29

Plane wave scattering by an object in an unbounded, homogeneous, isotropic, lossless embedding

In this chapter, the simplest scattering configuration is investigated in more detail. It consists of an unbounded, homogeneous, isotropic, lossless embedding in which a plane wave is incident upon a scattering object of bounded extent. First, the reciprocity properties of the amplitudes of the scattered wave in the far-field region are investigated. Next, an energy theorem ("extinction cross-section theorem") is derived that relates the sum of the energy carried by the scattered wave and the energy absorbed by the scattering object to the amplitude of the scattered wave in the far-field region when observed in the forward scattering direction. Finally, the first term in the Neumann solution to the relevant system of integral equations (the so-called "Rayleigh–Gans–Born approximation") is determined for penetrable, homogeneous scatterers of different shapes. The analysis is carried out in the time domain as well as in the complex frequency domain.

29.1 The scattering configuration, the incident plane wave and the far-field scattering amplitudes

The scattering configuration consists of a homogeneous, isotropic, lossless *embedding* that occupies the entire $\mathcal{R}^3$. The electromagnetic properties of the embedding are characterised by its permittivity ε and its permeability μ, which are posititve constants. The associated electromagnetic wave speed is $c = (\varepsilon\mu)^{-1/2}$, which is also a positive constant. In the embedding, an electromagnetic *scatterer* is present that occupies the bounded domain $\mathcal{D}^{\rm s}$. The boundary surface of $\mathcal{D}^{\rm s}$ is denoted by $\partial\mathcal{D}^{\rm s}$ and ν is the unit vector along the normal to $\partial\mathcal{D}^{\rm s}$ oriented away from $\mathcal{D}^{\rm s}$. The complement of $\mathcal{D}^{\rm s}\cup\partial\mathcal{D}^{\rm s}$ in $\mathcal{R}^3$ is denoted by $\mathcal{D}^{\rm s'}$ (Figure 29.1-1).

Time-domain analysis

In the time-domain analysis of the problem, the electromagnetic properties of the scatterer are, if the scatterer is an electromagnetically penetrable object, characterised by the relaxation

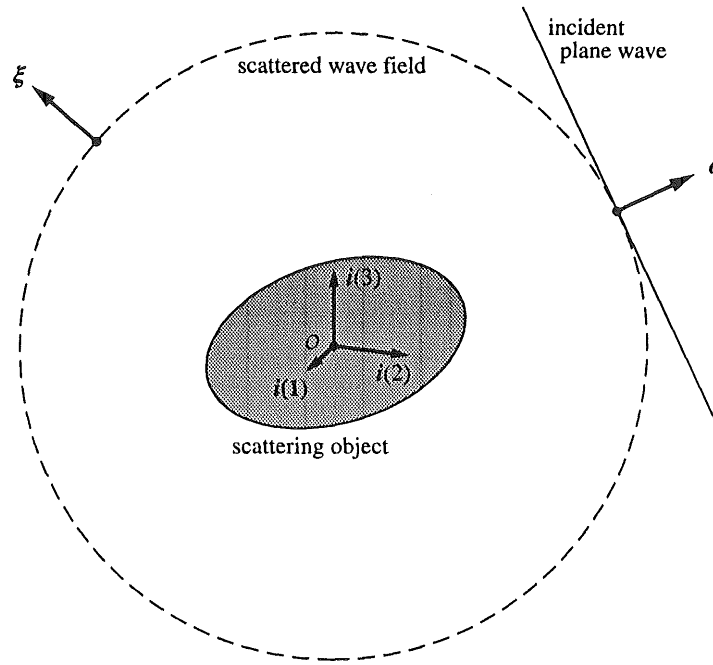

Figure 29.1-1 Scattering object occupying the bounded domain $\mathcal{D}^s$ in an unbounded electromagnetically homogeneous, isotropic, lossless embedding with permittivity ε and permeability μ.

functions $\{\varepsilon^s_{k,r},\mu^s_{j,p}\} = \{\varepsilon^s_{k,r},\mu^s_{j,p}\}(x,t)$, which are causal functions of time. The equivalent contrast volume source densities of electric and magnetic current are then given by Equations (28.9-18) and (28.9-19))

$$J^s_k = \partial_t C_t(\varepsilon^s_{k,r} - \varepsilon\delta(t)\delta_{k,r},E_r;x;t) \qquad \text{for } x\in\mathcal{D}^s, \tag{29.1-1}$$

$$K^s_j = \partial_t C_t(\mu^s_{j,p} - \mu\delta(t)\delta_{j,p},H_p;x;t) \qquad \text{for } x\in\mathcal{D}^s, \tag{29.1-2}$$

in which the *total electromagnetic field* $\{E_r,H_p\}$ is the sum of the *incident field* $\{E^i_r,H^i_p\}$ and the *scattered field* $\{E^s_r,H^s_p\}$ (see Equation (28.9-5)). If the scatterer is *electromagnetically impenetrable*, either of the two boundary conditions

$$\lim_{h\downarrow 0}\varepsilon_{j,n,r}\nu_n E_r(x+h\nu,t) = 0 \qquad \text{for } x\in\partial\mathcal{D}^s \tag{29.1-3}$$

or

$$\lim_{h\downarrow 0}\varepsilon_{k,m,p}\nu_m H_p(x+h\nu,t) = 0 \qquad \text{for } x\in\partial\mathcal{D}^s \tag{29.1-4}$$

applies.

For the incident wave we now take the *uniform plane wave* (see Equations (27.4-7) and (27.4-18))

$$\{E^i_r,H^i_p\} = \{e_r,h_p\}a(t-\alpha_s x_s/c)\ , \tag{29.1-5}$$

that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ (i.e. $\alpha_s \alpha_s = 1$) and has the normalised pulse shape $a(t)$. Its electric and magnetic field amplitudes are related through (see Equations (27.4-8) and (27.4-9))

$$e_r = -Z\varepsilon_{r,m,p}\alpha_m h_p \,, \tag{29.1-6}$$

in which (see Equation (27.4-23))

$$Z = (\mu/\varepsilon)^{1/2} \qquad \text{with } (\dots)^{1/2} > 0 \tag{29.1-7}$$

is the electromagnetic plane wave impedance of the wave and

$$h_p = Y\varepsilon_{p,n,r}\alpha_n e_r \,, \tag{29.1-8}$$

in which (see Equation (27.4-22))

$$Y = (\varepsilon/\mu)^{1/2} \qquad \text{with } (\dots)^{1/2} > 0 \tag{29.1-9}$$

is the electromagnetic plane wave admittance of the wave.

For an *electromagnetically penetrable scatterer* we use for the scattered wave the constrast volume source integral representation (see Equations (28.9-20) and (28.9-21))

$$E_r^{\mathrm{s}}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G_{r,k}^{EJ}, J_k^{\mathrm{s}}; \boldsymbol{x}', \boldsymbol{x}, t) + \mathrm{C}_t(G_{r,j}^{EK}, K_j^{\mathrm{s}}; \boldsymbol{x}', \boldsymbol{x}, t)\right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}' \in \mathcal{R}^3 \tag{29.1-10}$$

and

$$H_p^{\mathrm{s}}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G_{p,k}^{HJ}, J_k^{\mathrm{s}}; \boldsymbol{x}', \boldsymbol{x}, t) + \mathrm{C}_t(G_{p,j}^{HK}, K_j^{\mathrm{s}}; \boldsymbol{x}', \boldsymbol{x}, t)\right] \mathrm{d}V \qquad \text{for } \boldsymbol{x}' \in \mathcal{R}^3 \,, \tag{29.1-11}$$

in which (see Exercise 28.8-9, with $\boldsymbol{x}$ and $\boldsymbol{x}'$ interchanged)

$$G_{r,k}^{EJ}(\boldsymbol{x}',\boldsymbol{x},t) = -\mu\partial_t G(\boldsymbol{x}',\boldsymbol{x},t)\delta_{r,k} + \varepsilon^{-1}\partial_r'\partial_k'\,\mathrm{I}_t G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{29.1-12}$$

$$G_{r,j}^{EK}(\boldsymbol{x}',\boldsymbol{x},t) = -\varepsilon_{r,n,j}\partial_n' G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{29.1-13}$$

$$G_{p,k}^{HJ}(\boldsymbol{x}',\boldsymbol{x},t) = \varepsilon_{p,m,k}\partial_m' G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{29.1-14}$$

$$G_{p,j}^{HK}(\boldsymbol{x}',\boldsymbol{x},t) = -\varepsilon\partial_t G(\boldsymbol{x}',\boldsymbol{x},t)\delta_{p,j} + \mu^{-1}\partial_p'\partial_j'\,\mathrm{I}_t G(\boldsymbol{x}',\boldsymbol{x},t) \,, \tag{29.1-15}$$

where ∂_m' means differentiation with respect to x_m' and

$$G(\boldsymbol{x}',\boldsymbol{x},t) = \frac{\delta(t - |\boldsymbol{x}' - \boldsymbol{x}|/c)}{4\pi|\boldsymbol{x}' - \boldsymbol{x}|} \qquad \text{for } \boldsymbol{x}' \neq \boldsymbol{x} \,. \tag{29.1-16}$$

In the *far-field region*, the expansion

$$\{E_r^{\mathrm{s}}, H_p^{\mathrm{s}}\}(\boldsymbol{x}',t) = \frac{\{E_r^{\mathrm{s};\infty}, H_p^{\mathrm{s};\infty}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1 + O(|\boldsymbol{x}'|^{-1})\right]$$

$$\text{as } |\boldsymbol{x}'| \to \infty \quad \text{with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi} \tag{29.1-17}$$

holds, where (see Equations (26.12-5)–(26.12-10))

$$E_r^{\mathrm{s};\infty} = \mu(\xi_r\xi_k - \delta_{r,k})\partial_t\Phi_k^{J^{\mathrm{s}};\infty} + \varepsilon_{r,n,j}(\xi_n/c)\partial_t\Phi_j^{K^{\mathrm{s}};\infty}, \tag{29.1-18}$$

$$H_p^{\mathrm{s};\infty} = \varepsilon(\xi_p\xi_j - \delta_{p,j})\partial_t\Phi_j^{K^{\mathrm{s}};\infty} - \varepsilon_{p,m,k}(\xi_m/c)\partial_t\Phi_k^{J^{\mathrm{s}};\infty}, \tag{29.1-19}$$

in which

$$\Phi_k^{J^{s;\infty}}(\xi,t) = \int_{x\in\mathcal{D}^s} J_k^s(x,t+\xi_s x_s/c)\,\mathrm{d}V\,, \tag{29.1-20}$$

$$\Phi_j^{K^{s;\infty}}(\xi,t) = \int_{x\in\mathcal{D}^s} K_j^s(x,t+\xi_s x_s/c)\,\mathrm{d}V\,. \tag{29.1-21}$$

For an *electromagnetically impenetrable scatterer* the electromagnetic field is not defined in the interior $\mathcal{D}^s$ of the scatterer and we have to resort to an equivalent surface source integral representation that expresses the scattered wave field in the exterior $\mathcal{D}^{s'}$ of the scatterer in terms of the wave-field values on the boundary surface $\partial\mathcal{D}^s$ of $\mathcal{D}^s$. This representation is, on account of Equations (28.12-38) and (28.12-39),

$$E_r^s(x',t)\chi_{\mathcal{D}^{s'}}(x') = \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G_{r,k}^{EJ},\partial J_k^s;x',x,t) + \mathrm{C}_t(G_{r,j}^{EK},\partial K_j^s;x',x,t)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-22}$$

and

$$H_p^s(x',t)\chi_{\mathcal{D}^{s'}}(x') = \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G_{p,k}^{HJ},\partial J_k^s;x',x,t) + \mathrm{C}_t(G_{p,j}^{HK},\partial K_j^s;x',x,t)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3\,, \tag{29.1-23}$$

in which (note the orientation of ν)

$$\partial J_k^s = \varepsilon_{k,m,p}\nu_m H_p^s\,, \tag{29.1-24}$$

$$\partial K_j^s = -\varepsilon_{j,n,r}\nu_n E_r^s\,. \tag{29.1-25}$$

In the *far-field region*, the expansion given in Equation (29.1-17) holds, where, based upon Equations (29.1-22)–(29.1-25), we have

$$E_r^{s;\infty} = \mu(\xi_r\xi_k - \delta_{r,k})\partial_t\Phi_k^{\partial J^{s;\infty}} + \varepsilon_{r,n,j}(\xi_n/c)\partial_t\Phi_j^{\partial K^{s;\infty}}\,, \tag{29.1-26}$$

$$H_p^{s;\infty} = \varepsilon(\xi_p\xi_j - \delta_{p,j})\partial_t\Phi_j^{\partial K^{s;\infty}} - \varepsilon_{p,m,k}(\xi_m/c)\partial_t\Phi_k^{\partial J^{s;\infty}}\,, \tag{29.1-27}$$

in which

$$\Phi_k^{\partial J^{s;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial J_k^s(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.1-28}$$

$$\Phi_j^{\partial K^{s;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial K_j^s(x,t+\xi_s x_s/c)\,\mathrm{d}A\,. \tag{29.1-29}$$

However, upon applying Equations (28.12-12) and (28.12-19) to the incident wave field $\{E_r^i,H_p^i\}$ and to the domain $\mathcal{D}^s$, we have (note that the incident wave field is source-free in $\mathcal{D}^s$)

$$E_r^i(x',t)\chi_{\mathcal{D}^s}(x') = \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G_{r,k}^{EJ},\partial J_k^i;x',x,t) + \mathrm{C}_t(G_{r,j}^{EK},\partial K_j^i;x',x,t)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-30}$$

and

$$H_p^{\rm i}(x',t)\chi_{\mathcal{D}^{\rm s}}(x') = \int_{x\in\partial\mathcal{D}^{\rm s}} \left[\mathrm{C}_t(G_{p,k}^{HJ},\partial J_k^{\rm i};x',x,t) + \mathrm{C}_t(G_{p,j}^{HK},\partial K_j^{\rm i};x',x,t)\right] \mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3, \tag{29.1-31}$$

in which (note the orientation of ν),

$$\partial J_k^{\rm i} = -\varepsilon_{k,m,p}\nu_m H_p^{\rm i}, \tag{29.1-32}$$

$$\partial K_j^{\rm i} = \varepsilon_{j,n,r}\nu_n E_r^{\rm i}. \tag{29.1-33}$$

Subtraction of Equation (29.1-30) from Equation (29.1-22) and of Equation (29.1-31) from Equation (29.1-23) leads to

$$E_r^{\rm s}(x',t)\chi_{\mathcal{D}^{\rm s\prime}}(x') - E_r^{\rm i}(x',t)\chi_{\mathcal{D}^{\rm s}}(x') = \int_{x\in\partial\mathcal{D}^{\rm s}} \left[\mathrm{C}_t(G_{r,k}^{EJ},\partial J_k;x',x,t) + \mathrm{C}_t(G_{r,j}^{EK},\partial K_j;x',x,t)\right] \mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-34}$$

and

$$H_p^{\rm s}(x',t)\chi_{\mathcal{D}^{\rm s\prime}}(x') - H_p^{\rm i}(x',t)\chi_{\mathcal{D}^{\rm s}}(x') = \int_{x\in\partial\mathcal{D}^{\rm s}} \left[\mathrm{C}_t(G_{p,k}^{HJ},\partial J_k;x',x,t) + \mathrm{C}_t(G_{p,j}^{HK},\partial K_j;x',x,t)\right] \mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3, \tag{29.1-35}$$

in which

$$\partial J_k = \varepsilon_{k,m,p}\nu_m H_p, \tag{29.1-36}$$

$$\partial K_j = -\varepsilon_{j,n,r}\nu_n E_r. \tag{29.1-37}$$

In the *far-field region*, again the expansion given in Equation (29.1-17) holds, in which, based upon Equations (29.1-34)–(29.1-37), we now have

$$E_r^{\rm s;\infty} = \mu(\xi_r\xi_k - \delta_{r,k})\partial_t\Phi_k^{\partial J;\infty} + \varepsilon_{r,n,j}(\xi_n/c)\partial_t\Phi_j^{\partial K;\infty}, \tag{29.1-38}$$

$$H_p^{\rm s;\infty} = \varepsilon(\xi_p\xi_j - \delta_{p,j})\partial_t\Phi_j^{\partial K;\infty} - \varepsilon_{p,m,k}(\xi_m/c)\partial_t\Phi_k^{\partial J;\infty}, \tag{29.1-39}$$

in which

$$\Phi_k^{\partial J;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^{\rm s}} \partial J_k(x,t+\xi_s x_s/c)\,\mathrm{d}A, \tag{29.1-40}$$

$$\Phi_j^{\partial K;\infty}(\xi,t) = \int_{x\in\partial\mathcal{D}^{\rm s}} \partial K_j(x,t+\xi_s x_s/c)\,\mathrm{d}A. \tag{29.1-41}$$

Of course, the equivalent surface source representations also apply to the case of an electromagnetically penetrable scatterer. For $x'\in\mathcal{D}^{\rm s\prime}$, Equations (29.1-10) and (29.1-11) must then yield the same result as Equations (29.1-22) and (29.1-23), and (29.1-34) and (29.1-35). Similarly, in the far-field region, Equations (29.1-18) and (29.1-19) must yield the same result as Equations (29.1-26) and (29.1-27) and Equations (29.1-38) and (29.1-39). Note, however, that the results for $x'\in\mathcal{D}^{\rm s}$ differ.

Equations (29.1-10) and (29.1-11), when taken for $x' \in \mathcal{D}^s$, provide the basis for the *time-domain domain integral equation method* to solve problems of the scattering by penetrable objects. For solving problems of the scattering by impenetrable objects, Equations (29.1-34) and (29.1-35) provide, when taken for $x' \in \partial\mathcal{D}^s$, the basis for the *time-domain boundary integral equation method* and, when taken for $x' \in \mathcal{D}^s$, the basis for the *time-domain null-field method.* For general scatterers, all three methods need numerical implementation.

Complex frequency-domain analysis

In the complex frequency-domain analysis of the problem, the electromagnetic properties of the scatterer are, if the scatterer is an electromagnetically penetrable object, characterised by the functions $\{\hat{\eta}^s_{k,r}, \hat{\zeta}^s_{j,p}\} = \{\hat{\eta}^s_{k,r}, \hat{\zeta}^s_{j,p}\}(x,s)$. The equivalent contrast volume source densities of electric and magnetic current are then given by (see Equations (28.9-41) and (28.9-42))

$$\hat{J}^s_k = (\hat{\eta}^s_{k,r} - s\varepsilon\delta_{k,r})\hat{E}_r \qquad \text{for } x \in \mathcal{D}^s, \tag{29.1-42}$$

$$\hat{K}^s_j = (\hat{\zeta}^s_{j,p} - s\mu\delta_{j,p})\hat{H}_p \qquad \text{for } x \in \mathcal{D}^s, \tag{29.1-43}$$

in which the *total electromagnetic field* $\{\hat{E}_r, \hat{H}_p\}$ is the sum of the *incident field* $\{\hat{E}^i_r, \hat{H}^i_p\}$ and the *scattered field* $\{\hat{E}^s_r, \hat{H}^s_p\}$ (see Equation (28.9-28)). If the scatterer is *electromagnetically impenetrable*, either of the two boundary conditions

$$\lim_{h \downarrow 0} \varepsilon_{j,n,r} \nu_n \hat{E}_r(x + h\nu, s) = 0 \qquad \text{for } x \in \partial\mathcal{D}^s \tag{29.1-44}$$

or

$$\lim_{h \downarrow 0} \varepsilon_{k,m,p} \nu_m \hat{H}_p(x + h\nu, s) = 0 \qquad \text{for } x \in \partial\mathcal{D}^s \tag{29.1-45}$$

applies.

For the incident wave we now take the *uniform plane wave* (see Equations (27.1-3), (27.1-21) and (27.2-39))

$$\{\hat{E}^i_r, \hat{H}^i_p\} = \{e_r, h_p\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{29.1-46}$$

that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ (i.e. $\alpha_s\alpha_s = 1$) and has the complex frequency-domain normalised pulse shape $\hat{a}(s)$. Its electric and magnetic field strength amplitudes are related through Equations (29.1-6)–(29.1-9).

For an *electromagnetically penetrable scatterer* we use for the scattered wave the contrast volume source representation (see Equations (28.9-43) and (28.9-44))

$$\hat{E}^s_r(x', s) = \int_{x \in \mathcal{D}^s} \left[\hat{G}^{EJ}_{r,k}(x', x, s)\hat{J}^s_k(x, s) + \hat{G}^{EK}_{r,j}(x', x, s)\hat{K}^s_j(x, s)\right] \mathrm{d}V \quad \text{for } x' \in \mathcal{R}^3 \tag{29.1-47}$$

and

$$\hat{H}^s_p(x', s) = \int_{x \in \mathcal{D}^s} \left[\hat{G}^{HJ}_{p,k}(x', x, s)\hat{J}^s_k(x, s) + \hat{G}^{HK}_{p,j}(x', x, s)\hat{K}^s_j(x, s)\right] \mathrm{d}V \quad \text{for } x' \in \mathcal{R}^3, \tag{29.1-48}$$

in which (see Exercise 28.8-10, with x and x' interchanged)

$$\hat{G}^{EJ}_{r,k}(x',x,s) = -s\mu\hat{G}(x',x,s)\delta_{r,k} + (s\varepsilon)^{-1}\partial'_r\partial'_k\hat{G}(x',x,s) , \tag{29.1-49}$$

$$\hat{G}^{EK}_{r,j}(x',x,s) = -\varepsilon_{r,n,j}\partial'_n\hat{G}(x',x,s) , \tag{29.1-50}$$

$$\hat{G}^{HJ}_{p,k}(x',x,s) = \varepsilon_{p,m,k}\partial'_m\hat{G}(x',x,s) , \tag{29.1-51}$$

$$\hat{G}^{HK}_{p,j}(x',x,s) = -s\varepsilon\hat{G}(x',x,s)\delta_{p,j} + (s\mu)^{-1}\partial'_p\partial'_j\hat{G}(x',x,s) , \tag{29.1-52}$$

where ∂'_m means differentiation with respect to x'_m and

$$\hat{G}(x',x,s) = \frac{\exp(-s|x'-x|/c)}{4\pi|x'-x|} \qquad \text{for } x' \neq x . \tag{29.1-53}$$

In the *far-field region*, the expansion

$$\{\hat{E}^{s}_r,\hat{H}^{s}_p\}(x',s) = \{\hat{E}^{s;\infty}_r,\hat{H}^{s;\infty}_p\}(\xi,s)\,\frac{\exp(-s|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$
$$\text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi \tag{29.1-54}$$

holds, where (see Equations (26.11-8)–(26.11-12))

$$\hat{E}^{s;\infty}_r = s\mu(\xi_r\xi_k - \delta_{r,k})\hat{\Phi}^{J^{s};\infty}_k + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}^{K^{s};\infty}_j , \tag{29.1-55}$$

$$\hat{H}^{s;\infty}_r = s\varepsilon(\xi_p\xi_j - \delta_{p,j})\hat{\Phi}^{K^{s};\infty}_j - \varepsilon_{p,m,k}(s\xi_m/c)\hat{\Phi}^{J^{s};\infty}_k , \tag{29.1-56}$$

in which

$$\hat{\Phi}^{J^{s};\infty}_k(\xi,s) = \int_{x\in\mathcal{D}^{s}} \hat{J}^{s}_k(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}V , \tag{29.1-57}$$

$$\hat{\Phi}^{K^{s};\infty}_j(\xi,s) = \int_{x\in\mathcal{D}^{s}} \hat{K}^{s}_j(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}V . \tag{29.1-58}$$

For an *electromagnetically impenetrable scatterer* the electromagnetic field is not defined in the interior $\mathcal{D}^{s}$ of the scatterer and we have to resort to an equivalent surface source integral representation that expresses the scattered field in the exterior $\mathcal{D}^{s'}$ of the scatterer in terms of the field values on the boundary surface $\partial\mathcal{D}^{s}$ of $\mathcal{D}^{s}$. This representation is, on account of Equations (28.12-40) and (28.12-41),

$$\hat{E}^{s}_r(x',s)\chi_{\mathcal{D}^{s'}}(x') = \int_{x\in\partial\mathcal{D}^{s}} \left[\hat{G}^{EJ}_{r,k}(x',x,s)\partial\hat{J}^{s}_k(x,s) + \hat{G}^{EK}_{r,j}(x',x,s)\partial\hat{K}^{s}_j(x,s)\right]\mathrm{d}A$$
$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-59}$$

and

$$\hat{H}^{s}_p(x',s)\chi_{\mathcal{D}^{s'}}(x') = \int_{x\in\partial\mathcal{D}^{s}} \left[\hat{G}^{HJ}_{p,k}(x',x,s)\partial\hat{J}^{s}_k(x,s) + \hat{G}^{HK}_{p,j}(x',x,s)\partial\hat{K}^{s}_j(x,s)\right]\mathrm{d}A$$
$$\text{for } x'\in\mathcal{R}^3 , \tag{29.1-60}$$

in which (note the orientation of ν)

$$\partial\hat{J}_k^{\mathrm{s}} = \varepsilon_{k,m,p}\nu_m\hat{H}_p^{\mathrm{s}}, \tag{29.1-61}$$

$$\partial\hat{K}_j^{\mathrm{s}} = -\varepsilon_{j,n,r}\nu_n\hat{E}_r^{\mathrm{s}}. \tag{29.1-62}$$

