functions are $\hat{G}_{r,k}^{EJ} = \hat{G}_{r,k}^{EJ}(x',x,s)$, the *electric-field/electric-current source Green's function*, and the *electric-field/magnetic-current source Green's function* $\hat{G}_{r,j}^{EK} = \hat{G}_{r,j}^{EK}(x',x,s)$. These Green's functions are the electric field strength at x', radiated in the actual medium with constitutive parameters $\{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s)$, by a point source of electric current at x and a point source of magnetic current at x, respectively.

Secondly, we choose

$$\hat{J}_r^{\mathrm{B}} = 0 \quad \text{and} \quad \hat{K}_p^{\mathrm{B}} = \hat{b}_p(s)\delta(x-x') \,, \tag{28.8-36}$$

where $\hat{b}_p = \hat{b}_p(s)$ is an arbitrary vector function of s. The electromagnetic field causally radiated by this source is denoted as

$$\{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\} = \{\hat{E}_k^{K;\mathrm{B}},\hat{H}_j^{K;\mathrm{B}}\}(x,x',s) \,, \tag{28.8-37}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. Now, Equation (28.4-7) is applied to $\mathcal{R}^3$. The standard provisions of Section 28.1 for the handling of an unbounded domain yield

$$\varepsilon_{m,r,p} \int_{x\in\mathcal{S}(O,\Delta)} \nu_m \left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s) - \hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right] \mathrm{d}A$$
$$= \varepsilon_{m,r,p} \int_{x\in\mathcal{S}(O,\Delta)} \nu_m \left[\hat{E}_r^{\mathrm{T}}(x,s)\hat{H}_p^{K;\mathrm{B}}(x,x',s) - \hat{E}_r^{K;\mathrm{B}}(x,x',s)\hat{H}_p^{\mathrm{T}}(x,s)\right] \mathrm{d}A \to 0$$
$$\text{as } \Delta\to\infty \,. \tag{28.8-38}$$

Furthermore, in view of Equation (28.8-36) and the properties of $\delta(x-x')$,

$$\int_{x\in\mathcal{R}^3}\left[\hat{J}_r^{\mathrm{B}}(x,s)\hat{E}_r^{\mathrm{A}}(x,s)-\hat{K}_p^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}V$$

$$=-\int_{x\in\mathcal{R}^3}\hat{b}_p(s)\delta(x-x')\hat{H}_p^{\mathrm{T}}(x,s)\,\mathrm{d}V=-\hat{b}_p(s)\hat{H}_p^{\mathrm{T}}(x',s)\,. \tag{28.8-39}$$

Since, further, the sources have the support $\mathcal{D}^T$,

$$\int_{x\in\mathcal{R}^3}\left[\hat{J}_k^{\mathrm{A}}(x,s)\hat{E}_k^{\mathrm{B}}(x,s)-\hat{K}_j^{\mathrm{A}}(x,s)\hat{H}_j^{\mathrm{B}}(x,s)\right]\mathrm{d}V$$

$$=\int_{x\in\mathcal{D}^T}\left[\hat{J}_k^{\mathrm{T}}(x,s)\hat{E}_k^{K;\mathrm{B}}(x,x',s)-\hat{K}_j^{\mathrm{T}}(x,s)\hat{H}_j^{K;\mathrm{B}}(x,x',s)\right]\mathrm{d}V\,. \tag{28.8-40}$$

Collecting the results, we arrive at

$$\hat{b}_p(s)\hat{H}_p^{\mathrm{T}}(x',s)=\int_{x\in\mathcal{D}^T}\left[-\hat{E}_k^{K;\mathrm{B}}(x,x',s)\hat{J}_k^{\mathrm{T}}(x,s)+\hat{H}_j^{K;\mathrm{B}}(x,x',s)\hat{K}_j^{\mathrm{T}}(x,s)\right]\mathrm{d}V$$

$$\text{for }\ x'\in\mathcal{R}^3\,. \tag{28.8-41}$$

From Equation (28.8-41) a representation for $\hat{H}_p^{\mathrm{T}}(x',s)$ is obtained by taking into account that $\hat{E}_k^{K;\mathrm{B}}$ and $\hat{H}_j^{K;\mathrm{B}}$ are linearly related to $\hat{b}_p(s)$. The latter relationship is expressed by

$$\{\hat{E}_k^{K;\mathrm{B}},\hat{H}_j^{K;\mathrm{B}}\}(x,x',s)=\{\hat{G}_{k,p}^{EK;\mathrm{B}},\hat{G}_{j,p}^{HK;\mathrm{B}}\}(x,x',s)\hat{b}_p(s)\,. \tag{28.8-42}$$

Since, however, for the right-hand side the reciprocity relations (see Exercises 28.8-6 and 28.8-8)

$$\{\hat{G}_{k,p}^{EK;\mathrm{B}},\hat{G}_{j,p}^{HK;\mathrm{B}}\}(x,x',s)=\{-\hat{G}_{p,k}^{HJ},\hat{G}_{p,j}^{HK}\}(x',x,s) \tag{28.8-43}$$

hold, Equation (28.8-41) leads, with Equations (28.8-42) and (28.8-43), and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{b}_p(s)$, to the final result

$$\hat{H}_p^{\mathrm{T}}(x',s)=\int_{x\in\mathcal{D}^T}\left[\hat{G}_{p,k}^{HJ}(x',x,s)\hat{J}_k^{\mathrm{T}}(x,s)+\hat{G}_{p,j}^{HK}(x',x,s)\hat{K}_j^{\mathrm{T}}(x,s)\right]\mathrm{d}V\quad\text{for } x'\in\mathcal{R}^3\,. \tag{28.8-44}$$

Equation (28.8-44) expresses the magnetic field strength $\hat{H}_p^{\mathrm{T}}$ of the generated electromagnetic field at x' as the superposition of the contributions from the elementary distributed sources $\hat{J}_k^{\mathrm{T}}\,\mathrm{d}V$ and $\hat{K}_j^{\mathrm{T}}\,\mathrm{d}V$ at x. The intervening kernel functions are $\hat{G}_{p,k}^{HJ}=\hat{G}_{p,k}^{HJ}(x',x,s)$, the *magnetic-field/electric-current source Green's function*, and the *magnetic-field/magnetic-current source Green's function* $\hat{G}_{p,j}^{HK}=\hat{G}_{p,j}^{HK}(x',x,s)$. These Green's functions are the magnetic field strength at x', radiated in the actual medium with constitutive parameters $\{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}=\{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s)$, by a point source of electric current at x and a point source of magnetic current at x, respectively.

Exercises

Exercise 28.8-1

Let $\{E_r^A, H_p^A\} = \{E_r^A, H_p^A\}(x,x',t)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{J_k^A, K_j^A\} = \{a_k^A\delta(x - x',t), 0\}$ and let $\{E_k^B, H_j^B\} = \{E_k^B, H_j^B\}(x,x'',t)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{J_r^B, K_p^B\} = \{a_r^B\delta(x - x'',t), 0\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.2-7)) to $\mathcal{R}^3$. (b) Write

$$E_r^A = G_{r,k}^{EJ;A}(x,x',t)a_k^A, \qquad H_p^A = G_{p,k}^{HJ;A}(x,x',t)a_k^A,$$
$$E_k^B = G_{k,r}^{EJ;B}(x,x'',t)a_r^B, \qquad H_j^B = G_{j,r}^{HJ;B}(x,x'',t)a_r^B,$$

invoke the condition that the result should hold for arbitrary a_k^A and a_r^B, and show that $G_{r,k}^{EJ;A}(x'',x',t) = G_{k,r}^{EJ;B}(x',x'',t)$.

Exercise 28.8-2

Let $\{E_r^A, H_p^A\} = \{E_r^A, H_p^A\}(x,x',t)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{J_k^A, K_j^A\} = \{a_k^A\delta(x - x',t), 0\}$ and let $\{E_k^B, H_j^B\} = \{E_k^B, H_j^B\}(x,x'',t)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{J_r^B, K_p^B\} = \{0, b_p^B\delta(x - x'',t)\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.2-7)) to $\mathcal{R}^3$. (b) Write

$$E_r^A = G_{r,k}^{EJ;A}(x,x',t)a_k^A, \qquad H_p^A = G_{p,k}^{HJ;A}(x,x',t)a_k^A,$$
$$E_k^B = G_{k,p}^{EK;B}(x,x'',t)b_p^B, \qquad H_j^B = G_{j,p}^{HK;B}(x,x'',t)b_p^B,$$

invoke the condition that the result should hold for arbitrary a_k^A and b_p^B, and show that $G_{p,k}^{HJ;A}(x'',x',t) = -G_{k,p}^{EK;B}(x',x'',t)$.

Exercise 28.8-3

Let $\{E_r^A, H_p^A\} = \{E_r^A, H_p^A\}(x,x',t)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{J_k^A, K_j^A\} = \{0, b_j^A\delta(x - x',t)\}$ and let $\{E_k^B, H_j^B\} = \{E_k^B, H_j^B\}(x,x'',t)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{J_r^B, K_p^B\} = \{a_r^B\delta(x - x'',t), 0\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.2-7)) to $\mathcal{R}^3$. (b) Write

$$E_r^A = G_{r,j}^{EK;A}(x,x',t)b_j^A, \qquad H_p^A = G_{p,j}^{HK;A}(x,x',t)b_j^A,$$
$$E_k^B = G_{k,r}^{EJ;B}(x,x'',t)a_r^B, \qquad H_j^B = G_{j,r}^{HJ;B}(x,x'',t)a_r^B,$$

invoke the condition that the result should hold for arbitrary b_j^{A} and a_r^{B}, and show that $G_{r,j}^{EK;\mathrm{A}}(x'',x',t) = -G_{j,r}^{HJ;\mathrm{B}}(x',x'',t)$. (Note that this result is consistent with the result of Exercise 28.8-2.)

Exercise 28.8-4

Let $\{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\} = \{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\}(x,x',t)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{J_k^{\mathrm{A}},K_j^{\mathrm{A}}\} = \{0,b_j^{\mathrm{A}}\delta(x-x',t)\}$ and let $\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\} = \{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\}(x,x'',t)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{J_r^{\mathrm{B}},K_p^{\mathrm{B}}\} = \{0,b_p^{\mathrm{B}}\delta(x-x'',t)\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.2-7)) to $\mathcal{R}^3$. (b) Write

$$E_r^{\mathrm{A}} = G_{r,j}^{EK;\mathrm{A}}(x,x',t)b_j^{\mathrm{A}}, \qquad H_p^{\mathrm{A}} = G_{p,j}^{HK;\mathrm{A}}(x,x',t)b_j^{\mathrm{A}},$$
$$E_k^{\mathrm{B}} = G_{k,p}^{EK;\mathrm{B}}(x,x'',t)b_p^{\mathrm{B}}, \qquad H_j^{\mathrm{B}} = G_{j,p}^{HK;\mathrm{B}}(x,x'',t)b_p^{\mathrm{B}},$$

invoke the condition that the result should hold for arbitrary b_j^{A} and b_p^{B}, and show that $G_{p,j}^{HK;\mathrm{A}}(x'',x',t) = G_{j,p}^{HK;\mathrm{B}}(x',x'',t)$.

Exercise 28.8-5

Let $\{\hat{E}_r^{\mathrm{A}},\hat{H}_p^{\mathrm{A}}\} = \{\hat{E}_r^{\mathrm{A}},\hat{H}_p^{\mathrm{A}}\}(x,x',s)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{\hat{J}_k^{\mathrm{A}},\hat{K}_j^{\mathrm{A}}\} = \{\hat{a}_k^{\mathrm{A}}(s)\delta(x-x'),0\}$ and let $\{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\} = \{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\}(x,x'',s)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{J}_r^{\mathrm{B}},\hat{K}_p^{\mathrm{B}}\} = \{\hat{a}_r^{\mathrm{B}}(s)\delta(x-x''),0\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.4-7)) to the entire $\mathcal{R}^3$. (b) Write

$$\hat{E}_r^{\mathrm{A}} = \hat{G}_{r,k}^{EJ;\mathrm{A}}(x,x',s)\hat{a}_k^{\mathrm{A}}(s), \qquad \hat{H}_p^{\mathrm{A}} = \hat{G}_{p,k}^{HJ;\mathrm{A}}(x,x',s)\hat{a}_k^{\mathrm{A}}(s),$$
$$\hat{E}_k^{\mathrm{B}} = \hat{G}_{k,r}^{EJ;\mathrm{B}}(x,x'',s)\hat{a}_r^{\mathrm{B}}(s), \qquad \hat{H}_j^{\mathrm{B}} = \hat{G}_{j,r}^{HJ;\mathrm{B}}(x,x'',s)\hat{a}_r^{\mathrm{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}_k^{\mathrm{A}}(s)$ and $\hat{a}_r^{\mathrm{B}}(s)$, and show that $\hat{G}_{r,k}^{EJ;\mathrm{A}}(x'',x',s) = \hat{G}_{k,r}^{EJ;\mathrm{B}}(x',x'',s)$.

Exercise 28.8-6

Let $\{\hat{E}_r^{\mathrm{A}},\hat{H}_p^{\mathrm{A}}\} = \{\hat{E}_r^{\mathrm{A}},\hat{H}_p^{\mathrm{A}}\}(x,x',s)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{\hat{J}_k^{\mathrm{A}},\hat{K}_j^{\mathrm{A}}\} = \{\hat{a}_k^{\mathrm{A}}(s)\delta(x-x'),0\}$ and let $\{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\} = \{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\}(x,x'',s)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{J}_r^{\mathrm{B}},\hat{K}_p^{\mathrm{B}}\} = \{0,\hat{b}_p^{\mathrm{B}}(s)\delta(x-x'')\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.4-7)) to the entire $\mathcal{R}^3$. (b) Write

$$\hat{E}_r^{\mathrm{A}} = \hat{G}_{r,k}^{EJ;\mathrm{A}}(x,x',s)\hat{a}_k^{\mathrm{A}}(s), \qquad \hat{H}_p^{\mathrm{A}} = \hat{G}_{p,k}^{HJ;\mathrm{A}}(x,x',s)\hat{a}_k^{\mathrm{A}}(s),$$

$$\hat{E}_k^{\text{B}} = \hat{G}_{k,p}^{EK;\text{B}}(x,x'',s)\hat{b}_p^{\text{B}}(s), \qquad \hat{H}_j^{\text{B}} = \hat{G}_{j,p}^{HK;\text{B}}(x,x'',s)\hat{b}_p^{\text{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}_k^{\text{A}}(s)$ and $\hat{b}_p^{\text{B}}(s)$, and show that $\hat{G}_{p,k}^{HJ;\text{A}}(x'',x',s) = -\hat{G}_{k,p}^{EK;\text{B}}(x',x'',s)$.

