Chapter 19
Similarity in Wave Propagation Under a Global Relaxation Law*

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Abstract

The propagation of acoustic, electromagnetic, and elastodynamic waves in inhomogeneous media is usually, in the first instance, mathematically modeled under the assumption of negligible medium losses. In practice, however, often attenuation and dispersion in the wave phenomena are observed. These must be attributed to some unknown relaxation phenomenon. As a first try to model these phenomena, the use of a global relaxation law is suggested by which the time derivatives in the first-order coupled wave equations are replaced by a relaxation operator with a constant coefficient. Via the Schouten - Van der Pol theorem of the one-sided Laplace transformation with respect to time, the wave motion in such a medium can directly be expressed in terms of the one present in the medium's lossless counterpart (for which the relaxation coefficients have the value zero). The example for acoustic waves in a fluid is worked out in detail.

1 Introduction

The mathematical modeling of wave phenomena is usually, in the first instance, carried out under the assumption of negligibly small losses in the medium in which the waves propagate. In practice, however, often attenuation and dispersion in the wave phenomena are observed. These must be attributed to some, in many cases as yet unknown, dissipation mechanism. Now, for each particular dissipation mechanism to be considered in the mathematical description of the wave propagation, the calculations or computations have to be carried out anew. Especially if one is still at the initial stage of investigating whether or not a particular dissipation mechanism can explain the observed attenuation and dispersion phenomena, this is an awkward situation. Out of this situation the question arose whether there exists some dissipation mechanism, with possibly one or two adjustable parameters, for which the propagated waves could be evaluated by some simple additional operation to be carried out on the already obtained answer for the lossless case. In the present contribution this is shown to be the case for a particular global relaxation law. The case of linearized acoustic waves generated by arbitrary sources in a inhomogeneous, anisotropic fluid is worked out in detail.

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2 Basic Equations for Acoustic Wave Motion

The basic equations governing the linearized acoustic wave motion generated by arbitrary sources in an inhomogeneous, anisotropic, dissipative fluid are

$$\partial_k p + \partial_t \mathcal{C}_t(\mu_{k,r}, v_r; \mathbf{x}, t) = f_k, \tag{1}$$
$$\partial_r v_r + \partial_t \mathcal{C}_t(\chi, p; \mathbf{x}, t) = q, \tag{2}$$

Here, $\mathbf{x} = \{x_1, x_2, x_3\}$ are the Cartesian coordinates of a point of observation in three-dimensional Euclidean space $\mathcal{R}^3$, t is the time coordinate, ∂_k denotes differentiation with respect to x_k, ∂_t is a reserved symbol denoting differentiation with respect to t, and $\mathcal{C}_t$ denotes time convolution:

$$\mathcal{C}_t(f, g; \mathbf{x}, t) = \int_{t' \epsilon \mathcal{R}} f(\mathbf{x}, t') g(\mathbf{x}, t - t') \mathrm{d}t'. \tag{3}$$

The quantities in Equations (1) - (2) have the following meaning:

- p = acoustic pressure,
- v_r = particle velocity,
- f_k = volume source density of force,
- q = volume source density of injection rate,
- $\mu_{k,r}$ = inertia relaxation function,
- χ = compliance relaxation function.

The subscript notation of Cartesian vectors and tensors is used and the summation convention (with subscript range 1:3) applies. The causality of the reaction of the medium entails the causality property

$$\mu_{k,r}(\mathbf{x}, t) = 0 \quad \text{and} \quad \chi(\mathbf{x}, t) = 0 \quad \text{for } t < 0 \text{ and all } \mathbf{x}. \tag{4}$$

For a *lossless* fluid, we have

$$\mu_{k,r}(\mathbf{x}, t) = \rho_{k,r}(\mathbf{x})\delta(t), \tag{5}$$
$$\chi(\mathbf{x}, t) = \kappa(\mathbf{x})\delta(t), \tag{6}$$

where $\delta(t)$ is the one-dimensional Dirac distribution operative at $t = 0$, and

- $\rho_{k,r}$ = volume density of mass,
- κ = compressibility.

For a fluid with *frictional force/bulk viscosity* losses, we have

$$\mu_{k,r}(\mathbf{x}, t) = \rho_{k,r}(\mathbf{x})\delta(t) + K_{k,r}(\mathbf{x})H(t), \tag{7}$$
$$\chi(\mathbf{x}, t) = \kappa(\mathbf{x})\delta(t) + \Gamma(\mathbf{x})H(t), \tag{8}$$

where $H(t)$ is the Heaviside unit step function, and

- $K_{k,r}$ = coefficient of frictional force,
- Γ = coefficient of bulk viscosity.