In the *far-field region*, the expansion given in Equation (29.1-54) holds, where, based upon Equations (29.1-59)–(29.1-62), we have

$$\hat{E}_r^{\mathrm{s};\infty} = s\mu(\xi_r\xi_k - \delta_{r,k})\hat{\Phi}_k^{\partial J^{\mathrm{s}};\infty} + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}_j^{\partial K^{\mathrm{s}};\infty}, \tag{29.1-63}$$

$$\hat{H}_r^{\mathrm{s};\infty} = s\varepsilon(\xi_p\xi_j - \delta_{p,j})\hat{\Phi}_j^{\partial K^{\mathrm{s}};\infty} - \varepsilon_{p,m,k}(s\xi_m/c)\hat{\Phi}_k^{\partial J^{\mathrm{s}};\infty}, \tag{29.1-64}$$

in which

$$\hat{\Phi}_k^{\partial J^{\mathrm{s}};\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{J}_k^{\mathrm{s}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.1-65}$$

$$\hat{\Phi}_j^{\partial K^{\mathrm{s}};\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{K}_j^{\mathrm{s}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,. \tag{29.1-66}$$

However, upon applying Equations (28.12-30) and (28.12-37) to the incident wave field $\{\hat{E}_r^{\mathrm{i}},\hat{H}_p^{\mathrm{i}}\}$ and to the domain $\mathcal{D}^{\mathrm{s}}$, we have (note that the incident wave field is source-free in $\mathcal{D}^{\mathrm{s}}$)

$$\hat{E}_r^{\mathrm{i}}(x',s)\chi_{\mathcal{D}^{\mathrm{s}}}(x') = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\left[\hat{G}_{r,k}^{EJ}(x',x,s)\partial\hat{J}_k^{\mathrm{i}}(x,s) + \hat{G}_{r,j}^{EK}(x',x,s)\partial\hat{K}_j^{\mathrm{i}}(x,s)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-67}$$

and

$$\hat{H}_p^{\mathrm{i}}(x',s)\chi_{\mathcal{D}^{\mathrm{s}}}(x') = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\left[\hat{G}_{p,k}^{HJ}(x',x,s)\partial\hat{J}_k^{\mathrm{i}}(x,s) + \hat{G}_{p,j}^{HK}(x',x,s)\partial\hat{K}_j^{\mathrm{i}}(x,s)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3, \tag{29.1-68}$$

in which (note the orientation of ν),

$$\partial\hat{J}_k^{\mathrm{i}} = -\varepsilon_{k,m,p}\nu_m\hat{H}_p^{\mathrm{i}}, \tag{29.1-69}$$

$$\partial\hat{K}_j^{\mathrm{i}} = \varepsilon_{j,n,r}\nu_n\hat{E}_r^{\mathrm{i}}. \tag{29.1-70}$$

Subtraction of Equation (29.1-67) from Equation (29.1-59) and of Equation (29.1-68) from Equation (29.1-60) leads to

$$\hat{E}_r^{\mathrm{s}}(x',s)\chi_{\mathcal{D}^{\mathrm{s}\prime}}(x') - \hat{E}_r^{\mathrm{i}}(x',s)\chi_{\mathcal{D}^{\mathrm{s}}}(x') = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\left[\hat{G}_{r,k}^{EJ}(x',x,s)\partial\hat{J}_k(x,s) + \hat{G}_{r,j}^{EK}(x',x,s)\partial\hat{K}_j(x,s)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-71}$$

and

$$\hat{H}_p^{\mathrm{s}}(x',s)\chi_{\mathcal{D}^{\mathrm{s}\prime}}(x') - \hat{H}_p^{\mathrm{i}}(x',s)\chi_{\mathcal{D}^{\mathrm{s}}}(x') = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\left[\hat{G}_{p,k}^{HJ}(x',x,s)\partial\hat{J}_k(x,s) + \hat{G}_{p,j}^{HK}(x',x,s)\partial\hat{K}_j(x,s)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3, \tag{29.1-72}$$

in which (note the orientation of ν),

$$\partial\hat{J}_k = \varepsilon_{k,m,p}\nu_m\hat{H}_p \,, \tag{29.1-73}$$

$$\partial\hat{K}_j = -\varepsilon_{j,n,r}\nu_n\hat{E}_r \,. \tag{29.1-74}$$

In the *far-field region*, again the expansion given in Equation (29.1-54) holds, in which, based upon Equations (29.1-71)–(29.1-74), we now have

$$\hat{E}_r^{\mathrm{s};\infty} = s\mu(\xi_r\xi_k - \delta_{r,k})\hat{\Phi}_k^{\partial J;\infty} + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}_j^{\partial K;\infty}, \tag{29.1-75}$$

$$\hat{H}_r^{\mathrm{s};\infty} = s\varepsilon(\xi_p\xi_j - \delta_{p,j})\hat{\Phi}_j^{\partial K;\infty} - \varepsilon_{p,m,k}(s\xi_m/c)\hat{\Phi}_k^{\partial J;\infty}, \tag{29.1-76}$$

in which

$$\hat{\Phi}_k^{\partial J;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{J}_k(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A \,, \tag{29.1-77}$$

$$\hat{\Phi}_j^{\partial K;\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{K}_j(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A \,. \tag{29.1-78}$$

Of course, the equivalent surface source representations also apply to the case of an electromagnetically penetrable scatterer. For $x'\in\mathcal{D}^{\mathrm{s}\,\prime}$, Equations (29.1-47) and (29.1-48) must then yield the same result as Equations (29.1-59) and (29.1-60), and (29.1-71) and (29.1-72). Similarly, in the far-field region, Equations (29.1-55) and (29.1-56) must yield the same result as Equations (29.1-63) and (29.1-64), and Equations (29.1-75) and (29.1-76). Note, however, that the results for $x'\in\mathcal{D}^{\mathrm{s}}$ differ.

Equations (29.1-47) and (29.1-48), when taken for $x'\in\mathcal{D}^{\mathrm{s}}$, provide the basis for the *complex frequency-domain domain integral equation method* to solve problems of the scattering by penetrable objects. For solving problems of the scattering by impenetrable objects, Equations (29.1-71) and (29.1-72) provide, when taken for $x'\in\partial\mathcal{D}^{\mathrm{s}}$, the basis for the *complex frequency-domain boundary integral equation method* and, when taken for $x'\in\mathcal{D}^{\mathrm{s}}$, the basis for the *complex frequency-domain null-field method.* For general scatterers, all three methods need numerical implementation.

The different representations in this section will be needed in the analysis in the remainder of this chapter.

Exercises

Exercise 29.1-1

Show that from Equations (29.1-34) and (29.1-35) it follows that

$$E_r(x',t)\chi_{\mathcal{D}^{\mathrm{s}}}{}'(x') = E_r^{\mathrm{i}}(x',t) + \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \left[\mathrm{C}_t(G_{r,k}^{EJ},\partial J_k;x',x,t) + \mathrm{C}_t(G_{r,j}^{EK},\partial K_j;x',x,t)\right]\mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-79}$$

and

$$H_p(x',t)\chi_{\mathcal{D}^{s\prime}}(x') = H_p^{\mathrm{i}}(x',t) + \int_{x\in\partial\mathcal{D}^s} \left[\mathrm{C}_t(G_{p,k}^{HJ},\partial J_k;x',x,t) + \mathrm{C}_t(G_{p,j}^{HK},\partial K_j;x',x,t)\right] \mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3. \tag{29.1-80}$$

(*Hint*: Consider the cases $x'\in\mathcal{D}^{s\prime}$, $x'\in\partial\mathcal{D}^s$ and $x'\in\mathcal{D}^s$.)

Exercise 29.1-2

Show that from Equations (29.1-71) and (29.1-72) it follows that

$$\hat{E}_r(x',s)\chi_{\mathcal{D}^{s\prime}}(x') = \hat{E}_r^{\mathrm{i}}(x,s) + \int_{x\in\partial\mathcal{D}^s} \left[\hat{G}_{r,k}^{EJ}(x',x,s)\partial\hat{J}_k(x,s) + \hat{G}_{r,j}^{EK}(x',x,s)\partial\hat{K}_j(x,s)\right] \mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3 \tag{29.1-81}$$

and

$$\hat{H}_p(x',s)\chi_{\mathcal{D}^{s\prime}}(x') = \hat{H}_p^{\mathrm{i}}(x,s) + \int_{x\in\partial\mathcal{D}^s} \left[\hat{G}_{p,k}^{HJ}(x',x,s)\partial\hat{J}_k(x,s) + \hat{G}_{p,j}^{HK}(x',x,s)\partial\hat{K}_j(x,s)\right] \mathrm{d}A$$

$$\text{for } x'\in\mathcal{R}^3. \tag{29.1-82}$$

(*Hint*: Consider the cases $x'\in\mathcal{D}^{s\prime}$, $x'\in\partial\mathcal{D}^s$ and $x'\in\mathcal{D}^s$.)

29.2 Far-field scattered wave amplitude reciprocity of the time convolution type

In this section we investigate the reciprocity relation of the time convolution type that applies to the far-field scattered wave amplitude reciprocity for plane wave incidence upon an electromagnetically penetrable or impenetrable object. The scattering configuration of Figure 29.1-1 applies. Two states in this configuration are considered; they are denoted as state A and state B, respectively. In state A, a uniform plane electromagnetic wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ is incident upon the scattering object; in state B, a uniform plane electromagnetic wave that propagates in the direction of the unit vector $\boldsymbol{\beta}$ is incident upon the scattering object. It will be shown that the far-field scattered wave amplitude in state A when observed in the direction of observation $\boldsymbol{\xi} = -\boldsymbol{\beta}$ is related, via reciprocity, to the far-field scattered wave amplitude in state B when observed in the direction of observation $\boldsymbol{\xi} = -\boldsymbol{\alpha}$ (Figure 29.2-1).

The corresponding relationships in the time domain and in the complex frequency domain will be derived separately below.

Time-domain analysis

In the time-domain analysis, the incident uniform plane wave in state A is taken as

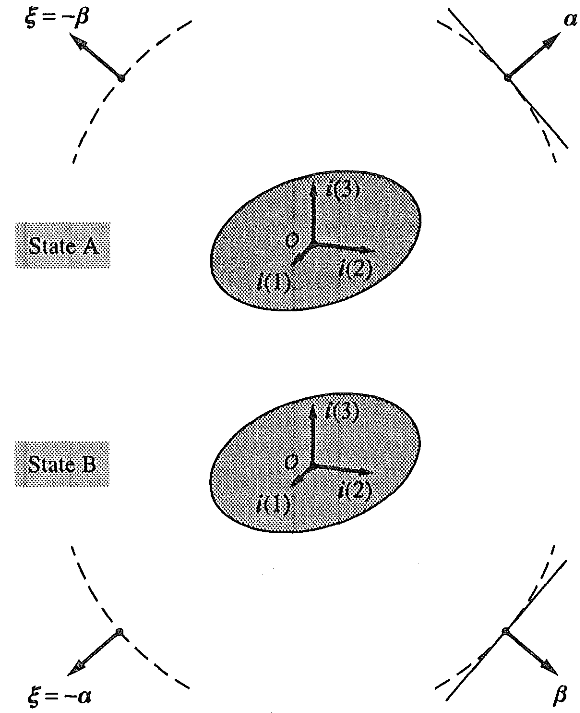

Figure 29.2-1 Configuration for the far-field scattered wave amplitude reciprocity of the time convolution type.

$$\{E_r^{\mathrm{i;A}}, H_p^{\mathrm{i;A}}\} = \{e_r^{\mathrm{A}}, h_p^{\mathrm{A}}\} a(t - \alpha_s x_s / c) , \tag{29.2-1}$$

with (see Equation (29.1-8))

$$h_p^{\mathrm{A}} = Y \varepsilon_{p,m,k} \alpha_m e_k^{\mathrm{A}} , \tag{29.2-2}$$

in which Y is given by Equation (29.1-9). In the far-field region, the scattered wave in state A is represented as

$$\{E_r^{\mathrm{s;A}}, H_p^{\mathrm{s;A}}\}(\boldsymbol{x}', t) = \frac{\{E_r^{\mathrm{s;A;\infty}}, H_p^{\mathrm{s;A;\infty}}\}(\boldsymbol{\xi}, t - |\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|} \left[1 + O(|\boldsymbol{x}'|^{-1})\right]$$

$$\text{as } |\boldsymbol{x}'| \to \infty \quad \text{with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi} , \tag{29.2-3}$$

in which, on account of Equations (29.1-24)–(29.1-29),

$$E_r^{\mathrm{s;A;\infty}} = \mu(\xi_r \xi_k - \delta_{r,k}) \partial_t \Phi_k^{\partial J^{\mathrm{s;A;\infty}}} + \varepsilon_{r,n,j} (\xi_n / c) \partial_t \Phi_j^{\partial K^{\mathrm{s;A;\infty}}} , \tag{29.2-4}$$

$$H_p^{\mathrm{s;A;\infty}} = Y \varepsilon_{p,m,k} \xi_m E_k^{\mathrm{s;A;\infty}} , \tag{29.2-5}$$

with

$$\Phi_k^{\partial J^{s;A;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial J_k^{s;A}(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-6}$$

$$\Phi_j^{\partial K^{s;A;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial K_j^{s;A}(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-7}$$

in which (note the orientation of ν)

$$\partial J_k^{s;A} = \varepsilon_{k,m,p}\nu_m H_p^{s;A}, \tag{29.2-8}$$

$$\partial K_j^{s;A} = -\varepsilon_{j,n,r}\nu_n E_r^{s;A}. \tag{29.2-9}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{E_r^{i;B}, H_p^{i;B}\} = \{e_r^B, h_p^B\}\,b(t-\beta_s x_s/c)\,, \tag{29.2-10}$$

with

$$h_p^B = Y\varepsilon_{p,m,k}\beta_m e_k^B, \tag{29.2-11}$$

In the far-field region, the scattered wave in state B is represented as

$$\{E_r^{s;B}, H_p^{s;B}\}(x',t) = \frac{\{E_r^{s;B;\infty}, H_p^{s;B;\infty}\}(\xi,t-|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$
$$\text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi\,, \tag{29.2-12}$$

in which, on account of Equations (29.1-24)–(29.1-29),

$$E_r^{s;B;\infty} = \mu(\xi_r\xi_k - \delta_{r,k})\partial_t\Phi_k^{\partial J^{s;B;\infty}} + \varepsilon_{r,n,j}(\xi_n/c)\partial_t\Phi_j^{\partial K^{s;B;\infty}}, \tag{29.2-13}$$

$$H_p^{s;B;\infty} = Y\varepsilon_{p,m,k}\xi_m E_k^{s;B;\infty}, \tag{29.2-14}$$

with

$$\Phi_k^{\partial J^{s;B;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial J_k^{s;B}(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-15}$$

$$\Phi_j^{\partial K^{s;B;\infty}}(\xi,t) = \int_{x\in\partial\mathcal{D}^s} \partial K_j^{s;B}(x,t+\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-16}$$

in which (note the orientation of ν)

$$\partial J_k^{s;B} = \varepsilon_{k,m,p}\nu_m H_p^{s;B}, \tag{29.2-17}$$

$$\partial K_j^{s;B} = -\varepsilon_{j,n,r}\nu_n E_r^{s;B}. \tag{29.2-18}$$

If the scatterer is penetrable, its electromagnetic properties in state B are assumed to be the adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (29.1-3) or (29.1-4) applies. These boundary conditions apply to both state A and state B, and are, therefore, self-adjoint.

To establish the desired reciprocity relation, we first apply the time-domain reciprocity theorem of the convolution type Equation (28.2-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^s$ occupied by the scatterer. For a penetrable scatterer this yields

$$\varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[C_t(E_r^A, H_p^B; x, t) - C_t(E_r^B H_p^A; x, t) \right] dA = 0 \,, \tag{29.2-19}$$

since in the interior of the scatterer the total wave field is source-free. For an impenetrable scatterer, Equation (29.2-19) holds in view of the boundary conditions upon approaching $\partial \mathcal{D}^s$ via $\mathcal{D}^{s'}$. In Equation (29.2-19) we substitute

$$\{E_r^A, H_p^A\} = \{E_r^{i;A} + E_r^{s;A}, H_p^{i;A} + H_p^{s;A}\} \tag{29.2-20}$$

and

$$\{E_r^B, H_p^B\} = \{E_r^{i;B} + E_r^{s;B}, H_p^{i;B} + H_p^{s;B}\} \,. \tag{29.2-21}$$

Next, the time-domain reciprocity theorem of the convolution type is applied to the incident wave field and to the domain $\mathcal{D}^s$. Since the incident wave field is source-free in the interior of the scatterer and the embedding is self-adjoint in its electromagnetic properties, this leads to

$$\varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[C_t(E_r^{i;A}, H_p^{i;B}; x, t) - C_t(E_r^{i;B} H_p^{i;A}; x, t) \right] dA = 0 \,. \tag{29.2-22}$$

Finally, the time-domain reciprocity theorem of the convolution type is applied to the scattered wave field and to the domain $\mathcal{D}^{s'}$. Since the embedding is self-adjoint in its electromagnetic properties and the scattered wave field is source-free in the exterior of the scatterer and satisfies the condition of causality at infinity, this leads to

$$\varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[C_t(E_r^{s;A}, H_p^{s;B}; x, t) - C_t(E_r^{s;B} H_p^{s;A}; x, t) \right] dA = 0 \,. \tag{29.2-23}$$

From Equations (29.2-19)–(29.2-23) we conclude that

$$\begin{aligned} \varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \Big[& C_t(E_r^{i;A}, H_p^{s;B}; x, t) + C_t(E_r^{s;A}, H_p^{i;B}; x, t) \\ & - C_t(E_r^{i;B} H_p^{s;A}; x, t) - C_t(E_r^{s;B} H_p^{i;A}; x, t) \Big] dA = 0 \,. \end{aligned} \tag{29.2-24}$$

However, on account of Equations (29.2-10) and (29.2-11), and (29.2-4)–(29.2-9) we have

$$\begin{aligned} & \varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[C_t(E_r^{s;A}, H_p^{i;B}; x, t) - C_t(E_r^{i;B}, H_p^{s;A}; x, t) \right] dA \\ & = \int_{t' \in \mathcal{R}} dt' \, \varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[E_r^{s;A}(x,t') h_p^B - e_r^B H_p^{s;A}(x,t') \right] b(t - \beta_s x_s / c - t') \, dA \\ & = \int_{t'' \in \mathcal{R}} b(t - t'') \, dt'' \, \varepsilon_{m,r,p} \int_{x \in \partial \mathcal{D}^s} \nu_m \left[E_r^{s;A}(x, t'' - \beta_s x_s / c) h_p^B - e_r^B H_p^{s;A}(x, t'' - \beta_s x_s / c) \right] dA \\ & = -\mu^{-1} e_r^B \int_{t'' \in \mathcal{R}} b(t - t'') I_t E_r^{s;A;\infty}(-\beta, t'') \, dt'' \end{aligned} \tag{29.2-25}$$

and on account of Equations (29.2-1) and (29.2-2), and (29.2-13)–(29.2-18)

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s;B}},H_p^{\mathrm{i;A}};x,t)-\mathrm{C}_t(E_r^{\mathrm{i;A}},H_p^{\mathrm{s;B}};x,t)\right]\mathrm{d}A$$

$$=\int_{t'\in\mathcal{R}}\mathrm{d}t'\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\left[E_r^{\mathrm{s;B}}(x,t')h_p^{\mathrm{A}}-e_r^{\mathrm{A}}H_p^{\mathrm{s;B}}(x,t')\right]a(t-\alpha_s x_s/c-t')\,\mathrm{d}A$$

$$=\int_{t''\in\mathcal{R}}a(t-t'')\,\mathrm{d}t''\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}}\nu_m\left[E_r^{\mathrm{s;B}}(x,t''-\alpha_s x_s/c)h_p^{\mathrm{A}}-e_r^{\mathrm{A}}H_p^{\mathrm{s;B}}(x,t''-\alpha_s x_s/c)\right]\mathrm{d}A$$

$$=-\mu^{-1}e_r^{\mathrm{A}}\int_{t''\in\mathcal{R}}a(t-t'')\mathrm{I}_t E_r^{\mathrm{s;B;\infty}}(-\alpha,t'')\,\mathrm{d}t''\,. \tag{29.2-26}$$

Equations (29.2-24)–(29.2-26) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$e_r^{\mathrm{B}}\int_{t''\in\mathcal{R}}b(t-t'')\mathrm{I}_t E_r^{\mathrm{s;A;\infty}}(-\beta,t'')\,\mathrm{d}t''$$

$$=e_r^{\mathrm{A}}\int_{t''\in\mathcal{R}}a(t-t'')\mathrm{I}_t E_r^{\mathrm{s;B;\infty}}(-\alpha,t'')\,\mathrm{d}t''. \tag{29.2-27}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, both in state A and state B. To this end, we write (adapting the subscripts for later convenience)

$$E_r^{\mathrm{s;A;\infty}}(\xi,t)=e_k^{\mathrm{A}}\int_{t'\in\mathcal{R}}a(t')S_{r,k}^{\mathrm{A}}(\xi,\alpha,t-t')\,\mathrm{d}t' \tag{29.2-28}$$

and

$$E_k^{\mathrm{s;B;\infty}}(\xi,t)=e_r^{\mathrm{B}}\int_{t'\in\mathcal{R}}b(t')S_{k,r}^{\mathrm{B}}(\xi,\beta,t-t')\,\mathrm{d}t'\,, \tag{29.2-29}$$

where $S_{r,k}^{\mathrm{A}}$ and $S_{k,r}^{\mathrm{B}}$ are the configurational *time-domain electric far-field scattering tensors.* Substitution of Equations (29.2-28) and (29.2-29) in Equation (29.2-27) and rewriting the convolutions, we obtain

$$e_r^{\mathrm{B}}e_k^{\mathrm{A}}\mathrm{I}_t\int_{t''\in\mathcal{R}}b(t'')\,\mathrm{d}t''\int_{t'\in\mathcal{R}}a(t')S_{r,k}^{\mathrm{A}}(-\beta,\alpha,t-t''-t')\,\mathrm{d}t'$$

$$=e_k^{\mathrm{A}}e_r^{\mathrm{B}}\mathrm{I}_t\int_{t''\in\mathcal{R}}a(t'')\,\mathrm{d}t''\int_{t'\in\mathcal{R}}b(t')S_{k,r}^{\mathrm{B}}(-\alpha,\beta,t-t''-t')\,\mathrm{d}t', \tag{29.2-30}$$

where, in accordance with the rules applying to the time convolution, the operator I_t has been brought in front of the integral signs. Taking into account that Equation (29.2-30) has to hold for arbitrary values of $e_k^{\mathrm{A}},e_r^{\mathrm{B}}$, $a(t)$ and $b(t)$, and using the causality of the scattered wave, we end up with

$$S_{r,k}^{\mathrm{A}}(-\beta,\alpha,t)=S_{k,r}^{\mathrm{B}}(-\alpha,\beta,t) \tag{29.2-31}$$

as the final expression of the time-domain reciprocity property under consideration.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the incident uniform plane wave in state A is taken as

$$\{\hat{E}_r^{\mathrm{i;A}},\hat{H}_p^{\mathrm{i;A}}\} = \{e_r^{\mathrm{A}},h_p^{\mathrm{A}}\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{29.2-32}$$

with (see Equation (29.1-8))

$$h_p^{\mathrm{A}} = Y\varepsilon_{p,m,k}\alpha_m e_k^{\mathrm{A}}\,, \tag{29.2-33}$$

in which Y is given by Equation (29.1-9). In the far-field region, the scattered wave in state A is represented as

$$\{\hat{E}_r^{\mathrm{s;A}},\hat{H}_p^{\mathrm{s;A}}\}(\boldsymbol{x}',s) = \{\hat{E}_r^{\mathrm{s;A;\infty}},\hat{H}_p^{\mathrm{s;A;\infty}}\}(\boldsymbol{\xi},s)\,\frac{\exp(-s|\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1+O(|\boldsymbol{x}'|^{-1})\right]$$
$$\text{as } |\boldsymbol{x}'|\to\infty \quad \text{with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi}\,, \tag{29.2-34}$$

where, on account of Equations (29.1-61)–(29.1-66),

$$\hat{E}_r^{\mathrm{s;A;\infty}} = s\mu(\xi_r\xi_k-\delta_{r,k})\hat{\Phi}_k^{\partial J^{\mathrm{s}};\mathrm{A};\infty} + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}_j^{\partial K^{\mathrm{s}};\mathrm{A};\infty}\,, \tag{29.2-35}$$

$$\hat{H}_p^{\mathrm{s;A;\infty}} = Y\varepsilon_{p,m,k}\xi_m\hat{E}_k^{\mathrm{s;A;\infty}}\,, \tag{29.2-36}$$

with

$$\hat{\Phi}_k^{\partial J^{\mathrm{s}};\mathrm{A};\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{J}_k^{\mathrm{s;A}}(\boldsymbol{x},s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-37}$$

$$\hat{\Phi}_j^{\partial K^{\mathrm{s}};\mathrm{A};\infty}(\boldsymbol{\xi},s) = \int_{\boldsymbol{x}\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{K}_j^{\mathrm{s;A}}(\boldsymbol{x},s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-38}$$

in which (note the orientation of ν)

$$\partial\hat{J}_k^{\mathrm{s;A}} = \varepsilon_{k,m,p}\nu_m\hat{H}_p^{\mathrm{s;A}}\,, \tag{29.2-39}$$

$$\partial\hat{K}_j^{\mathrm{s;A}} = -\varepsilon_{j,n,r}\nu_n\hat{E}_r^{\mathrm{s;A}}\,. \tag{29.2-40}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{\hat{E}_r^{\mathrm{i;B}},\hat{H}_p^{\mathrm{i;B}}\} = \{e_r^{\mathrm{B}},h_p^{\mathrm{B}}\}\hat{b}(s)\exp(-s\beta_s x_s/c)\,, \tag{29.2-41}$$

with

$$h_p^{\mathrm{B}} = Y\varepsilon_{p,m,k}\beta_m e_k^{\mathrm{B}}\,. \tag{29.2-42}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\hat{E}_r^{\mathrm{s;B}},\hat{H}_p^{\mathrm{s;B}}\}(\boldsymbol{x}',s) = \{\hat{E}_r^{\mathrm{s;B;\infty}},\hat{H}_p^{\mathrm{s;B;\infty}}\}(\boldsymbol{\xi},s)\,\frac{\exp(-s|\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1+O(|\boldsymbol{x}'|^{-1})\right]$$
$$\text{as } |\boldsymbol{x}'|\to\infty \quad \text{with } \boldsymbol{x}' = |\boldsymbol{x}'|\boldsymbol{\xi}\,, \tag{29.2-43}$$

where, on account of Equations (29.1-61)–(29.1-66),

$$\hat{E}_r^{\mathrm{s;B;\infty}} = s\mu(\xi_r\xi_k - \delta_{r,k})\hat{\Phi}_k^{\partial J^{\mathrm{s}};\mathrm{B};\infty} + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}_j^{\partial K^{\mathrm{s}};\mathrm{B};\infty}, \tag{29.2-44}$$

$$\hat{H}_p^{\mathrm{s;B;\infty}} = Y\varepsilon_{p,m,k}\xi_m\hat{E}_k^{\mathrm{s;B;\infty}}, \tag{29.2-45}$$

with

$$\hat{\Phi}_k^{\partial J^{\mathrm{s}};\mathrm{B};\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{J}_k^{\mathrm{s;B}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-46}$$

$$\hat{\Phi}_j^{\partial K^{\mathrm{s}};\mathrm{B};\infty}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{K}_j^{\mathrm{s;B}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.2-47}$$

in which (note the orientation of ν)

$$\partial\hat{J}_k^{\mathrm{s;B}} = \varepsilon_{k,m,p}\nu_m\hat{H}_p^{\mathrm{s;B}}, \tag{29.2-48}$$

$$\partial\hat{K}_j^{\mathrm{s;B}} = -\varepsilon_{j,n,r}\nu_n\hat{E}_r^{\mathrm{s;B}}. \tag{29.2-49}$$

If the scatterer is penetrable, its electromagnetic properties in state B are assumed to be the adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (29.1-44) or (29.1-45) applies. These boundary conditions apply to both state A and state B, and are, therefore, self-adjoint.