Exercise 28.8-7

Let $\{\hat{E}_r^{\text{A}},\hat{H}_p^{\text{A}}\} = \{\hat{E}_r^{\text{A}},\hat{H}_p^{\text{A}}\}(x,x',s)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{\hat{J}_k^{\text{A}},\hat{K}_j^{\text{A}}\} = \{0,\hat{b}_j^{\text{A}}(s)\delta(x-x')\}$ and let $\{\hat{E}_k^{\text{B}},\hat{H}_j^{\text{B}}\} = \{\hat{E}_k^{\text{B}},\hat{H}_j^{\text{B}}\}(x,x'',s)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{J}_r^{\text{B}},\hat{K}_p^{\text{B}}\} = \{\hat{a}_r^{\text{B}}(s)\delta(x-x''),0\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.4-7)) to the entire $\mathcal{R}^3$. (b) Write

$$\hat{E}_r^{\text{A}} = \hat{G}_{r,j}^{EK;\text{A}}(x,x',s)\hat{b}_j^{\text{A}}(s), \qquad \hat{H}_p^{\text{A}} = \hat{G}_{p,j}^{HK;\text{A}}(x,x',s)\hat{b}_j^{\text{A}}(s),$$
$$\hat{E}_k^{\text{B}} = \hat{G}_{k,r}^{EJ;\text{B}}(x,x'',s)\hat{a}_r^{\text{B}}(s), \qquad \hat{H}_j^{\text{B}} = \hat{G}_{j,r}^{HJ;\text{B}}(x,x'',s)\hat{a}_r^{\text{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{a}_k^{\text{A}}(s)$ and $\hat{b}_p^{\text{B}}(s)$, and show that $\hat{G}_{r,j}^{EK;\text{A}}(x'',x',s) = -\hat{G}_{j,r}^{HJ;\text{B}}(x',x'',s)$. (Note that this result is consistent with the result of Exercise 28.8-6.)

Exercise 28.8-8

Let $\{\hat{E}_r^{\text{A}},\hat{H}_p^{\text{A}}\} = \{\hat{E}_r^{\text{A}},\hat{H}_p^{\text{A}}\}(x,x',s)$ be the electromagnetic field at x that is causally radiated by the point source at x' with volume source density $\{\hat{J}_k^{\text{A}},\hat{K}_j^{\text{A}}\} = \{0,\hat{b}_j^{\text{A}}(s)\delta(x-x')\}$ and let $\{\hat{E}_k^{\text{B}},\hat{H}_j^{\text{B}}\} = \{\hat{E}_k^{\text{B}},\hat{H}_j^{\text{B}}\}(x,x'',s)$ be the electromagnetic field at x that is causally radiated by the point source at x'' with volume source density $\{\hat{J}_r^{\text{B}},\hat{K}_p^{\text{B}}\} = \{0,\hat{b}_p^{\text{B}}(s)\delta(x-x'')\}$, with $x' \neq x''$. The two sources radiate in adjoint media occupying the entire $\mathcal{R}^3$. The standard provisions given in Section 28.1 for handling an unbounded domain are made. (a) Apply the reciprocity theorem (Equation (28.4-7)) to the entire $\mathcal{R}^3$. (b) Write

$$\hat{E}_r^{\text{A}} = \hat{G}_{r,j}^{EK;\text{A}}(x,x',s)\hat{b}_j^{\text{A}}(s), \qquad \hat{H}_p^{\text{A}} = \hat{G}_{p,j}^{HK;\text{A}}(x,x',s)\hat{b}_j^{\text{A}}(s),$$
$$\hat{E}_k^{\text{B}} = \hat{G}_{k,p}^{EK;\text{B}}(x,x'',s)\hat{b}_p^{\text{B}}(s), \qquad \hat{H}_j^{\text{B}} = \hat{G}_{j,p}^{HK;\text{B}}(x,x'',s)\hat{b}_p^{\text{B}}(s),$$

invoke the condition that the result should hold for arbitrary $\hat{b}_j^{\text{A}}(s)$ and $\hat{b}_p^{\text{B}}(s)$, and show that $\hat{G}_{p,j}^{HK;\text{A}}(x'',x',s) = \hat{G}_{j,p}^{HK;\text{B}}(x',x'',s)$.

Exercise 28.8-9

Give the expressions for the time-domain Green's functions (a) $G_{r,k}^{EJ}(x,x',t)$, (b) $G_{r,j}^{EK}(x,x',t)$, (c) $G_{p,k}^{HJ}(x,x',t)$, (d) $G_{p,j}^{HK}(x,x',t)$, for a homogeneous isotropic, lossless medium with permittivity ε and permeability μ that occupies the entire $\mathcal{R}^3$. (*Hint*: Use Equations (26.4-7)–(26.4-11).)

Answers:

(a) $G_{r,k}^{EJ} = -\mu\partial_t G(x,x',t)\delta_{r,k} + \varepsilon^{-1}\partial_r\partial_k \mathrm{I}_t G(x,x',t)$,

(b) $G_{r,j}^{EK} = -\varepsilon_{r,n,j}\partial_n G(x,x',t)$,

(c) $G^{HJ}_{p,k} = \epsilon_{p,m,k}\partial_m G(x,x',t)$,

(d) $G^{HK}_{p,j} = -\varepsilon\partial_t G(x,x',t)\delta_{p,j} + \mu^{-1}\partial_p\partial_j \mathrm{I}_t G(x,x',t)$,

in which

$$G(x,x',t) = \frac{\delta(t - |x - x'|/c)}{4\pi|x - x'|} \quad \text{for} \quad |x - x'| \neq 0, \quad \text{with } c = (\varepsilon\mu)^{-1/2}.$$

Exercise 28.8-10

Give the expressions for the complex frequency-domain Green's functions (a) $\hat{G}^{EJ}_{r,k}(x,x',s)$, (b) $\hat{G}^{EK}_{r,j}(x,x',s)$, (c) $\hat{G}^{HJ}_{p,k}(x,x',s)$, (d) $\hat{G}^{HK}_{p,j}(x,x',s)$ for a homogeneous, isotropic, medium with transverse admittance per length $\hat{\eta}$ and longitudinal impedance per length $\hat{\zeta}$, that occupies the entire $\mathcal{R}^3$. (*Hint*: Use Equations (26.3-1)–(26.3-5).)

Answers:

(a) $\hat{G}^{EJ}_{r,k} = -\hat{\zeta}\hat{G}(x,x',s)\delta_{r,k} + \hat{\eta}^{-1}\partial_r\partial_k\hat{G}(x,x',s)$,

(b) $\hat{G}^{EK}_{r,j} = -\epsilon_{r,n,j}\partial_n\hat{G}(x,x',s)$,

(c) $\hat{G}^{HJ}_{p,k} = \epsilon_{p,m,k}\partial_m\hat{G}(x,x',s)$,

(d) $\hat{G}^{HK}_{p,j} = -\hat{\eta}\hat{G}(x,x',s)\delta_{p,j} + \hat{\zeta}^{-1}\partial_p\partial_j\hat{G}(x,x',s)$,

in which

$$G(x,x',s) = \frac{\exp(-\hat{\gamma}|x - x'|)}{4\pi|x - x'|} \quad \text{for} \quad |x - x'| \neq 0, \quad \text{with } \hat{\gamma} = (\hat{\eta}\hat{\zeta})^{1/2}.$$

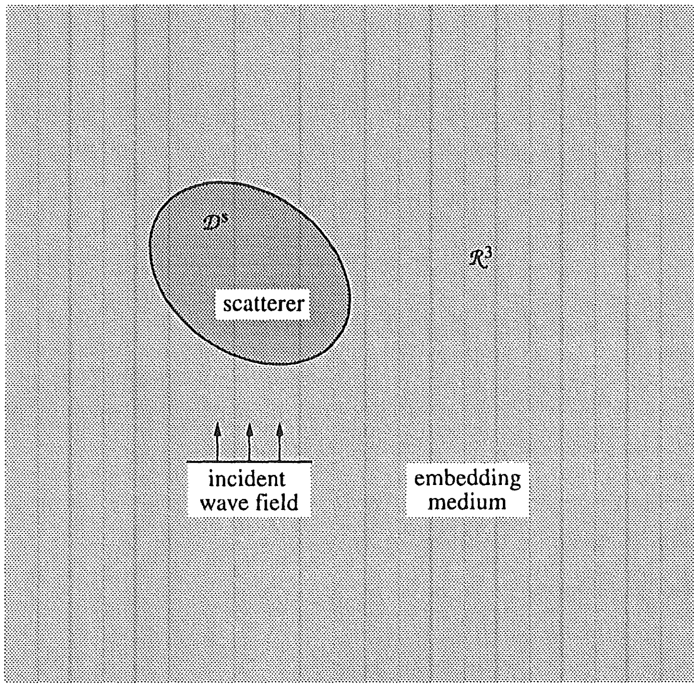
Exercise 28.8-11

Show that Equation (28.8-35) follows from Equation (28.8-13) and Equation (28.8-44) from Equation (28.8-22) by taking the Laplace transform with respect to time.

28.9 The direct (forward) scattering problem

The configuration in an electromagnetic scattering problem generally consists of a background medium with known electromagnetic properties, occupying the entire $\mathcal{R}^3$ (the "*embedding*"), in which, in principle, the radiation from given, arbitrarily distributed electromagnetic sources can be calculated with the aid of the theory developed in Section 28.8. In the embedding, an electromagnetically penetrable object of bounded support $\mathcal{D}^s$ (the "*scatterer*") is present, whose known electromagnetic properties differ from the ones of the embedding (Figure 28.9-1).

The scatterer is electromagnetically irradiated by given sources located in the embedding, in a subdomain outside the scatterer. The problem is to determine the total electromagnetic field in the configuration. The standard procedure is to calculate first the so-called "*incident*" electromagnetic field, i.e. the wave field that would be present in the entire configuration if the object showed *no contrast* with respect to its embedding. (This can be done by employing the representations derived in Section 28.8.) Next, the total wave field is written as the superposition



28.9-1 Scattering configuration with embedding $\mathcal{R}^3$ and scattering object $\mathcal{D}^{\rm s}$.

of the incident wave field and the "*scattered*" wave field, and, through a particular reasoning, the problem of determining the scattered wave field is reduced to calculating its *equivalent contrast source distributions*, whose common support will be shown to be the domain $\mathcal{D}^{\rm s}$ occupied by the scatterer. The standard provisions of Section 28.1 for handling an unbounded domain are made. Both the incident wave field and the scattered wave field are causally related to the action of their respective sources.

Time-domain analysis

In the time-domain analysis of the problem, the electromagnetic properties of the embedding are characterised by the relaxation functions $\{\varepsilon_{k,r},\mu_{j,p}\} = \{\varepsilon_{k,r},\mu_{j,p}\}(\boldsymbol{x},t)$, which are causal functions of time. The electromagnetic properties of the scatterer are characterised by the relaxation functions $\{\varepsilon^{\rm s}_{k,r},\mu^{\rm s}_{j,p}\} = \{\varepsilon^{\rm s}_{k,r},\mu^{\rm s}_{j,p}\}(\boldsymbol{x},t)$, which are causal functions of time as well. The cases of an instantaneously reacting embedding and/or an instantaneously reacting scatterer easily follow from the more general cases for media with relaxation. The contrast in the medium properties only differs from zero in $\mathcal{D}^{\rm s}$, and hence

$$\{\varepsilon^{\rm s}_{k,r}-\varepsilon_{k,r},\mu^{\rm s}_{j,p}-\mu_{j,p}\} = \{0,0\} \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\rm s\,\prime}, \tag{28.9-1}$$

where $\mathcal{D}^{\rm s\prime}$ is the complement of $\mathcal{D}^{\rm s}\cup\partial\mathcal{D}^{\rm s}$ in $\mathcal{R}^3$, i.e. the part of $\mathcal{R}^3$ that is exterior to $\mathcal{D}^{\rm s}$. The *incident wave field* is denoted by

$$\{E_r^{\mathrm{i}},H_p^{\mathrm{i}}\} = \{E_r^{\mathrm{i}},H_p^{\mathrm{i}}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.9-2}$$

and is considered to be known. (Once its generating sources are given, the expressions of the type derived in Section 28.8 yield the wave-field values at any $x\in\mathcal{R}^3$.) The *total wave field* is denoted by

$$\{E_r,H_p\} = \{E_r,H_p\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.9-3}$$

and the *scattered wave field* by

$$\{E_r^{\mathrm{s}},H_p^{\mathrm{s}}\} = \{E_r^{\mathrm{s}},H_p^{\mathrm{s}}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.9-4}$$

Then,

$$\{E_r,H_p\} = \{E_r^{\mathrm{i}}+E_r^{\mathrm{s}},H_p^{\mathrm{i}}+H_p^{\mathrm{s}}\} \qquad \text{for } x\in\mathcal{R}^3. \tag{28.9-5}$$

First, we investigate the structure of the electromagnetic field equations in the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. Since the sources that generate the total wave field are located in the domain exterior to the scatterer, the total wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$-\varepsilon_{k,m,p}\partial_m H_p + \partial_t C_t(\varepsilon_{k,r}^{\mathrm{s}},E_r;x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.9-6}$$

$$\varepsilon_{j,n,r}\partial_n E_r + \partial_t C_t(\mu_{j,p}^{\mathrm{s}},H_p;x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}. \tag{28.9-7}$$

Since the sources that generate the total wave field would also generate the incident wave field, also this part of the wave field is source-free in $\mathcal{D}^{\mathrm{s}}$, and hence

$$-\varepsilon_{k,m,p}\partial_m H_p^{\mathrm{i}} + \partial_t C_t(\varepsilon_{k,r},E_r^{\mathrm{i}};x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.9-8}$$

$$\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{i}} + \partial_t C_t(\mu_{j,p},H_p^{\mathrm{i}};x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}. \tag{28.9-9}$$

In view of Equation (28.9-5), Equations (28.9-6)–(28.9-9) lead to equations with the scattered wave field on the left-hand side that can, alternatively, be written as

$$-\varepsilon_{k,m,p}\partial_m H_p^{\mathrm{s}} + \partial_t C_t(\varepsilon_{k,r}^{\mathrm{s}},E_r^{\mathrm{s}};x,t) = -\partial_t C_t(\varepsilon_{k,r}^{\mathrm{s}}-\varepsilon_{k,r},E_r^{\mathrm{i}};x,t) \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.9-10}$$

$$\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{s}} + \partial_t C_t(\mu_{j,p}^{\mathrm{s}},H_p^{\mathrm{s}};x,t) = -\partial_t C_t(\mu_{j,p}^{\mathrm{s}}-\mu_{j,p},H_p^{\mathrm{i}};x,t) \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.9-11}$$

or as

$$-\varepsilon_{k,m,p}\partial_m H_p^{\mathrm{s}} + \partial_t C_t(\varepsilon_{k,r},E_r^{\mathrm{s}};x,t) = -\partial_t C_t(\varepsilon_{k,r}^{\mathrm{s}}-\varepsilon_{k,r},E_r;x,t) \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.9-12}$$

$$\varepsilon_{j,n,r}\partial_n E_r^{\mathrm{s}} + \partial_t C_t(\mu_{j,p},H_p^{\mathrm{s}};x,t) = -\partial_t C_t(\mu_{j,p}^{\mathrm{s}}-\mu_{j,p},H_p;x,t) \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}. \tag{28.9-13}$$

Equations (28.9-10) and (28.9-11) express that the scattered wave field in $\mathcal{D}^{\mathrm{s}}$ can be envisaged as to be excited through both the presence of a contrast in the medium properties and the presence of an incident wave field. If either of the two is absent, the scattered wave field vanishes in $\mathcal{D}^{\mathrm{s}}$. This system of equations customarily serves as the starting point for the wave-field computation via a numerical discretisation procedure applied to the pertaining differential equations (finite-difference or finite-element techniques).