In any subdomain of the configuration where the constitutive coefficients $\{\mu_{k,r}, \chi\}$ and the volume source densities $\{q, f_k\}$ are continuous, the acoustic wave quantities $\{p, v_r\}$ are continuously differentiable. Across an interface of discontinuity in fluid properties, the acoustic wave quantities satisfy the boundary conditions

$$p \quad \text{and} \quad \nu_r v_r \quad \text{continuous across interface,} \tag{9}$$

where ν_r is the unit vector along the normal to the interface. On the boundary surface of an acoustically impenetrable object, either of the following boundary conditions holds:

$$\lim_{h \downarrow 0} p(\mathbf{x} + h\boldsymbol{\nu}, t) = 0 \tag{10}$$

on the boundary of a perfectly compliant object, or

$$\lim_{h \downarrow 0} \nu_r v_r(\mathbf{x} + h\boldsymbol{\nu}, t) = 0 \tag{11}$$

on the boundary of a immovable perfectly rigid object, where $\boldsymbol{\nu}$ is the unit vector along the outward normal to the boundary of the impenetrable object. We assume that the sources start to act at the instant $t = 0$. The acoustic wavefield that is causally related to the action of the sources then vanishes throughout the configuration for $t < 0$.

Through the point-source solutions (Green's functions) of the acoustic wave problem, the wavefield quantities can be expressed in terms of the source densities. Let the latter have the bounded support $\mathcal{D}$, then

$$p(\mathbf{x}', t) = \int_{\mathbf{x} \epsilon \mathcal{D}} [\mathcal{C}_t(G^{pq}, q; \mathbf{x}', \mathbf{x}, t) + \mathcal{C}_t(G^{pf}_k, f_k; \mathbf{x}', \mathbf{x}, t)] \mathrm{d}V, \tag{12}$$

$$v_r(\mathbf{x}', t) = \int_{\mathbf{x} \epsilon \mathcal{D}} [\mathcal{C}_t(G^{vq}_r, q; \mathbf{x}', \mathbf{x}, t) + \mathcal{C}_t(G^{vf}_{r,k}, f_k; \mathbf{x}', \mathbf{x}, t)] \mathrm{d}V, \tag{13}$$

in which $\{G^{pq}, G^{vq}_r\} = \{G^{pq}, G^{vq}_r\}(\mathbf{x}', \mathbf{x}, t)$ satisfy the system of equations

$$\partial'_k G^{pq} + \partial_t \mathcal{C}_t(\mu_{k,r}, G^{vq}_r; \mathbf{x}', \mathbf{x}, t) = 0, \tag{14}$$

$$\partial'_r G^{vq}_r + \partial_t \mathcal{C}_t(\chi, G^{pq}; \mathbf{x}', \mathbf{x}, t) = \delta(\mathbf{x}' - \mathbf{x}, t), \tag{15}$$

with a point source of volume injection, and $\{G^{pf}_{k'}, G^{vf}_{r,k'}\} = \{G^{pf}_{k'}, G^{vf}_{r,k'}\}(\mathbf{x}', \mathbf{x}, t)$ satisfy the system of equations

$$\partial'_k G^{pf}_{k'} + \partial_t \mathcal{C}_t(\mu_{k,r}, G^{vf}_{r,k'}; \mathbf{x}', \mathbf{x}, t) = \delta_{k,k'} \delta(\mathbf{x}' - \mathbf{x}, t), \tag{16}$$

$$\partial'_r G^{vf}_{r,k'} + \partial_t \mathcal{C}_t(\chi, G^{pf}_{k'}; \mathbf{x}', \mathbf{x}, t) = 0, \tag{17}$$

with a point source of force. In these equations ∂'_k means differentiation with respect to x'_k, $\delta(\mathbf{x}' - \mathbf{x}, t)$ is the four-dimensional Dirac distribution operative at $\mathbf{x}' = \mathbf{x}$ and $t = 0$, and $\delta_{k,k'}$ is the Kronecker tensor: $\delta_{k,k'} = 1$ if $k = k'$, $\delta_{k,k'} = 0$ if $k \neq k'$.