To establish the desired reciprocity relation, we first apply the complex frequency-domain reciprocity theorem of the time convolution type Equation (28.4-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\hat{E}_r^{\mathrm{A}}(x,s)\hat{H}_p^{\mathrm{B}}(x,s) - \hat{E}_r^{\mathrm{B}}(x,s)\hat{H}_p^{\mathrm{A}}(x,s)\right]\mathrm{d}A = 0\,, \tag{29.2-50}$$

since in the interior of the scatterer the total wave field is source-free. For an impenetrable scatterer, Equation (29.2-50) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\mathrm{s}}$ via $\mathcal{D}^{\mathrm{s}\prime}$. In Equation (29.2-50) we substitute

$$\{\hat{E}_r^{\mathrm{A}},\hat{H}_p^{\mathrm{A}}\} = \{\hat{E}_r^{\mathrm{i;A}} + \hat{E}_r^{\mathrm{s;A}},\hat{H}_p^{\mathrm{i;A}} + \hat{H}_p^{\mathrm{s;A}}\} \tag{29.2-51}$$

and

$$\{\hat{E}_r^{\mathrm{B}},\hat{H}_p^{\mathrm{B}}\} = \{\hat{E}_r^{\mathrm{i;B}} + \hat{E}_r^{\mathrm{s;B}},\hat{H}_p^{\mathrm{i;B}} + \hat{H}_p^{\mathrm{s;B}}\}\,. \tag{29.2-52}$$

Next, the complex frequency-domain reciprocity theorem of the time convolution type is applied to the incident wave field and to the domain $\mathcal{D}^{\mathrm{s}}$. Since the incident wave field is source-free in the interior of the scatterer and the embedding is self-adjoint in its electromagnetic properties, this leads to

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\hat{E}_r^{\mathrm{i;A}}(x,s)\hat{H}_p^{\mathrm{i;B}}(x,s) - \hat{E}_r^{\mathrm{i;B}}(x,s)\hat{H}_p^{\mathrm{i;A}}(x,s)\right]\mathrm{d}A = 0\,. \tag{29.2-53}$$

Finally, the complex frequency-domain reciprocity theorem of the time convolution type is applied to the scattered wave field and to the domain $\mathcal{D}^{\mathrm{s}\prime}$. Since the embedding is self-adjoint in its electromagnetic properties and the scattered wave field is source-free in the exterior of the scatterer and satisfies the condition of causality at infinity, this leads to

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{s;A}(x,s)\hat{H}_p^{s;B}(x,s)-\hat{E}_r^{s;B}(x,s)\hat{H}_p^{s;A}(x,s)\right]\mathrm{d}A=0\,. \tag{29.2-54}$$

From Equations (29.2-50)–(29.2-54) we conclude that

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{i;A}(x,s)\hat{H}_p^{s;B}(x,s)+\hat{E}_r^{s;A}(x,s)\hat{H}_p^{i;B}(x,s)\right.$$
$$\left.-\hat{E}_r^{i;B}(x,s)\hat{H}_p^{s;A}(x,s)-\hat{E}_r^{s;B}(x,s)\hat{H}_p^{i;A}(x,s)\right]\mathrm{d}A=0\,. \tag{29.2-55}$$

However, on account of Equations (29.2-41) and (29.2-42), and (29.2-35)–(29.2-40) we have

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{s;A}(x,s)\hat{H}_p^{i;B}(x,s)-\hat{E}_r^{i;B}(x,s)\hat{H}_p^{s;A}(x,s)\right]\mathrm{d}A$$
$$=\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{s;A}(x,s)h_p^B-e_r^B\hat{H}_p^{s;A}(x,s)\right]\hat{b}(s)\exp(-s\beta_s x_s/c)\,\mathrm{d}A$$
$$=-(s\mu)^{-1}e_r^B\hat{b}(s)\hat{E}_r^{s;A;\infty}(-\beta,s) \tag{29.2-56}$$

and on account of Equations (29.2-32) and (29.2-33), and Equations (29.2-44)–(29.2-49)

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{s;B}(x,s)\hat{H}_p^{i;A}(x,s)-\hat{E}_r^{i;A}(x,s)\hat{H}_p^{s;B}(x,s)\right]\mathrm{d}A$$
$$=\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{s;B}(x,s)h_p^A-e_r^A\hat{H}_p^{s;B}(x,s)\right]\hat{a}(s)\exp(-s\alpha_s x_s/c)\,\mathrm{d}A$$
$$=-(s\mu)^{-1}e_r^A\hat{a}(s)\hat{E}_r^{s;B;\infty}(-\alpha,s)\,. \tag{29.2-57}$$

Equations (29.2-55)–(29.2-57) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$e_r^B\hat{b}(s)\hat{E}_r^{s;A;\infty}(-\beta,s)=e_r^A\hat{a}(s)\hat{E}_r^{s;B;\infty}(-\alpha,s)\,. \tag{29.2-58}$$

At this point it is, again, elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, both in state A and in state B. To this end, we write, in accordance with Equations (29.2-28) and (29.2-29),

$$\hat{E}_r^{s;A;\infty}(\xi,s)=\hat{S}_{r,k}^A(\xi,\alpha,s)e_k^A\hat{a}(s) \tag{29.2-59}$$

and

$$\hat{E}_k^{s;B;\infty}(\xi,s)=\hat{S}_{k,r}^B(\xi,\beta,s)e_r^B\hat{b}(s)\,, \tag{29.2-60}$$

where $\hat{S}_{r,k}^A$ and $\hat{S}_{k,r}^B$ are the configurational *complex frequency-domain electric field far-field scattering tensors*. Substitution of Equations (29.2-59) and (29.2-60) in Equation (29.2-58) yields

$$\hat{S}_{r,k}^A(-\beta,\alpha,s)e_k^A\hat{a}(s)e_r^B\hat{b}(s)=\hat{S}_{k,r}^B(-\alpha,\beta,s)e_r^B\hat{b}(s)e_k^A\hat{a}(s)\,. \tag{29.2-61}$$

Taking into account that Equation (29.2-61) has to hold for arbitrary values of e_k^{A}, e_r^{B}, $\hat{a}(s)$ and $\hat{b}(s)$, we end up with

$$\hat{S}_{r,k}^{\mathrm{A}}(-\boldsymbol{\beta},\boldsymbol{\alpha},s) = \hat{S}_{k,r}^{\mathrm{B}}(-\boldsymbol{\alpha},\boldsymbol{\beta},s) \tag{29.2-62}$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

In a theoretical analysis, the reciprocity relations derived in this section serve as an important check on the correctness of the analytic solutions as well as on the accuracy of numerical solutions to scattering problems. Note, however, that the reciprocity relations are necessary conditions to be satisfied by the scattered wave field (in the far-field region), but their satisfaction does not guarantee correctness of a total analytic solution or a certain accuracy of a total numerical solution. In a physical experiment, the redundancy induced by the reciprocity relations can be exploited to reduce the influence of noise on the quality of the observed data.

References to the earlier literature on the reciprocity relations of the type discussed in this section can be found in De Hoop (1960).

Exercises

Exercise 29.2-1

To what form reduces Equation (29.2-27) if $a(t) = b(t)$? (*Hint*: Use the fact that the resulting indentity has to hold for any pulse shape and employ the causality of the scattered wave).

Answer:

$$e_r^{\mathrm{B}} E_r^{\mathrm{s;A;\infty}}(-\boldsymbol{\beta},t) = e_r^{\mathrm{A}} E_r^{\mathrm{s;B;\infty}}(-\boldsymbol{\alpha},t) \, .$$

Exercise 29.2-2

To what form reduces Equation (29.2-58) if $\hat{a}(s) = \hat{b}(s)$?

Answer:

$$e_r^{\mathrm{B}} \hat{E}_r^{\mathrm{s;A;\infty}}(-\boldsymbol{\beta},s) = e_r^{\mathrm{A}} \hat{E}_r^{\mathrm{s;B;\infty}}(-\boldsymbol{\alpha},s) \, .$$

Exercise 29.2-3

Give the compatibility relations satisfied by the electric field far-field scattering tensor $S_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},t)$ that result from the scattering of an incident uniform plane wave propagating in the direction of the unit vector $\boldsymbol{\alpha}$ and apply to observation in the direction of the unit vector $\boldsymbol{\xi}$.

Answer:

(a) $\xi_r S_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},t) = 0$, (b) $S_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},t)\alpha_k = 0$.

(Hence, $S_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},t)$ only contains components transverse to both $\boldsymbol{\xi}$ and $\boldsymbol{\alpha}$.)

Exercise 29.2-4

Give the compatibility relations satisfied by the electric field far-field scattering tensor $\hat{S}_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},s)$ that result from the scattering of an incident uniform plane wave propagating in the direction of the unit vector $\boldsymbol{\alpha}$ and apply to observation in the direction of the unit vector $\boldsymbol{\xi}$.

Answer:

(a) $\xi_r \hat{S}_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},s) = 0$, (b) $\hat{S}_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},s)\alpha_k = 0$.

(Hence, $\hat{S}_{r,k}(\boldsymbol{\xi},\boldsymbol{\alpha},s)$ only contains components transverse to both $\boldsymbol{\xi}$ and $\boldsymbol{\alpha}$.)

Exercise 29.2-5

Show that Equation (29.2-62) follows from Equation (29.2-31) by taking the time Laplace transform.

29.3 Far-field scattered wave amplitude reciprocity of the time correlation type

In this section we investigate the reciprocity relation of the time correlation type that applies to the far-field scattered wave amplitude reciprocity for plane wave incidence upon an electromagnetically penetrable or impenetrable object. The scattering configuration of Figure 29.1-1 applies. Two states in this configuration are considered; they are denoted as state A and state B. In state A, a uniform plane electromagnetic wave that propagates in the direction of the unit vector $\boldsymbol{\alpha}$ is incident upon the scattering object; in state B, a uniform plane electromagnetic wave that propagates in the direction of the unit vector $\boldsymbol{\beta}$ is incident upon the scattering object. It will be shown that the far-field scattered wave amplitude in state A when observed in the direction of observation $\boldsymbol{\xi}=\boldsymbol{\beta}$ is related, via reciprocity, to the far-field scattered wave amplitude in state B when observed in the direction of observation $\boldsymbol{\xi}=\boldsymbol{\alpha}$ (Figure 29.3-1).

The corresponding relationships in the time domain and in the complex frequency domain will be derived separately below.

Time-domain analysis

In the time-domain analysis, the incident uniform plane wave in state A is taken as

$$\{E_r^{\mathrm{i;A}},H_p^{\mathrm{i;A}}\} = \{e_r^{\mathrm{A}},h_p^{\mathrm{A}}\}a(t-\alpha_s x_s/c)\,, \tag{29.3-1}$$

with (see Equation (29.1-8))

$$h_p^{\mathrm{A}} = Y\varepsilon_{p,m,k}\alpha_m e_k^{\mathrm{A}}, \tag{29.3-2}$$

in which Y is given by Equation (29.1-9). In the far-field region, the scattered wave in state A is represented as

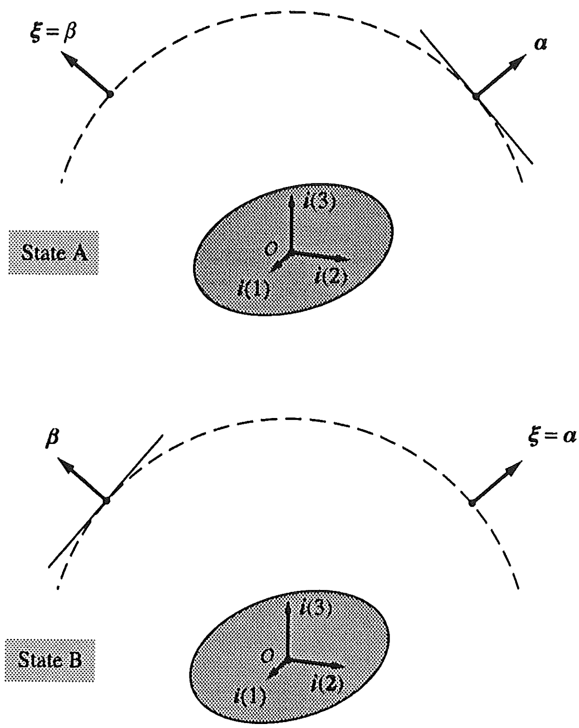

Figure 29.3-1 Configuration for the far-field scattered wave amplitude reciprocity of the time correlation type.

$$\{E_r^{\mathrm{s;A}}, H_p^{\mathrm{s;A}}\}(x',t) = \frac{\{E_r^{\mathrm{s;A};\infty}, H_p^{\mathrm{s;A};\infty}\}(\xi, t - |x'|/c)}{4\pi|x'|}\left[1 + O(|x'|^{-1})\right]$$

$$\text{as } |x'| \to \infty \quad \text{with } x' = |x'|\xi , \tag{29.3-3}$$

in which, on account of Equations (29.1-24)–(29.1-29),

$$E_r^{\mathrm{s;A};\infty} = \mu(\xi_r\xi_k - \delta_{r,k})\partial_t \Phi_k^{\partial J^{\mathrm{s;A}};\infty} + \varepsilon_{r,n,j}(\xi_n/c)\partial_t \Phi_j^{\partial K^{\mathrm{s;A}};\infty}, \tag{29.3-4}$$

$$H_p^{\mathrm{s;A};\infty} = Y\varepsilon_{p,m,k}\xi_m E_k^{\mathrm{s;A};\infty}, \tag{29.3-5}$$

with

$$\Phi_k^{\partial J^{\mathrm{s;A}};\infty}(\xi,t) = \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \partial J_k^{\mathrm{s;A}}(x, t + \xi_s x_s/c)\,\mathrm{d}A , \tag{29.3-6}$$

$$\Phi_j^{\partial K^{\mathrm{s;A}};\infty}(\xi,t) = \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \partial K_j^{\mathrm{s;A}}(x, t + \xi_s x_s/c)\,\mathrm{d}A , \tag{29.3-7}$$

in which (note the orientation of ν)

$$\partial J_k^{\mathrm{s;A}} = \varepsilon_{k,m,p}\nu_m H_p^{\mathrm{s;A}}, \tag{29.3-8}$$

$$\partial K_j^{s;A} = -\varepsilon_{j,n,r}\nu_n E_r^{s;A}. \tag{29.3-9}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{E_r^{i;B}, H_p^{i;B}\} = \{e_r^B, h_p^B\} b(t - \beta_s x_s/c), \tag{29.3-10}$$

with

$$h_p^B = Y\varepsilon_{p,m,k}\beta_m e_k^B. \tag{29.3-11}$$

In the far-field region, the scattered wave in state B is represented as

$$\{E_r^{s;B}, H_p^{s;B}\}(x', t) = \frac{\{E_r^{s;B;\infty}, H_p^{s;B;\infty}\}(\xi, t - |x'|/c)}{4\pi|x'|}\left[1 + O(|x'|^{-1})\right]$$
$$\text{as } |x'| \to \infty \quad \text{with } x' = |x'|\xi, \tag{29.3-12}$$

in which, on account of Equations (29.1-24)-(29.1-29),

$$E_r^{s;B;\infty} = \mu(\xi_r\xi_k - \delta_{r,k})\partial_t \Phi_k^{\partial J^{s;B;\infty}} + \varepsilon_{r,n,j}(\xi_n/c)\partial_t \Phi_j^{\partial K^{s;B;\infty}}, \tag{29.3-13}$$

$$H_p^{s;B;\infty} = Y\varepsilon_{p,m,k}\xi_m E_k^{s;B;\infty}, \tag{29.3-14}$$

with

$$\Phi_k^{\partial J^{s;B;\infty}}(\xi, t) = \int_{x\in\partial\mathcal{D}^s} \partial J_k^{s;B}(x, t + \xi_s x_s/c)\,\mathrm{d}A, \tag{29.3-15}$$

$$\Phi_j^{\partial K^{s;B;\infty}}(\xi, t) = \int_{x\in\partial\mathcal{D}^s} \partial K_j^{s;B}(x, t + \xi_s x_s/c)\,\mathrm{d}A, \tag{29.3-16}$$

in which (note the orientation of ν)

$$\partial J_k^{s;B} = \varepsilon_{k,m,p}\nu_m H_p^{s;B}, \tag{29.3-17}$$

$$\partial K_j^{s;B} = -\varepsilon_{j,n,r}\nu_n E_r^{s;B}. \tag{29.3-18}$$

If the scatterer is penetrable, its electromagnetic properties in state B are assumed to be the time-reverse adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (29.1-3) or (29.1-4) applies. These boundary conditions apply to both state A and state B, and are, therefore, time reverse self-adjoint.

To establish the desired reciprocity relation, we first apply the time-domain reciprocity theorem of the correlation type Equation (28.3-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^s$ occupied by the scatterer. For a penetrable scatterer this yields

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s} \nu_m\left[\mathrm{C}_t(E_r^A, \mathrm{J}_t(H_p^B); x, t) - \mathrm{C}_t(\mathrm{J}_t(E_r^B)H_p^A; x, t)\right]\mathrm{d}A = 0, \tag{29.3-19}$$

since in the interior of the scatterer the total wave field is source-free. For an impenetrable scatterer, Equation (29.3-19) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^s$ via $\mathcal{D}^{s'}$. In Equation (29.3-19) we substitute

$$\{E_r^A, H_p^A\} = \{E_r^{i;A} + E_r^{s;A}, H_p^{i;A} + H_p^{s;A}\}. \tag{29.3-20}$$

and

$$\{E_r^B, H_p^B\} = \{E_r^{i;B} + E_r^{s;B}, H_p^{i;B} + H_p^{s;B}\}. \tag{29.3-21}$$

Next, the time-domain reciprocity theorem of the correlation type is applied to the incident wave field and to the domain $\mathcal{D}^{\mathrm{s}}$. Since the incident wave field is source-free in the interior of the scatterer and the embedding is time reverse self-adjoint in its electromagnetic properties, this leads to

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{i;A}},\mathrm{J}_t(H_p^{\mathrm{i;B}});x,t)+\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{i;B}}),H_p^{\mathrm{i;A}};x,t)\right]\mathrm{d}A=0\,. \qquad (29.3\text{-}22)$$

Finally, the time-domain reciprocity theorem of the correlation type is applied to the scattered wave field and to the domain $\mathcal{D}^{\mathrm{s}'}$. Since the embedding is time reverse self-adjoint in its electromagnetic properties and the scattered wave field is source-free in the exterior of the scatterer and satisfies the condition of causality, this leads to

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s;A}},\mathrm{J}_t(H_p^{\mathrm{s;B}});x,t)+\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{s;B}}),H_p^{\mathrm{s;A}};x,t)\right]\mathrm{d}A$$

$$=\lim_{\Delta\to\infty}\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(\mathcal{O},\Delta)} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s;A}},\mathrm{J}_t(H_p^{\mathrm{s;B}});x,t)\right.$$

$$\left.+\,\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{s;B}}),H_p^{\mathrm{s;A}};x,t)\right]\mathrm{d}A\,, \qquad (29.3\text{-}23)$$

where $\mathcal{S}(\mathcal{O},\Delta)$ is the sphere of radius Δ with centre at the origin $\mathcal{O}$ of the chosen reference frame. From Equations (29.3-19)–(29.3-23) we conclude that

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{i;A}},\mathrm{J}_t(H_p^{\mathrm{s;B}});x,t)+\mathrm{C}_t(E_r^{\mathrm{s;A}},\mathrm{J}_t(H_p^{\mathrm{i;B}});x,t)\right.$$

$$\left.+\,\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{i;B}}),H_p^{\mathrm{s;A}};x,t)+\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{s;B}})H_p^{\mathrm{i;A}};x,t)\right]\mathrm{d}A$$

$$+\lim_{\Delta\to\infty}\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(\mathcal{O},\Delta)} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s;A}},\mathrm{J}_t(H_p^{\mathrm{s;B}});x,t)\right.$$

$$\left.+\,\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{s;B}})H_p^{\mathrm{s;A}};x,t)\right]\mathrm{d}A=0\,. \qquad (29.3\text{-}24)$$

However, on account of Equations (29.3-10) and (29.3-11), and (29.3-4)–(29.3-9) we have

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\mathrm{C}_t(E_r^{\mathrm{s;A}},\mathrm{J}_t(H_p^{\mathrm{i;B}});x,t)+\mathrm{C}_t(\mathrm{J}_t(E_r^{\mathrm{i;B}}),H_p^{\mathrm{s;A}};x,t)\right]\mathrm{d}A$$

$$=\int_{t'\in\mathcal{R}}\mathrm{d}t'\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[E_r^{\mathrm{s;A}}(x,t')h_p^{\mathrm{B}}+e_r^{\mathrm{B}}H_p^{\mathrm{s;A}}(x,t')\right]b(t'-\beta_s x_s/c-t)\,\mathrm{d}A$$

$$=\int_{t''\in\mathcal{R}} b(t''-t)\,\mathrm{d}t''\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[E_r^{\mathrm{s;A}}(x,t''+\beta_s x_s/c)h_p^{\mathrm{B}}+e_r^{\mathrm{B}}H_p^{\mathrm{s;A}}(x,t''+\beta_s x_s/c)\right]\mathrm{d}A$$

$$=-\mu^{-1}e_r^{\mathrm{B}}\int_{t''\in\mathcal{R}} b(t''-t)\mathrm{I}_t E_r^{\mathrm{s;A;\infty}}(\beta,t'')\,\mathrm{d}t'' \qquad (29.3\text{-}25)$$

and on account of Equations (29.3-1) and (29.3-2), and (29.3-13)–(29.3-18)