Equations (28.9-12) and (28.9-13) express that the scattered wave field can be envisaged as to be generated by contrast sources (with support $\mathcal{D}^{\mathrm{s}}$) radiating into the embedding. This system of equations customarily serves as the starting point for the wave-field computation via an integral equation approach. This aspect, for which we also need the electromagnetic field equations that govern the wave field in $\mathcal{D}^{\mathrm{s}\prime}$, will be further discussed below. Now, in $\mathcal{D}^{\mathrm{s}\prime}$ the scattered wave field is source-free since the (actual) total wave field and the (calculated) incident wave field are assumed to be generated by the same source distributions. Consequently (note that in $\mathcal{D}^{s'}$ the medium parameters are the ones of the embedding),

$$-\varepsilon_{k,m,p}\partial_m H_p^{s} + C_t(\varepsilon_{k,r},E_r^{s};x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{s'}, \tag{28.9-14}$$

$$\varepsilon_{j,n,r}\partial_n E_r^{s} + C_t(\mu_{j,p},H_p^{s};x,t) = 0 \qquad \text{for } x\in\mathcal{D}^{s'}. \tag{28.9-15}$$

Equations (28.9-12) and (28.9-13), and (28.9-14) and (28.9-15) can be combined to

$$-\varepsilon_{k,m,p}\partial_m H_p^{s} + C_t(\varepsilon_{k,r},E_r^{s};x,t) = -\{J_k^{s},0\} \qquad \text{for } x\in\{\mathcal{D}^{s},\mathcal{D}^{s'}\}, \tag{28.9-16}$$

$$\varepsilon_{j,n,r}\partial_n E_r^{s} + C_t(\mu_{j,p},H_p^{s};x,t) = -\{K_j^{s},0\} \qquad \text{for } x\in\{\mathcal{D}^{s},\mathcal{D}^{s'}\}, \tag{28.9-17}$$

where

$$J_k^{s} = \partial_t C_t(\varepsilon_{k,r}^{s} - \varepsilon_{k,r},E_r;x,t) \qquad \text{for } x\in\mathcal{D}^{s} \tag{28.9-18}$$

is the *equivalent contrast volume source density of electric current* and

$$K_j^{s} = \partial_t C_t(\mu_{j,p}^{s} - \mu_{j,p},H_p;x,t) \qquad \text{for } x\in\mathcal{D}^{s} \tag{28.9-19}$$

is the *equivalent contrast volume source density of magnetic current.* If the contrast volume source densities J_k^s and K_j^s were known, Equations (28.9-16) and (28.9-17) would constitute a direct (forward) source problem in the embedding of the type discussed in Section 28.8. As yet, however, these contrast volume source densities are unknown.

To construct a system of equations from which the scattering problem can be solved, we employ the source type integral representations for the scattered wave field (see Equations (28.8-13) and (28.8-22)), viz.

$$E_r^{s}(x',t) = \int_{x\in\mathcal{D}^{s}} \left[C_t(G_{r,k}^{EJ},J_k^{s};x',x,t) + C_t(G_{r,j}^{EK},K_j^{s};x',x,t)\right]\mathrm{d}V \qquad \text{for } x'\in\mathcal{R}^3, \tag{28.9-20}$$

$$H_p^{s}(x',t) = \int_{x\in\mathcal{D}^{s}} \left[C_t(G_{p,k}^{HJ},J_k^{s};x',x,t) + C_t(G_{p,j}^{HK},K_j^{s};x',x,t)\right]\mathrm{d}V \qquad \text{for } x'\in\mathcal{R}^3, \tag{28.9-21}$$

in which the Green's functions apply to a medium with the same electromagnetic properties as the embedding. Writing Equations (28.9-18) and (28.9-19) with the aid of Equation (28.9-5) as

$$J_k^{s} = \partial_t C_t(\varepsilon_{k,r}^{s} - \varepsilon_{k,r},E_r^{i} + E_r^{s};x,t) \qquad \text{for } x\in\mathcal{D}^{s}, \tag{28.9-22}$$

$$K_j^{s} = \partial_t C_t(\mu_{j,p}^{s} - \mu_{j,p},H_p^{i} + H_p^{s};x,t) \qquad \text{for } x\in\mathcal{D}^{s}, \tag{28.9-23}$$

and invoking Equations (28.9-20) and (28.9-21) for $x'\in\mathcal{D}^s$, a system of integral equations result from which J_k^s and K_j^s can be solved. Once these quantities have been determined, the scattered wave field can be calculated in the entire configuration by reusing Equations (28.9-20) and (28.9-21) for all $x\in\mathcal{R}^3$, and since the incident wave field was presumably known already, the total wave field follows.

Except for some simple geometries, where analytic methods can be employed, the integral equations for the scattering of electromagnetic waves have to be solved with the aid of numerical methods. The circumstance that the Green's tensors are singular when $x' = x$ presents difficulties, in the sense that in the neighbourhood of x' the integrations with respect to x cannot be evaluated by a simple numerical formula (such as the tetrahedral formula, which is the three-dimensional equivalent of the one-dimensional trapezoidal formula), but have to be evaluated by a limiting analytic procedure. For the rest, the application of numerical methods to the relevant integral equations presents no essential difficulties.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the electromagnetic properties of the embedding medium are characterised by the functions $\{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s)$. The electromagnetic properties of the scatterer are characterised by the functions $\{\hat{\eta}^{s}_{k,r},\hat{\zeta}^{s}_{j,p}\} = \{\hat{\eta}^{s}_{k,r},\hat{\zeta}^{s}_{j,p}\}(x,s)$. The contrast in medium properties only differs from zero in $\mathcal{D}^{s}$, and hence

$$\{\hat{\eta}^{s}_{k,r} - \hat{\eta}_{k,r},\hat{\zeta}^{s}_{j,p} - \hat{\zeta}_{j,p}\} = \{0,0\} \qquad \text{for } x\in\mathcal{D}^{s\prime}, \tag{28.9-24}$$

where $\mathcal{D}^{s\prime}$ is the complement of $\mathcal{D}^{s}\cup\partial\mathcal{D}^{s}$ in $\mathcal{R}^3$, i.e. the part of $\mathcal{R}^3$ that is exterior to $\mathcal{D}^{s}$. The *incident wave field* is denoted by

$$\{\hat{E}^{i}_{r},\hat{H}^{i}_{p}\} = \{\hat{E}^{i}_{r},\hat{H}^{i}_{p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3, \tag{28.9-25}$$

and is considered to be known. (Once its generating sources are given, the expressions of the type derived in Section 28.8 yield the wave-field values at any $x\in\mathcal{R}^3$.) The *total wave field* is denoted by

$$\{\hat{E}_{r},\hat{H}_{p}\} = \{\hat{H}_{r},\hat{E}_{p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3, \tag{28.9-26}$$

and the *scattered wave field* by

$$\{\hat{E}^{s}_{r},\hat{H}^{s}_{p}\} = \{\hat{E}^{s}_{r},\hat{H}^{s}_{p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3. \tag{28.9-27}$$

Then,

$$\{\hat{E}_{r},\hat{H}_{p}\} = \{\hat{E}^{i}_{r} + \hat{E}^{s}_{r},\hat{H}^{i}_{p} + \hat{H}^{s}_{p}\} \qquad \text{for } x\in\mathcal{R}^3. \tag{28.9-28}$$

First, we investigate the structure of the complex frequency-domain electromagnetic field equations in the domain $\mathcal{D}^{s}$ occupied by the scatterer. Since the sources that generate the total wave field are located in the domain exterior to the scatterer, the total wave field is source-free in $\mathcal{D}^{s}$, and hence

$$-\varepsilon_{k,m,p}\partial_m\hat{H}_p + \hat{\eta}^{s}_{k,r}\hat{E}_r = 0 \qquad \text{for } x\in\mathcal{D}^{s}, \tag{28.9-29}$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}_r + \hat{\zeta}^{s}_{j,p}\hat{H}_p = 0 \qquad \text{for } x\in\mathcal{D}^{s}. \tag{28.9-30}$$

Since the sources that generate the total wave field would also generate the incident wave field, also this part of the wave field is source-free in $\mathcal{D}^{s}$, and hence

$$-\varepsilon_{k,m,p}\partial_m\hat{H}^{i}_p + \hat{\eta}_{k,r}\hat{E}^{i}_r = 0 \qquad \text{for } x\in\mathcal{D}^{s}, \tag{28.9-31}$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}^{i}_r + \hat{\zeta}_{j,p}\hat{H}^{i}_p = 0 \qquad \text{for } x\in\mathcal{D}^{s}. \tag{28.9-32}$$

In view of Equation (28.9-28), Equations (28.9-29)–(28.9-32) lead to equations with the scattered wave field on the left-hand side that can, alternatively, be written as

$$-\varepsilon_{k,m,p}\partial_m\hat{H}^{s}_p + \hat{\eta}^{s}_{k,r}\hat{E}^{s}_r = -(\hat{\eta}^{s}_{k,r} - \hat{\eta}_{k,r})\hat{E}^{i}_r \qquad \text{for } x\in\mathcal{D}^{s}, \tag{28.9-33}$$

$$\varepsilon_{j,n,r}\partial_n\hat{E}^{s}_r + \hat{\zeta}^{s}_{j,p}\hat{H}^{s}_p = -(\hat{\zeta}^{s}_{j,p} - \hat{\zeta}_{j,p})\hat{H}^{i}_p \qquad \text{for } x\in\mathcal{D}^{s}, \tag{28.9-34}$$

or as

$$-\epsilon_{k,m,p}\partial_m \hat{H}_p^{\mathrm{s}} + \hat{\eta}_{k,r}\hat{E}_r^{\mathrm{s}} = -(\hat{\eta}_{k,r}^{\mathrm{s}} - \hat{\eta}_{k,r})\hat{E}_r \qquad \text{for } x \in \mathcal{D}^{\mathrm{s}}, \tag{28.9-35}$$

$$\epsilon_{j,n,r}\partial_n \hat{E}_r^{\mathrm{s}} + \hat{\zeta}_{j,p}\hat{H}_p^{\mathrm{s}} = -(\hat{\zeta}_{j,p}^{\mathrm{s}} - \hat{\zeta}_{j,p})\hat{H}_p \qquad \text{for } x \in \mathcal{D}^{\mathrm{s}}. \tag{28.9-36}$$

Equations (28.9-33) and (28.9-34) express that the scattered wave field in $\mathcal{D}^{\mathrm{s}}$ can be envisaged as to be excited through both the presence of a contrast in the medium properties and the presence of an incident wave field. If either of the two is absent, the scattered wave field vanishes in $\mathcal{D}^{\mathrm{s}}$. This system of equations customarily serves as the starting point for the wave-field computation via a numerical discretisation procedure applied to the pertaining differential equations (finite-difference or finite-element techniques).

Equations (28.9-35) and (28.9-36) express that the scattered wave field can be envisaged as to be generated by contrast sources (with support $\mathcal{D}^{\mathrm{s}}$) radiating into the embedding. This system of equations customarily serves as the starting point for the wave-field computation via an integral equation approach. This aspect, for which we also need the electromagnetic field equations that govern the wave field in $\mathcal{D}^{\mathrm{s}\prime}$, will be further discussed below. Now, in $\mathcal{D}^{\mathrm{s}\prime}$ the scattered wave field is source-free since the (actual) total wave field and the (calculated) incident wave field are assumed to be generated by the same source distributions. Consequently (note that in $\mathcal{D}^{\mathrm{s}\prime}$ the medium parameters are the ones of the embedding),

$$-\epsilon_{k,m,p}\partial_m \hat{H}_p^{\mathrm{s}} + \hat{\eta}_{k,r}\hat{E}_r^{\mathrm{s}} = 0 \qquad \text{for } x \in \mathcal{D}^{\mathrm{s}\prime}, \tag{28.9-37}$$

$$\epsilon_{j,n,r}\partial_n \hat{E}_r^{\mathrm{s}} + \hat{\zeta}_{j,p}\hat{H}_p^{\mathrm{s}} = 0 \qquad \text{for } x \in \mathcal{D}^{\mathrm{s}\prime}. \tag{28.9-38}$$

Equations (28.9-35) and (28.9-36), and (28.9-37) and (28.9-38), can be combined to

$$-\epsilon_{k,m,p}\partial_m \hat{H}_p^{\mathrm{s}} + \hat{\eta}_{k,r}\hat{E}_r^{\mathrm{s}} = -\{\hat{J}_k^{\mathrm{s}}, 0\} \qquad \text{for } x \in \{\mathcal{D}^{\mathrm{s}}, \mathcal{D}^{\mathrm{s}\prime}\}, \tag{28.9-39}$$

$$\epsilon_{j,n,r}\partial_n \hat{E}_r^{\mathrm{s}} + \hat{\zeta}_{j,p}\hat{H}_p^{\mathrm{s}} = -\{\hat{K}_j^{\mathrm{s}}, 0\} \qquad \text{for } x \in \{\mathcal{D}^{\mathrm{s}}, \mathcal{D}^{\mathrm{s}\prime}\}. \tag{28.9-40}$$

where

$$\hat{J}_k^{\mathrm{s}} = (\hat{\eta}_{k,r}^{\mathrm{s}} - \hat{\eta}_{k,r})\hat{E}_r \qquad \text{for } x \in \mathcal{D}^{\mathrm{s}} \tag{28.9-41}$$

is the *equivalent contrast volume source density of electric current* and

$$\hat{K}_j^{\mathrm{s}} = (\hat{\zeta}_{j,p}^{\mathrm{s}} - \hat{\zeta}_{j,p})\hat{H}_p \qquad \text{for } x \in \mathcal{D}^{\mathrm{s}} \tag{28.9-42}$$

is the *equivalent contrast volume source density of magnetic current.* If the contrast volume source densities $\hat{J}_k^{\mathrm{s}}$ and $\hat{K}_j^{\mathrm{s}}$ were known, Equations (28.9-39) and (28.9-40) would constitute a direct (forward) source problem in the embedding of the type discussed in Section 28.8. As yet, however, these contrast volume source densities are unknown.