3 The Complex Frequency Domain Acoustic Equations

The key issue of the similarity analysis to be carried out is found in the time Laplace transform domain or complex frequency domain. Therefore, we need the complex frequency domain counterparts of the equations of Section 2. With

$$\hat{p}(\mathbf{x}, s) = \int_{t=0}^{\infty} \exp(-st) p(\mathbf{x}, t) \mathrm{d}t \tag{18}$$

and similar relations for the other quantities, and the properties $\hat{\partial}_t = s$ and

$$\hat{C}_t(f, g; \mathbf{x}, t) = \hat{f}(\mathbf{x}, s)\hat{g}(\mathbf{x}, s), \tag{19}$$

Equations (1) and (2) change into

$$\partial_k \hat{p} + \hat{\zeta}_{k,r}\hat{v}_r = \hat{f}_k, \tag{20}$$

$$\partial_r \hat{v}_r + \hat{\eta}\hat{p} = \hat{q}, \tag{21}$$

where

$$\hat{\zeta}_{k,r}(\mathbf{x}, s) = s\hat{\mu}_{k,r}(\mathbf{x}, s), \tag{22}$$

$$\hat{\eta}(\mathbf{x}, s) = s\hat{\chi}(\mathbf{x}, s), \tag{23}$$

while Equation (9) changes into

$$\hat{p} \quad \text{and} \quad \nu_r \hat{v}_r \quad \text{continuous across interface}, \tag{24}$$

and Equations (10) and (11) into

$$\lim_{h \downarrow 0} \hat{p}(\mathbf{x} + h\boldsymbol{\nu}, s) = 0 \tag{25}$$

on the boundary of a perfectly compliant object, and

$$\lim_{h \downarrow 0} \nu_r \hat{v}_r(\mathbf{x} + h\boldsymbol{\nu}, s) = 0, \tag{26}$$

on the boundary of an immovable perfectly rigid object, respectively.

Further, Equations (12) and (13) change into

$$\hat{p}(\mathbf{x}', s) = \int_{\mathbf{x} \epsilon \mathcal{D}} [\hat{G}^{pq}(\mathbf{x}', \mathbf{x}, s)\hat{q}(\mathbf{x}, s) + \hat{G}^{pf}_k(\mathbf{x}', \mathbf{x}, s)\hat{f}_k(\mathbf{x}, s)]\mathrm{d}V, \tag{27}$$

$$\hat{v}_r(\mathbf{x}', s) = \int_{\mathbf{x} \epsilon \mathcal{D}} [\hat{G}^{vq}_r(\mathbf{x}', \mathbf{x}, s)\hat{q}(\mathbf{x}, s) + \hat{G}^{vf}_{r,k}(\mathbf{x}', \mathbf{x}, s)\hat{f}_k(\mathbf{x}, s)]\mathrm{d}V, \tag{28}$$

in which $\{\hat{G}^{pq}, \hat{G}^{vq}_r\} = \{\hat{G}^{pq}, \hat{G}^{vq}_r\}(\mathbf{x}', \mathbf{x}, s)$ satisfy the system of equations

$$\partial'_k \hat{G}^{pq} + \hat{\zeta}_{k,r}\hat{G}^{vq}_r = 0, \tag{29}$$

$$\partial'_r \hat{G}^{vq}_r + \hat{\eta}G^{pq} = \delta(\mathbf{x}' - \mathbf{x}), \tag{30}$$

with a point source of volume injection, and $\{\hat{G}^{pf}_{k'}, \hat{G}^{vf}_{r,k'}\} = \{\hat{G}^{pf}_{k'}, \hat{G}^{vf}_{r,k'}\}(\mathbf{x}', \mathbf{x}, s)$ satisfy the system of equations

$$\partial'_k \hat{G}^{pf}_{k'} + \hat{\zeta}_{k,r}\hat{G}^{vf}_{r,k'} = \delta_{k,k'}\delta(\mathbf{x}' - \mathbf{x}), \tag{31}$$

$$\partial'_r \hat{G}^{vf}_{r,k'} + \hat{\eta}G^{pf}_{k'} = 0, \tag{32}$$

with a point source of force. In these equations ∂'_k means differentiation with respect to x'_k, $\delta(\mathbf{x} - \mathbf{x}')$ is the three-dimensional Dirac distribution operative at $\mathbf{x}' = \mathbf{x}$, and the property $\hat{\delta}(\mathbf{x} - \mathbf{x}', t) = \delta(\mathbf{x}' - \mathbf{x})$ has been used.