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\mathrm{C}_t(\mathrm{J}_t(E_r^{s;B}),H_p^{i;A};x,t)+\mathrm{C}_t(E_r^{i;A},\mathrm{J}_t(H_p^{s;B});x,t)\right]\mathrm{d}A$$

$$=\int_{t'\in\mathcal{R}}\mathrm{d}t'\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[E_r^{s;B}(x,t'-t)h_p^A+e_r^AH_p^{s;B}(x,t'-t)\right]a(t'-\alpha_sx_s/c)\,\mathrm{d}A$$

$$=\int_{t''\in\mathcal{R}}a(t'')\,\mathrm{d}t''\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[E_r^{s;B}(x,t''+\alpha_sx_s/c-t)h_p^A+e_r^AH_p^{s;B}(x,t''+\alpha_sx_s/c-t)\right]\mathrm{d}A$$

$$=-\mu^{-1}e_r^A\int_{t''\in\mathcal{R}}a(t'')\mathrm{I}_tE_r^{s;B;\infty}(\alpha,t''-t)\,\mathrm{d}t''. \tag{29.3-26}$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\left[\mathrm{C}_t(E_r^{s;A},\mathrm{J}_t(H_p^{s;B});x,t)+\mathrm{C}_t(\mathrm{J}_t(E_r^{s;B}),H_p^{s;A};x,t)\right]\mathrm{d}A$$

$$=(4\pi)^{-2}\int_{t'\in\mathcal{R}}\mathrm{d}t'\,\varepsilon_{m,r,p}\int_{\xi\in\Omega}\xi_m\left[E_r^{s;A;\infty}(\xi,t')H_p^{s;B;\infty}(\xi,t'-t)\right.$$

$$\left.+E_r^{s;B;\infty}(\xi,t'-t)H_p^{s;A;\infty}(\xi,t')\right]\mathrm{d}A$$

$$=(8\pi^2)^{-1}(\mu c)^{-1}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{\xi\in\Omega}E_r^{s;A;\infty}(\xi,t')E_r^{s;B;\infty}(\xi,t'-t)\,\mathrm{d}A\,, \tag{29.3-27}$$

where Ω is the sphere of unit radius and center at O. Equations (29.3-24)–(29.3-27) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$\mu^{-1}e_r^B\int_{t''\in\mathcal{R}}b(t''-t)\mathrm{I}_tE_r^{s;A;\infty}(\beta,t'')\,\mathrm{d}t''$$

$$+\mu^{-1}e_r^A\int_{t''\in\mathcal{R}}a(t'')\mathrm{I}_tE_r^{s;B;\infty}(\alpha,t''-t)\,\mathrm{d}t''$$

$$=-(8\pi^2\mu c)^{-1}\int_{t'\in\mathcal{R}}\mathrm{d}t'\int_{\xi\in\Omega}E_r^{s;A;\infty}(\xi,t')E_r^{s;B;\infty}(\xi,t'-t)\,\mathrm{d}A\,. \tag{29.3-28}$$

At this point it is elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, both in state A and state B. Substitution of Equations (29.2-28) and (29.2-29) in Equation (29.3-28) and rewriting the convolutions and the correlation, we obtain

$$e_r^Be_k^A\int_{t''\in\mathcal{R}}b(t''-t)\,\mathrm{d}t''\int_{t'\in\mathcal{R}}a(t')\mathrm{I}_tS_{r,k}^A(\beta,\alpha,t''-t')\,\mathrm{d}t'$$

$$+e_k^Ae_r^B\int_{t''\in\mathcal{R}}a(t'')\,\mathrm{d}t''\int_{t'\in\mathcal{R}}b(t')\mathrm{I}_tS_{k,r}^B(\alpha,t''-t-t')\,\mathrm{d}t'$$

$$= -(8\pi^2 c)^{-1} e_k^{\mathrm{A}} e_r^{\mathrm{B}} \int_{\tau\in\mathcal{R}} \mathrm{d}\tau \int_{\xi\in\Omega} \left[\int_{t'\in\mathcal{R}} a(t') S^{\mathrm{A}}_{r',k}(\xi,\tau-t')\,\mathrm{d}t' \right.$$

$$\left. \int_{t''\in\mathcal{R}} b(t'') S^{\mathrm{B}}_{r',r}(\xi,\tau-t-t'')\,\mathrm{d}t'' \right] \mathrm{d}A\,, \tag{29.3-29}$$

or

$$e_k^{\mathrm{A}} e_r^{\mathrm{B}} \int_{t'\in\mathcal{R}} a(t')\,\mathrm{d}t' \int_{t''\in\mathcal{R}} b(t'') \mathrm{I}_t S^{\mathrm{A}}_{r,k}(\beta,\alpha,t+t''-t')\,\mathrm{d}t''$$

$$+ e_r^{\mathrm{B}} e_k^{\mathrm{A}} \int_{t'\in\mathcal{R}} a(t')\,\mathrm{d}t' \int_{t''\in\mathcal{R}} b(t'') \mathrm{I}_t S^{\mathrm{B}}_{r,k}(\alpha,\beta,t'-t-t'')\,\mathrm{d}t''$$

$$= -(8\pi^2 c)^{-1} e_k^{\mathrm{A}} e_r^{\mathrm{B}} \int_{t\in\mathcal{R}} a(t')\,\mathrm{d}t' \int_{t''\in\mathcal{R}} b(t'')\,\mathrm{d}t''$$

$$\int_{\xi\in\Omega} \left[\int_{\tau\in\mathcal{R}} S^{\mathrm{A}}_{r',k}(\xi,\tau-t') S^{\mathrm{B}}_{r',r}(\xi,\tau-t-t'')\,\mathrm{d}\tau \right] \mathrm{d}A\,. \tag{29.3-30}$$

Since Equation (29.3-30) has to hold for arbitrary values of e_k^{A}, e_r^{B}, $a(t)$ and $b(t)$, we end up with

$$\mathrm{I}_t S^{\mathrm{A}}_{r,k}(\beta,\alpha,t) + \mathrm{I}_t S^{\mathrm{B}}_{k,r}(\alpha,\beta,-t)$$

$$= -(8\pi^2 c)^{-1} \int_{\xi\in\Omega} \left[\int_{\tau\in\mathcal{R}} S^{\mathrm{A}}_{r',k}(\xi,\tau) S^{\mathrm{B}}_{r',r}(\xi,\tau-t)\,\mathrm{d}\tau \right] \mathrm{d}A \tag{29.3-31}$$

as the final expression of the reciprocity relation under consideration.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the incident uniform plane wave in state A is taken as

$$\{\hat{E}_r^{\mathrm{i;A}}, \hat{H}_p^{\mathrm{i;A}}\} = \{e_r^{\mathrm{A}}, h_p^{\mathrm{A}}\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{29.3-32}$$

with

$$h_p^{\mathrm{A}} = Y\varepsilon_{p,m,k}\alpha_m e_k^{\mathrm{A}}\,, \tag{29.3-33}$$

in which Y is given by Equation (29.1-9). In the far-field region, the scattered wave in state A is represented as

$$\{\hat{E}_r^{\mathrm{s;A}}, \hat{H}_p^{\mathrm{s;A}}\}(\boldsymbol{x}',s) = \{\hat{E}_r^{\mathrm{s;A;\infty}}, \hat{H}_p^{\mathrm{s;A;\infty}}\}(\xi,s)\,\frac{\exp(-s|\boldsymbol{x}'|/c)}{4\pi|\boldsymbol{x}'|}\left[1+O(|\boldsymbol{x}'|^{-1})\right]$$

$$\text{as } |\boldsymbol{x}'|\to\infty \quad \text{with } \boldsymbol{x}' = |\boldsymbol{x}'|\xi\,, \tag{29.3-34}$$

where, on account of Equations (29.1-61)–(29.1-66),

$$\hat{E}_r^{\mathrm{s;A;\infty}} = s\mu(\xi_r\xi_k - \delta_{r,k})\hat{\Phi}_k^{\partial J^{\mathrm{s;A;\infty}}} + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}_j^{\partial K^{\mathrm{s;A;\infty}}}\,, \tag{29.3-35}$$

$$\hat{H}_p^{\mathrm{s;A;\infty}} = Y\varepsilon_{p,m,k}\xi_m\hat{E}_k^{\mathrm{s;A;\infty}}, \tag{29.3-36}$$

with

$$\hat{\Phi}_k^{\partial J^{\mathrm{s;A;\infty}}}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{J}_k^{\mathrm{s;A}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.3-37}$$

$$\hat{\Phi}_j^{\partial K^{\mathrm{s;A;\infty}}}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{K}_j^{\mathrm{s;A}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.3-38}$$

in which (note the orientation of ν)

$$\partial\hat{J}_k^{\mathrm{s;A}} = \varepsilon_{k,m,p}\nu_m\hat{H}_p^{\mathrm{s;A}}, \tag{29.3-39}$$

$$\partial\hat{K}_j^{\mathrm{s;A}} = -\varepsilon_{j,n,r}\nu_n\hat{E}_r^{\mathrm{s;A}}. \tag{29.3-40}$$

Similarly, the incident uniform plane wave in state B is taken as

$$\{\hat{E}_r^{\mathrm{i;B}},\hat{H}_p^{\mathrm{i;B}}\} = \{e_r^{\mathrm{B}},h_p^{\mathrm{B}}\}\hat{b}(s)\exp(-s\beta_s x_s/c)\,, \tag{29.3-41}$$

with

$$h_p^{\mathrm{B}} = Y\varepsilon_{p,m,k}\beta_m e_k^{\mathrm{B}}. \tag{29.3-42}$$

In the far-field region, the scattered wave in state B is represented as

$$\{\hat{E}_r^{\mathrm{s;B}},\hat{H}_p^{\mathrm{s;B}}\}(x',s) = \{\hat{E}_r^{\mathrm{s;B;\infty}},\hat{H}_p^{\mathrm{s;B;\infty}}\}(\xi,s)\,\frac{\exp(-s|x'|/c)}{4\pi|x'|}\left[1+O(|x'|^{-1})\right]$$

$$\text{as } |x'|\to\infty \quad \text{with } x' = |x'|\xi\,, \tag{29.3-43}$$

where, on account of Equations (29.1-61)–(29.1-66),

$$\hat{E}_r^{\mathrm{s;B;\infty}} = s\mu(\xi_r\xi_k - \delta_{r,k})\hat{\Phi}_k^{\partial J^{\mathrm{s;B;\infty}}} + \varepsilon_{r,n,j}(s\xi_n/c)\hat{\Phi}_j^{\partial K^{\mathrm{s;B;\infty}}}, \tag{29.3-44}$$

$$\hat{H}_p^{\mathrm{s;B;\infty}} = Y\varepsilon_{p,m,k}\xi_m\hat{E}_k^{\mathrm{s;B;\infty}}, \tag{29.3-45}$$

with

$$\hat{\Phi}_k^{\partial J^{\mathrm{s;B;\infty}}}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{J}_k^{\mathrm{s;B}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.3-46}$$

$$\hat{\Phi}_j^{\partial K^{\mathrm{s;B;\infty}}}(\xi,s) = \int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \partial\hat{K}_j^{\mathrm{s;B}}(x,s)\exp(s\xi_s x_s/c)\,\mathrm{d}A\,, \tag{29.3-47}$$

in which (note the orientation of ν)

$$\partial\hat{J}_k^{\mathrm{s;B}} = \varepsilon_{k,m,p}\nu_m\hat{H}_p^{\mathrm{s;B}}, \tag{29.3-48}$$

$$\partial\hat{K}_j^{\mathrm{s;B}} = -\varepsilon_{j,n,r}\nu_n\hat{E}_r^{\mathrm{s;B}}. \tag{29.3-49}$$

If the scatterer is penetrable, its electromagnetic properties in state B are assumed to be the time-reverse adjoint of the ones pertaining to state A. If the scatterer is impenetrable, either of the two boundary conditions given in Equations (29.1-44) or (29.1-45) applies. These boundary conditions apply to both state A and state B, and are, therefore, time reverse self-adjoint.

To establish the desired reciprocity relation, we first apply the complex frequency-domain reciprocity theorem of the time correlation type Equation (28.5-7) to the total wave fields in the states A and B, and to the domain $\mathcal{D}^{\rm s}$ occupied by the scatterer. For a penetrable scatterer this yields

$$\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\rm s}} \nu_m \left[\hat{E}_r^{\rm A}(x,s)\hat{H}_p^{\rm B}(x,-s) + \hat{E}_r^{\rm B}(x,-s)\hat{H}_p^{\rm A}(x,s)\right] {\rm d}A = 0 \,, \tag{29.3-50}$$

since in the interior of the scatterer the total wave field is source-free. For an impenetrable scatterer, Equation (29.3-50) holds in view of the boundary conditions upon approaching $\partial\mathcal{D}^{\rm s}$ via $\mathcal{D}^{\rm s'}$. In Equation (29.3-50) we substitute

$$\{\hat{E}_r^{\rm A},\hat{H}_p^{\rm A}\} = \{\hat{E}_r^{\rm i;A} + \hat{E}_r^{\rm s;A},\hat{H}_p^{\rm i;A} + \hat{H}_p^{\rm s;A}\} \tag{29.3-51}$$

and

$$\{\hat{E}_r^{\rm B},\hat{H}_p^{\rm B}\} = \{\hat{E}_r^{\rm i;B} + \hat{E}_r^{\rm s;B},\hat{H}_p^{\rm i;B} + \hat{H}_p^{\rm s;B}\} \,. \tag{29.3-52}$$

Next, the complex frequency-domain reciprocity theorem of the time correlation type is applied to the incident wave field and to the domain $\mathcal{D}^{\rm s}$. Since the incident wave field is source-free in the interior of the scatterer and the embedding is time reverse self-adjoint in its electromagnetic properties, this leads to

$$\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\rm s}} \nu_m \left[\hat{E}_r^{\rm i;A}(x,s)\hat{H}_p^{\rm i;B}(x,-s) + \hat{E}_r^{\rm i;B}(x,-s)\hat{H}_p^{\rm i;A}(x,s)\right] {\rm d}A = 0 \,. \tag{29.3-53}$$

Finally, the complex frequency-domain reciprocity theorem of the time correlation type is applied to the scattered wave field and to the domain $\mathcal{D}^{\rm s'}$. Since the embedding is time reverse self-adjoint in its electromagnetic properties and the scattered wave field is source-free in the exterior of the scatterer and satisfies the condition of causality, this leads to

$$\begin{aligned} &\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\rm s}} \nu_m \left[\hat{E}_r^{\rm s;A}(x,s)\hat{H}_p^{\rm s;B}(x,-s) + \hat{E}_r^{\rm s;B}(x,-s)\hat{H}_p^{\rm s;A}(x,s)\right] {\rm d}A \\ &= \lim_{\Delta\to\infty} \varepsilon_{m,r,p} \int_{x \in \mathcal{S}(\mathcal{O},\Delta)} \nu_m \left[\hat{E}_r^{\rm s;A}(x,s)\hat{H}_p^{\rm s;B}(x,-s) + \hat{E}_r^{\rm s;B}(x,-s)\hat{H}_p^{\rm s;A}(x,s)\right] {\rm d}A \,, \end{aligned} \tag{29.3-54}$$

where $\mathcal{S}(\mathcal{O},\Delta)$ is the sphere of radius Δ with centre at the origin $\mathcal{O}$ of the chosen reference frame. From Equations (29.3-50)–(29.3-54) we conclude that

$$\begin{aligned} &\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\rm s}} \nu_m \Big[\hat{E}_r^{\rm i;A}(x,s)\hat{H}_p^{\rm s;B})(x,-s) + \hat{E}_r^{\rm s;A}(x,s)\hat{H}_p^{\rm i;B}(x,-s) \\ &\qquad + \hat{E}_r^{\rm i;B}(x,-s)\hat{H}_p^{\rm s;A}(x,s) + \hat{E}_r^{\rm s;B}(x,-s)\hat{H}_p^{\rm i;A}(x,s)\Big] {\rm d}A \\ &+ \lim_{\Delta\to\infty} \varepsilon_{m,r,p} \int_{x \in \mathcal{S}(\mathcal{O},\Delta)} \nu_m \Big[\hat{E}_r^{\rm s;A}(x,s)\hat{H}_p^{\rm s;B}(x,-s) \\ &\qquad + \hat{E}_r^{\rm s;B}(x,-s)\hat{H}_p^{\rm s;A}(x,s)\Big] {\rm d}A = 0 \,. \end{aligned} \tag{29.3-55}$$

However, on account of Equations (29.3-41) and (29.3-42), and (29.3-35)–(29.3-40) we have

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{\mathrm{s;A}}(x,s)\hat{H}_p^{\mathrm{i;B}}(x,-s)+\hat{E}_r^{\mathrm{i;B}}(x,-s)\hat{H}_p^{\mathrm{s;A}}(x,s)\right]\mathrm{d}A$$

$$=\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{\mathrm{s;A}}(x,s)h_p^{\mathrm{B}}+e_r^{\mathrm{B}}\hat{H}_p^{\mathrm{s;A}}(x,s)\right]\hat{b}(-s)\exp(s\beta_s x_s/c)\,\mathrm{d}A$$

$$=(s\mu)^{-1}e_r^{\mathrm{B}}\hat{b}(-s)\hat{E}_r^{\mathrm{s;A;\infty}}(\beta,s) \tag{29.3-56}$$

and on account of Equations (29.3-32) and (29.3-33), and (29.3-44)–(29.3-49)

$$\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{\mathrm{s;B}}(x,-s)\hat{H}_p^{\mathrm{i;A}}(x,s)+\hat{E}_r^{\mathrm{i;A}}(x,s)\hat{H}_p^{\mathrm{s;B}}(x,-s)\right]\mathrm{d}A$$

$$=\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^s}\nu_m\left[\hat{E}_r^{\mathrm{s;B}}(x,-s)h_p^{\mathrm{A}}+e_r^{\mathrm{A}}\hat{H}_p^{\mathrm{s;B}}(x,-s)\right]\hat{a}(s)\exp(-s\alpha_s x_s/c)\,\mathrm{d}A$$

$$=(s\mu)^{-1}e_r^{\mathrm{A}}\hat{a}(s)\hat{E}_r^{\mathrm{s;B;\infty}}(\alpha,-s)\,. \tag{29.3-57}$$

Furthermore, we have

$$\lim_{\Delta\to\infty}\varepsilon_{m,r,p}\int_{x\in\mathcal{S}(O,\Delta)}\nu_m\left[\hat{E}_r^{\mathrm{s;A}}(x,s)\hat{H}_p^{\mathrm{s;B}}(x,-s)+\hat{E}_r^{\mathrm{s;B}}(x,-s)\hat{H}_p^{\mathrm{s;A}}(x,s)\right]\mathrm{d}A$$

$$=(4\pi)^{-2}\,\varepsilon_{m,r,p}\int_{\xi\in\Omega}\xi_m\left[\hat{E}_r^{\mathrm{s;A;\infty}}(\xi,s)\hat{H}_p^{\mathrm{s;B;\infty}}(\xi,-s)+\hat{E}_r^{\mathrm{s;B;\infty}}(\xi,-s)\hat{H}_p^{\mathrm{s;A;\infty}}(\xi,s)\right]\mathrm{d}A$$

$$=(8\pi^2)^{-1}(\mu c)^{-1}\int_{\xi\in\Omega}\hat{E}_r^{\mathrm{s;A;\infty}}(\xi,s)\hat{E}_r^{\mathrm{s;B;\infty}}(\xi,-s)\,\mathrm{d}A\,, \tag{29.3-58}$$

where Ω is the sphere of unit radius and centre at O. Equations (29.3-55)–(29.3-58) lead to the desired reciprocity relation for the far-field scattered wave amplitudes:

$$e_r^{\mathrm{B}}\hat{b}(-s)\hat{E}_r^{\mathrm{s;A;\infty}}(\beta,s)+e_r^{\mathrm{A}}\hat{a}(s)\hat{E}_r^{\mathrm{s;B;\infty}}(\alpha,-s)$$

$$=-(8\pi^2)^{-1}(s/c)\int_{\xi\in\Omega}\hat{E}_r^{\mathrm{s;A;\infty}}(\xi,s)\hat{E}_r^{\mathrm{s;B;\infty}}(\xi,-s)\,\mathrm{d}A\,. \tag{29.3-59}$$

At this point it is, again, elegant to express the linear relationship that exists between the far-field scattered wave amplitude and the incident wave field, both in state A and state B. Substitution of Equations (29.2-59) and (29.2-60) in Equation (29.3-59) yields

$$\hat{S}_{r,k}^{\mathrm{A}}(\beta,\alpha,s)e_r^{\mathrm{B}}e_k^{\mathrm{A}}\hat{b}(-s)\hat{a}(s)+\hat{S}_{k,r}^{\mathrm{B}}(\alpha,\beta,-s)e_k^{\mathrm{A}}e_r^{\mathrm{B}}\hat{a}(s)\hat{b}(-s)$$

$$=-(8\pi^2)^{-1}(s/c)e_k^{\mathrm{A}}e_r^{\mathrm{B}}\hat{a}(s)\hat{b}(-s)\int_{\xi\in\Omega}\hat{S}_{r',k}^{\mathrm{A}}(\xi,\alpha,s)\hat{S}_{r',r}^{\mathrm{B}}(\xi,\beta,-s)\,\mathrm{d}A\,. \tag{29.3-60}$$

Taking into account that Equation (29.3-60) has to hold for arbitrary values of e_k^{A}, e_r^{B}, $\hat{a}(s)$ and $\hat{b}(-s)$, we end up with

$$\hat{S}^{A}_{r,k}(\beta,\alpha,s) + \hat{S}^{B}_{k,r}(\alpha,\beta,-s) = -(s/8\pi^2 c) \int_{\xi \in \Omega} \hat{S}^{A}_{r',k}(\xi,\alpha,s)\hat{S}^{B}_{r',r}(\xi,\beta,-s)\,\mathrm{d}A \tag{29.3-61}$$

as the final expression of the complex frequency-domain reciprocity property under consideration.

In a theoretical analysis the reciprocity relations derived in this section serve as an important check on the correctness of the analytic solutions as well as on the accuracy of numerical solutions to scattering problems. Note, however, that the reciprocity relations are necessary conditions to be satisfied by the scattered wave field (in the far-field region), but their satisfaction does not guarantee correctness of a total analytic solution or a certain accuracy of a total numerical solution. In a physical experiment, the redundancy induced by the reciprocity relations can be exploited to reduce the influence of noise on the quality of the observed data.

Exercises

Exercise 29.3-1

Show that Equation (29.3-61) follows from Equation (29.3-27) by taking the time Laplace transform.

29.4 An energy theorem about the far-field forward scattered wave amplitude

A special case arises when in the reciprocity relations of the time correlation type derived in Section 29.3, state A and state B are taken to be identical states. Since the superscripts A and B are then superfluous, they will be omitted in the present section.

Time-domain version of the energy theorem

In the time-domain version of the theorem we start from Equation (29.3-19), take state A identical to state B, and consider the result at zero correlation time shift. Furthermore, for the case of an electromagnetically penetrable scatterer, the medium occupying the scattering domain $\mathcal{D}^{\mathrm{s}}$ is no longer assumed to be time reverse self-adjoint, i.e. it may have non-zero electromagnetic losses. Thus, we are led to consider the expression

$$\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \mathrm{C}_t(E_r, \mathrm{J}_t(H_p); x, 0)\,\mathrm{d}A = \varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \mathrm{C}_t(\mathrm{J}_t(E_r), H_p; x, 0)\,\mathrm{d}A$$

$$= \int_{t' \in \mathcal{R}} \mathrm{d}t'\, \varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m E_r(x,t') H_p(x,t')\,\mathrm{d}A = -W^{\mathrm{a}}, \tag{29.4-1}$$

where

$$W^{\mathrm{a}} = \int_{t' \in \mathcal{R}} P^{\mathrm{a}}(t')\mathrm{d}t' \tag{29.4-2}$$

is the *total electromagnetic energy absorbed by the scatterer* and

$$P^{\mathrm{a}}(t') = -\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m E_r(x,t') H_p(x,t')\, \mathrm{d}A \tag{29.4-3}$$

is the instantaneous electromagnetic power absorbed by the scatterer. (Note that the minus sign in front of the integral sign on the right-hand side of Equation (29.4-3) is due to the fact that power absorption by the scatterer is effected by an inward power flow, while ν_m points away from the scatterer.)

Next, we substitute in the right-hand side of Equation (29.4-3) the relation

$$\{E_r, H_p\} = \{E_r^{\mathrm{i}} + E_r^{\mathrm{s}}, H_p^{\mathrm{i}} + H_p^{\mathrm{s}}\}\,, \tag{29.4-4}$$

and observe that the incident wave dissipates no net energy upon traversing the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer, which domain has then the electromagnetic medium properties of the lossless embedding. Hence, with

$$P^{\mathrm{i}}(t') = -\varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m E_r^{\mathrm{i}}(x,t') H_p^{\mathrm{i}}(x,t')\, \mathrm{d}A \tag{29.4-5}$$

as the instantaneous electromagnetic power that the incident wave carries across $\partial\mathcal{D}^{\mathrm{s}}$ towards the domain $\mathcal{D}^{\mathrm{s}}$, we have

$$W^{\mathrm{i}} = \int_{t' \in \mathcal{R}} P^{\mathrm{i}}(t')\mathrm{d}t' = 0\,. \tag{29.4-6}$$

Furthermore, with the uniform incident plane wave

$$\{E_r^{\mathrm{i}}, H_p^{\mathrm{i}}\}(x,t) = \{e_r, h_p\} a(t - \alpha_s x_s / c)\,, \tag{29.4-7}$$

for which

$$h_p = Y \varepsilon_{p,m,r} \alpha_m e_r\,, \tag{29.4-8}$$

we have, upon using Equations (29.3-4)–(29.3-9),

$$\int_{t' \in \mathcal{R}} \mathrm{d}t'\, \varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m \left[E_r^{\mathrm{i}}(x,t') H_p^{\mathrm{s}}(x,t') + E_r^{\mathrm{s}}(x,t') H_p^{\mathrm{i}}(x,t')\right] \mathrm{d}A$$

$$= \mu^{-1} e_r \int_{t' \in \mathcal{R}} a(t') \mathrm{I}_t E_r^{\mathrm{s};\infty}(\alpha,t')\, \mathrm{d}t'. \tag{29.4-9}$$

Finally, the *total electromagnetic energy carried by the scattered wave* across $\partial\mathcal{D}^{\mathrm{s}}$ towards the embedding is introduced as

$$W^{\mathrm{s}} = \int_{t' \in \mathcal{R}} P^{\mathrm{s}}(t')\mathrm{d}t', \tag{29.4-10}$$

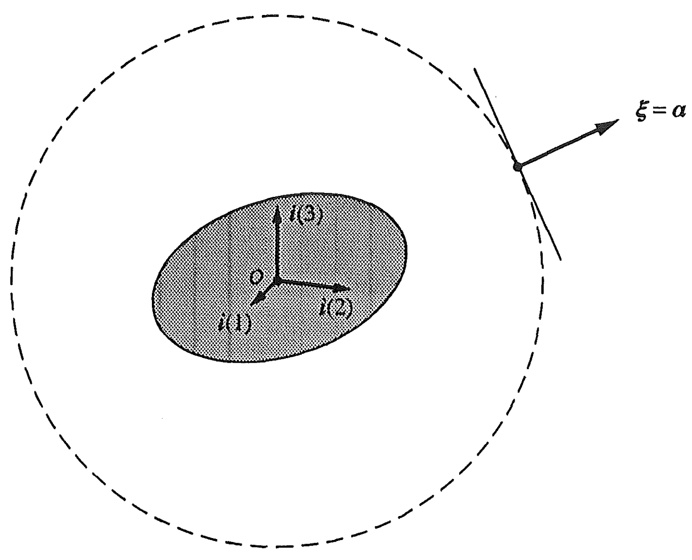

Figure 29.4-1 Electromagnetic scattering configuration for the time-domain energy theorem about the far-field forward scattered wave amplitude.

where

$$P^{\mathrm{s}}(t') = \varepsilon_{m,r,p} \int_{x \in \partial\mathcal{D}^{\mathrm{s}}} \nu_m E_r^{\mathrm{s}}(x,t') H_p^{\mathrm{s}}(x,t')\,\mathrm{d}A \tag{29.4-11}$$

is the instantaneous electromagnetic power that the scattered wave carries across $\partial\mathcal{D}^{\mathrm{s}}$ towards the embedding.