To construct a system of equations from which the scattering problem can be solved, we employ the source type integral representations for the scattered wave field (see Equations (28.8-35) and (28.8-44)), viz.

$$\hat{E}_r^{\mathrm{s}}(x',s) = \int_{x \in \mathcal{D}^{\mathrm{s}}} \left[\hat{G}_{r,k}^{EJ}(x',x,s)\hat{J}_k^{\mathrm{s}}(x,s) + \hat{G}_{r,j}^{EK}(x',x,s)\hat{K}_j^{\mathrm{s}}(x,s)\right] \mathrm{d}V \qquad \text{for } x' \in \mathcal{R}^3, \tag{28.9-43}$$

$$\hat{H}_p^{\mathrm{s}}(x',s) = \int_{x \in \mathcal{D}^{\mathrm{s}}} \left[\hat{G}_{p,k}^{HJ}(x',x,s)\hat{J}_k^{\mathrm{s}}(x,s) + \hat{G}_{p,j}^{HK}(x',x,s)\hat{K}_j^{\mathrm{s}}(x,s)\right] \mathrm{d}V \qquad \text{for } x' \in \mathcal{R}^3, \tag{28.9-44}$$

in which the Green's functions apply to a medium with the same electromagnetic properties as the embedding. Writing Equations (28.9-41) and (28.9-42) with the aid of Equation (28.9-28) as

$$\hat{J}_k^{\mathrm{s}} = (\hat{\eta}_{k,r}^{\mathrm{s}} - \hat{\eta}_{k,r})(\hat{E}_r^{\mathrm{i}} + \hat{E}_r^{\mathrm{s}}) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}}, \tag{28.9-45}$$

$$\hat{K}_j^{\mathrm{s}} = (\hat{\zeta}_{j,p}^{\mathrm{s}} - \hat{\zeta}_{j,p})(\hat{H}_p^{\mathrm{i}} + \hat{H}_p^{\mathrm{s}}) \qquad \text{for } \boldsymbol{x} \in \mathcal{D}^{\mathrm{s}}, \tag{28.9-46}$$

and invoking Equations (28.9-43) and (28.9-44) for $\boldsymbol{x}' \in \mathcal{D}^{\mathrm{s}}$, a system of integral equations result from which $\hat{J}_k^{\mathrm{s}}$ and $\hat{K}_j^{\mathrm{s}}$ can be solved. Once these quantities have been determined, the scattered wave field can be calculated in the entire configuration by reusing Equations (28.9-43) and (28.9-44) for all $\boldsymbol{x} \in \mathcal{R}^3$, and since the incident wave field was presumably known already, the total wave field follows.

Except for some simple geometries (see, for example, Bowman, Senior and Uslenghi 1969), where analytic methods can be employed, the complex frequency-domain integral equations for the scattering of electromagnetic waves have to be solved with the aid of numerical methods. The circumstance that the Green's tensors are singular when $\boldsymbol{x}' = \boldsymbol{x}$ presents difficulties, in the sense that in the neighbourhood of $\boldsymbol{x}'$ the integrations with respect to $\boldsymbol{x}$ cannot be evaluated by a simple numerical formula (such as the tetrahedral formula, which is the three-dimensional equivalent of the one-dimensional trapezoidal formula), but have to be evaluated by a limiting analytic procedure. For the rest, the application of numerical methods to the relevant integral equations presents no essential difficulties. Recent advances on this subject can be found in Van den Berg (1991) and in Fokkema and Van den Berg (1993).

28.10 The inverse source problem

The configuration in an electromagnetic inverse source problem generally consists of a background medium with known electromagnetic properties, occupying the entire $\mathcal{R}^3$ (the "*embedding*"), in which, in principle, the radiation from given, arbitrarily distributed electromagnetic sources can be calculated with the aid of the theory developed in Section 28.8. In the embedding an, either known or guessed, bounded domain $\mathcal{D}^T$ is present in which electromagnetically radiating sources of unknown nature and unknown spatial distribution are present. The presence of these sources manifests itself in the entire embedding. In some bounded subdomain $\mathcal{D}^\Omega$ of $\mathcal{R}^3$, and exterior to $\mathcal{D}^T$, the radiated electromagnetic field is accessible to measurement (Figure 28.10-1).

We assume that the action of the radiating sources can be modelled by volume source densities of electric current and magnetic current. The objective is to reconstruct these volume source densities with support $\mathcal{D}^T$ from (a set of) measured values of the electric field strength and/or the magnetic field strength in $\mathcal{D}^\Omega$. Since the inverse source problem is, by necessity, a *remote sensing problem*, the global reciprocity theorems of Sections 28.2–28.5 can be expected to provide a means for interrelating the known, measured wave-field data with the unknown source distributions. The standard provisions of Section 28.1 for handling an unbounded domain are made. The radiated wave field is, by its nature, causally related to the sources by which it is generated. For gathering maximum information, the reciprocity theorems are applied to the domain interior to a closed surface $\mathcal{S}^\Omega$ that completely surrounds both $\mathcal{D}^T$ and $\mathcal{D}^\Omega$. If necessary, measurement on $\mathcal{S}^\Omega$ can also be carried out.

Time-domain analysis

In the time-domain analysis of the problem, the electromagnetic properties of the embedding medium are characterised by the relaxation functions $\{\varepsilon_{k,r},\mu_{j,p}\} = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t)$, which are causal functions of time. The case of an instantaneously reacting embedding medium easily follows from the more general case of a medium with relaxation. The causally radiated electromagnetic field is denoted by $\{E_r^{\mathrm{T}},H_p^{\mathrm{T}}\} = \{E_r^{\mathrm{T}},H_p^{\mathrm{T}}\}(x,t)$.

First, the measured electromagnetic field data are interrelated with the unknown source distributions $\{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\} = \{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\}(x,t)$ via the global time-domain reciprocity theorem of the *convolution type*, Equation (28.2-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\}(x,t) = \{E_r^{\mathrm{T}},H_p^{\mathrm{T}}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.10-1}$$

$$\{J_k^{\mathrm{A}},K_j^{\mathrm{A}}\}(x,t) = \{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\}(x,t) \qquad \text{for } x\in\mathcal{D}^T, \tag{28.10-2}$$

and

$$\{\varepsilon_{k,r}^{\mathrm{A}},\mu_{j,p}^{\mathrm{A}}\}(x,t) = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-3}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\}(x,t) = \{E_k^{\Omega},H_j^{\Omega}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.10-4}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{J_r^{\mathrm{B}},K_p^{\mathrm{B}}\}(x,t) = \{J_r^{\Omega},K_p^{\Omega}\}(x,t) \qquad \text{for } x\in\mathcal{D}^{\Omega} . \tag{28.10-5}$$

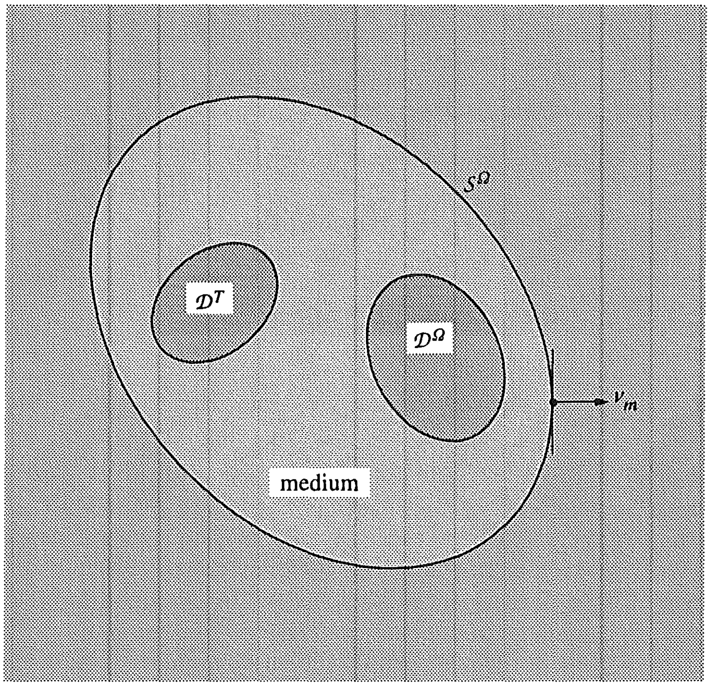
Furthermore, the medium properties of the embedding in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\varepsilon_{r,k}^{\mathrm{B}},\mu_{p,j}^{\mathrm{B}}\}(x,t) = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-6}$$

Then, application of Equation (28.2-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(J_k^{\mathrm{T}},E_k^{\Omega};x,t) - \mathrm{C}_t(K_j^{\mathrm{T}},H_j^{\Omega};x,t)\right]\mathrm{d}V \\
&= \int_{x\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_t(J_r^{\Omega},E_r^{\mathrm{T}};x,t) - \mathrm{C}_t(K_p^{\Omega},H_p^{\mathrm{T}};x,t)\right]\mathrm{d}V \\
&\quad + \varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{T}},H_p^{\Omega};x,t) - \mathrm{C}_t(E_r^{\Omega},H_p^{\mathrm{T}};x,t)\right]\mathrm{d}A .
\end{aligned} \tag{28.10-7}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are causal as well and the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the radiated or computational wave



28.10-1 Configuration of the inverse source problem: $\mathcal{D}^T$ is the support of the unknown radiating sources; on $\mathcal{D}^{\Omega}$ and $\mathcal{S}^{\Omega}$ the transmitted wavefield is accessible to measurement.

fields are present. If, however, the wave-field generation in state Ω is taken to be anti-causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are not causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.2-2).

Secondly, the measured electromagnetic field data are interrelated with the unknown source distributions $\{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\} = \{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\}(\boldsymbol{x},t)$ via the global time-domain reciprocity theorem of the *correlation type*, Equation (28.3-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\}(\boldsymbol{x},t) = \{E_r^{\mathrm{T}},H_p^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3\,, \tag{28.10-8}$$

$$\{J_k^{\mathrm{A}},K_j^{\mathrm{A}}\}(\boldsymbol{x},t) = \{J_k^{\mathrm{T}},K_j^{\mathrm{T}}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^T, \tag{28.10-9}$$

and

$$\{\varepsilon_{k,r}^{\mathrm{A}},\mu_{j,p}^{\mathrm{A}}\}(\boldsymbol{x},t) = \{\varepsilon_{k,r},\mu_{j,p}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3\,. \tag{28.10-10}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\}(\boldsymbol{x},t) = \{E_k^{\Omega},H_j^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3\,, \tag{28.10-11}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{J_r^{\mathrm{B}},K_p^{\mathrm{B}}\}(\boldsymbol{x},t) = \{J_r^{\Omega},K_p^{\Omega}\}(\boldsymbol{x},t) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}\,. \tag{28.10-12}$$

Furthermore, the medium properties of the embedding in state B will be taken to be the time-reverse adjoint of ones in state A, i.e.

$$\{\varepsilon^{\mathrm{B}}_{r,k},\mu^{\mathrm{B}}_{p,j}\}(x,t) = \{\mathrm{J}_t(\varepsilon_{k,r}),\mathrm{J}_t(\mu_{j,p})\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-13}$$

Then, application of Equation (28.3-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$-\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(J_k^{\mathrm{T}},\mathrm{J}_t(E_k^{\Omega});x,t) + \mathrm{C}_t(K_j^{\mathrm{T}},\mathrm{J}_t(H_j^{\Omega});x,t)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_t(\mathrm{J}_t(J_r^{\Omega}),E_r^{\mathrm{T}};x,t) + \mathrm{C}_t(\mathrm{J}_t(K_p^{\Omega}),H_p^{\mathrm{T}};x,t)\right]\mathrm{d}V$$

$$+\,\varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{T}},\mathrm{J}_t(H_p^{\Omega});x,t) + \mathrm{C}_t(\mathrm{J}_t(E_r^{\Omega}),H_p^{\mathrm{T}};x,t)\right]\mathrm{d}A . \tag{28.10-14}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are non-causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.3-2). If, however, the wave-field generation in state Ω is taken to be anti-causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are causal and the surface integral over $\mathcal{S}^{\Omega}$ does vanish, since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the radiated or computational wave fields are present.

For additional literature on the subject, see De Hoop (1987).

Complex frequency-domain analysis

The electromagnetic properties of the embedding medium are characterised by the relaxation functions $\{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s)$ in the complex frequency-domain analysis of the problem. The causally radiated electromagnetic field is denoted by $\{\hat{E}_r^{\mathrm{T}},\hat{H}_p^{\mathrm{T}}\} = \{\hat{E}_r^{\mathrm{T}},\hat{H}_p^{\mathrm{T}}\}(x,s)$.

First, the measured electromagnetic field data are interrelated with the unknown source distributions $\{\hat{J}_k^{\mathrm{T}},\hat{K}_j^{\mathrm{T}}\} = \{\hat{J}_k^{\mathrm{T}},\hat{K}_j^{\mathrm{T}}\}(x,s)$ via the global complex frequency-domain reciprocity theorem of the *time convolution type*, Equation (28.4-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\hat{E}_p^{\mathrm{A}},\hat{H}_r^{\mathrm{A}}\}(x,s) = \{\hat{E}_p^{\mathrm{T}},\hat{H}_r^{\mathrm{T}}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.10-15}$$

$$\{\hat{J}_j^{\mathrm{A}},\hat{K}_k^{\mathrm{A}}\}(x,s) = \{\hat{J}_j^{\mathrm{T}},\hat{K}_k^{\mathrm{T}}\}(x,s) \qquad \text{for } x\in\mathcal{D}^T, \tag{28.10-16}$$

and

$$\{\hat{\eta}^{\mathrm{A}}_{k,r},\hat{\zeta}^{\mathrm{A}}_{j,p}\}(x,s) = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-17}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\}(x,s) = \{\hat{E}_k^{\Omega},\hat{H}_j^{\Omega}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.10-18}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{J}_r^{\mathrm{B}},\hat{K}_p^{\mathrm{B}}\}(x,s) = \{\hat{J}_r^{\Omega},\hat{K}_p^{\Omega}\}(x,s) \qquad \text{for } x\in\mathcal{D}^{\Omega}. \tag{28.10-19}$$

Furthermore, the medium properties in the embedding in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\hat{\eta}_{r,k}^{\mathrm{B}},\hat{\zeta}_{p,j}^{\mathrm{B}}\}(x,s) = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-20}$$

Then, application of Equation (28.4-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{x\in\mathcal{D}^T}\left[\hat{J}_k^{\mathrm{T}}(x,s)\hat{E}_k^{\Omega}(x,s) - \hat{K}_j^{\mathrm{T}}(x,s)\hat{H}_j^{\Omega}(x,s)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^{\Omega}}\left[\hat{J}_r^{\Omega}(x,s)\hat{E}_r^{\mathrm{T}}(x,s) - \hat{K}_p^{\Omega}(x,s)\hat{H}_p^{\mathrm{T}}(x,s)\right]\mathrm{d}V$$

$$+ \varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\hat{E}_r^{\mathrm{T}}(x,s)\hat{H}_p^{\Omega}(x,s) - \hat{E}_r^{\Omega}(x,s)\hat{H}_p^{\mathrm{T}}(x,s)\right]\mathrm{d}A . \tag{28.10-21}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known, provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the radiated or computational wave fields are present. If, however, the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.4-4).