For a *lossless* fluid, we have, on account of Equations (5) and (6)

$$\hat{\zeta}_{k,r}(\mathbf{x}, s) = s\rho_{k,r}(\mathbf{x}), \tag{33}$$

$$\hat{\eta}(\mathbf{x}, s) = s\kappa(\mathbf{x}), \tag{34}$$

while for a fluid with *frictional force/bulk viscosity losses* we have, on account of Equations (7) and (8),

$$\hat{\zeta}_{k,r}(\mathbf{x}, s) = s\rho_{k,r}(\mathbf{x}) + K_{k,r}(\mathbf{x}), \tag{35}$$
$$\hat{\eta}(\mathbf{x}, s) = s\kappa(\mathbf{x}) + \Gamma(\mathbf{x}). \tag{36}$$

For a global relaxation law, the coefficients in Equations (35) and (36) are interrelated through

$$K_{k,r}(\mathbf{x}) = \alpha\rho_{k,r}(\mathbf{x}) \quad \text{for all } \mathbf{x}, \tag{37}$$
$$\Gamma(\mathbf{x}) = \beta\kappa(\mathbf{x}) \quad \text{for all } \mathbf{x}, \tag{38}$$

where α and β are arbitrary, non-negative constants. Under these conditions, Equations (35) and (36) change into

$$\hat{\zeta}_{k,r} = (s+\alpha)\rho_{k,r}(\mathbf{x}), \tag{39}$$
$$\hat{\eta} = (s+\beta)\kappa(\mathbf{x}). \tag{40}$$

These expressions will be used in our further similarity analysis. Obviously, $\alpha = \beta = 0$ corresponds to the lossless case.

4 Similarity Analysis in a Fluid with Global Frictional Force/Bulk Viscosity Relaxation Parameters

The similarity analysis will be carried out for the wave motion governed by the two-parameter system of acoustic wave equations

$$\partial_k p + \rho_{k,r}(\alpha + \partial_t)v_r = f_k, \tag{41}$$
$$\partial_r v_r + \kappa(\beta + \partial_t)p = q. \tag{42}$$

The pertaining source-type integral representations for the wavefield quantities are written as (cf. Equations (12) and (13))

$$p(\mathbf{x}', t; \alpha, \beta) = \int_{\mathbf{x}\epsilon\mathcal{D}} [C_t(G^{pq}, q; \mathbf{x}', \mathbf{x}, t; \alpha, \beta) + C_t(G^{pf}_k, f_k; \mathbf{x}', \mathbf{x}, t; \alpha, \beta)]\mathrm{d}V, \tag{43}$$
$$v_r(\mathbf{x}', t; \alpha, \beta) = \int_{\mathbf{x}\epsilon\mathcal{D}} [C_t(G^{vq}_r, q; \mathbf{x}', \mathbf{x}, t; \alpha, \beta) + C_t(G^{vf}_{r,k}, f_k; \mathbf{x}', \mathbf{x}, t; \alpha, \beta)]\mathrm{d}V, \tag{44}$$

in which the Green's function satisfy the equations (cf. Equations (14) - (17))

$$\partial'_k G^{pq} + \rho_{k,r}(\alpha + \partial_t)G^{vq}_r = 0, \tag{45}$$
$$\partial'_r G^{vq}_r + \kappa(\beta + \partial_t)G^{pq} = \delta(\mathbf{x}' - \mathbf{x}, t), \tag{46}$$

and

$$\partial'_k G^{pf}_{k'} + \rho_{k,r}(\alpha + \partial_t)G^{vq}_{r,k'} = \delta_{k,k'}\delta(\mathbf{x}' - \mathbf{x}, t), \tag{47}$$
$$\partial'_r G^{vf}_{r,k'} + \kappa(\beta + \partial_t)G^{pf}_{k'} = 0. \tag{48}$$

The complex frequency domain counterparts of Equations (45) - (48) are obtained upon taking the time Laplace transform of these equations, under which operation ∂_t is replaced by the factor s. The relevant result is rewritten as

$$\partial'_k[(s+\beta)^{1/2}\hat{G}^{pq}] + (s+\beta)^{1/2}(s+\alpha)^{1/2}\rho_{k,r}[(s+\alpha)^{1/2}\hat{G}^{vq}_r] = 0, \tag{49}$$

$$\partial'_r[(s+\alpha)^{1/2}\hat{G}^{vq}_r] + (s+\alpha)^{1/2}(s+\beta)^{1/2}\kappa[(s+\beta)^{1/2}\hat{G}^{pq}] = (s+\alpha)^{1/2}\delta(\mathbf{x}'-\mathbf{x}), \tag{50}$$