Combining Equations (29.4-1)–(29.4-6) and (29.4-9)–(29.4-11) we end up with

$$W^{\mathrm{a}} + W^{\mathrm{s}} = -\mu^{-1} e_r \int_{t' \in \mathcal{R}} a(t') \mathrm{I}_t E_r^{\mathrm{s};\infty}(\alpha,t')\,\mathrm{d}t'. \tag{29.4-12}$$

Equation (29.4-12) is the desired *time-domain energy relation.* It relates the sum of the electromagnetic energies absorbed and scattered by the object to the scattered wave amplitude in the far-field region, for observation of this wave in the direction $\xi = \alpha$ of propagation of the incident plane wave, i.e. in the "forward" direction, or "behind" the scatterer (Figure 29.4-1).

It is noted that for a lossless electromagnetically penetrable scatterer we have $W^{\mathrm{a}} = 0$. Also, $W^{\mathrm{a}} = 0$ for an impenetrable scatterer, since the right-hand side of Equation (29.4-3) then vanishes in view of the pertaining boundary conditions (Equation (29.1-3) or Equation (29.1-4)). Note also that in the derivation of the result we have nowhere used the linearity in the electromagnetic behaviour of the scatterer. Therefore, Equation (29.4-12) also holds for non-linear electromagnetic scatterers, subject to the condition, of course, that the embedding retains its linear properties.

Complex frequency-domain version of the energy theorem

In the complex frequency-domain version of the theorem we start from Equation (29.3-50) and take state A identical to state B. Furthermore, for the case of an electromagnetically penetrable scatterer the medium occupying the scattering domain $\mathcal{D}^{\mathrm{s}}$ is no longer assumed to be time reverse self-adjoint, i.e. it may have non-zero electromagnetic losses. Thus, we are led to consider the expression

$$\frac{1}{4}\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\,[\hat{E}_r(x,s)\hat{H}_p(x,-s)+\hat{E}_r(x,-s)\hat{H}_p(x,s)]\,\mathrm{d}A=-\hat{P}^{\mathrm{a}}(s)\,, \tag{29.4-13}$$

where the symbol on the right-hand side has been chosen for reasons of equivalence with Equation (29.4-3) and the factor (¼) has been included because of the equivalence with the time-averaged power flow over a period for sinusoidally in time varying wave fields. It must be stressed, however, that $\hat{P}^{\mathrm{a}}(s)$ is *not* the time Laplace transform of $P^{\mathrm{a}}(t)$.

In the left-hand side of Equation (29.4-13) we now substitute the relation

$$\{\hat{E}_r,\hat{H}_p\}=\{\hat{E}_r^{\mathrm{i}}+\hat{E}_r^{\mathrm{s}},\hat{H}_p^{\mathrm{i}}+\hat{H}_p^{\mathrm{s}}\}\,, \tag{29.4-14}$$

and observe that

$$\frac{1}{4}\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\hat{E}_r^{\mathrm{i}}(x,s)\hat{H}_p^{\mathrm{i}}(x,-s)+\hat{E}_r^{\mathrm{i}}(x,-s)\hat{H}_p^{\mathrm{i}}(x,s)\right]\mathrm{d}A=0\,, \tag{29.4-15}$$

since the medium in the embedding has been assumed to be time reverse self-adjoint.

Furthermore, with the uniform incident plane wave

$$\{\hat{E}_r^{\mathrm{i}},\hat{H}_p^{\mathrm{i}}\}(x,s)=\{e_r,h_p\}\hat{a}(s)\exp(-s\alpha_s x_s/c)\,, \tag{29.4-16}$$

for which

$$h_p=Y\varepsilon_{p,m,r}\alpha_m e_r\,, \tag{29.4-17}$$

we have, upon using Equations (29.3-35)–(29.3-40),

$$\frac{1}{4}\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\hat{E}_r^{\mathrm{i}}(x,s)\hat{H}_p^{\mathrm{s}}(x,-s)+\hat{E}_r^{\mathrm{s}}(x,-s)\hat{H}_p^{\mathrm{i}}(x,s)\right]\mathrm{d}A$$

$$=\frac{1}{4}\,(s\mu)^{-1}e_r\hat{a}(s)\hat{E}_r^{\mathrm{s};\infty}(\alpha,-s)\,, \tag{29.4-18}$$

with a similar result for s replaced by $-s$. Finally, we introduce, by analogy with Equation (29.4-10), the quantity

$$\hat{P}^{\mathrm{s}}(s)=\frac{1}{4}\,\varepsilon_{m,r,p}\int_{x\in\partial\mathcal{D}^{\mathrm{s}}} \nu_m\left[\hat{E}_r^{\mathrm{s}}(x,s)\hat{H}_p^{\mathrm{s}}(x,-s)+\hat{E}_r^{\mathrm{s}}(x,-s)\hat{H}_p^{\mathrm{s}}(x,s)\right]\mathrm{d}A \tag{29.4-19}$$

that is associated with the electromagnetic power carried by the scattered wave.

Combining Equations (29.4-13)–(29.4-15) and (29.4-18) and (29.4-19), we end up with

$$\hat{P}^{\mathrm{a}}(s)+\hat{P}^{\mathrm{s}}(s)=-\frac{1}{4}\,(s\mu)^{-1}e_r\left[\hat{a}(s)\hat{E}_r^{\mathrm{s};\infty}(\alpha,-s)-\hat{a}(-s)\hat{E}_r^{\mathrm{s};\infty}(\alpha,s)\right]. \tag{29.4-20}$$

Equation (29.4-20) is the desired complex frequency-domain energy relation. It relates the sum of the quantities $\hat{P}^{a}(s)$ and $\hat{P}^{s}(s)$ to the scattered wave amplitude in the far-field region for observation of this wave in the direction of propagation of the incident plane wave, i.e. in the "forward" direction, or "behind" the scatterer.

It is noted that for a lossless electromagnetically penetrable scatterer we have $\hat{P}^{a} = 0$. Also, $\hat{P}^{a} = 0$ for an impenetrable scatterer, since the right-hand side of Equations (29.4-13) vanishes in view of the pertaining boundary conditions (Equations (29.1-44) or Equation (29.1-45)). For imaginary values of s, i.e. for $s = \mathrm{j}\omega$, with $\omega \in \mathcal{R}$,, Equation (29.4-20) is known as the "extinction cross-section theorem". Note that in the complex frequency-domain result (contrary to the corresponding time-domain result) the linearity in the electromagnetic behaviour of the scatterer has implicitly been used since the space–time wave quantities have been represented, through the Bromwich integral, as a (linear) superposition of exponential time functions.

References to the earlier literature on the subject can be found in De Hoop (1984) and De Hoop (1959).

Exercises

Exercise 29.4-1

Consider, in the complex frequency-domain energy relation Equation (29.4-20), the case $s = \mathrm{j}\omega$. Observe that the quantities $\hat{P}^{a}(s)$ as introduced in Equation (29.4-13) and $\hat{P}^{s}(s)$ as introduced in Equation (29.4-19) have the property $\hat{P}^{a}(s) = \hat{P}^{a}(-s)$ and $\hat{P}^{s}(s) = \hat{P}^{s}(-s)$ in the common domain of regularity of both the left-hand and the right-hand sides. As a consequence, $\hat{P}^{a}(\mathrm{j}\omega)$ and $\hat{P}^{s}(\mathrm{j}\omega)$ are real-valued for $\omega \in \mathcal{R}$. Next, introduce the quantity

$$\hat{S}^{i}(s) = \frac{1}{2} c\varepsilon e_r e_r \hat{a}(s)\hat{a}(-s) \tag{29.4-21}$$

that is associated with the electromagnetic power flow density in the incident wave. Also, $\hat{S}^{i}(s) = \hat{S}^{i}(-s)$ in the common domain of regularity of both the left-hand and the right-hand sides, and hence, also $\hat{S}^{i}(\mathrm{j}\omega)$ is real-valued for $\omega \in \mathcal{R}$. Furthermore, let

$$\hat{\sigma}^{a}(s) = \hat{P}^{a}(s)/\hat{S}^{i}(s) \tag{29.4-22}$$

denote the complex frequency-domain *absorption cross-section* of the scattering object and

$$\hat{\sigma}^{s}(s) = \hat{P}^{s}(s)/\hat{S}^{i}(s) \tag{29.4-23}$$

its *scattering cross-section.* Note that $\hat{\sigma}^{a}(s) = \hat{\sigma}^{a}(-s)$ and $\hat{\sigma}^{s}(s) = \hat{\sigma}^{s}(-s)$ in the common domain of regularity of both the left-hand and the right-hand sides, which entails that $\hat{\sigma}^{a}(\mathrm{j}\omega)$ and $\hat{\sigma}^{s}(\mathrm{j}\omega)$ are real-valued for $\omega \in \mathcal{R}$. Show that, for $s = \mathrm{j}\omega$, with $\omega \in \mathcal{R}$, Equation (29.4-20) leads to

$$\hat{\sigma}^{a}(\mathrm{j}\omega) + \hat{\sigma}^{s}(\mathrm{j}\omega) = \frac{c}{\omega}\,\frac{\mathrm{Im}\left[e_r \hat{a}(-\mathrm{j}\omega)\hat{E}_r^{s;\infty}(\boldsymbol{\alpha},\mathrm{j}\omega)\right]}{e_r e_r |\hat{a}(\mathrm{j}\omega)|^2}\,. \tag{29.4-24}$$

Equation (29.4-24) is known as the *extinction cross-section theorem* (De Hoop 1959). (*Note*: Extinction cross-section = absorption cross-section + scattering cross-section.)

29.5 The Neumann expansion in the integral equation formulation of the scattering by a penetrable object

In this section we discuss the Neumann expansion in the integral equation formulation of the scattering problem. The expansion is an analytic procedure that applies to a *penetrable scatterer*. The procedure is *iterative* in nature and is expected to converge for sufficiently low contrast of the scatterer with respect to its embedding.

Time-domain analysis

In the time-domain presentation of the method we start from Equations (28.9-5) and (28.9-20)–(28.9-23), which, through combination of the time convolutions, we write for the present configuration as

$$E_r(\boldsymbol{x}',t) = E_r^{\rm i}(\boldsymbol{x}',t) + \int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big[\partial_t C_t(G^{EJ}_{r,k},\varepsilon^{\rm s}_{k,r'} - \varepsilon\delta(t)\delta_{k,r'},E_{r'};\boldsymbol{x}',\boldsymbol{x},t)$$
$$+\,\partial_t C_t(G^{EK}_{r,j},\mu^{\rm s}_{j,p'} - \mu\delta(t)\delta_{j,p'},H_{p'};\boldsymbol{x}',\boldsymbol{x},t)\Big]\,{\rm d}V \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3, \qquad (29.5\text{-}1)$$

and

$$H_p(\boldsymbol{x}',t) = H_p^{\rm i}(\boldsymbol{x}',t) + \int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big[\partial_t C_t(G^{HJ}_{p,k},\varepsilon^{\rm s}_{k,r'} - \varepsilon\delta(t)\delta_{k,r'},E_{r'};\boldsymbol{x}',\boldsymbol{x},t)$$
$$+\,\partial_t C_t(G^{HK}_{p,j},\mu^{\rm s}_{j,p'} - \mu\delta(t)\delta_{j,p'},H_{p'};\boldsymbol{x}',\boldsymbol{x},t)\Big]\,{\rm d}V \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3. \qquad (29.5\text{-}2)$$

For $\boldsymbol{x}'\in\mathcal{D}^{\rm s}$, Equations (29.5-1) and (29.5-2) constitute a system of linear integral equations of the second kind to be solved for $\{E_r,H_p\}$ for $\boldsymbol{x}\in\mathcal{D}^{\rm s}$ and $t\in\mathcal{R}$, and with $\{E_r^{\rm i},H_p^{\rm i}\}$ as forcing terms. To solve these equations analytically, an iterative procedure, known as the *Neumann expansion* is set up. The successive steps in this procedure will be labelled by integer superscripts enclosed by brackets ([...]). The procedure is *initialised* by putting

$$E_r^{[0]} = E_r^{\rm i} \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3, \qquad (29.5\text{-}3)$$

$$H_p^{[0]} = H_p^{\rm i} \quad \text{for } \boldsymbol{x}'\in\mathcal{R}^3. \qquad (29.5\text{-}4)$$

Next, the procedure is *updated* through

$$E_r^{[n+1]}(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^{\rm s}} \Big[\partial_t C_t(G^{EJ}_{r,k},\varepsilon^{\rm s}_{k,r'} - \varepsilon\delta(t)\delta_{k,r'},E^{[n]}_{r'};\boldsymbol{x}',\boldsymbol{x},t)$$
$$+\,\partial_t C_t(G^{EK}_{r,j},\mu^{\rm s}_{j,p'} - \mu\delta(t)\delta_{j,p'},H^{[n]}_{p'};\boldsymbol{x}',\boldsymbol{x},t)\Big]\,{\rm d}V$$
$$\text{for } \boldsymbol{x}'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{etc.} \qquad (29.5\text{-}5)$$

and

$$H_p^{[n+1]}(x',t) = \int_{x\in\mathcal{D}^s} \left[\partial_t \mathrm{C}_t(G_{p,k}^{HJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, E_{r'}^{[n]}; x', x, t)\right.$$
$$\left. + \partial_t \mathrm{C}_t(G_{p,j}^{HK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, H_{p'}^{[n]}; x', x, t)\right] \mathrm{d}V$$
$$\text{for } x'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{ etc.} \tag{29.5-6}$$

As can be inferred from these updating equations, the terms of order $[n + 1]$ can be expected to be "smaller" than their counterparts of order $[n]$, provided that the contrast quantities are "small enough". On account of this, it can be conjectured that for sufficiently small contrast of the scatterer with respect to its embedding the *procedure is convergent* and we can put

$$E_r = \sum_{n=0}^{\infty} E_r^{[n]} \qquad \text{for } x'\in\mathcal{R}^3, \tag{29.5-7}$$

$$H_p = \sum_{n=0}^{\infty} H_p^{[n]} \qquad \text{for } x'\in\mathcal{R}^3. \tag{29.5-8}$$

Assuming that the series on the right-hand sides of Equations (29.5-7) and (29.5-8) are uniformly convergent, it can easily be proved that $\{E_r, H_p\}$ as defined by these equations indeed satisfy Equations (29.5-1) and (29.5-2). To this end we observe that

$$\int_{x\in\mathcal{D}^s} \left[\partial_t \mathrm{C}_t(G_{r,k}^{EJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, E_{r'}; x', x, t) + \partial_t \mathrm{C}_t(G_{r,j}^{EK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, H_{p'}; x', x, t)\right] \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^s} \left[\partial_t \mathrm{C}_t(G_{r,k}^{EJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, \sum_{n=0}^{\infty} E_{r'}^{[n]}; x', x, t)\right.$$
$$\left. + \partial_t \mathrm{C}_t(G_{r,j}^{EK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, \sum_{n=0}^{\infty} H_{p'}^{[n]}; x', x, t)\right] \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \int_{x\in\mathcal{D}^s} \left[\partial_t \mathrm{C}_t(G_{r,k}^{EJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, E_{r'}^{[n]}; x', x, t)\right.$$
$$\left. + \partial_t \mathrm{C}_t(G_{r,j}^{EK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, H_{p'}^{[n]}; x', x, t)\right] \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} E_r^{[n+1]}(x',t) = \sum_{m=0}^{\infty} E_r^{[m]}(x',t) - E_r^{[0]}(x',t) = E_r(x',t) - E_r^{\mathrm{i}}(x,t) \quad \text{for } x'\in\mathcal{R}^3, \tag{29.5-9}$$

and

$$\int_{x\in\mathcal{D}^s} \left[\partial_t \mathrm{C}_t(G_{p,k}^{HJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, E_{r'}; x', x, t) + \partial_t \mathrm{C}_t(G_{p,j}^{HK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, H_{p'}; x', x, t)\right] \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^s} \left[\partial_t \mathrm{C}_t(G_{p,k}^{HJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, \sum_{n=0}^{\infty} E_{r'}^{[n]}; x', x, t)\right.$$
$$\left. + \partial_t \mathrm{C}_t(G_{p,j}^{HK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, \sum_{n=0}^{\infty} H_{p'}^{[n]}; x', x, t)\right] \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \int_{x \in \mathcal{D}^s} \left[\partial_t C_t(G_{p,k}^{HJ}, \varepsilon_{k,r'}^{s} - \varepsilon\delta(t)\delta_{k,r'}, E_{r'}^{[n]}; x', x, t) \right.$$

$$\left. + \partial_t C_t(G_{p,j}^{HK}, \mu_{j,p'}^{s} - \mu\delta(t)\delta_{j,p'}, H_{p'}^{[n]}; x', x, t) \right] \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} H_p^{[n+1]}(x',t) = \sum_{m=0}^{\infty} H_p^{[m]}(x',t) - H_p^{[0]}(x',t) = H_p(x',t) - H_p^{\mathrm{i}}(x,t) \quad \text{for } x' \in \mathcal{R}^3, \quad (29.5\text{-}10)$$

where Equations (29.5-3)–(29.5-8) have been used and the interchange of the summations with respect to n and the integrations with respect to x is justified by the assumed uniform convergence of the series expansions. Equations (29.5-9) and (29.5-10) are evidently identical to Equations (29.5-1) and (29.5-2), and, hence, the expansions given in Equations (29.5-7) and (29.5-8) indeed solve the problem.

Complex frequency-domain analysis

In the complex frequency-domain presentation of the method we start from Equations (28.9-28) and (28.9-43)–(28.9-46), which are combined to

$$\hat{E}_r(x',s) = \hat{E}_r^{\mathrm{i}}(x',s) \int_{x \in \mathcal{D}^s} \left\{ \hat{G}_{r,k}^{EJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'} \right] \hat{E}_{r'}(x,s) \right.$$

$$\left. + \hat{G}_{r,j}^{EK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'} \right] \hat{H}_{p'}(x,s) \right\} \mathrm{d}V \quad \text{for } x' \in \mathcal{R}^3, \quad (29.5\text{-}11)$$

and

$$\hat{H}_p(x',s) = \hat{H}_p^{\mathrm{i}}(x,s) \int_{x \in \mathcal{D}^s} \left\{ \hat{G}_{p,k}^{HJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'} \right] \hat{E}_{r'}(x,s) \right.$$

$$\left. + \hat{G}_{p,j}^{HK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'} \right] \hat{H}_{p'}(x,s) \right\} \mathrm{d}V \quad \text{for } x' \in \mathcal{R}^3. \quad (29.5\text{-}12)$$

For $x' \in \mathcal{D}^s$, Equations (29.5-11) and (29.5-12) constitute a system of linear integral equations of the second kind to be solved for $\{\hat{E}_r, \hat{H}_p\}$ for $x \in \mathcal{D}^s$, and with $\{\hat{E}_r^{\mathrm{i}}, \hat{H}_p^{\mathrm{i}}\}$ as forcing terms. The Neumann procedure to solve these equations is *initialised* by putting

$$\hat{E}_r^{[0]} = \hat{E}_r^{\mathrm{i}} \qquad \text{for } x' \in \mathcal{R}^3, \qquad (29.5\text{-}13)$$

$$\hat{H}_p^{[0]} = \hat{H}_p^{\mathrm{i}} \qquad \text{for } x' \in \mathcal{R}^3. \qquad (29.5\text{-}14)$$

Next, the procedure is *updated* through

$$\hat{E}_r^{[n+1]}(x',s) = \int_{x \in \mathcal{D}^s} \left\{ \hat{G}_{r,k}^{EJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'} \right] \hat{E}_{r'}^{[n]}(x,s) \right.$$

$$\left. + \hat{G}_{r,j}^{EK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'} \right] \hat{H}_{p'}^{[n]}(x,s) \right\} \mathrm{d}V$$

$$\text{for } x' \in \mathcal{R}^3 \text{ and } n = 0,1,2, \text{ etc.} \qquad (29.5\text{-}15)$$

and

$$\hat{H}_p^{[n+1]}(x',s) = \int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{p,k}^{HJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \hat{E}_{r'}^{[n]}(x,s)$$

$$+ \hat{G}_{p,j}^{HK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'}\right] \hat{H}_{p'}^{[n]}(x,s) \Big\}\, \mathrm{d}V$$

$$\text{for } x'\in\mathcal{R}^3 \text{ and } n = 0,1,2, \text{etc.} \tag{29.5-16}$$

Assuming that the *procedure is convergent*, we can put

$$\hat{E}_r = \sum_{n=0}^{\infty} \hat{E}_r^{[n]} \qquad \text{for } x'\in\mathcal{R}^3, \tag{29.5-17}$$

$$\hat{H}_p = \sum_{n=0}^{\infty} \hat{H}_p^{[n]} \qquad \text{for } x'\in\mathcal{R}^3. \tag{29.5-18}$$

Assuming that the series on the right-hand sides of Equations (29.5-17) and (29.5-18) are uniformly convergent, it can easily be proved that $\{\hat{E}_r, \hat{H}_p\}$ as defined by these equations indeed satisfy Equations (29.5-11) and (29.5-12). To this end we observe that

$$\int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{r,k}^{EJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \hat{E}_{r'}(x,s) + \hat{G}_{r,j}^{EK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'}\right] \hat{H}_{p'}(x,s) \Big\}\, \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{r,k}^{EJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \sum_{n=0}^{\infty} \hat{E}_{r'}^{[n]}(x,s)$$

$$+ \hat{G}_{r,j}^{EK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'}\right] \sum_{n=0}^{\infty} \hat{H}_{p'}^{[n]}(x,s) \Big\}\, \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{r,k}^{EJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \hat{E}_{r'}^{[n]}(x,s)$$

$$+ \hat{G}_{r,j}^{EK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'}\right] \hat{H}_{p'}^{[n]}(x,s) \Big\}\, \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \hat{E}_r^{[n+1]}(x,s) = \sum_{m=0}^{\infty} \hat{E}_r^{[m]}(x,s) - \hat{E}_r^{[0]}(x,s) = \hat{E}_r(x,s) - \hat{E}_r^{\mathrm{i}}(x,s) \quad \text{for } x'\in\mathcal{R}^3 \tag{29.5-19}$$

and

$$\int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{p,k}^{HJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \hat{E}_{r'}(x,s) + \hat{G}_{p,j}^{HK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'}\right] \hat{H}_{p'}(x,s) \Big\}\, \mathrm{d}V$$

$$= \int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{p,k}^{HJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \sum_{m=0}^{\infty} \hat{E}_{r'}^{[n]}(x,s)$$

$$+ \hat{G}_{p,j}^{HK}(x',x,s) \left[\hat{\zeta}_{j,p'}^{s}(x,s) - s\mu\delta_{j,p'}\right] \sum_{m=0}^{\infty} \hat{H}_{p'}^{[n]}(x,s) \Big\}\, \mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \int_{x\in\mathcal{D}^s} \Big\{ \hat{G}_{p,k}^{HJ}(x',x,s) \left[\hat{\eta}_{k,r'}^{s}(x,s) - s\varepsilon\delta_{k,r'}\right] \hat{E}_{r'}^{[n]}(x,s)$$

$$+ \hat{G}^{HK}_{p,j}(x',x,s)\left[\hat{\xi}^{s}_{j,p'}(x,s) - s\mu\delta_{j,p'}\right]\hat{H}^{[n]}_{p'}(x,s)\Big\}\,\mathrm{d}V$$

$$= \sum_{n=0}^{\infty} \hat{H}^{[n+1]}_{p}(x,s) = \sum_{m=0}^{\infty} \hat{H}^{[m]}_{p}(x,s) - \hat{H}^{[0]}_{p}(x,s) = \hat{H}_{p}(x,s) - \hat{H}^{i}_{p}(x,s) \quad \text{for } x' \in \mathcal{R}^3, \qquad (29.5\text{-}20)$$

where Equations (29.5-13) – (29.5-18) have been used and the interchange of the summations with respect to n and the integrations with respect to x is justified by the assumed uniform convergence of the series expansions. Equations (29.5-19) and (29.5-20) are evidently identical to Equations (29.5-11) and (29.5-12), and, hence, the expansions given in Equations (29.5-17) and (29.5-18) indeed solve the problem.