Secondly, the measured electromagnetic field data are interrelated with the unknown source distributions $\{\hat{J}_k^{\mathrm{T}},\hat{K}_j^{\mathrm{T}}\} = \{\hat{J}_k^{\mathrm{T}},\hat{K}_j^{\mathrm{T}}\}(x,s)$, via the global complex frequency-domain reciprocity theorem of the *time correlation type*, Equation (28.5-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual state present in the configuration, i.e.

$$\{\hat{E}_p^{\mathrm{A}},\hat{H}_r^{\mathrm{A}}\}(x,s) = \{\hat{E}_p^{\mathrm{T}},\hat{H}_r^{\mathrm{T}}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.10-22}$$

$$\{\hat{J}_k^{\mathrm{A}},\hat{K}_j^{\mathrm{A}}\}(x,s) = \{\hat{J}_k^{\mathrm{T}},\hat{K}_j^{\mathrm{T}}\}(x,s) \qquad \text{for } x\in\mathcal{D}^T, \tag{28.10-23}$$

and

$$\{\hat{\eta}_{k,r}^{\mathrm{A}},\hat{\zeta}_{j,p}^{\mathrm{A}}\}(x,s) = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-24}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\}(x,s) = \{\hat{E}_k^{\Omega},\hat{H}_j^{\Omega}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.10-25}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{J}_r^{\mathrm{B}},\hat{K}_p^{\mathrm{B}}\}(x,s) = \{\hat{J}_r^{\Omega},\hat{K}_p^{\Omega}\}(x,s) \qquad \text{for } x\in\mathcal{D}^{\Omega}. \tag{28.10-26}$$

Furthermore, the medium properties in the embedding in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\hat{\eta}_{r,k}^{\mathrm{B}},\hat{\zeta}_{p,j}^{\mathrm{B}}\}(x,s) = \{-\hat{\eta}_{k,r},-\hat{\zeta}_{j,p}\}(x,-s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.10-27}$$

Then, application of Equation (28.5-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$-\int_{x\in\mathcal{D}^T}\left[\hat{J}_k^{\mathrm{T}}(x,s)\hat{E}_k^{\Omega}(x,-s)+\hat{K}_j^{\mathrm{T}}(x,s)\hat{H}_j^{\Omega}(x,-s)\right]\mathrm{d}V$$

$$=\int_{x\in\mathcal{D}^{\Omega}}\left[\hat{J}_r^{\Omega}(x,-s)\hat{E}_r^{\mathrm{T}}(x,s)+\hat{K}_p^{\Omega}(x,-s)\hat{H}_p^{\mathrm{T}}(x,s)\right]\mathrm{d}V$$

$$+\,\varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\hat{E}_r^{\mathrm{T}}(x,s)\hat{H}_p^{\Omega}(x,-s)+\hat{E}_r^{\Omega}(x,-s)\hat{H}_p^{\mathrm{T}}(x,s)\right]\mathrm{d}A\,. \tag{28.10-28}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state T and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.5-4). If, however, the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the radiated or computational wave fields are present.

A solution to the inverse source problem is commonly constructed as follows. For the source distributions in the computational state Ω we take a sequence of M linearly independent spatial distributions with the common spatial support $\mathcal{D}^{\Omega}$. The corresponding sequence of electromagnetic field distributions (in the medium adjoint, or time-reverse adjoint, to the actual one) is computed. Next, the unknown source distributions are expanded into an appropriate sequence of N expansion functions with the common spatial support $\mathcal{D}^T$ or a subset of it; the corresponding expansion coefficients are unknown. Substitution of the results in Equations (28.10-7), (28.10-14), (28.10-21) or (28.10-28) and evaluation of the relevant integrals leads to a system of M linear algebraic equations with N unknowns. When $M < N$, the system is underdetermined and cannot be solved. When $M = N$, the system can be solved, unless the pertaining matrix of coefficients is singular. However, even if this matrix is non-singular, it turns out to be ill-conditioned in most practical cases. Therefore, one usually takes $M > N$, and a best fit of the expanded source distributions to the measured data is obtained by the application of minimisation techniques (for example, least-square minimisation). Note that each of the Equations (28.10-7), (28.10-14), (28.10-21) or (28.10-28) leads to an associated inversion algorithm.

The computational state Ω is representative for the manner in which the measured data are processed in the inversion algorithms. Since a computational state does not have to meet the physical condition of causality, there is no objection against its being anti-causal. Which of the two possibilities (causal or anti-causal) leads to the best results as far as accuracy and amount of computational effort are concerned, is difficult to judge. Research on this aspect is still in full progress (see Fokkema and Van den Berg 1993).

It is to be noted that the solution to an inverse source problem is not unique because of the existence of non-radiating source distributions (i.e. non-zero distributions with the support $\mathcal{D}^T$ that yield a vanishing wave field in the domain exterior to $\mathcal{D}^T$). Therefore, a numerically constructed solution to an inverse source problem is always *a* solution (and *not the* solution) that depends on the solution method employed.

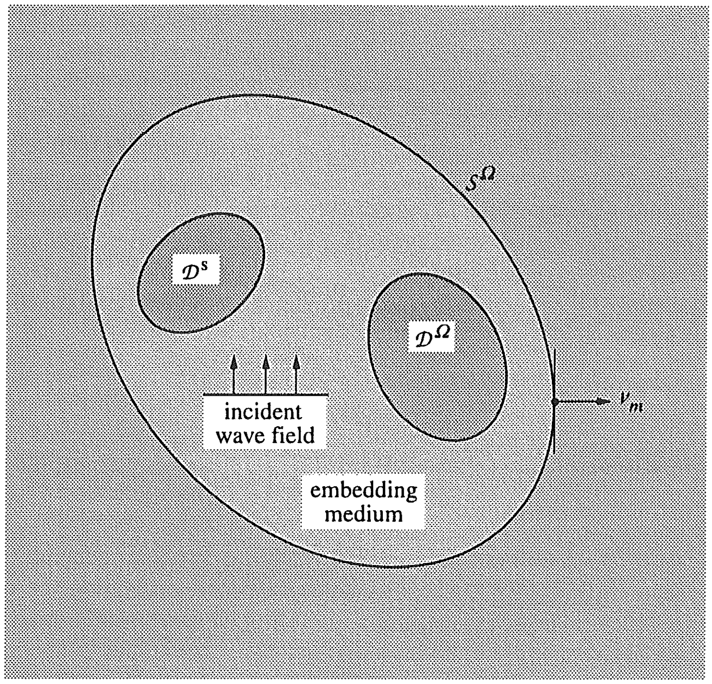
Examples of electromagnetic inverse source problems are found in radio-astronomy where radio stars serve as a source of electromagnetic radiation.

28.11 The inverse scattering problem

The configuration in an electromagnetic inverse scattering problem generally consists of a background medium with known electromagnetic properties, occupying the entire $\mathcal{R}^3$ (the "*embedding*"), in which, in principle, the radiation from given, arbitrarily distributed electromagnetic sources can be calculated with the aid of the theory developed in Section 28.8. In the embedding an, either known or guessed, bounded domain $\mathcal{D}^s$ (the "*scatterer*") is present in which the medium properties show an unknown contrast with the ones of the embedding. The contrasting domain is irradiated by an incident electromagnetic field that is generated by sources in some subdomain $\mathcal{D}^i$ of $\mathcal{R}^3$ and that propagates in the embedding. The presence of the contrasting domain manifests itself through the presence of a non-vanishing scattered electromagnetic field in the entire embedding. In some bounded subdomain $\mathcal{D}^\Omega$ of $\mathcal{R}^3$, and exterior to $\mathcal{D}^s$, the scattered electromagnetic field is accessible to measurement (Figure 28.11-1).

The objective is to reconstruct the medium parameters (or their contrasts with the ones of the embedding) from (a set of) measured values of the electric field strength and/or the magnetic field strength in $\mathcal{D}^\Omega$. Since the inverse scattering problem is, by necessity, a *remote sensing problem*, the global reciprocity theorems of Sections 28.2–28.5 can be expected to provide a means for interrelating the known, measured wave-field data with the unknown medium properties in the scattering region. The standard provisions of Section 28.1 for handling an unbounded domain are made. The scattered wave field is, by its nature, causally related to the contrast sources by which it is generated. For gathering maximum information, the reciprocity theorems are applied to the domain interior to a closed surface $\mathcal{S}^\Omega$ that completely surrounds both $\mathcal{D}^s$ and $\mathcal{D}^\Omega$. If necessary, also measurements on $\mathcal{S}^\Omega$ can be carried out. In general, $\mathcal{D}^s$ and $\mathcal{D}^i$ are disjoint, as well as $\mathcal{D}^s$ and $\mathcal{D}^\Omega$. This need not be the case for $\mathcal{D}^i$ and $\mathcal{D}^\Omega$; these domains may have a non-empty cross-section.

The incident, scattered and total wave fields are introduced as in Section 28.9. Now, the easiest way to address the inverse scattering problem is to consider it partly as an inverse source problem with the contrast volume source densities as the unknowns, where the non-uniqueness of the contrast volume source distributions is to be removed by invoking the remaining conditions to be satisfied. In the latter, the condition that the reconstructed contrast-in-medium parameters must be independent of the incident wave field plays a crucial role. Once the contrast volume source distributions have been determined, the scattered wave field is, following the



28.11-1 Configuration of the inverse scattering problem: $\mathcal{D}^s$ is the support of the unknown contrast in medium properties; on $\mathcal{D}^\Omega$ and $\mathcal{S}^\Omega$ the scattered wavefield is accessible to measurement.

procedures of Section 28.9, calculated in the domain $\mathcal{D}^s$ and since the incident wave field and the medium parameters of the embedding are known, the parameters of the medium in $\mathcal{D}^s$ follow.

Time-domain analysis

In the time-domain analysis of the problem, the electromagnetic properties of the embedding medium are characterised by the relaxation functions $\{\varepsilon_{k,r},\mu_{j,p}\} = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t)$, which are causal functions of time. The case of an instantaneously reacting embedding medium easily follows from the more general case of a medium with relaxation. The unknown electromagnetic properties of the scatterer are characterised by the relaxation functions $\{\varepsilon^s_{k,r},\mu^s_{j,p}\} = \{\varepsilon^s_{k,r},\mu^s_{j,p}\}(x,t)$, which are causal functions of time as well. The incident wave field is $\{E^i_r,H^i_p\} = \{E^i_r,H^i_p\}(x,t)$, the scattered wave field is $\{E^s_r,H^s_p\} = \{E^s_r,H^s_p\}(x,t)$ and the total wave field is $\{E_r,H_p\} = \{E_r,H_p\}(x,t)$, with $\{E_r,H_p\} = \{E^i_r + E^s_r, H^i_p + H^s_p\}$. The equivalent contrast volume source distributions that generate the scattered wave field are then (see Equations (28.9-18) and (28.9-19))

$$J^s_k = C_t(\varepsilon^s_{k,r} - \varepsilon_{k,r}, E_r; x,t) \quad \text{for } x \in \mathcal{D}^s, \tag{28.11-1}$$

$$K^s_j = C_t(\mu^s_{j,p} - \mu_{j,p}, H_p; x,t) \quad \text{for } x \in \mathcal{D}^s. \tag{28.11-2}$$

First, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{J_k^{\mathrm{s}},K_j^{\mathrm{s}}\} = \{J_k^{\mathrm{s}},K_j^{\mathrm{s}}\}(x,t)$ via the global time-domain reciprocity theorem of the *convolution type*, Equation (28.2-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\}(x,t) = \{E_r^{\mathrm{s}},H_p^{\mathrm{s}}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.11-3}$$

$$\{J_k^{\mathrm{A}},K_j^{\mathrm{A}}\}(x,t) = \{J_k^{\mathrm{s}},K_j^{\mathrm{s}}\}(x,t) \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.11-4}$$

and

$$\{\varepsilon_{k,r}^{\mathrm{A}},\mu_{j,p}^{\mathrm{A}}\}(x,t) = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.11-5}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\}(x,t) = \{E_k^{\Omega},H_j^{\Omega}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 , \tag{28.11-6}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{J_r^{\mathrm{B}},K_p^{\mathrm{B}}\}(x,t) = \{J_r^{\Omega},K_p^{\Omega}\}(x,t) \qquad \text{for } x\in\mathcal{D}^{\Omega}. \tag{28.11-7}$$

Furthermore, the medium properties in the embedding in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\varepsilon_{r,k}^{\mathrm{B}},\mu_{p,j}^{\mathrm{B}}\}(x,t) = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.11-8}$$

Then, application of Equation (28.2-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(J_k^{\mathrm{s}},E_k^{\Omega};x,t) - \mathrm{C}_t(K_j^{\mathrm{s}},H_j^{\Omega};x,t)\right]\mathrm{d}V \\
&= \int_{x\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_t(J_r^{\Omega},E_r^{\mathrm{s}};x,t) - \mathrm{C}_t(K_p^{\Omega},H_p^{\mathrm{s}};x,t)\right]\mathrm{d}V \\
&\quad + \varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s}},H_p^{\Omega};x,t) - \mathrm{C}_t(E_r^{\Omega},H_p^{\mathrm{s}};x,t)\right]\mathrm{d}A .
\end{aligned} \tag{28.11-9}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are causal as well and the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the scattered or computational wave fields are present. If, however, the wave-field generation in state Ω is taken to be anti-causal, the convolutions occurring in the integral over $\mathcal{S}^{\Omega}$ are not causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.2-2).

Secondly, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{J_k^{\mathrm{s}},K_j^{\mathrm{s}}\} = \{J_k^{\mathrm{s}},K_j^{\mathrm{s}}\}(x,t)$ via the global time-domain reciprocity theorem of the *correlation type*, Equation (28.3-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\}(x,t) = \{E_r^{\mathrm{s}},H_p^{\mathrm{s}}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3\,, \tag{28.11-10}$$

$$\{J_k^{\mathrm{A}},K_j^{\mathrm{A}}\}(x,t) = \{J_k^{\mathrm{s}},K_j^{\mathrm{s}}\}(x,t) \qquad \text{for } x\in\mathcal{D}^{\mathrm{s}}, \tag{28.11-11}$$

and

$$\{\varepsilon_{k,r}^{\mathrm{A}},\mu_{j,p}^{\mathrm{A}}\}(x,t) = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3\,. \tag{28.11-12}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\}(x,t) = \{E_k^{\Omega},H_j^{\Omega}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3\,, \tag{28.11-13}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{J_r^{\mathrm{B}},K_p^{\mathrm{B}}\}(x,t) = \{J_r^{\Omega},K_p^{\Omega}\}(x,t) \qquad \text{for } x\in\mathcal{D}^{\Omega}. \tag{28.11-14}$$

Furthermore, the medium properties of the embedding in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\varepsilon_{r,k}^{\mathrm{B}},\mu_{p,j}^{\mathrm{B}}\}(x,t) = \{\mathrm{J}_t(\varepsilon_{k,r}),\mathrm{J}_t(\mu_{j,p})\}(x,t) \qquad \text{for } x\in\mathcal{R}^3\,. \tag{28.11-15}$$

Then, application of Equation (28.3-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\begin{aligned}
&-\int_{x\in\mathcal{D}^T}\left[\mathrm{C}_t(J_k^{\mathrm{s}},\mathrm{J}_t(E_k^{\Omega});x,t) + \mathrm{C}_t(K_j^{\mathrm{s}},\mathrm{J}_t(H_j^{\Omega});x,t)\right]\mathrm{d}V \\
&= \int_{x\in\mathcal{D}^{\Omega}}\left[\mathrm{C}_t(\mathrm{J}_t(J_r^{\Omega}),E_r^{\mathrm{s}};x,t) + \mathrm{C}_t(\mathrm{J}_t(K_p^{\Omega}),H_p^{\mathrm{s}};x,t)\right]\mathrm{d}V \\
&\quad + \varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s}},\mathrm{J}_t(H_p^{\Omega});x,t) + \mathrm{C}_t(\mathrm{J}_t(E_r^{\Omega}),H_p^{\mathrm{s}};x,t)\right]\mathrm{d}A\,.
\end{aligned} \tag{28.11-16}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are non-causal and the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.3-2). If, however, the wave-field generation in state Ω is taken to be anti-causal, the correlations occurring in the integral over $\mathcal{S}^{\Omega}$ are causal and the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the scattered or computational wave fields are present.