and

$$\partial'_k[(s+\beta)^{1/2}\hat{G}^{pf}_{k'}] + (s+\beta)^{1/2}(s+\alpha)^{1/2}\rho_{k,r}[(s+\alpha)^{1/2}\hat{G}^{vf}_{r,k'}] = (s+\beta)^{1/2}\delta_{k,k'}\delta(\mathbf{x}'-\mathbf{x}), \tag{51}$$

$$\partial'_r[(s+\alpha)^{1/2}\hat{G}^{vf}_{r,k'}] + (s+\alpha)^{1/2}(s+\beta)^{1/2}\kappa[(s+\beta)^{1/2}\hat{G}^{pf}_{k'} = 0. \tag{52}$$

Upon comparing Equations (49) - (52) with the ones for $\alpha = \beta = 0$ (i.e., the case of wave propagation in a lossless fluid), it follows that

$$(s+\beta)^{1/2}\hat{G}^{pq}(\mathbf{x}',\mathbf{x},s;\alpha,\beta) = (s+\alpha)^{1/2}\hat{G}^{pq}[\mathbf{x}',\mathbf{x},(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0], \tag{53}$$

$$\hat{G}^{vq}_r(\mathbf{x}',\mathbf{x},s;\alpha,\beta) = \hat{G}^{vq}_r[\mathbf{x}',\mathbf{x},(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0], \tag{54}$$

$$\hat{G}^{pf}_{k'}(\mathbf{x}',\mathbf{x},s;\alpha,\beta) = \hat{G}^{pf}_{k'}[\mathbf{x}',\mathbf{x},(s+\beta)^{1/2}(s+\alpha)^{1/2};0,0], \tag{55}$$

$$(s+\alpha^{1/2})\hat{G}^{vf}_{r,k'}(\mathbf{x}',\mathbf{x},s;\alpha,\beta) = (s+\beta)^{1/2}\hat{G}^{vf}_{r,k'}[\mathbf{x}',\mathbf{x},(s+\beta)^{1/2}(s+\alpha)^{1/2};0,0]. \tag{56}$$

In the right-hand sides, the complex frequency domain Green's fucntions for the lossless fluid occur, but with the time Laplace transform parameter s replaced by $(s+\alpha)^{1/2}(s+\beta)^{1/2}$. The time-domain counterparts of the latter functions follow from an application of the Schouten - Van der Pol theorem of the time Laplace transformation that relates two time functions whose Laplace transforms are interrelated through the operations of replacing the transform parameter s by a suitable function of s [1], [2], [3], [4], [5]. For the present case, the consequences of this theorem are worked out in Appendix A. Using the results of this appendix, we obtain

$$G^{pq}(\mathbf{x}',\mathbf{x},t;\alpha,\beta) = (\alpha+\partial_t)\left[\int_{\tau=0}^{t} U_0(t,\tau;\alpha,\beta)G^{pq}(\mathbf{x}',\mathbf{x},\tau;0,0)\mathrm{d}\tau\right]H(t), \tag{57}$$

$$G^{vq}_r(\mathbf{x}',\mathbf{x},t;\alpha,\beta) = \left[\int_{\tau=0}^{t} U_1(t,\tau;\alpha,\beta)G^{vq}_r(\mathbf{x}',\mathbf{x},\tau;0,0)\mathrm{d}\tau\right]H(t), \tag{58}$$

$$G^{pf}_{k'}(\mathbf{x}',\mathbf{x},t;\alpha,\beta) = \left[\int_{\tau=0}^{t} U_1(t,\tau;\alpha,\beta)G^{pf}_{k'}(\mathbf{x}',\mathbf{x},\tau;0,0)\mathrm{d}\tau\right]H(t), \tag{59}$$

$$G^{vf}_{r,k'}(\mathbf{x}',\mathbf{x},t;\alpha,\beta) = (\beta+\partial_t)\left[\int_{\tau=0}^{t} U_0(t,\tau;\alpha,\beta)G^{vf}_{r,k'}(\mathbf{x}',\mathbf{x},\tau;0,0)\mathrm{d}\tau\right]H(t), \tag{60}$$

in which

$$U_1(t,\tau;\alpha,\beta) = -\partial_\tau U_0(t,\tau;\alpha,\beta), \tag{61}$$

with

(62) $$U_0(t,\tau;\alpha,\beta) = \exp[-(\alpha+\beta)t/2]I_0[(|\beta-\alpha|/2)(t^2-\tau^2)^{1/2}]H(t-\tau),$$