The construction of convergence criteria for the Neumann expansion is complicated by the singularities of the Green's functions. For the simpler case of the scattering problem associated with the scalar wave equation, a convergence criterion has been derived (De Hoop 1991).

The nth term in the Neumann expansion is also known as the nth *Rayleigh–Gans–Born approximation.*

29.6 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation; time-domain analysis and complex frequency-domain analysis for canonical geometries of the scattering object

In this section the far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation is investigated further. In particular, closed-form analytic expressions are derived for the far-field scattered wave amplitude associated with the plane wave scattering by a *homogeneous object* in the shape of an ellipsoid, a rectangular block, an elliptical cylinder of finite height, an elliptical cone of finite height, or a tetrahedron. A structure consisting of the union of the listed objects can, in the first-order Rayleigh–Gans–Born approximation, be dealt with by superposition.

Time-domain analysis

In the time-domain analysis, the expressions for the scattered wave amplitude in the far-field region in the first-order Rayleigh–Gans–Born approximation follow, with the use of Equations (29.1-5), (29.1-17)–(29.1-21), and (29.5-3) and (29.5-4) as (Figure 29.6-1)

$$E^{s;\infty}_{r}(\xi,t) = c^{-2}(\xi_r\xi_k - \delta_{r,k})A^{\varepsilon}_{k,r'}(\xi/c - \alpha/c,t)e_{r'}$$
$$+ \varepsilon_{r,n,j}(\mu/\varepsilon)^{1/2}c^{-2}\xi_n A^{\mu}_{j,p'}(\xi/c - \alpha/c,t)h_{p'}\,, \qquad (29.6\text{-}1)$$

$$H^{s;\infty}_{p}(\xi,t) = Y\varepsilon_{p,m,k}\xi_m E^{s;\infty}_{k}(\xi,t)\,, \qquad (29.6\text{-}2)$$

where

$$A^{\varepsilon}_{k,r'}(u,t) = \int_{x\in\mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[\varepsilon^{s}_{k,r'}(x,t')/\varepsilon - \delta(t')\delta_{k,r'}\right]\partial_t^2 a(t - t' + u_s x_s)\,\mathrm{d}t' \qquad (29.6\text{-}3)$$

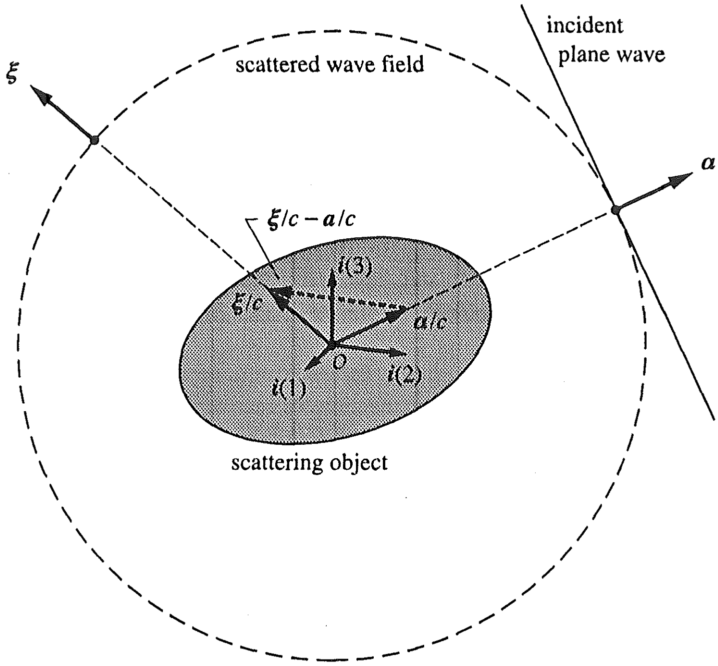

Figure 29.6-1 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation.

and

$$\mathcal{A}^{\mu}_{j,p'}(u,t) = \int_{x \in \mathcal{D}^s} \mathrm{d}V \int_{t'=0}^{\infty} \left[\mu^{\mathrm{s}}_{j,p'}(x,t')/\mu - \delta(t')\delta_{j,p'}\right] \partial_t^2 a(t - t' + u_s x_s)\, \mathrm{d}t'. \tag{29.6-4}$$

Note that these scattered wave amplitudes depend in their directional characteristics only on the difference $\xi/c - a/c$ between the slowness ξ/c in the direction of observation ξ and the slowness a/c in the direction of propagation of the incident uniform plane wave. This property only holds in the first-order Rayleigh–Gans–Born approximation and is not exact.

For a *homogeneous object*, Equations (29.6-3) and (29.6-4) reduce to

$$\mathcal{A}^{\varepsilon}_{k,r'}(u,t) = \int_{t'=0}^{\infty} \left[\varepsilon^{\mathrm{s}}_{k,r'}(t')/\varepsilon - \delta(t')\delta_{k,r'}\right] \Upsilon(u,t-t')\, \mathrm{d}t' \tag{29.6-5}$$

and

$$\mathcal{A}^{\mu}_{j,p'}(u,t) = \int_{t'=0}^{\infty} \left[\mu^{\mathrm{s}}_{j,p'}(t')/\mu - \delta(t')\delta_{j,p'}\right] \Upsilon(u,t-t')\, \mathrm{d}t', \tag{29.6-6}$$

in which

$$\Upsilon(u,t) = \int_{x \in \mathcal{D}^s} \partial_t^2 a(t + u_s x_s)\, \mathrm{d}V \tag{29.6-7}$$

is the *time-domain shape factor* corresponding to the domain $\mathcal{D}^s$ occupied by the scatterer. From Equation (29.6-7) it immediately follows that for $\boldsymbol{\xi}/c = \boldsymbol{\alpha}/c$, i.e. for observation "behind" the scatterer or "forward scattering", we have

$$\Upsilon(\mathbf{0},t) = V^{s}\partial_t^2 a(t) , \tag{29.6-8}$$

where V^s is the *volume of the scatterer*. Note, again, that Equation (29.6-8) only holds in the first-order Rayleigh–Gans–Born approximation, and is not exact.

Below, we shall derive for a number of canonical geometries of the scatterer, closed-form analytic expressions for the shape factor $\Upsilon = \Upsilon(\boldsymbol{u},t)$.

Ellipsoid

Let the scattering ellipsoid be defined by (see Equation (A.9-21) and Figure 29.6-2)

$$\mathcal{D}^s = \left\{x\in\mathcal{R}^3 ; 0 \leqslant (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 < 1\right\} . \tag{29.6-9}$$

Its volume is

$$V^s = (4\pi/3)a_1 a_2 a_3 . \tag{29.6-10}$$

In the integral on the right-hand side of Equation (29.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{29.6-11}$$

as the variables of integration. In y-space, the domain of integration is then the unit ball $\{y\in\mathcal{R}^3;\ 0\leqslant y_1^2 + y_2^2 + y_3^2 < 1\}$. The integration over this unit ball is carried out with the aid of spherical polar coordinates $\{r,\theta,\phi\}$, with $0\leqslant r<1$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, about the vector $u_1 a_1 \boldsymbol{i}(1) + u_2 a_2 \boldsymbol{i}(2) + u_3 a_3 \boldsymbol{i}(3)$ as polar axis. Then

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3 = (u_1 a_1)y_1 + (u_2 a_2)y_2 + (u_3 a_3)y_3 = Ur\cos(\theta) , \tag{29.6-12}$$

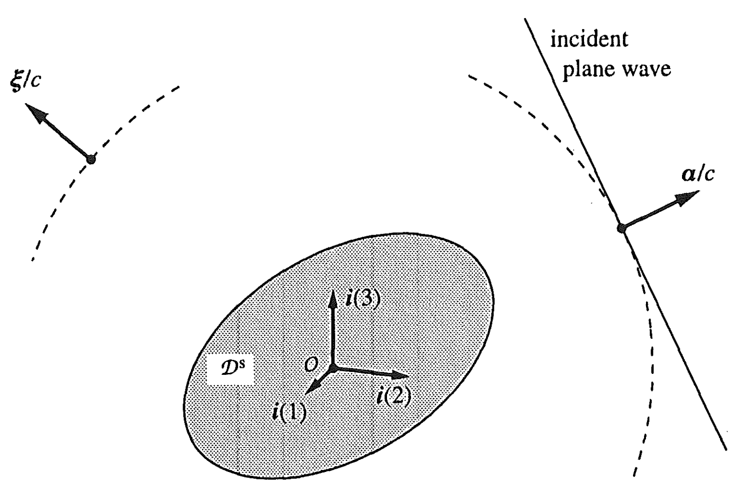

Figure 29.6-2 Scatterer in the shape of an ellipsoid.

where

$$U = \left[(u_1a_1)^2 + (u_2a_2)^2 + (u_3a_3)^2\right]^{1/2} \geqslant 0 \,, \tag{29.6-13}$$

while

$$\mathrm{d}V = a_1a_2a_3r^2 \sin(\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \,. \tag{29.6-14}$$

The integration then runs as follows:

$$\begin{aligned}
\Upsilon(u,t) &= a_1a_2a_3 \int_{r=0}^{1} r^2\,\mathrm{d}r \int_{\theta=0}^{\pi} \sin(\theta)\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \partial_t^2 a\,[t + Ur\cos(\theta)]\,\mathrm{d}\phi \\
&= 2\pi a_1a_2a_3 \int_{r=0}^{1} r^2\,\mathrm{d}r \int_{\theta=0}^{\pi} \partial_t^2 a\,[t + Ur\cos(\theta)]\sin(\theta)\,\mathrm{d}\theta \\
&= 2\pi a_1a_2a_3U^{-1} \int_{r=0}^{1} \left[\partial_t^2 a\,(t + Ur) - \partial_t a(t - Ur)\right] r\,\mathrm{d}r \\
&= 2\pi a_1a_2a_3 \left\{U^{-2}a(t + U) - U^{-3}\left[\mathrm{I}_t a(t + U) - \mathrm{I}_t a(t)\right]\right. \\
&\quad \left. + U^{-2}a(t - U) + U^{-3}\left[\mathrm{I}_t a(t - U) - \mathrm{I}_t a(t)\right]\right\} \\
&= (3V^{\mathrm{s}}/2)\left\{U^{-2}\left[a(t + U) + a(t - U)\right] - U^{-3}\left[\mathrm{I}_t a(t + U) - \mathrm{I}_t a(t - U)\right]\right\} .
\end{aligned} \tag{29.6-15}$$

By using the Taylor expansion of the right-hand side about $U = 0$ and taking the limit $U \downarrow 0$, it can be verified that the result is in accordance with Equation (29.6-8).

Rectangular block

Let the scattering domain be the rectangular block defined by (see Equation (A.9-14) and Figure 29.6-3)

$$\mathcal{D}^{\mathrm{s}} = \left\{x \in \mathcal{R}^3 \,;\, -a_1 < x_1 < a_1,\ -a_2 < x_2 < a_2,\ -a_3 < x_3 < a_3\right\} . \tag{29.6-16}$$

Its volume is given by

$$V^{\mathrm{s}} = 8a_1a_2a_3 \,. \tag{29.6-17}$$

In the integral on the right-hand side of Equation (29.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{29.6-18}$$

as the variables of integration. In y-space the domain of integration is then the cube $\{y \in \mathcal{R}^3 \,;\, -1 < y_1 < 1,\ -1 < y_2 < 1,\ -1 < y_3 < 1\}$ with edge lengths 2. With

$$U_1 = u_1a_1,\ U_2 = u_2a_2,\ U_3 = u_3a_3 \,, \tag{29.6-19}$$

furthermore, we have

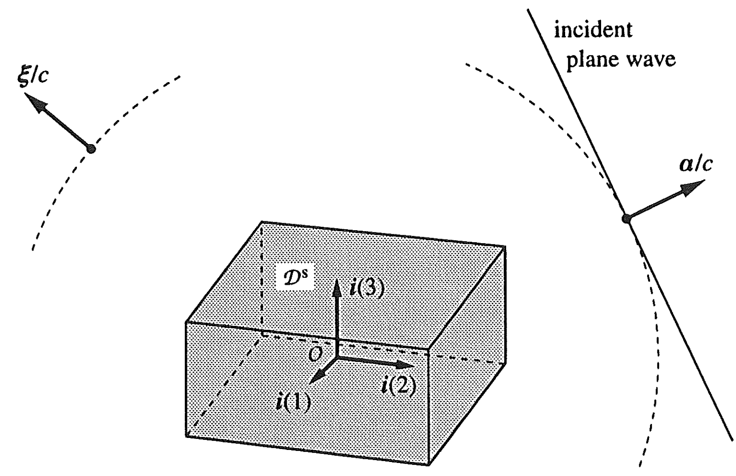

Figure 29.6-3 Scatterer in the shape of a rectangular block.

$$u_s x_s = u_1 x_1 + u_2 x_2 + u_3 x_3$$
$$= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 a_3) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3 , \tag{29.6-20}$$

while

$$\mathrm{d}V = a_1 a_2 a_3 \,\mathrm{d}y_1 \,\mathrm{d}y_2 \,\mathrm{d}y_3 . \tag{29.6-21}$$

The integration then runs as follows:

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 a_3 \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \mathrm{d}y_2 \int_{y_1=-1}^{1} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_1$$
$$= a_1 a_2 a_3 U_1^{-1} \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} [\partial_t a(t + U_1 + U_2 y_2 + U_3 y_3)$$
$$- \partial_t a(t - U_1 + U_2 y_2 + U_3 y_3)]\,\mathrm{d}y_2$$
$$= a_1 a_2 a_3 (U_1 U_2)^{-1} \int_{y_3=-1}^{1} [a(t + U_1 + U_2 + U_3 y_3) - a(t + U_1 - U_2 + U_3 y_3)$$
$$- a(t - U_1 + U_2 + U_3 y_3) + a(t - U_1 - U_2 + U_3 y_3)]\,\mathrm{d}y_3$$
$$= a_1 a_2 a_3 (U_1 U_2 U_3)^{-1} [\mathrm{I}_t a(t + U_1 + U_2 + U_3) - \mathrm{I}_t a(t + U_1 + U_2 - U_3)$$
$$- \mathrm{I}_t a(t + U_1 - U_2 + U_3) + \mathrm{I}_t a(t + U_1 - U_2 - U_3) - \mathrm{I}_t a(t - U_1 + U_2 + U_3)$$
$$+ \mathrm{I}_t a(t - U_1 + U_2 - U_3) + \mathrm{I}_t a(t - U_1 - U_2 + U_3) - \mathrm{I}_t a(t - U_1 - U_2 - U_3)] . \tag{29.6-22}$$

Special cases occur for either $U_1 \to 0$, $U_2 \to 0$, and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (29.6-22) by using the pertaining Taylor expansions in the right-hand side. In particular, it can be verified that for $U_1 \to 0$ and $U_2 \to 0$ and $U_3 \to 0$ the result is in accordance with Equation (29.6-8).

Elliptical cylinder of finite height

Let the elliptical cylinder of finite height be defined by (Figure 29.6-4)

$$\mathcal{D}^s = \left\{ x \in \mathcal{R}^3;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\ -h < x_3 < h \right\}. \tag{29.6-23}$$

Its volume is

$$V^s = 2\pi a_1 a_2 h. \tag{29.6-24}$$

In the integral on the right-hand side of Equation (29.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{29.6-25}$$

as the variables of integration. In y-space, the domain of integration is then the Cartesian product of the unit disk $\Delta^2 = \{(y_1,y_2) \in \mathcal{R}^2;\ 0 \leqslant y_1^2 + y_2^2 < 1\}$ and the interval $\{y_3 \in \mathcal{R};\ -1 < y_3 < 1\}$ along the axis of the cylinder. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h, \tag{29.6-26}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3, \end{aligned} \tag{29.6-27}$$

while

$$\mathrm{d}V = a_1 a_2 h\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3. \tag{29.6-28}$$

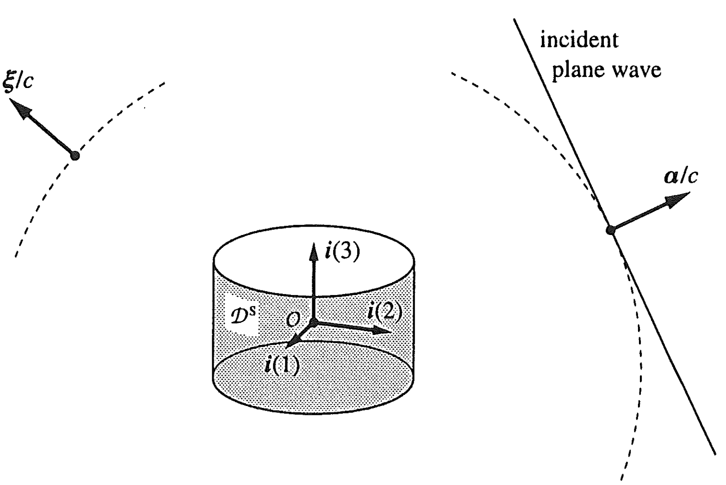

Figure 29.6-4 Scatterer in the shape of an elliptic cylinder of finite height.

The integration then runs as follows:

$$\mathit{\Upsilon}(\boldsymbol{u},t) = a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} \mathrm{d}y_1\,\mathrm{d}y_2 \int_{y_3=-1}^{1} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_3$$

$$= a_1 a_2 h U_3^{-1} \int_{(y_1,y_2)\in\Delta^2} [\partial_t a(t + U_1 y_1 + U_2 y_2 + U_3)$$

$$- \partial_t a(t + U_1 y_1 + U_2 y_2 - U_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2 . \qquad (29.6\text{-}29)$$

Next, we observe that

$$\partial_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) = \partial_t^2 \mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)$$

$$= (U_1^2 + U_2^2)^{-1}(\partial_{y_1}^2 + \partial_{y_2}^2)\mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3) \qquad \text{for } U_1^2 + U_2^2 \neq 0 . \qquad (29.6\text{-}30)$$

Now, applying Gauss' divergence theorem to the integration over Δ^2, we obtain

$$\int_{(y_1,y_2)\in\Delta^2} (\partial_{y_1}^2 + \partial_{y_2}^2)\mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}y_1\,\mathrm{d}y_2$$

$$= \int_{(y_1,y_2)\in C^2} (y_1 \partial_{y_1} + y_2 \partial_{y_2})\mathrm{I}_t a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}\sigma$$

$$= \int_{(y_1,y_2)\in C^2} (U_1 y_1 + U_2 y_2) a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}\sigma, \qquad (29.6\text{-}31)$$

where $\mathrm{d}\sigma$ is the elementary arc length along the unit circle C^2 that forms the closed boundary of the unit disk Δ^2, and where we have used the property that the unit vector along the normal to C^2 pointing away from Δ^2 is given by $\boldsymbol{\nu} = y_1 \boldsymbol{i}(1) + y_2 \boldsymbol{i}(2)$. In the integral on the right-hand side of Equation (29.6-31) we introduce the polar coordinates $\{r,\phi\}$, with $r = 1$ and $0 \leqslant \phi < 2\pi$, about the vector $U_1 \boldsymbol{i}(1) + U_2 \boldsymbol{i}(2)$ as polar axis, as the variables of integration. This yields

$$\int_{(y_1,y_2)\in C^2} (U_1 y_1 + U_2 y_2) a(t + U_1 y_1 + U_2 y_2 \pm U_3)\,\mathrm{d}\sigma$$

$$= \int_{\phi=0}^{2\pi} U \cos(\phi) a\left[t + U\cos(\phi) \pm U_3\right]\mathrm{d}\phi , \qquad (29.6\text{-}32)$$

where

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0 . \qquad (29.6\text{-}33)$$

Collecting the results, we end up with

$$\mathit{\Upsilon}(\boldsymbol{u},t) = a_1 a_2 h U^{-1} U_3^{-1} \int_{\phi=0}^{2\pi} \cos(\phi)\left\{ a\left[t + U\cos(\phi) + U_3\right]\right.$$

$$\left. - a\left[t + U\cos(\phi) - U_3\right]\right\}\mathrm{d}\phi . \qquad (29.6\text{-}34)$$

Special cases occur for $U{\downarrow}0$ and/or $U_3{\to}0$. The corresponding limits easily follow from Equation (29.6-34) by using the pertaining Taylor expansions in the right-hand side. In particular, it can be verified that for $U{\downarrow}0$ and $U_3{\to}0$ the result is in accordance with Equation (29.6-8).

Elliptical cone of finite height

Let the elliptical cone of finite height be defined by (Figure 29.6-5)

$$\mathcal{D}^{\rm s} = \left\{ x \in \mathcal{R}^3 ;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < x_3^2/h^2,\ 0 < x_3 < h \right\} . \tag{29.6-35}$$

Its volume is

$$V^{\rm s} = \pi a_1 a_2 h/3 . \tag{29.6-36}$$

In the integral on the right-hand side of Equation (29.6-7) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{29.6-37}$$

as the variables of integration. In y-space, the domain of integration is then $\{y \in \mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2,\ 0 < y_3 < 1\}$. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h , \tag{29.6-38}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3 , \end{aligned} \tag{29.6-39}$$

while

$$\mathrm{d}V = a_1 a_2 h \,\mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3 . \tag{29.6-40}$$

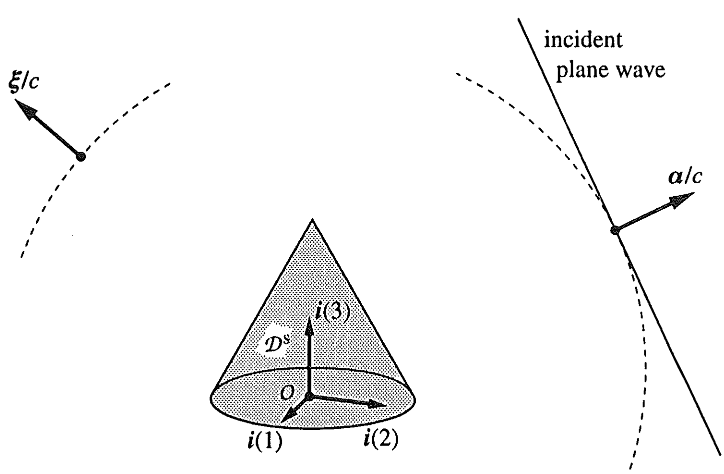

Figure 29.6-5 Scatterer in the shape of an elliptic cone of finite height.

The integration then runs as follows:

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h \int_{y_3=0}^{1} \mathrm{d}y_3 \int_{(y_1,y_2)\in\Delta^2(y_3)} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_1\,\mathrm{d}y_2\,, \quad (29.6\text{-}41)$$

where $\Delta^2(y_3) = \{(y_1,y_2)\in\mathcal{R}^2;\, 0 \leqslant y_1^2 + y_2^2 < y_3^2\}$ is the circular disk of radius y_3. With a reasoning similar to that used in Equations (29.6-30)–(29.6-32), we obtain

$$\int_{(y_1,y_2)\in\Delta^2(y_3)} \partial_t^2 a(t + U_1 y_1 + U_2 y_2 + U_3 y_3)\,\mathrm{d}y_1\,\mathrm{d}y_2$$

$$= U^{-1} y_3 \int_{\phi=0}^{2\pi} \cos(\phi)\partial_t a\left[t + U y_3 \cos(\phi) + U_3 y_3\right] \mathrm{d}\phi\,, \quad (29.6\text{-}42)$$

in which

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0\,. \quad (29.6\text{-}43)$$

Furthermore,

$$\int_{y_3=0}^{1} y_3 \partial_t a\left[t + U y_3 \cos(\phi) + U_3 y_3\right] \mathrm{d}y_3$$

$$= \left[U\cos(\phi) + U_3\right]^{-1} a\left[t + U\cos(\phi) + U_3\right]$$

$$- \left[U\cos(\phi) + U_3\right]^{-2} \left\{\mathrm{I}_t a\left[t + U\cos(\phi) + U_3\right] - \mathrm{I}_t a(t)\right\}. \quad (29.6\text{-}44)$$

Collecting the results, we end up with

$$\Upsilon(\boldsymbol{u},t) = a_1 a_2 h U^{-1} \int_{\phi=0}^{2\pi} \cos(\phi) \left\{\left[U\cos(\phi) + U_3\right]^{-1} a\left[t + U\cos(\phi) + U_3\right]\right.$$

$$\left. - \left[U\cos(\phi) + U_3\right]^{-2} \left\{\mathrm{I}_t a\left[t + U\cos(\phi) + U_3\right] - \mathrm{I}_t a(t)\right\}\right\} \mathrm{d}\phi\,. \quad (29.6\text{-}45)$$

Special cases occur for $U \downarrow 0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (29.6-45) by using the pertaining Taylor expansions in the right-hand side. In particular, it can be verified that for $U \downarrow 0$ and $U_3 \to 0$ the result is in accordance with Equation (29.6-8).