A solution to the time-domain inverse scattering problem is commonly constructed as follows. First, the contrast-in-medium parameters are discretised by writing them as a linear combination of M expansion functions with unknown expansion coefficients. Each of the expansion functions has $\mathcal{D}^{\rm s}$, or a subset of it, as its support. Next, for each given incident wave field, the scattered wave field is measured in N subdomains of $\mathcal{D}^{\Omega}$. The latter discretisation induces the choice of the N supports of the source distributions of the observational state. Finally, a number I of different incident wave fields is selected, where "different" may involve different choices in temporal behaviour, in location in space, or in both. With $NI \geqslant M$, the non-linear problem of evaluating the M expansion coefficients of the contrast-in-medium discretisation is solved by some iterative procedure (for example, by iterative minimisation of the global error over all domains where equality signs should pointwise hold, and all time intervals involved). In this procedure, Equations (28.11-1), (28.11-2), (28.11-9) or (28.11-16) and the source type integral representations (28.9-20) and (28.9-21) are used simultaneously.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the electromagnetic properties of the embedding medium are characterised by the relaxation functions $\{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(\boldsymbol{x},s)$. The unknown electromagnetic properties of the scatterer are characterised by the relaxation functions $\{\hat{\eta}^{\rm s}_{k,r},\hat{\zeta}^{\rm s}_{j,p}\} = \{\hat{\eta}^{\rm s}_{k,r},\hat{\zeta}^{\rm s}_{j,p}\}(\boldsymbol{x},s)$. The incident wave field is $\{\hat{E}^{\rm i}_r,\hat{H}^{\rm i}_p\} = \{\hat{E}^{\rm i}_r,\hat{H}^{\rm i}_p\}(\boldsymbol{x},s)$, the scattered wave field is $\{\hat{E}^{\rm s}_r,\hat{H}^{\rm s}_p\} = \{\hat{E}^{\rm s}_r,\hat{H}^{\rm s}_p\}(\boldsymbol{x},s)$ and the total wave field is $\{\hat{E}_r,\hat{H}_p\} = \{\hat{E}_r,\hat{H}_p\}(\boldsymbol{x},s)$, with $\{\hat{E}_r,\hat{H}_p\} = \{\hat{E}^{\rm i}_r + \hat{E}^{\rm s}_r,\hat{H}^{\rm i}_p + \hat{H}^{\rm s}_p\}$. The equivalent contrast volume source distributions that generate the scattered wave field are then (see Equations (28.9-41) and (28.9-42)).

$$\hat{J}^{\rm s}_k = (\hat{\eta}^{\rm s}_{k,r} - \hat{\eta}_{k,r})\hat{E}_r \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\rm s}, \tag{28.11-17}$$

$$\hat{K}^{\rm s}_j = (\hat{\zeta}^{\rm s}_{j,p} - \hat{\zeta}_{j,p})\hat{H}_p \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\rm s}. \tag{28.11-18}$$

First, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{\hat{J}^{\rm s}_k,\hat{K}^{\rm s}_j\}$ via the global complex frequency-domain reciprocity theorem of the *time convolution type*, Equation (28.4-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\hat{E}^{\rm A}_r,\hat{H}^{\rm A}_p\}(\boldsymbol{x},s) = \{\hat{E}^{\rm s}_r,\hat{H}^{\rm s}_p\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3, \tag{28.11-19}$$

$$\{\hat{J}^{\rm A}_k,\hat{K}^{\rm A}_j\}(\boldsymbol{x},s) = \{\hat{J}^{\rm s}_k,\hat{K}^{\rm s}_j\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\rm s}, \tag{28.11-20}$$

and

$$\{\hat{\eta}^{\rm A}_{k,r},\hat{\zeta}^{\rm A}_{j,p}\}(\boldsymbol{x},s) = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3. \tag{28.11-21}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{E}^{\rm B}_k,\hat{H}^{\rm B}_j\}(\boldsymbol{x},s) = \{\hat{E}^{\Omega}_k,\hat{H}^{\Omega}_j\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3, \tag{28.11-22}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{J}^{\rm B}_r,\hat{K}^{\rm B}_p\}(\boldsymbol{x},s) = \{\hat{J}^{\Omega}_r,\hat{K}^{\Omega}_p\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{28.11-23}$$

Furthermore, the medium properties of the embedding in state B will be taken to be the adjoint of the ones in state A, i.e.

$$\{\hat{\eta}^{\mathrm{B}}_{r,k},\hat{\zeta}^{\mathrm{B}}_{p,j}\}(\boldsymbol{x},s) = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3 . \tag{28.11-24}$$

Then, application of Equation (28.4-7) to the domain interior to $\mathcal{S}^{\Omega}$ yields

$$\int_{\boldsymbol{x}\in\mathcal{D}^T}\left[\hat{J}^{\mathrm{s}}_k(\boldsymbol{x},s)\hat{E}^{\Omega}_k(\boldsymbol{x},s) - \hat{K}^{\mathrm{s}}_j(\boldsymbol{x},s)\hat{H}^{\Omega}_j(\boldsymbol{x},s)\right]\mathrm{d}V$$

$$= \int_{\boldsymbol{x}\in\mathcal{D}^{\Omega}}\left[\hat{J}^{\Omega}_r(\boldsymbol{x},s)\hat{E}^{\mathrm{s}}_r(\boldsymbol{x},s) - \hat{K}^{\Omega}_p(\boldsymbol{x},s)\hat{H}^{\mathrm{s}}_p(\boldsymbol{x},s)\right]\mathrm{d}V$$

$$+\,\varepsilon_{m,r,p}\int_{\boldsymbol{x}\in\mathcal{S}^{\Omega}}\nu_m\left[\hat{E}^{\mathrm{s}}_r(\boldsymbol{x},s)\hat{H}^{\Omega}_p(\boldsymbol{x},s) - \hat{E}^{\Omega}_r(\boldsymbol{x},s)\hat{H}^{\mathrm{s}}_p(\boldsymbol{x},s)\right]\mathrm{d}A\,. \tag{28.11-25}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^{\Omega}$. In case the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^{\Omega}$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^{\Omega}$ and that sphere no sources of the scattered or computational wave fields are present. If, however, the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^{\Omega}$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.4-4).

Secondly, the measured scattered wave-field data are interrelated with the unknown contrast source distributions $\{\hat{J}^{\mathrm{s}}_k,\hat{K}^{\mathrm{s}}_j\} = \{\hat{J}^{\mathrm{s}}_k,\hat{K}^{\mathrm{s}}_j\}(\boldsymbol{x},s)$ via the global complex frequency-domain reciprocity theorem of the *time correlation type*, Equation (28.5-7). This theorem is applied to the domain interior to the closed surface $\mathcal{S}^{\Omega}$. In it, we take for state A the actual scattered state present in the configuration, i.e.

$$\{\hat{E}^{\mathrm{A}}_r,\hat{H}^{\mathrm{A}}_p\}(\boldsymbol{x},s) = \{\hat{E}^{\mathrm{s}}_r,\hat{H}^{\mathrm{s}}_p\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3 , \tag{28.11-26}$$

$$\{\hat{J}^{\mathrm{A}}_k,\hat{K}^{\mathrm{A}}_j\}(\boldsymbol{x},s) = \{\hat{J}^{\mathrm{s}}_k,\hat{K}^{\mathrm{s}}_j\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}, \tag{28.11-27}$$

and

$$\{\hat{\eta}^{\mathrm{A}}_{k,r},\hat{\zeta}^{\mathrm{A}}_{j,p}\}(\boldsymbol{x},s) = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3 . \tag{28.11-28}$$

For state B, we take a "computational" or "observational" one; this state will be denoted by the superscript Ω. The corresponding wave field is

$$\{\hat{E}^{\mathrm{B}}_k,\hat{H}^{\mathrm{B}}_j\}(\boldsymbol{x},s) = \{\hat{E}^{\Omega}_k,\hat{H}^{\Omega}_j\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{R}^3 , \tag{28.11-29}$$

and its source distributions will be taken to have the support $\mathcal{D}^{\Omega}$, i.e.

$$\{\hat{J}^{\mathrm{B}}_r,\hat{K}^{\mathrm{B}}_p\}(\boldsymbol{x},s) = \{\hat{J}^{\Omega}_r,\hat{K}^{\Omega}_p\}(\boldsymbol{x},s) \qquad \text{for } \boldsymbol{x}\in\mathcal{D}^{\Omega}. \tag{28.11-30}$$

Furthermore, the medium properties of the embedding in state B will be taken to be the time-reverse adjoint of the ones in state A, i.e.

$$\{\hat{\eta}^{\mathrm{B}}_{r,k},\hat{\zeta}^{\mathrm{B}}_{p,j}\}(x,s) = \{-\hat{\eta}_{k,r},-\hat{\zeta}_{j,p}\}(x,-s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.11-31}$$

Then, application of Equation (28.5-7) to the domain interior to $\mathcal{S}^\Omega$ yields

$$-\int_{x\in\mathcal{D}^{\mathrm{s}}}\left[\hat{J}^{\mathrm{s}}_k(x,s)\hat{E}^{\Omega}_k(x,-s)+\hat{K}^{\mathrm{s}}_j(x,s)\hat{H}^{\Omega}_j(x,-s)\right]\mathrm{d}V$$

$$=\int_{x\in\mathcal{D}^{\Omega}}\left[\hat{J}^{\Omega}_r(x,-s)\hat{E}^{\mathrm{s}}_r(x,s)+\hat{K}^{\Omega}_p(x,-s)\hat{H}^{\mathrm{s}}_p(x,s)\right]\mathrm{d}V$$

$$+\,\varepsilon_{m,r,p}\int_{x\in\mathcal{S}^{\Omega}}\nu_m\left[\hat{E}^{\mathrm{s}}_r(x,s)\hat{H}^{\Omega}_p(x,-s)+\hat{E}^{\Omega}_r(x,-s)\hat{H}^{\mathrm{s}}_p(x,s)\right]\mathrm{d}A\,. \tag{28.11-32}$$

The left-hand side of this equation contains the unknown quantities, while the right-hand side is known provided that the necessary measurements pertaining to the state s and the wave-field evaluations pertaining to the state Ω are carried out. For the latter (computational) state we can choose between either causal or anti-causal generation of the wave field by its sources. This makes a difference for the surface contribution over $\mathcal{S}^\Omega$. In case the wave-field generation in state Ω is taken to be causal, the surface integral over $\mathcal{S}^\Omega$ does not vanish, although its value is a constant for each choice of the source distributions in state Ω (see Exercise 28.5-4). If, however, the wave-field generation in state Ω is taken to be anti-causal, the surface integral over $\mathcal{S}^\Omega$ vanishes since the one over a sphere with infinitely large radius does and in between $\mathcal{S}^\Omega$ and that sphere no sources of the scattered or computational wave fields are present.

A solution to the complex frequency-domain inverse scattering problem is commonly constructed as follows. First, the contrast-in-medium parameters are discretised by writing them as a linear combination of M expansion functions with unknown expansion coefficients. Each of the expansion functions has $\mathcal{D}^{\mathrm{s}}$, or a subset of it, as its support. Next, for each given incident wave field, the scattered wave field is measured in N subdomains of $\mathcal{D}^\Omega$. The latter discretisation induces the choice of the N supports of the source distributions of the observational state. Finally, a number I of different incident wave fields is selected, where "different" may involve different choices in complex frequency content, in location in space, or in both. With $NI \geqslant M$, the non-linear problem of evaluating the M expansion coefficients of the contrast-in-medium discretisation is solved by some iterative procedure (for example, by iterative minimisation of the global error over all domains where equality signs should pointwise hold, and all time intervals or complex frequency values involved). In this procedure, Equations (28.11-17) and (28.11-18), (28.11-25) or (28.11-32) and the source type integral representations (28.9-43) and (28.9-44) are used simultaneously.

The computational state Ω is representative for the manner in which the measured data are processed in the inversion algorithms. Since a computational state does not have to meet the physical condition of causality, there is no objection against its being anti-causal. Which of the two possibilities (causal or anti-causal) leads to the best results as far as accuracy and amount of computational effort are concerned, is difficult to say. Research on this aspect is still in full progress (see Fokkema and Van den Berg 1993).

Examples of electromagnetic inverse scattering problems are found in the electromagnetic environmental monitoring of waste disposal in the ground, in geophysical exploration for minerals and fossil energy resources, and in the non-destructive evaluation of mechanical structures.

28.12 Electromagnetic wave-field representations in a subdomain of the configuration space; equivalent surface sources; Huygens' principle and the Ewald–Oseen extinction theorem

In Section 28.8, wave-field representations have been derived that express the electric field strength and the magnetic field strength at any point of a configuration in terms of the volume source distributions of electric current and magnetic current that generate the wave field. In them, the point-source solutions (Green's functions) to the radiation problem play a crucial role. In a number of cases we are, however, only interested in the values of the wave-field quantities in some subdomain of the configuration, and a wave-field representation pertaining to that subdomain would suffice. In the present section it is shown, how the reciprocity theorem of the time convolution type leads to the desired expressions, be it that now, in addition to the volume integrals over the volume source distributions (as far as present in the subdomain of interest), surface integrals over the boundary surface of this subdomain occur. In these representations, again the point-source solution (Green's functions) are the intervening kernels.

We assume that the Green's functions introduced in Section 28.8 are defined in the entire $\mathcal{R}^3$. The standard provisions of Section 28.1 for the handling of an unbounded domain are made. Let, further, $\mathcal{D}$ be the subdomain of $\mathcal{R}^3$ in which expressions for the generated electromagnetic field are to be found. The boundary surface of $\mathcal{D}$ is $\partial\mathcal{D}$ and the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$ is denoted by $\mathcal{D}'$ (Figure 28.12-1). In fact, the relevant wave field need only be defined in $\mathcal{D}$ and on $\partial\mathcal{D}$. The constitutive properties must, however, be defined in the entire $\mathcal{R}^3$ in order that the necessary point-source solutions can be defined in $\mathcal{R}^3$. In this sense, $\mathcal{R}^3$ serves as an embedding of $\mathcal{D}$. The Green's functions of the embedding are assumed to satisfy the causality conditions at infinity.

Since the generated electromagnetic field is a physical wave field, it is causally related to its source distributions.

Time-domain analysis

For the time-domain analysis of the problem the electromagnetic properties of the medium present in $\mathcal{R}^3$ are characterised by the relaxation functions $\{\varepsilon_{k,r},\mu_{j,p}\} = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t)$, which are causal functions of time, and the global reciprocity theorem of the time convolution type, Equation (28.2-7), is applied to the subdomain $\mathcal{D}$ of $\mathcal{R}^3$. In the theorem, state A is taken to be the generated electromagnetic field under consideration, i.e.