where I_0 denotes the modified Bessel function of the first kind and order zero and H is the Heaviside unit step function. In Equations (57) - (60) care has to be taken to include the Dirac distribution at $\tau = t$ due to the differentiation of the Heaviside unit step function ocurring in Equation (62). Equations (57) - (60) show, how the Green's functions for the dissipative case are related to the Green's functions of the lossless case. For the case considered, the mutual relationship is fairly elementary and involves, after carrying out the differentiations analytically, only single integrations. In this respect it is important to notice that the changes in wave forms that do occur in the presence of losses are, in general, different for the different wave field quantities and for the different types of sources. Only G_r^{vq} and $G_{k'}^{pf}$ undergo a common modification as Equations (58) and (59) indicate.

5 Conclusion

Through the Schouten - Van der Pol theorem of the time Laplace transformation, a relationship has been derived between the Green's functions of the acoustic wave generation by known sources in an arbitrarily inhomogeneous and anisotropic, lossless fluid and the Green's functions pertaining to the corresponding dissipative fluid with two global relaxation parameters related to frictional force/bulk viscosity losses. The two parameters are adjustable and can be used in the explanation of attenuation and dispersion phenomena occurring during the acoustic wave propagation in a dissipative fluid with an as yet unknown loss mechanism.

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A The Schouten - Van der Pol theorem for the replacement of s by $(s+\alpha)^{1/2}(s+\beta)^{1/2}$

Let $G(t;0,0)$ be a known, causal function of time with support $\{t\epsilon\mathcal{R}; t > 0\}$ and let

(63) $$\hat{G}(s;0,0) = \int_{\tau=0}^{\infty} \exp(-s\tau)G(\tau,0,0)d\tau$$

be its Laplace transform. Let, further,

(64) $$\hat{G}(s;\alpha,\beta) = \hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2}; 0.0],$$

then

$$\hat{G}(s;\alpha,\beta) = \int_{\tau=0}^{\infty} \exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau]G(\tau,0,0)d\tau. \tag{65}$$

To arrive at the time-domain counterpart $G(t;\alpha,\beta)$ of $\hat{G}(s;\alpha,\beta)$, we observe that [6]

$$\exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau] = \int_{t=\tau}^{\infty} \exp(-st)U_1(t,\tau;\alpha,\beta)dt, \tag{66}$$

in which

$$U_1(t,\tau;\alpha,\beta) = -\partial_\tau U_0(t,\tau;\alpha,\beta), \tag{67}$$

with

$$U_0(t,\tau;\alpha,\beta) = \exp[-(\alpha+\beta)t/2]I_0[(|\beta-\alpha|/2)(t^2-\tau^2)^{1/2}]H(t-\tau), \tag{68}$$

where I_0 denotes the modified Bessel function of the first kind and order zero and H is the Heaviside unit step function. Using Equation (66) in Equation(65) and employing the uniqueness of the time Laplace transform (Lerch's theorem [7]), we end up with

$$G(t;\alpha,\beta) = \left[\int_{\tau=0}^{t} U_1(t,\tau,\alpha,\beta)G(\tau;0,0)d\tau\right]H(t), \tag{69}$$

where care has to be taken to include the Dirac distribution at $\tau = t$ due to the differentiation in Equation (67) of the Heaviside unit step function occurring in Equation (68). Further we use the results that

$$\begin{aligned} \frac{(s+\alpha)^{1/2}}{(s+\beta)^{1/2}}\hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0] &= (\alpha+s)\frac{\hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0]}{(s+\alpha)^{1/2}(s+\beta)^{1/2}} \\ &\Rightarrow (\alpha+\partial_t)U_0(t,\tau;\alpha,\beta) \end{aligned} \tag{70}$$

and

$$\begin{aligned} \frac{(s+\beta)^{1/2}}{(s+\alpha)^{1/2}}\hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0] &= (\beta+s)\frac{\hat{G}[(s+\alpha)^{1/2}(s+\beta)^{1/2};0,0]}{(s+\alpha)^{1/2}(s+\beta)^{1/2}} \\ &\Rightarrow (\beta+\partial_t)U_0(t,\tau;\alpha,\beta), \end{aligned} \tag{71}$$

where the property

$$\hat{U}_0(t,\tau;\alpha,\beta) = \frac{\exp[-(s+\alpha)^{1/2}(s+\beta)^{1/2}\tau]}{(s+\alpha)^{1/2}(s+\beta)^{1/2}}$$

has been used. These results are used in the main text.