Tetrahedron

Let the tetrahedron be defined by (see Equation (A.9-17) and Figure 29.6-6)

$$\mathcal{D}^{\mathrm{s}} = \left\{ \boldsymbol{x}\in\mathcal{R}^3;\, \boldsymbol{x} = \sum_{I=0}^{3} \lambda(I)\boldsymbol{x}(I),\quad 0 < \lambda(I) < 1,\quad \sum_{I=0}^{3} \lambda(I) = 1 \right\}, \quad (29.6\text{-}46)$$

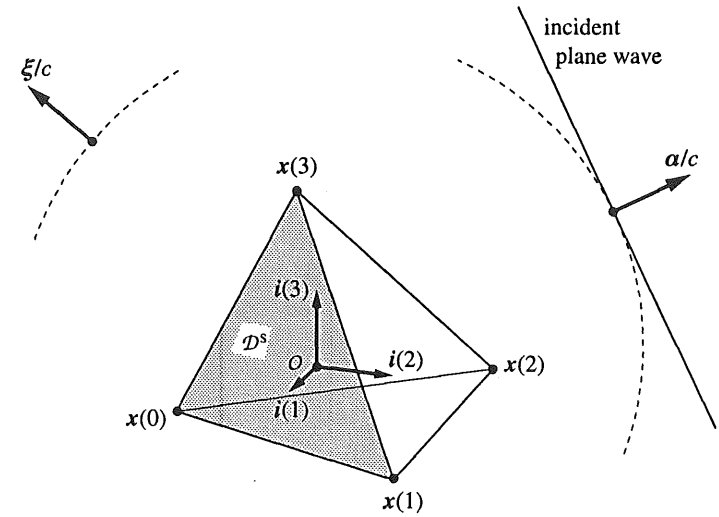

Figure 29.6-6 Scatterer in the shape of a tetrahedron (3-simplex).

in which $\{x(0),x(1),x(2),x_3(3)\}$ are the position vectors of the vertices and $\{\lambda(0),\ \lambda(1),\ \lambda(2),\lambda(3)\}$ are the barycentric coordinates. Its volume is (see Equations (A.10-29) and (A.10-33))

$$V^{\mathrm{s}} = \det\left[x(1)-x(0),x(2)-x(1),x(3)-x(2)\right]/6\,. \qquad (29.6\text{-}47)$$

In the integral on the right-hand side of Equation (29.6-7) we replace $\lambda(0)$ by $1-\lambda(1)-\lambda(2)-\lambda(3)$ and introduce $\{\lambda(1),\lambda(2),\lambda(3)\}$ as the (dimensionless) variables of integration. In $\{\lambda(1),\lambda(2),\lambda(3)\}$ space the domain of integration is then $\{0<\lambda(1)<1,\ 0<\lambda(2)<1-\lambda(1),\ 0<\lambda(3)<1-\lambda(1)-\lambda(2)\}$. Then, with

$$U(I) = u_s x_s(I) \qquad \text{for } I = 0,\ 1,\ 2,\ 3\,, \qquad (29.6\text{-}48)$$

we have

$$\begin{aligned} u_s x_s &= \lambda(0)U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3) \\ &= [1-\lambda(1)-\lambda(2)-\lambda(3)]\ U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3) \\ &= U(0) + [U(1)-U(0)]\ \lambda(1) + [U(2)-U(0)]\ \lambda(2) + [U(3)-U(0)]\ \lambda(3)\,, \end{aligned} \qquad (29.6\text{-}49)$$

while, with the Jacobian (see Equation (A.10-31))

$$\frac{\partial(x_1,x_2,x_3)}{\partial[\lambda(1),\lambda(2),\lambda(3)]} = 6V^{\mathrm{s}}, \qquad (29.6\text{-}50)$$

the elementary volume is expressed as

$$\mathrm{d}V = 6V^{\mathrm{s}}\,\mathrm{d}\lambda(1)\,\mathrm{d}\lambda(2)\,\mathrm{d}\lambda(3)\,. \qquad (29.6\text{-}51)$$

After some lengthy, but elementary, calculations it is found that

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left\{\frac{1}{U(0)-U(1)}\,\frac{1}{U(0)-U(2)}\,\frac{1}{U(0)-U(3)}\,\mathrm{I}_t a\,[t+U(0)]\right.$$
$$+\frac{1}{U(1)-U(0)}\,\frac{1}{U(1)-U(2)}\,\frac{1}{U(1)-U(3)}\,\mathrm{I}_t a\,[t+U(1)]$$
$$+\frac{1}{U(2)-U(0)}\,\frac{1}{U(2)-U(1)}\,\frac{1}{U(2)-U(3)}\,\mathrm{I}_t a\,[t+U(2)]$$
$$\left.+\frac{1}{U(3)-U(0)}\,\frac{1}{U(3)-U(1)}\,\frac{1}{U(3)-U(2)}\,\mathrm{I}_t a\,[t+U(3)]\right\}. \qquad (29.6\text{-}52)$$

In a symmetrical fashion, this result can be written as

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}}\sum_{I=0}^{3}\frac{1}{U(I)-U(J)}\,\frac{1}{U(I)-U(K)}\,\frac{1}{U(I)-U(L)}\,\mathrm{I}_t a\,[t+U(I)]\,, \qquad (29.6\text{-}53)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$.

Special cases occur for $U(I)=U(J)$ and/or $U(I)=U(K)$ and/or $U(I)=U(L)$. The easiest way to arrive at the expressions for the relevant cases is to redo the integrations that need modifications.

Complex frequency-domain analysis

In the complex frequency-domain analysis, the expressions for the scattered wave amplitude in the far-field region in the first-order Rayleigh–Gans–Born approximation follow, with the use of Equations (29.1-46), (29.1-54)–(29.1-58) and (29.5-13) and (29.5-14) as (Figure 29.6-7)

$$\hat{E}_r^{\mathrm{s};\infty}(\boldsymbol{\xi},s) = c^{-2}(\xi_r\xi_k-\delta_{r,k})\hat{A}^{\varepsilon}_{k,r'}(\boldsymbol{\xi}/c-\boldsymbol{\alpha}/c,s)e_{r'}$$
$$+\,\varepsilon_{r,n,j}(\mu/\varepsilon)^{1/2}c^{-2}\xi_n\hat{A}^{\mu}_{j,p'}(\boldsymbol{\xi}/c-\boldsymbol{\alpha}/c,s)h_{p'}\,, \qquad (29.6\text{-}54)$$

$$\hat{H}_p^{\mathrm{s};\infty}(\boldsymbol{\xi},s) = Y\varepsilon_{p,m,k}\xi_m\hat{E}_k^{\mathrm{s};\infty}(\boldsymbol{\xi},s)\,, \qquad (29.6\text{-}55)$$

where

$$\hat{A}^{\varepsilon}_{k,r'}(\boldsymbol{u},s) = s^2\hat{a}(s)\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}}\left[\hat{\eta}^{\mathrm{s}}_{k,r'}(\boldsymbol{x},s)/s\varepsilon-\delta_{k,r'}\right]\exp(su_sx_s)\,\mathrm{d}V\,, \qquad (29.6\text{-}56)$$

and

$$\hat{A}^{\mu}_{j,p'}(\boldsymbol{u},s) = s^2\hat{a}(s)\int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}}\left[\hat{\zeta}^{\mathrm{s}}_{j,p'}(\boldsymbol{x},s)/s\mu-\delta_{j,p'}\right]\exp(su_sx_s)\,\mathrm{d}V\,. \qquad (29.6\text{-}57)$$

Note that these scattered wave amplitudes depend in their directional characteristics only on the difference $\boldsymbol{\xi}/c-\boldsymbol{\alpha}/c$ between the slowness $\boldsymbol{\xi}/c$ in the direction of observation $\boldsymbol{\xi}$ and the slowness $\boldsymbol{\alpha}/c$ in the direction of propagation of the incident uniform plane wave. This property only holds in the Rayleigh–Gans–Born approximation and is not exact.

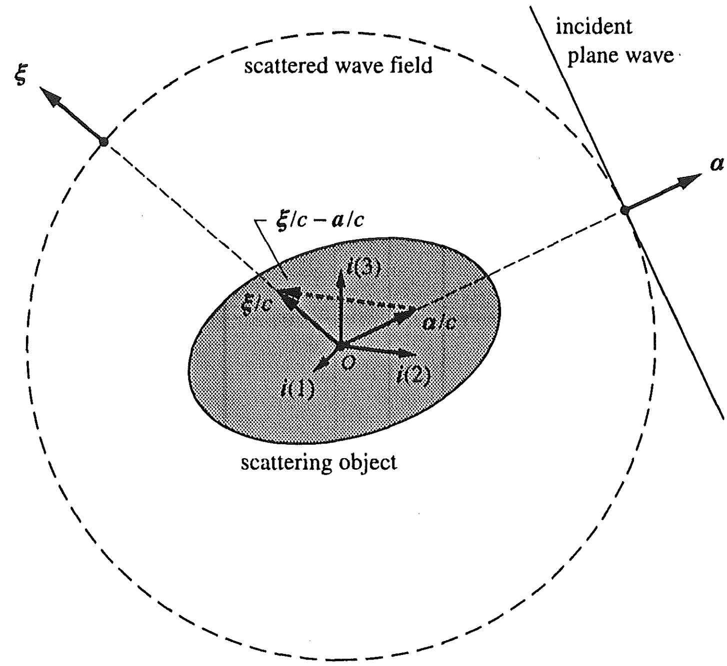

Figure 29.6-7 Far-field plane wave scattering in the first-order Rayleigh–Gans–Born approximation.

For a *homogeneous object*, Equations (29.6-56) and (29.6-57) reduce to

$$\hat{\Lambda}^{\varepsilon}_{k,r'}(\boldsymbol{u},s) = s^2\hat{a}(s)\left[\hat{\eta}^{\mathrm{s}}_{k,r'}(s)/s\varepsilon - \delta_{k,r'}\right]\hat{\Upsilon}(\boldsymbol{u},s)\,, \tag{29.6-58}$$

and

$$\hat{\Lambda}^{\mu}_{j,p'}(\boldsymbol{u},s) = s^2\hat{a}(s)\left[\hat{\zeta}^{\mathrm{s}}_{j,p'}(s)/s\mu - \delta_{j,p'}\right]\hat{\Upsilon}(\boldsymbol{u},s)\,, \tag{29.6-59}$$

in which

$$\hat{\Upsilon}(\boldsymbol{u},s) = \int_{\boldsymbol{x}\in\mathcal{D}^{\mathrm{s}}} \exp(su_s x_s)\,\mathrm{d}V \tag{29.6-60}$$

is the *complex frequency-domain shape factor* corresponding to the domain $\mathcal{D}^{\mathrm{s}}$ occupied by the scatterer. Note that Equation (29.6-60) differs by a factor of $s^2\hat{a}(s)$ from the time Laplace transform of the time-domain shape factor as given by Equation (29.6-7). From Equation (29.6-60) it immediately follows that for $\boldsymbol{\xi}/c = \boldsymbol{a}/c$ i.e. for observation "behind" the scatterer or "forward scattering", we have

$$\hat{\Upsilon}(\boldsymbol{0},s) = V^{\mathrm{s}}, \tag{29.6-61}$$

where V^{s} is the *volume of the scatterer*. Note, again, that Equation (29.6-61) only holds in the first-order Rayleigh–Gans–Born approximation, and is not exact.

Below, we shall derive for a number of canonical geometries of the scatterer, closed-form analytic expressions for the shape factor $\hat{\Upsilon}(\boldsymbol{u},s)$.

Ellipsoid

Let the scattering ellipsoid be defined by (see Equation (A.9-21) and Figure 29.6-8.

$$\mathcal{D}^{s} = \left\{x\in\mathcal{R}^{3};\ 0 \leqslant (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 < 1\right\}. \tag{29.6-62}$$

Its volume is

$$V^{s} = (4\pi/3)a_1a_2a_3 \,. \tag{29.6-63}$$

In the integral on the right-hand side of Equation (29.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{29.6-64}$$

as the variables of integration. In y-space, the domain of integration is then the unit ball $\{y\in\mathcal{R}^3;\ 0\leqslant y_1^2 + y_2^2 + y_3^2 < 1\}$. The integration over this unit ball is carried out with the aid of spherical polar coordinates $\{r,\theta,\phi\}$, with $0\leqslant r<1$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, about the vector $u_1a_1\boldsymbol{i}(1) + u_2a_2\boldsymbol{i}(2) + u_3a_3\boldsymbol{i}(3)$ as polar axis. Then

$$u_s x_s = u_1x_1 + u_2x_2 + u_3x_3 = (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3a_3)y_3 = Ur\cos(\theta)\,, \tag{29.6-65}$$

where

$$U = \left[(u_1a_1)^2 + (u_2a_2)^2 + (u_3a_3)^2\right]^{1/2} \geqslant 0\,, \tag{29.6-66}$$

while

$$\mathrm{d}V = a_1a_2a_3r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{29.6-67}$$

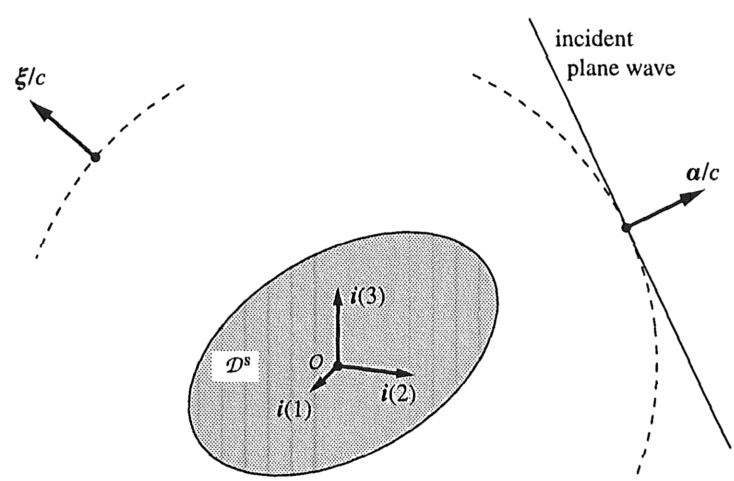

Figure 29.6-8 Scatterer in the shape of an ellipsoid.

The integration then runs as follows:

$$\hat{\Upsilon}(u,s) = a_1a_2a_3 \int_{r=0}^{1} r^2\,\mathrm{d}r \int_{\theta=0}^{\pi} \sin(\theta)\,\mathrm{d}\theta \int_{\phi=0}^{2\pi} \exp\,[sUr\cos(\theta)]\,\mathrm{d}\phi$$

$$= 2\pi a_1a_2a_3 \int_{r=0}^{1} r^2\,\mathrm{d}r \int_{\theta=0}^{\pi} \exp\,[sUr\cos(\theta)]\sin(\theta)\,\mathrm{d}\theta$$

$$= 2\pi a_1a_2a_3(sU)^{-1} \int_{r=0}^{1} [\exp(sUr) - \exp(-sUr)]\,r\,\mathrm{d}r$$

$$= 2\pi a_1a_2a_3(sU)^{-2} \left\{\exp(sU) + \exp(-sU) - \int_{r=0}^{1} [\exp(sUr) + \exp(-sUr)]\,\mathrm{d}r\right\}$$

$$= 2\pi a_1a_2a_3(sU)^{-2} \left\{\exp(sU) + \exp(-sU) - (sU)^{-1}\,[\exp(sU) - \exp(-sU)]\right\}$$

$$= 3V^{\mathrm{s}}\,\frac{sU\cosh(sU) - \sinh(sU)}{(sU)^3}\,. \tag{29.6-68}$$

By using the Taylor expansion of the right-hand side about $U = 0$ and taking the limit $U{\downarrow}0$, it can be verified that the result is in accordance with Equation (29.6-61).

Rectangular block

Let the scattering domain be the rectangular block defined by (see Equation (A.9-14) and Figure 29.6-9)

$$\mathcal{D}^{\mathrm{s}} = \left\{x{\in}\mathcal{R}^3;\, -a_1<x_1<a_1,\ -a_2<x_2<a_2,\ -a_3<x_3<a_3\right\}. \tag{29.6-69}$$

Its volume is given by

$$V^{\mathrm{s}} = 8a_1a_2a_3\,. \tag{29.6-70}$$

In the integral on the right-hand side of Equations (29.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/a_3 \tag{29.6-71}$$

as the variables of integration. In y-space the domain of integration is then the cube $\{y{\in}\mathcal{R}^3;\ -1<y_1<1,\ -1<y_2<1,\ -1<y_3<1\}$ with edge lengths 2. With

$$U_1 = u_1a_1,\ U_2 = u_2a_2,\ U_3 = u_3a_3\,, \tag{29.6-72}$$

furthermore, we have

$$\begin{aligned} u_sx_s &= u_1x_1 + u_2x_2 + u_3x_3 \\ &= (u_1a_1)y_1 + (u_2a_2)y_2 + (u_3a_3)y_3 = U_1y_1 + U_2y_2 + U_3y_3\,, \end{aligned} \tag{29.6-73}$$

while

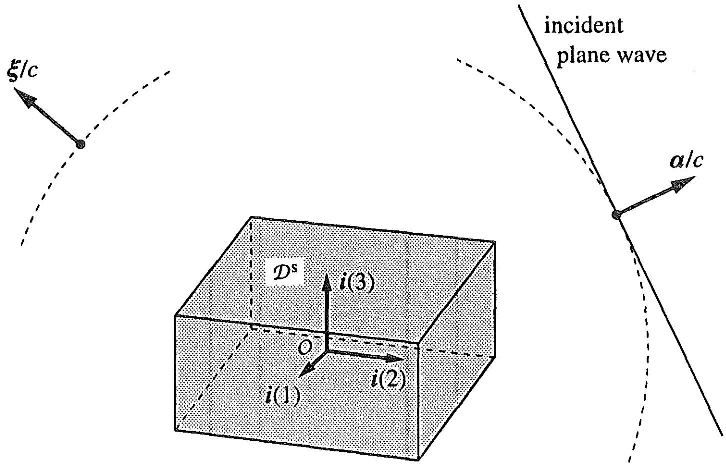

Figure 29.6-9 Scatterer in the shape of a rectangular block.

$$\mathrm{d}V = a_1 a_2 a_3 \,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3 \,. \tag{29.6-74}$$

The integration then runs as follows:

$$\begin{aligned}
\hat{\Upsilon}(\boldsymbol{u},s) &= a_1 a_2 a_3 \int_{y_3=-1}^{1} \mathrm{d}y_3 \int_{y_2=-1}^{1} \mathrm{d}y_2 \int_{y_1=-1}^{1} \exp\,[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\,\mathrm{d}y_1 \\
&= a_1 a_2 a_3 \int_{y_3=-1}^{1} \exp(sU_3 y_3)\,\mathrm{d}y_3 \int_{y_2=-1}^{1} \exp(sU_2 y_2)\,\mathrm{d}y_2 \int_{y_1=-1}^{1} \exp(sU_1 y_1)\,\mathrm{d}y_1 \\
&= a_1 a_2 a_3 \frac{\exp(sU_3) - \exp(-sU_3)}{sU_3}\,\frac{\exp(sU_2) - \exp(-sU_2)}{sU_2}\,\frac{\exp(sU_1) - \exp(-sU_1)}{sU_1} \\
&= V^{\mathrm{s}} \frac{\sinh(sU_3)}{sU_3}\,\frac{\sinh(sU_2)}{sU_2}\,\frac{\sinh(sU_1)}{sU_1} \,.
\end{aligned} \tag{29.6-75}$$

Special cases occur for either $U_1 \to 0$, $U_2 \to 0$, and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (29.6-75) by using the relevant Taylor expansions in the right-hand side. In particular, it can be verified that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$ the result is in accordance with Equation (29.6-61).

Elliptical cylinder of finite height

Let the elliptical cylinder of finite height be defined by (Figure 29.6-10)

$$\mathcal{D}^{\mathrm{s}} = \left\{ \boldsymbol{x} \in \mathcal{R}^3 ;\ 0 \leqslant x_1^2/a_1^2 + x_2^2/a_2^2 < 1,\ -h < x_3 < h \right\} . \tag{29.6-76}$$

Its volume is

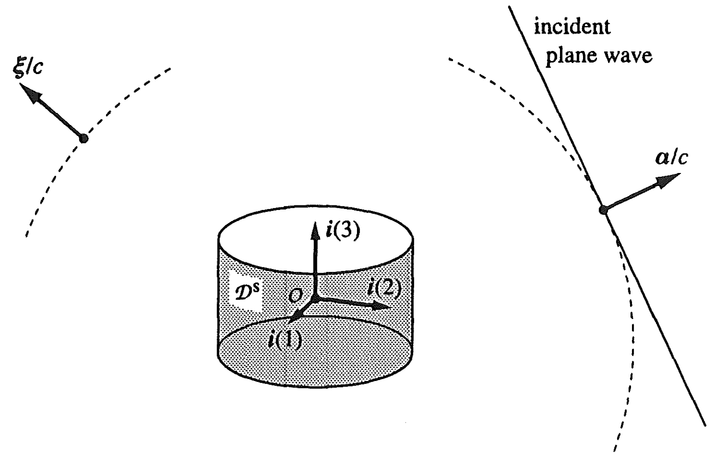

Figure 29.6-10 Scatterer in the shape of an elliptic cylinder of finite height.

$$V^s = 2\pi a_1 a_2 h\,. \tag{29.6-77}$$

In the integral on the right-hand side of Equation (29.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{29.6-78}$$

as the variables of integration. In y space, the domain of integration is then the Cartesian product of the unit disk $\Delta^2 = \{(y_1,y_2)\in\mathcal{R}^2;\ 0\leqslant y_1^2 + y_2^2 < 1\}$ and the interval $\{y_3\in\mathcal{R};\ -1 < y_3 < 1\}$ along the axis of the cylinder. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \tag{29.6-79}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3\,, \end{aligned} \tag{29.6-80}$$

while

$$\mathrm{d}V = a_1 a_2 h\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3\,. \tag{29.6-81}$$

The integration then runs as follows:

$$\begin{aligned} \hat{\Upsilon}(u,s) &= a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} \mathrm{d}y_1\, \mathrm{d}y_2 \int_{y_3=-1}^{1} \exp\,[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\, \mathrm{d}y_3 \\ &= a_1 a_2 h \int_{(y_1,y_2)\in\Delta^2} (sU_3)^{-1} \Big\{\exp\,[s(U_1 y_1 + U_2 y_2 + U_3)] \\ &\quad - \exp\,[s(U_1 y_1 + U_2 y_2 - U_3)]\Big\}\, \mathrm{d}y_1\, \mathrm{d}y_2\,. \end{aligned} \tag{29.6-82}$$

Next, we observe that

$$\exp[s(U_1y_1+U_2y_2\pm U_3)] = (s^2U_1^2+s^2U_2^2)^{-1}(\partial_{y_1}^2+\partial_{y_2}^2)\exp[s(U_1y_1+U_2y_2\pm U_3)]$$
$$\text{for } U_1^2+U_2^2\neq 0 . \tag{29.6-83}$$

Now, applying Gauss' divergence theorem to the integration over Δ^2, we obtain

$$\int_{(y_1,y_2)\in\Delta^2}(\partial_{y_1}^2+\partial_{y_2}^2)\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}y_1\,\mathrm{d}y_2$$
$$=\int_{(y_1,y_2)\in C^2}(y_1\partial_{y_1}+y_2\partial_{y_2})\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}\sigma$$
$$=\int_{(y_1,y_2)\in C^2}s(U_1y_1+U_2y_2)\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}\sigma, \tag{29.6-84}$$

where $\mathrm{d}\sigma$ is the elementary arc length along the unit circle C^2 that forms the boundary of the unit disk Δ^2 and where we have used the property that the unit vector along the normal to C^2 pointing away from Δ^2 is given by $\boldsymbol{\nu}=y_1\boldsymbol{i}(1)+y_2\boldsymbol{i}(2)$. In the integral on the right-hand side of Equation (29.6-84) we introduce the polar coordinates $\{r,\phi\}$, with $r=1$ and $0\leqslant\phi<2\pi$, about the vector $U_1\boldsymbol{i}(1)+U_2\boldsymbol{i}(2)$ as polar axis, as the variables of integration. This yields

$$\int_{(y_1,y_2)\in C^2}(U_1y_1+U_2y_2)\exp[s(U_1y_1+U_2y_2\pm U_3)]\,\mathrm{d}\sigma$$
$$=\int_{\phi=0}^{2\pi}U\cos(\phi)\exp[sU\cos(\phi)\pm sU_3]\,\mathrm{d}\phi=2\pi U\exp(\pm sU_3)I_1(sU)\,, \tag{29.6-85}$$

where I_1 is the modified Bessel function of the first kind and order one (Abramowitz and Stegun 1964) and

$$U=(U_1^2+U_2^2)^{1/2}\geqslant 0\,. \tag{29.6-86}$$

Collecting the results, we end up with

$$\hat{\Upsilon}(\boldsymbol{u},s)=2\pi a_1a_2hs^{-2}U^{-1}U_3^{-1}I_1(sU)\,[\exp(sU_3)-\exp(-sU_3)]$$
$$=2V^{\mathrm{s}}s^{-2}U^{-1}U_3^{-1}I_1(sU)\sinh(sU_3)\,. \tag{29.6-87}$$

Special cases occur for $U\downarrow 0$ and/or $U_3\to 0$. The corresponding limits easily follow from Equation (29.6-87) by using the relevant Taylor expansions in the right-hand side. In particular, it can be verified that for $U\downarrow 0$ and $U_3\to 0$ the result is in accordance with Equation (29.6-61).