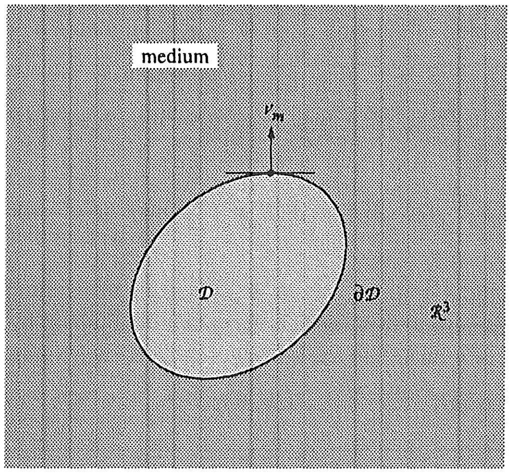
$$\{E_r^{\mathrm{A}},H_p^{\mathrm{A}}\} = \{E_r,H_p\}(x,t) \qquad \text{for } x\in\mathcal{D}, \tag{28.12-1}$$

$$\{J_k^{\mathrm{A}},K_j^{\mathrm{A}}\} = \{J_k,K_j\}(x,t) \qquad \text{for } x\in\mathcal{D}, \tag{28.12-2}$$

and

$$\{\varepsilon_{k,r}^{\mathrm{A}},\mu_{j,p}^{\mathrm{A}}\} = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.12-3}$$

Next, state B is chosen such that the application of Equation (28.2-7) to the subdomain $\mathcal{D}$ leads to the values of $\{E_r H_p\}$ at some arbitrary point $x'\in\mathcal{D}$. Inspection of the right-hand side of Equation (28.2-7) reveals that this is accomplished if we take for the source distributions of state B a point source at x' of electric current in case we want an expression for the electric field



28.12-1 Configuration for the wavefield representations in the subdomain $\mathcal{D}$ of the configuration space $\mathcal{R}^3$ for which the Green's functions are defined. $\partial\mathcal{D}$ is the (smooth) boundary surface of $\mathcal{D}$. The Green's functions satisfy causality conditions at infinity.

strength at x' and a point source at x' of magnetic current in case we want an expression for the magnetic field strength at x', while the medium in state B must be taken to be adjoint to the one in state A, i.e.

$$\{\varepsilon^{\mathrm{B}}_{r,k},\mu^{\mathrm{A}}_{p,j}\} = \{\varepsilon_{k,r},\mu_{j,p}\}(x,t) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.12-4}$$

The two choices for the source distribution will be discussed separately below.

First, we choose

$$J_r^{\mathrm{B}} = a_r\delta(x-x',t) \quad \text{and} \quad K_p^{\mathrm{B}} = 0 , \tag{28.12-5}$$

where $\delta(x-x',t)$ represents the four-dimensional unit impulse (Dirac distribution) operative at the point $x = x'$ and at the instant $t = 0$, while a_r is an arbitrary constant vector. The electromagnetic field that is, for the present application, radiated by this source and satisfies the causality condition at infinity, is denoted as

$$\{E_k^{\mathrm{B}},H_j^{\mathrm{B}}\} = \{E_k^{J;\mathrm{B}},H_j^{J;\mathrm{B}}\}(x,x',t) , \tag{28.12-6}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (28.12-5) and the properties of $\delta(x-x',t)$, we have

$$\int_{x\in\mathcal{D}} \left[\mathrm{C}_t(J_r^{\mathrm{B}},E_r^{\mathrm{A}};x,t) - \mathrm{C}_t(K_p^{\mathrm{B}},H_p^{\mathrm{A}};x,t)\right]\mathrm{d}V$$

$$= \int_{x\in\mathcal{D}} \mathrm{C}_t(a_r\delta(x-x',t),E_r;x,t)\,\mathrm{d}V = a_rE_r(x',t)\chi_{\mathcal{D}}(x') \qquad \text{for } x'\in\mathcal{R}^3 , \tag{28.12-7}$$

where

$$\chi_{\mathcal{D}}(x') = \{1,\tfrac{1}{2},0\} \qquad \text{for } x'\in\{\mathcal{D},\partial\mathcal{D},\mathcal{D}'\} \tag{28.12-8}$$

is the characteristic function of the set $\mathcal{D}$, and $\mathcal{D}'$ is the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. With this, we arrive at

$$a_r E_r(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(E_k^{J;\mathrm{B}}, J_k; x,x',t) - \mathrm{C}_t(H_j^{J;\mathrm{B}}, K_j; x,x',t)\right] \mathrm{d}V$$

$$+\, \varepsilon_{m,r,p} \int_{x\in\partial\mathcal{D}} \nu_m \left[\mathrm{C}_t(E_r^{J;\mathrm{B}}, H_p; x,x',t) - \mathrm{C}_t(E_r H_p^{J;\mathrm{B}}; x,x',t)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{R}^3 , \qquad (28.12\text{-}9)$$

where we have used the symmetry of the convolution in its functional arguments. From Equation (28.12-9) a representation for $E_r(x',t)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $E_k^{J;\mathrm{B}}$ and $H_j^{J;\mathrm{B}}$ are linearly related to a_r. Introducing the Green's functions through

$$\{E_k^{J;\mathrm{B}}, H_j^{J;\mathrm{B}}\}(x,x',t) = \{G_{k,r}^{EJ;\mathrm{B}}, G_{j,r}^{HJ;\mathrm{B}}\}(x,x',t)a_r , \qquad (28.12\text{-}10)$$

using the reciprocity relations for these functions (see Exercises 28.8-1 and 28.8-3)

$$\{G_{k,r}^{EJ;\mathrm{B}}, G_{j,r}^{HJ;\mathrm{B}}\}(x,x',t) = \{G_{r,k}^{EJ}, -G_{r,j}^{EK}\}(x',x,t) , \qquad (28.12\text{-}11)$$

and invoking the condition that the resulting equation has to hold for arbitrary values of a_r, Equation (28.12-9) leads to the final result

$$E_r(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(G_{r,k}^{EJ}, J_k; x',x,t) + \mathrm{C}_t(G_{r,j}^{EK}, K_j; x',x,t)\right] \mathrm{d}V$$

$$+ \int_{x\in\partial\mathcal{D}} \left[\mathrm{C}_t(G_{r,k}^{EJ}, -\varepsilon_{k,m,p}\nu_m H_p; x',x,t)\right.$$

$$\left. +\, \mathrm{C}_t(G_{r,j}^{EK}, \varepsilon_{j,n,r'}\nu_n E_{r'}; x',x,t)\right] \mathrm{d}A \qquad \text{for } x'\in\mathcal{R}^3 . \qquad (28.12\text{-}12)$$

Equation (28.12-12) expresses, for $x'\in\mathcal{D}$, the electric field strength E_r of the generated electromagnetic field at x' as the superposition of the contributions from the elementary volume sources $J_k\,\mathrm{d}V$ and $K_j\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the equivalent elementary surface sources $-\varepsilon_{k,m,p}\nu_m H_p\,\mathrm{d}A$ of electric current and $\varepsilon_{j,n,r'}\nu_n E_{r'}\,\mathrm{d}A$ of magnetic current at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Secondly, we choose

$$J_r^{\mathrm{B}} = 0 \quad \text{and} \quad K_p^{\mathrm{B}} = b_p\delta(x-x',t) , \qquad (28.12\text{-}13)$$

where b_p is an arbitrary constant vector. The electromagnetic field that is, for the present application, radiated by this source and satisfies the causality condition at infinity, is denoted as

$$\{E_k^{\mathrm{B}}, H_j^{\mathrm{B}}\} = \{E_k^{K;\mathrm{B}}, H_j^{K;\mathrm{B}}\}(x,x',t) , \qquad (28.12\text{-}14)$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (28.12-13) and the properties of $\delta(x-x',t)$, we have

$$\int_{x\in\mathcal{D}} \left[\mathrm{C}_t(J_r^{\mathrm{B}}, E_r^{\mathrm{A}}; x,t) - \mathrm{C}_t(K_p^{\mathrm{B}}, H_p^{\mathrm{A}}; x,t)\right] \mathrm{d}V$$

$$= -\int_{x\in\mathcal{D}} \mathrm{C}_t(b_p\delta(x-x',t), H_p; x,t)\,\mathrm{d}V = -b_p H_p(x',t)\chi_{\mathcal{D}}(x') \qquad \text{for } x'\in\mathcal{R}^3 . \qquad (28.12\text{-}15)$$

With this, we arrive at

$$-b_p H_p(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(E_k^{K;\mathrm{B}}, J_k;x,x',t) - \mathrm{C}_t(H_j^{K;\mathrm{B}}, K_j;x,x',t)\right] \mathrm{d}V$$

$$+ \varepsilon_{m,r,p} \int_{x\in\partial\mathcal{D}} \nu_m \left[\mathrm{C}_t(E_r^{K;\mathrm{B}}, H_p;x,x',t) - \mathrm{C}_t(E_r, H_p^{K;\mathrm{B}};x,x',t)\right] \mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 , \qquad (28.12\text{-}16)$$

where we have used the symmetry of the convolution in its functional arguments. From Equation (28.12-16) a representation for $H_p(x',t)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $E_j^{K;\mathrm{B}}$ and $H_k^{K;\mathrm{B}}$ are are linearly related to b_p. Introducing the Green's functions through

$$\{E_k^{K;\mathrm{B}}, H_j^{K;\mathrm{B}}\}(x,x',t) = \{G_{k,p}^{EK;\mathrm{B}}, G_{j,p}^{HK;\mathrm{B}}\}(x,x',t)\, b_p , \qquad (28.12\text{-}17)$$

using the reciprocity relations for these functions (see Exercises 28.8-2 and 28.8-4)

$$\{G_{k,p}^{EK;\mathrm{B}}, G_{j,p}^{HK;\mathrm{B}}\}(x,x',t) = \{-G_{p,k}^{HJ}, G_{p,j}^{HK}\}(x',x,t) , \qquad (28.12\text{-}18)$$

and invoking the condition the resulting equation has to hold for arbitrary values of b_p, Equation (28.12-16) leads to the final result

$$H_p(x',t)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\mathrm{C}_t(G_{p,k}^{HJ}, J_k;x',x,t) + \mathrm{C}_t(G_{p,j}^{HK}, K_j;x',x,t)\right] \mathrm{d}V$$

$$+ \int_{x\in\partial\mathcal{D}} \left[\mathrm{C}_t(G_{p,k}^{HJ}, -\varepsilon_{k,m,p'}\nu_m H_{p'};x',x,t)\right.$$

$$\left. + \mathrm{C}_t(G_{r,j}^{HK}, \varepsilon_{j,n,r}\nu_n E_r;x',x,t)\right] \mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 . \qquad (28.12\text{-}19)$$

Equation (28.12-9) expresses, for $x'\in\mathcal{D}$, the magnetic field strength H_p of the generated electromagnetic field at x' as the superposition of the contributions from the elementary volume sources $J_k\,\mathrm{d}V$ and $K_j\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the equivalent elementary surface sources $-\varepsilon_{k,m,p'}\nu_m H_{p'}\,\mathrm{d}A$ of electric current and $\varepsilon_{j,n,r}\nu_n E_r\,\mathrm{d}A$ of magnetic current at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Complex frequency-domain analysis

For the complex frequency-domain analysis of the problem the electromagnetic properties of the medium present in the embedding are characterised by the functions $\{\hat{\eta}_{k,r}, \hat{\zeta}_{j,p}\} = \{\hat{\eta}_{k,r}, \hat{\zeta}_{j,p}\}(x,s)$ and the global complex frequency-domain reciprocity theorem of the time-convolution type, Equation (28.4-7), is applied to the subdomain $\mathcal{D}$ of $\mathcal{R}^3$. In the theorem, state A is taken to be the generated electromagnetic field under consideration, i.e.

$$\{\hat{E}_r^{\mathrm{A}}, \hat{H}_p^{\mathrm{A}}\} = \{\hat{E}_r, \hat{H}_p\}(x,s) \qquad \text{for } x\in\mathcal{D} , \qquad (28.12\text{-}20)$$

$$\{\hat{J}_k^{\mathrm{A}}, \hat{K}_j^{\mathrm{A}}\} = \{\hat{J}_k, \hat{K}_j\}(x,s) \qquad \text{for } x\in\mathcal{D} , \qquad (28.12\text{-}21)$$

and

$$\{\hat{\eta}^{\mathrm{A}}_{k,r},\hat{\zeta}^{\mathrm{A}}_{j,p}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.12-22}$$

Next, state B is chosen such that the application of Equation (28.4-7) to the subdomain $\mathcal{D}$ leads to the values of $\{\hat{E}_r,\hat{H}_p\}$ at some arbitrary point $x'\in\mathcal{D}$. Inspection of the right-hand side of Equation (28.4-7) reveals that this is accomplished if we take for the source distributions of state B a point source at x' of electric current in case we want an expression for the electric field strength at x' and a point source at x' of magnetic current in case we want an expression for the magnetic field strength at x', while the medium in state B must be taken to be adjoint to the one in state A, i.e.

$$\{\hat{\eta}^{\mathrm{B}}_{r,k},\hat{\zeta}^{\mathrm{B}}_{p,j}\} = \{\hat{\eta}_{k,r},\hat{\zeta}_{j,p}\}(x,s) \qquad \text{for } x\in\mathcal{R}^3 . \tag{28.12-23}$$

The two choices for the source distributions will be discussed separately below.