Elliptical cone of finite height

Let the elliptical cone of finite height be defined by (Figure 29.6-11)

$$\mathcal{D}^{\mathrm{s}}=\left\{x\in\mathcal{R}^3;\ 0\leqslant x_1^2/a_1^2+x_2^2/a_2^2<x_3^2/h^2,\ 0<x_3<h\right\}. \tag{29.6-88}$$

Its volume is

$$V^{\mathrm{s}}=\pi a_1a_2h/3\,. \tag{29.6-89}$$

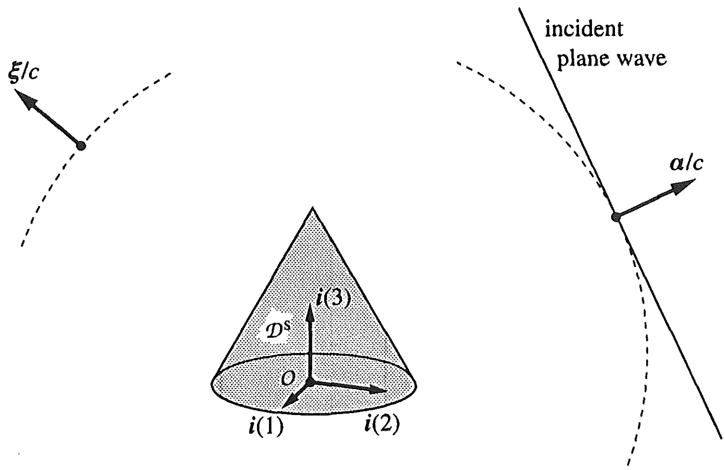

Figure 29.6-11 Scatterer in the shape of an elliptic cone of finite height.

In the integral on the right-hand side of Equation (29.6-60) we introduce the dimensionless variables

$$y_1 = x_1/a_1,\ y_2 = x_2/a_2,\ y_3 = x_3/h \tag{29.6-90}$$

as the variables of integration. In y-space, the domain of integration is then $\{y \in \mathcal{R}^3;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2\},\ 0 < y_3 < 1\}$. Then, with

$$U_1 = u_1 a_1,\ U_2 = u_2 a_2,\ U_3 = u_3 h\,, \tag{29.6-91}$$

we have

$$\begin{aligned} u_s x_s &= u_1 x_1 + u_2 x_2 + u_3 x_3 \\ &= (u_1 a_1) y_1 + (u_2 a_2) y_2 + (u_3 h) y_3 = U_1 y_1 + U_2 y_2 + U_3 y_3\,, \end{aligned} \tag{29.6-92}$$

while

$$\mathrm{d}V = a_1 a_2 h\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3\,. \tag{29.6-93}$$

The integration then runs as follows:

$$\hat{\Upsilon}(\boldsymbol{u},s) = a_1 a_2 h \int_{y_3=0}^{1} \mathrm{d}y_3 \int_{(y_1,y_2)\in\Delta^2(y_3)} \exp\left[s(U_1 y_1 + U_2 y_2 + U_3 y_3)\right] \mathrm{d}y_1\, \mathrm{d}y_2\,, \tag{29.6-94}$$

where $\Delta^2((y_3) = \{(y_1,y_2)\in\mathcal{R}^2;\ 0 \leqslant y_1^2 + y_2^2 < y_3^2\}$ is the circular disk of radius y_3. With a reasoning similar to the one as used in Equations (29.6-83)–(29.6-85), we obtain

$$\begin{aligned} &\int_{(y_1,y_2)\in\Delta^2(y_3)} \exp[s(U_1 y_1 + U_2 y_2 + U_3 y_3)]\, \mathrm{d}y_1\, \mathrm{d}y_2 \\ &= (sU)^{-1} y_3 \int_{\phi=0}^{2\pi} \cos(\phi) \exp[s(U y_3 \cos(\phi) + U_3 y_3)]\, \mathrm{d}\phi\,, \end{aligned} \tag{29.6-95}$$

in which

$$U = (U_1^2 + U_2^2)^{1/2} \geqslant 0 . \tag{29.6-96}$$

Furthermore,

$$\int_{y_3=0}^{1} y_3 \exp[s(U y_3 \cos(\phi) + U_3 y_3)]\,\mathrm{d}y_3$$

$$= [s(U\cos(\phi) + U_3)]^{-1} \left\{ \exp[s(U\cos(\phi) + U_3)] - \int_{y_3=0}^{1} \exp[s(U y_3 \cos(\phi) + U_3 y_3)]\,\mathrm{d}y_3 \right\}$$

$$= [s(U\cos(\phi) + U_3)]^{-1} \exp[s(U\cos(\phi) + U_3)]$$

$$- [s(U\cos(\phi) + U_3)]^{-2} \left\{ \exp[s(U y_3 \cos(\phi) + U_3)] - 1 \right\} . \tag{29.6-97}$$

Collecting the results, we end up with

$$\hat{\Gamma}(u,s) = 6V^{\mathrm{S}}(sU)^{-1} \int_{\phi=0}^{2\pi} \cos(\phi)$$

$$\times \frac{1}{2\pi} \left\{ \frac{\exp[s(U\cos(\phi) + U_3)]}{s(U\cos(\phi) + U_3)} - \frac{\exp[s(U\cos(\phi) + U_3)] - 1}{s^2(U\cos(\phi) + U_3)^2} \right\} \mathrm{d}\phi . \tag{29.6-98}$$

Special cases occur for $U{\downarrow}0$ and/or $U_3 \to 0$. The corresponding limits easily follow from Equation (29.6-98) by using the relevant Taylor expansions in the right-hand side. In particular, it can be verified that for $U{\downarrow}0$ and $U_3 \to 0$ the result is in accordance with Equation (29.6-61).

Tetrahedron

Let the tetrahedron be defined by (see Equation (A.9-17) and Figure 29.6-12)

$$\mathcal{D}^{\mathrm{S}} = \left\{ x \in \mathcal{R}^3 ;\ x = \sum_{I=0}^{3} \lambda(I) x(I),\ 0 < \lambda(I) < 1,\ \sum_{I=0}^{3} \lambda(I) = 1 \right\} , \tag{29.6-99}$$

in which $\{x(0), x(1), x(2), x(3)\}$ are the position vectors of the vertices and $\{\lambda(0), \lambda(1), \lambda(2), \lambda(3)\}$ are the barycentric coordinates. Its volume is given by (see Equations (A.10-29) and (A.10-33))

$$V^{\mathrm{S}} = \det[x(1) - x(0), x(2) - x(1), x(3) - x(2)] /6 . \tag{29.6-100}$$

In the integral on the right-hand side of Equation (29.6-60) we replace $\lambda(0)$ by $1 - \lambda(1) - \lambda(2) - \lambda(3)$ and introduce $\{\lambda(1), \lambda(2), \lambda(3)\}$ as the (dimensionless) variables of integration. In $\{\lambda(1), \lambda(2), \lambda(3)\}$ space the domain of integration is then $\{0 < \lambda(1) < 1,\ 0 < \lambda(2) < 1 - \lambda(1),\ 0 < \lambda(3) < 1 - \lambda(1) - \lambda(2)\}$. Then, with

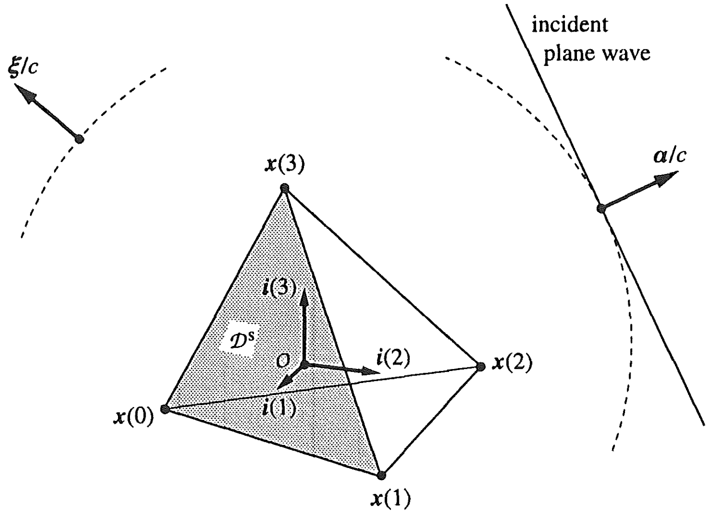

Figure 29.6-12 Scatterer in the shape of a tetrahedron (3-simplex).

$$U(I) = u_s x_s(I) \qquad \text{for } I = 0, 1, 2, 3\,, \tag{29.6-101}$$

we have

$$\begin{aligned} u_s x_s &= \lambda(0)U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3) \\ &= [1 - \lambda(1) - \lambda(2) - \lambda(3)]\, U(0) + \lambda(1)U(1) + \lambda(2)U(2) + \lambda(3)U(3) \\ &= U(0) + [U(1) - U(0)]\,\lambda(1) + [U(2) - U(0)]\,\lambda(2) + [U(3) - U(0)]\,\lambda(3)\,, \end{aligned} \tag{29.6-102}$$

while, with the Jacobian (see Equation (A.10-31))

$$\frac{\partial(x_1,x_2,x_3)}{\partial\,[\lambda(1),\lambda(2),\lambda(3)]} = 6V^{\mathrm{s}}, \tag{29.6-103}$$

the elementary volume is expressed as

$$\mathrm{d}V = 6V^{\mathrm{s}}\, \mathrm{d}\lambda(1)\, \mathrm{d}\lambda(2)\, \mathrm{d}\lambda(3)\,. \tag{29.6-104}$$

After some lengthy, but elementary, calculations it is found that

$$\begin{aligned} \hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{s}}s^{-3} \Bigg\{ & \frac{1}{U(0)-U(1)}\,\frac{1}{U(0)-U(2)}\,\frac{1}{U(0)-U(3)}\exp[sU(0)] \\ & + \frac{1}{U(1)-U(0)}\,\frac{1}{U(1)-U(2)}\,\frac{1}{U(1)-U(3)}\exp[sU(1)] \\ & + \frac{1}{U(2)-U(0)}\,\frac{1}{U(2)-U(1)}\,\frac{1}{U(2)-U(3)}\exp[sU(2)] \\ & + \frac{1}{U(3)-U(0)}\,\frac{1}{U(3)-U(1)}\,\frac{1}{U(3)-U(2)}\exp[sU(3)] \Bigg\}\,. \end{aligned} \tag{29.6-105}$$

In a symmetrical fashion, this result can be written as

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}}s^{-3}\sum_{I=0}^{3}\frac{1}{U(I)-U(J)}\,\frac{1}{U(I)-U(K)}\,\frac{1}{U(I)-U(L)}\exp[sU(I)]\,, \qquad (29.6\text{-}106)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$.

Special cases occur for $U(I) = U(J)$ and/or $U(I) = U(K)$ and/or $U(I) = U(L)$. The easiest way to arrive at the expressions for the relevant cases is to redo the integrations that need modifications.

(*Note*: Since the first-order Rayleigh–Gans–Born approximation is additive in the domains occupied by the scatterers, the scattering by an arbitrary union of canonical scatterers follows by superposition. In particular, the result for the tetrahedron is the building block for scatterers in the shape of an arbitrary polyhedron.)

The first-order Rayleigh–Gans–Born scattering finds numerous applications both in the forward (direct) and the inverse scattering theory. References to the earlier literature can be found in Quak and De Hoop (1986).

Exercises

Exercise 29.6-1

Show that Equation (29.6-60) follows from the time Laplace transform of Equation (29.6-7).

Exercise 29.6-2

Show that Equation (29.6-61) follows from the time Laplace transform of Equation (29.6-8).

Exercise 29.6-3

Show that Equation (29.6-68) follows from the time Laplace transform of Equation (29.6-15).

Exercise 29.6-4

Show that Equation (29.6-75) follows from the time Laplace transform of Equation (29.6-22).

Exercise 29.6-5

Show that Equation (29.6-87) follows from the time Laplace transform of Equation (29.6-34).

Exercise 29.6-6

Show that Equation (29.6-98) follows from the time Laplace transform of Equation (29.6-45).

Exercise 29.6-7

Show that Equation (29.6-106) follows from the time Laplace transform of Equation (29.6-53).

Exercise 29.6-8

Show that for $U \downarrow 0$, Equation (29.6-15) becomes Equation (29.6-8). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 29.6-9

Show that for $U_3 \to 0$, Equation (29.6-22) becomes

$$\Upsilon(\boldsymbol{u},t) = 2a_1a_2a_3(U_1U_2)^{-1}\,[a(t+U_1+U_2) - a(t+U_1-U_2) - a(t-U_1+U_2) + a(t-U_1-U_2)]\,. \tag{29.6-107}$$

(In this case, $\boldsymbol{u}$ is parallel to the x_1,x_2 plane.)

Exercise 29.6-10

Show that for $U_2 \to 0$ and $U_3 \to 0$, Equation (29.6-22) becomes

$$\Upsilon(\boldsymbol{u},t) = 4a_1a_2a_3U_1^{-1}\,[\partial_t a(t+U_1) - \partial_t a(t-U_1)]\,. \tag{29.6-108}$$

(In this case, $\boldsymbol{u}$ is parallel to the x_1 axis.)

Exercise 29.6-11

Show that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$, Equation (29.6-22) becomes Equation (29.6-8). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 29.6-12

Show that for $U \downarrow 0$, Equation (29.6-34) becomes

$$\Upsilon(\boldsymbol{u},t) = \pi a_1a_2hU_3^{-1}\,[\partial_t a(t+U_3) - \partial_t a(t-U_3)]\,. \tag{29.6-109}$$

(In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 29.6-13

Show that for $U_3 \to 0$, Equation (29.6-34) becomes

$$\Upsilon(\boldsymbol{u},t) = 2a_1a_2hU^{-1}\int_{\phi=0}^{2\pi} \cos(\phi)\partial_t a\,[t + U\cos(\phi)]\,\mathrm{d}\phi\,. \tag{29.6-110}$$

(In this case $\boldsymbol{u}$ is perpendicular to the axis of the cylinder.)

Exercise 29.6-14

Show that for $U{\downarrow}0$ and $U_3{\to}0$, Equation (29.6-34) becomes Equation (29.6-8). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 29.6-15

Show that for $U{\downarrow}0$, Equation (29.6-45) becomes

$$\Upsilon(\boldsymbol{u},t) = \pi a_1a_2h\left\{U_3^{-1}\,[\partial_t a(t + U_3) - 2U_3^{-2}a(t + U_3)] + 2U_3^{-3}\,[\mathrm{I}_t a(t + U_3) - \mathrm{I}_t a(t)]\right\}. \tag{29.6-111}$$

(In this case, $\boldsymbol{u}$ is parallel to the axis of the cone.)

Exercise 29.6-16

Show that for $U_3{\to}0$, Equation (29.6-45) becomes

$$\Upsilon(\boldsymbol{u},t) = a_1a_2hU^{-1}\int_{\phi=0}^{2\pi}\left\{[U\cos(\phi)\,]^{-1}\,a(t + U\cos(\phi)) - [U\cos(\phi)]^{-2}[\mathrm{I}_t a(t + U\cos(\phi)) - \mathrm{I}_t a(t)]\right\}\cos(\phi)\,\mathrm{d}\phi\,. \tag{29.6-112}$$

(In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cone.)

Exercise 29.6-17

Show that for $U{\downarrow}0$ and $U_3{\to}0$, Equation (29.6-45) becomes Equation (29.6-8). (In this case, $\boldsymbol{u} = \mathbf{0}$.)

Exercise 29.6-18

Show that for $U(J){\to}U(I)$, Equation (29.6-53) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{S}}\left\{\frac{1}{U(I) - U(K)}\,\frac{1}{U(I) - U(L)}\right\}a[t + U(I)] - \left\{\frac{1}{[U(I) - U(K)]^2}\,\frac{1}{U(I) - U(L)} + \frac{1}{U(I) - U(K)}\,\frac{1}{[U(I) - U(L)]^2}\right\}\mathrm{I}_t a[t + U(I)]$$

$$+\left\{\frac{1}{[U(K)-U(I)]^2}\,\frac{1}{U(K)-U(L)}\,\mathrm{I}_t a[t+U(K)]\right\}$$

$$+\left\{\frac{1}{[U(L)-U(I)]^2}\,\frac{1}{U(L)-U(K)}\,\mathrm{I}_t a[t+U(L)]\right\}, \tag{29.6-113}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$.)

Exercise 29.6-19

Show that for $U(J)\to U(I)$ and $U(L)\to U(K)$, Equation (29.6-53) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left(\frac{1}{[U(I)-U(K)]^2}\{a\,[t+U(I)]+a\,[t+U(K)]\}\right.$$

$$\left.-\frac{2}{[U(I)-U(K)]^3}\{\mathrm{I}_t a\,[t+U(I)]-\mathrm{I}_t a\,[t+U(K)]\}\right), \tag{29.6-114}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$, as well as perpendicular to the edge connecting the vertex $\boldsymbol{x}(K)$ with the vertex $\boldsymbol{x}(L)$.)

Exercise 29.6-20

Show that for $U(J)\to U(I)$ and $U(K)\to U(I)$, Equation (29.6-53) becomes

$$\Upsilon(\boldsymbol{u},t) = 6V^{\mathrm{s}}\left(\frac{1}{U(I)-U(L)}\,\partial_t a\,[t+U(I)] - \frac{1}{[U(I)-U(L)]^2}\,a\,[t+U(I)]\right.$$

$$\left.+\frac{1}{[U(I)-U(L)]^3}\left\{\mathrm{I}_t a\,[t+U(I)]-\mathrm{I}_t a\,[t+U(L)]\right\}\right), \tag{29.6-115}$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$. (In this case, $\boldsymbol{u}$ is perpendicular to the plane containing the triangle of which $\boldsymbol{x}(I)$, $\boldsymbol{x}(J)$ and $\boldsymbol{x}(K)$ are the vertices.)

Exercise 29.6-21

Show that for $\boldsymbol{u}=\boldsymbol{0}$, Equation (29.6-53) becomes Equation (29.6-8).

Exercise 29.6-22

Show that for $U\downarrow 0$, Equation (29.6-68) becomes Equation (29.6-61).

Exercise 29.6-23

Show that for $U_3 \to 0$, Equation (29.6-75) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = V^{s}\,\frac{\sinh(sU_2)}{sU_2}\,\frac{\sinh(sU_1)}{sU_1} \tag{29.6-116}$$

and show that the result follows from the time Laplace transform of Equation (29.6-107). (In this case, $\boldsymbol{u}$ is parallel to the x_1,x_2 plane.)

Exercise 29.6-24

Show that for $U_2 \to 0$ and $U_3 \to 0$, Equation (29.6-75) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = V^{s}\,\frac{\sinh(sU_1)}{sU_1} \tag{29.6-117}$$

and show that the result follows from the time Laplace transform of Equation (29.6-108). (In this case, $\boldsymbol{u}$ is parallel to the x_1 axis.)

Exercise 29.6-25

Show that for $U_1 \to 0$, $U_2 \to 0$ and $U_3 \to 0$, Equation (29.6-75) becomes Equation (29.6-61). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 29.6-26

Show that for $U \downarrow 0$, Equation (29.6-87) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 2\pi a_1 a_2 h s^{-1} U_3^{-1} \sinh(sU_3) \tag{29.6-118}$$

and show that the result follows from the time Laplace transform of Equation (29.6-109). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cylinder.)

Exercise 29.6-27

Show that for $U_3 \to 0$, Equation (29.6-87) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 4\pi a_1 a_2 h s^{-1} U^{-1} I_1(sU) \tag{29.6-119}$$

and show that the result follows from the time Laplace transform of Equation (29.6-110). (In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cylinder.)

Exercise 29.6-28

Show that for $U \downarrow 0$ and $U_3 \to 0$, Equation (29.6-87) becomes Equation (29.6-61). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 29.6-29

Show that for $U{\downarrow}0$, Equation (29.6-98) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = \pi a_1 a_2 h s^{-2} \left\{ \left[sU_3^{-1} - 2U_3^{-2} + 2s^{-1}U_3^{-3} \right] \exp(sU_3) - 2s^{-1}U_3^{-3} \right\} \qquad (29.6\text{-}120)$$

and show that the result follows from the time Laplace transform of Equation (29.6-111). (In this case, $\boldsymbol{u}$ is parallel to the axis of the cone.)

Exercise 29.6-30

Show that for $U_3 \rightarrow 0$, Equation (29.6-98) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = a_1 a_2 h s^{-2} U^{-1} \int_{\phi=0}^{2\pi} \left\{ [U\cos(\phi)]^{-1} \exp[sU\cos(\phi)] - s^{-1} [U\cos(\phi)]^{-2} [\exp(sU\cos(\phi)) - 1] \right\} \cos(\phi)\, \mathrm{d}\phi \qquad (29.6\text{-}121)$$

and show that the result follows from the time Laplace transform of Equation (29.6-112). (In this case, $\boldsymbol{u}$ is perpendicular to the axis of the cone.)

Exercise 29.6-31

Show that for $U{\downarrow}0$ and $U_3 \rightarrow 0$, Equation (29.6-98) becomes Equation (29.6-61). (In this case, $\boldsymbol{u} = \boldsymbol{0}$.)

Exercise 29.6-32

Show that for $U(J) \rightarrow U(I)$, Equation (29.6-105) becomes

$$\begin{aligned}\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{s}} s^{-2} \Bigg[& \left\{ \frac{1}{U(I)-U(K)} \, \frac{1}{U(I)-U(L)} \exp[sU(I)] \right\} \\ & - \left\{ \frac{1}{[U(I)-U(K)]^2} \, \frac{1}{U(I)-U(L)} + \frac{1}{U(I)-U(K)} \, \frac{1}{[U(I)-U(L)]^2} \right\} s^{-1} \exp[sU(I)] \\ & + \left\{ \frac{1}{[U(K)-U(L)]^2} \, \frac{1}{U(K)-U(L)} s^{-1} \exp[sU(K)] \right\} \\ & + \left\{ \frac{1}{[U(L)-U(I)]^2} \, \frac{1}{U(L)-U(K)} s^{-1} \exp[sU(L)] \right\} \Bigg], \end{aligned} \qquad (29.6\text{-}122)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$ and show that the result follows from the time Laplace transform of Equation (29.6-113). (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$.)

Exercise 29.6-33

Show that for $U(J) \to U(I)$ and $U(L) \to U(K)$, Equation (29.6-105) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}}s^{-2}\left(\frac{1}{[U(I)-U(K)]^2}\{\exp[sU(I)] + \exp[sU(K)]\}\right.$$

$$\left. - \frac{2s^{-1}}{[U(I)-U(K)]^3}\{\exp[sU(I)] - \exp[sU(L)]\}\right), \qquad (29.6\text{-}123)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$ and show that this result follows from the time Laplace transform of Equation (29.6-114). (In this case, $\boldsymbol{u}$ is perpendicular to the edge connecting the vertex $\boldsymbol{x}(I)$ with the vertex $\boldsymbol{x}(J)$ as well as perpendicular to the edge connecting the vertex $\boldsymbol{x}(K)$ with the vertex $\boldsymbol{x}(L)$.)

Exercise 29.6-34

Show that for $U(J) \to U(I)$ and $U(K) \to U(I)$, Equation (29.6-105) becomes

$$\hat{\Upsilon}(\boldsymbol{u},s) = 6V^{\mathrm{S}}s^{-2}\left(\frac{s}{U(I)-U(K)}\exp[sU(I)] - \frac{1}{[U(I)-U(K)]^2}\exp[sU(I)]\right.$$

$$\left. + \frac{s^{-1}}{[U(I)-U(K)]^3}\{\exp[sU(I)] - \exp[sU(L)]\}\right), \qquad (29.6\text{-}124)$$

where $\{I,J,K,L\}$ is a permutation of $\{0,1,2,3\}$ and show that this result follows from the time Laplace transform of Equation (29.6-115). (In this case, $\boldsymbol{u}$ is perpendicular to the plane containing the triangle of which $\boldsymbol{x}(I)$, $\boldsymbol{x}(J)$ and $\boldsymbol{x}(K)$ are the vertices.)

Exercise 29.6-35

Show that for $\boldsymbol{u} = \boldsymbol{0}$, Equation (29.6-105) becomes Equation (29.6-61).

References

Abromowitz, M., and Stegun, I.E. 1964, *Handbook of Mathematical Functions*, Washington DC: National Bureau of Standards, Formula 9.6.19, p. 376.

De Hoop, A.T., 1959, On the plane wave extinction cross-section of an obstacle. *Applied Scientific Research*, (B) **7**, 463–469.

De Hoop, A.T., 1960, A reciprocity theorem for the electromagnetic field scattered by an obstacle. *Applied Scientific Research*, (B) **8**, 135–140.

De Hoop, A.T., 1984, A time-domain energy theorem for scattering of plane electromagnetic waves. *Radio Science*, **19** (5), 1179–1184.

De Hoop, A.T., 1991, Convergence criterion for the time-domain iterative Born approximation to scattering by an inhomogeneous, dispersive object. *Journal of the Optical Society of America A*, **8** (8), 1256–1260.

Quak, D., and De Hoop, A.T., 1986, Time-domain Born approximation to the far field scattering of a plane electromagnetic wave by a penetrable object. *Radio Science*, **21** (5), 815–821.