First, we choose

$$\hat{J}^{\mathrm{B}}_r = \hat{a}_r(s)\delta(x-x') \quad \text{and} \quad \hat{K}^{\mathrm{B}}_p = 0 , \tag{28.12-24}$$

where $\delta(x-x')$ represents the three-dimensional unit impulse (Dirac distribution) operative at the point $x=x'$, while $\hat{a}_r = \hat{a}_r(s)$ is an arbitrary vector function of s. The electromagnetic field that is, for the present application, radiated by this source and satisfies the causality condition at infinity, is denoted as

$$\{\hat{E}^{\mathrm{B}}_k,\hat{H}^{\mathrm{B}}_j\} = \{\hat{E}^{J;\mathrm{B}}_k,\hat{H}^{J;\mathrm{B}}_j\}(x,x',s) , \tag{28.12-25}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (28.12-24) and the properties of $\delta(x-x')$, we have

$$\int_{x\in\mathcal{D}} \left[\hat{J}^{\mathrm{B}}_r(x,s)\hat{E}^{\mathrm{A}}_r(x,s) - \hat{K}^{\mathrm{B}}_p(x,s)\hat{H}^{\mathrm{A}}_p(x,s)\right] \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}} \hat{a}_r(s)\delta(x-x')\hat{E}_r(x,s)\,\mathrm{d}V = \hat{a}_r(s)\hat{E}_r(x',s)\chi_{\mathcal{D}}(x') \qquad \text{for } x'\in\mathcal{R}^3 . \tag{28.12-26}$$

With this, we arrive at

$$\hat{a}_r(s)\hat{E}_r(x',s)\chi_{\mathcal{D}}(x') = \int_{x\in\mathcal{D}} \left[\hat{E}^{J;\mathrm{B}}_k(x,x',s)\hat{J}_k(x,s) - \hat{H}^{J;\mathrm{B}}_j(x,x',s)\hat{K}_j(x,s)\right] \mathrm{d}V$$

$$+ \varepsilon_{m,r,p} \int_{x\in\partial\mathcal{D}} \nu_m \left[\hat{E}^{J;\mathrm{B}}_r(x,x',s)\hat{H}_p(x,s) - \hat{E}_r(x,x',s)\hat{H}^{J;\mathrm{B}}_p(x,s)\right] \mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 . \tag{28.12-27}$$

From Equation (28.12-27) a representation for $\hat{E}_r(x',s)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\hat{E}^{J;\mathrm{B}}_k$ and $\hat{H}^{J;\mathrm{B}}_j$ are linearly related to $\hat{a}_r(s)$. Introducing the Green's functions through

$$\{\hat{E}^{J;\mathrm{B}}_k,\hat{H}^{J;\mathrm{B}}_j\}(x,x',s) = \{\hat{G}^{EJ;\mathrm{B}}_{k,r},\hat{G}^{HJ;\mathrm{B}}_{j,r}\}(x,x',s)\hat{a}_r(s) , \tag{28.12-28}$$

using the reciprocity relations for these functions (see Exercises 28.8-5 and 28.8-7)

$$\{\hat{G}_{k,r}^{EJ;\mathrm{B}},\hat{G}_{j,r}^{HJ;\mathrm{B}}\}(x,x',s) = \{\hat{G}_{r,k}^{EJ},-\hat{G}_{r,j}^{EK}\}(x',x,s)\,, \tag{28.12-29}$$

and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{a}_r(s)$, Equation (28.12-27) leads to the final result

$$\begin{aligned}\hat{E}_r(x',s)\chi_{\mathcal{D}}(x') = &\int_{x\in\mathcal{D}}\left[\hat{G}_{r,k}^{EJ}(x',x,s)\hat{J}_k(x,s) + \hat{G}_{r,j}^{EK}(x',x,s)\hat{K}_j(x,s)\right]\mathrm{d}V \\ &+ \int_{x\in\partial\mathcal{D}}\Big\{\hat{G}_{r,k}^{EJ}(x',x,s)\left[-\varepsilon_{k,m,p}\nu_m\hat{H}_p(x,s)\right] \\ &+ \hat{G}_{r,j}^{EK}(x',x,s)\left[\varepsilon_{j,n,r'}\nu_n\hat{E}_{r'}(x,s)\right]\Big\}\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3\,. \end{aligned}\tag{28.12-30}$$

Equation (28.12-30) expresses, for $x'\in\mathcal{D}$, the electric field strength $\hat{E}_r$ of the generated electromagnetic field at x' as the superposition of the contributions from the elementary volume sources $\hat{J}_k\,\mathrm{d}V$ and $\hat{K}_j\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the elementary equivalent surface sources $-\varepsilon_{k,m,p}\nu_m\hat{H}_p\,\mathrm{d}A$ of electric current and $\varepsilon_{j,n,r'}\nu_n\hat{E}_{r'}\,\mathrm{d}A$ of magnetic current at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

Secondly, we choose

$$\hat{J}_r^{\mathrm{B}} = 0 \quad\text{and}\quad \hat{K}_p^{\mathrm{B}} = \hat{b}_p(s)\delta(x-x')\,, \tag{28.12-31}$$

where $\hat{b}_p = \hat{b}_p(s)$ is an arbitrary vector function of s. The electromagnetic field that is, for the present application, radiated by this source and satisfies the causality condition at infinity, is denoted as

$$\{\hat{E}_k^{\mathrm{B}},\hat{H}_j^{\mathrm{B}}\} = \{\hat{E}_k^{K;\mathrm{B}},\hat{H}_j^{K;\mathrm{B}}\}(x,x',s)\,, \tag{28.12-32}$$

where the first spatial argument indicates the position of the field point and the second spatial argument indicates the position of the source point. In view of Equation (28.12-31) and the properties of $\delta(x-x')$, we have

$$\begin{aligned}&\int_{x\in\mathcal{D}}\left[\hat{J}_r^{\mathrm{B}}(x,s)\hat{E}_r^{\mathrm{A}}(x,s) - \hat{K}_p^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}V \\ &= -\int_{x\in\mathcal{D}}\hat{b}_p(s)\delta(x-x')\hat{H}_p(x,s)\,\mathrm{d}V = -\hat{b}_p(s)\hat{H}_p(x',s)\chi_{\mathcal{D}}(x') \quad \text{for } x'\in\mathcal{R}^3\,.\end{aligned}\tag{28.12-33}$$

With this, we arrive at

$$\begin{aligned}-\hat{b}_p(s)\hat{H}_p(x',s)\chi_{\mathcal{D}}(x') = &\int_{x\in\mathcal{D}}\left[\hat{E}_k^{K;\mathrm{B}}(x,x',s)\hat{J}_k(x,s) - \hat{H}_j^{K;\mathrm{B}}(x,x',s)\hat{K}_j(x,s)\right]\mathrm{d}V \\ &+ \varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}}\nu_m\left[\hat{E}_r^{K;\mathrm{B}}(x,x',s)\hat{H}_p(x,s) - \hat{E}_r(x,x's)\hat{H}_p^{K;\mathrm{B}}(x,s)\right]\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3\,.\end{aligned}\tag{28.12-34}$$

From Equation (28.12-34) a representation for $\hat{H}_p(x',s)\chi_{\mathcal{D}}(x')$ is obtained by taking into account that $\hat{E}_k^{K;\mathrm{B}}$ and $\hat{H}_j^{K;\mathrm{B}}$ are linearly related to $\hat{b}_p(s)$. Introducing the Green's functions through

$$\{\hat{E}_k^{K;\mathrm{B}},\hat{H}_j^{K;\mathrm{B}}\}(x,x',s) = \{\hat{G}_{k,p}^{EK;\mathrm{B}},\hat{G}_{j,p}^{HK;\mathrm{B}}\}(x,x',s)\hat{b}_p(s)\,. \tag{28.12-35}$$

using the reciprocity relations for these functions (see Exercises 28.8-6 and 28.8-8)

$$\{\hat{G}_{k,p}^{EK;\mathrm{B}},\hat{G}_{j,p}^{HK;\mathrm{B}}\}(x,x',s) = \{-\hat{G}_{p,k}^{HJ},\hat{G}_{p,j}^{HK}\}(x,x',s)\,, \tag{28.12-36}$$

and invoking the condition that the resulting equation has to hold for arbitrary values of $\hat{b}_r(s)$, Equation (28.12-34) leads to the final result

$$\begin{aligned}\hat{H}_p(x',s)\chi_{\mathcal{D}}(x') = &\int_{x\in\mathcal{D}} \left[\hat{G}_{p,k}^{HJ}(x',x,s)\hat{J}_k(x,s) + \hat{G}_{p,j}^{HK}(x',x,s)\hat{K}_j(x,s)\right]\mathrm{d}V \\ &+\int_{x\in\partial\mathcal{D}} \left\{\hat{G}_{p,k}^{HJ}(x',x,s)\left[-\varepsilon_{k,m,p'}\nu_m\hat{H}_{p'}(x,s)\right]\right. \\ &\left.+\ \hat{G}_{p,j}^{HK}(x',x,s)\left[\varepsilon_{j,n,r}\nu_n\hat{E}_r(x,s)\right]\right\}\mathrm{d}A \qquad \text{for } x'\in\mathcal{R}^3\,.\end{aligned} \tag{28.12-37}$$

Equation (28.12-37) expresses, for $x'\in\mathcal{D}$, the magnetic field strength $\hat{H}_p$ of the generated electromagnetic field at x' as the superposition of the contributions from the elementary volume sources $\hat{J}_k\,\mathrm{d}V$ and $\hat{K}_j\,\mathrm{d}V$ at x as far as present in $\mathcal{D}$ and the contributions from the elementary equivalent surface sources $-\varepsilon_{k,m,p'}\nu_m\hat{H}_{p'}\,\mathrm{d}A$ of electric current and $\varepsilon_{j,n,r}\nu_n\hat{E}_r\,\mathrm{d}A$ of magnetic current at x on the boundary $\partial\mathcal{D}$ of the domain of interest.

For $x'\in\mathcal{D}$, Equations (28.12-12), (28.12-19), (28.12-30) and (28.12-37) express the values of the electric field strength and the magnetic field strength in some point of $\mathcal{D}$ as the sum of the contributions from the volume sources of electric current and magnetic current as far as these are present in $\mathcal{D}$, and the equivalent surface sources on $\partial\mathcal{D}$. Evidently, the equivalent surface sources yield, in the interior of $\mathcal{D}$, the contribution to the wave field insofar that arises from (unspecified) sources that are located in $\mathcal{D}'$, i.e. in the exterior of $\mathcal{D}$. In particular, the surface integrals in these expressions vanish in case the wave field is not only defined in $\mathcal{D}$ and on $\partial\mathcal{D}$, but also in $\mathcal{D}'$, if there are no sources exterior to $\partial\mathcal{D}$, and the wave fields in state B are causally related to their point source excitations (see Exercises 28.2-2, 28.3-2, 28.4-4 and 28.5-4). In the latter case, Equations (28.12-12), (28.12-19), (28.12-30) and (28.12-37) reduce to Equations (28.8-13), (28.8-22), (28.8-35) and (28.8-44), respectively.

Another property of Equations (28.12-12), (28.12-19), (28.12-30) and (28.12-37) is that the wave field emitted by the volume sources in $\mathcal{D}$ and the wave field emitted by the equivalent surface sources on $\partial\mathcal{D}$ apparently cancel each other when $x'\in\mathcal{D}'$. This property is known as the Ewald–Oseen extinction theorem (Oseen 1915, Ewald 1916).

Another special case arises when Equations (28.12-12), (28.12-19), (28.12-30) and (28.12-37) are used in a domain in which no volume source distributions are present. Then, they express Huygens' principle (Huygens 1690) that states that an electromagnetic wave field due to sources "behind" a closed surface that divides the configuration space into two disjoint regions and "in front" of which no volume sources are present, can be represented as due to equivalent surface sources located on that surface, while that representation yields the value zero "behind" that surface. In particular, Huygens stated his principle for the case where the relevant surface is a wave front of the wave motion in space–time. A number of historical details about the development of the mathematical theory of Huygens' principle can be found in Baker and Copson (1950). Additional literature on the subject can be found in Blok, Ferwerda and Kuiken (1992) and in De Hoop (1992).

Applications of the wave-field representations in a subdomain of space, are found in the integral equation formulation of scattering problems, while the Ewald–Oseen extinction theorem is at the basis of the so-called "null-field method" to solve such problems.

Exercises

Exercise 28.12-1

Let $\mathcal{D}$ be a bounded subdomain of three-dimensional Euclidean space $\mathcal{R}^3$. Let $\partial\mathcal{D}$ be the closed boundary surface of $\mathcal{D}$ and denote by $\mathcal{D}'$ the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$ (i.e. towards $\mathcal{D}'$), is denoted by ν (Figure 28.12-2).

In the domain $\mathcal{D}'$ an electromagnetic field $\{E_r, H_p\}$ is present whose sources are located in $\mathcal{D}$. Use Equations (28.12-12) and (28.12-19) to arrive at the equivalent surface source time-domain integral representations

$$E_r(x',t)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big[\mathrm{C}_t(G^{EJ}_{r,k}, \varepsilon_{k,m,p}\nu_m H_p; x',x,t)$$

$$+\,\mathrm{C}_t(G^{EK}_{r,j}, -\varepsilon_{j,n,r'}\nu_n E_{r'}; x',x,t)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{28.12-38}$$

and

$$H_p(x',t)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big[\mathrm{C}_t(G^{HJ}_{p,k}, \varepsilon_{k,m,p'}\nu_m H_{p'}; x',x,t)$$

$$+\,\mathrm{C}_t(G^{HK}_{p,j}, -\varepsilon_{j,n,r}\nu_n E_r; x',x,t)\Big]\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3\ . \tag{28.12-39}$$

Exercise 28.12-2

Let $\mathcal{D}$ be a bounded subdomain of three-dimensional Euclidean space $\mathcal{R}^3$. Let $\partial\mathcal{D}$ be the closed boundary surface of $\mathcal{D}$ and denote by $\mathcal{D}'$ the complement of $\mathcal{D}\cup\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$, pointing away from $\mathcal{D}$ (i.e. towards $\mathcal{D}'$), is denoted by ν (Figure 28.12-2).

In the domain $\mathcal{D}'$ an electromagnetic field $\{\hat{H}_p, \hat{E}_r\}$ is present whose sources are located in $\mathcal{D}$. Use Equations (28.12-30) and (28.12-37) to arrive at the equivalent surface source complex frequency-domain integral representations

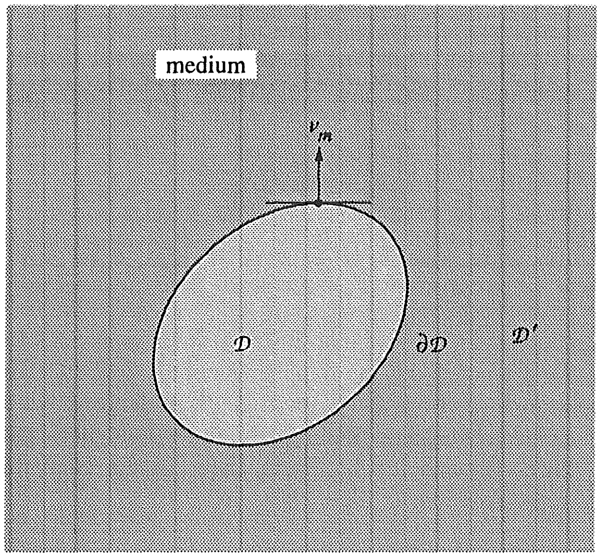
$$\hat{E}_r(x',s)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big\{\hat{G}^{EJ}_{r,k}(x',x,s)\big[\varepsilon_{k,m,p}\nu_m \hat{H}_p(x,s)\big]$$

$$+\,\hat{G}^{EK}_{r,j}(x',x,s)\big[-\varepsilon_{j,n,r'}\nu_n \hat{E}_{r'}(x,s)\big]\Big\}\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3 \tag{28.12-40}$$

and

$$\hat{H}_p(x',s)\chi_{\mathcal{D}'}(x') = \int_{x\in\partial\mathcal{D}} \Big\{\hat{G}^{HJ}_{p,k}(x',x,s)\big[\varepsilon_{k,m,p'}\nu_m \hat{H}_{p'}(x,s)\big]$$

$$+\,\hat{G}^{HK}_{p,j}(x',x,s)\big[-\varepsilon_{j,n,r}\nu_n \hat{E}_r(x,s)\big]\Big\}\,\mathrm{d}A \quad \text{for } x'\in\mathcal{R}^3\ . \tag{28.12-41}$$



28.12-2 Configuration for the equivalent surface source integral representation for an electromagnetic field in the source-free domain $\mathcal{D}'$ exterior to a bounded subdomain $\mathcal{D}$ of $\mathcal{R}^3$.

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