Transfer of transient signals between two fluid-filled boreholes embedded in an elastic formation — a closed-form solution

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A closed-form expression is obtained for the transient acoustic pressure in a borehole due to the action of a volume injection (acoustic monopole) source in another borehole in a typical cross-well seismic setting with a homogeneous isotropic solid formation. For the radiation of the elastic wavefield quantities into the formation, the Kirchhoff-Huygens integral representation is used. The acoustic pressure on the axis of the receiving borehole is evaluated through a suitable application of the fluid/solid acoustic reciprocity theorem. At the relatively low frequencies involved, the acoustic wave motion inside a fluid-filled borehole, which may be surrounded by a structure of perfectly bonded circularly cylindrical solid shells (casing), is dominated by tube waves. The excitation and propagation properties of the tube wave are modeled by regarding the borehole as an acoustic waveguide with a compliant inner wall. The corresponding elastic wavefield quantities at the outer borehole wall determine the surface sources. The entire analysis is carried out via an appropriate combination of a Laplace transformation with respect to time and Fourier transformations with respect to space. The closed-form representation for the received transient acoustic pressure is found by inspection. Various physical phenomena are described by the resulting expression, including post-critical conical waves for slow formations, in which the tube-wave speed exceeds the shear-wave speed in the formation, and tunneling-like phenomena for proximate boreholes in fast formations.

INTRODUCTION

Acoustic signals, as measured in cross-hole seismic experiments involving volume injection sources and/or acoustic pressure receivers, contain strong tube-wave related phenomena. This was first recognized by White and Sengbush (1963). For example, in so-called slow formations where the tube-wave speed exceeds the shear-wave speed in the solid formation, a tube wave propagating along a borehole excites strong conical shear-waves in the formation. De Bruin and Huizer (1992) have presented perhaps the most striking experimental observations of this phenomenon.

To model the acoustic wave motion inside a fluid-filled borehole, the acoustic radiation emanating from such a borehole and the full cross-hole transfer of acoustic signals, a wide variety of modeling methods has been employed. Here, we mention the far-field asymptotic methods to determine the wavefield radiated into the formation — Lee and Balch (1982) for fast formations, and Meredith (1990) for both fast and slow formations; the equivalent seismic source methods to determine the wavefield radiated into the formation — Ben Menahem and Kostek (1990) for fast formations and Kurkjian et al. (1992) for both fast and slow formations; the finite difference methods — Track and Daube (1992) for the full cross-hole acoustic signal transfer and Cheng et al. (1992) for the radiated wavefield in slow and fast formations; and the hybrid space-time-domain methods, in which the final expression is assembled out of the solutions to subproblems — White and Sengbush (1963) for the wavefield radiated into a fast formation and De Hoop et al. (1993) for the full cross-hole transfer of acoustic signals across slow and fast formations. As this brief overview does no justice to many other valuable contributions, we refer to the De Hoop et al. (1993) for a more detailed description of the literature.

In this paper, we present a method by which the transfer of transient tube-wave signals in cross-borehole experiments can completely be calculated in closed form. The corresponding space-time-domain received acoustic pressure follows by inspection. We perform the analysis analysis in the spectral

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domain via an appropriate combination of a Laplace transformation with respect to time and Fourier transformations with respect to the spatial coordinates. In doing so, we provide a unified short-cut with respect to the method employed by De Hoop et al. (1993).

DESCRIPTION OF THE CONFIGURATION AND FORMULATION OF THE PROBLEM

We investigate theoretically the signal transfer in a configuration consisting of two parallel, circularly cylindrical, fluid-filled boreholes embedded in a perfectly elastic, homogeneous, isotropic solid formation. To specify the position in the configuration, we employ the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian frame of reference, with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\underline{i}_1, \underline{i}_2, \underline{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. In accordance with the geophysical convention, $\underline{i}_3$ points vertically downwards. The subscript notation for Cartesian vectors and tensors is used. Lowercase Latin subscripts are used for this purpose; they are to be assigned the values 1, 2 and 3. Lowercase Greek subscripts are used to indicate the horizontal components of the Cartesian vectors and tensors; they are to be assigned the values 1 and 2. For the vertical component the subscript 3 is then written explicitly. To all repeated subscripts, the summation convention applies. The time coordinate is denoted by t. Partial differentiation with respect to x_p is denoted by ∂_p; ∂_t is a reserved symbol denoting partial differentiation with respect to time. Integration with respect to time is denoted by the symbol I_t.

At the instant $t = 0$ a point source of volume injection, located at $\underline{x} = \underline{x}^S$ on the axis of the source borehole, starts to generate the acoustic wave motion, which is initially at rest. An acoustic pressure point receiver located at $\underline{x} = \underline{x}^R$ on the axis of the receiving borehole measures the transferred acoustic signal. The domains occupied by the fluid columns inside the source and receiving boreholes are denoted as $\mathcal{B}_S^-$ and $\mathcal{B}_R^-$, while $b_{S;R}^-$ and $\Omega_{S;R}^-$ denote the pertaining radii and cross-sectional areas, respectively. In between the fluid columns and the solid formation a finite system of concentric circularly cylindrical shells, representing casing, cementing, etc, may be present. The domains occupied by the source and receiving holes, including these shells, are denoted as $\mathcal{B}_S^+$ and $\mathcal{B}_R^+$, while $b_{S;R}^+$ and $\Omega_{S;R}^+$ denote the pertaining radii and cross-sectional areas, respectively. The different layers in the shell structure are assumed to be perfectly bonded to one another and to the solid formation. The fluid/solid interfaces in the source and receiving holes are denoted as $\partial\mathcal{B}_S^-$ and $\partial\mathcal{B}_R^-$, respectively; the interfaces between the outermost shells of the source and receiving holes and the formation are denoted as $\partial\mathcal{B}_S^+$ and $\partial\mathcal{B}_R^+$, respectively. The unit vector $\underline{\nu}$ is oriented along the outward normal to the interfaces i.e., into the formation. Further, $\mathcal{L}_S$ and $\mathcal{L}_R$ are used to indicate the borehole axes of the source and receiving boreholes, respectively. The support of the homogeneous formation outside the boreholes is denoted as $\mathcal{F}$.

The vector $(\underline{x}^R - \underline{x}^S)$ has the Cartesian components $(d, 0, z)$, in which d and z are the horizontal and vertical offsets of the receiver with respect to the source, respectively. Further, we employ the spherical polar coordinates R and θ, where $d = R\sin(\theta)$ and $z = R\cos(\theta)$, in which θ is the angle between the vector $(\underline{x}^R - \underline{x}^S)$ and the vertical.

The linearized equation of motion and the deformation rate equation governing the acoustic wave motion in the borehole fluid in the presence of a point source of volume injection are given by

$$\partial_k p + \rho^f \partial_t w_k = 0, \tag{1a}$$

$$\partial_k w_k + \kappa^f \partial_t p = Q(t)\delta(\underline{x} - \underline{x}^S), \tag{1b}$$

in which p is the acoustic pressure (Pa), w_m is the particle velocity (m/s), ρ^f is the volume density of mass (kg/m^3), κ^f is the compressibility (Pa^{-1}) and $Q = Q(t)$ is the time-rate of volume injection (m^{3}s^{-1}) of the point source. The linearized source-free equation of motion and the deformation rate equation governing the wave motion in the source-free solid are given by

$$-\Delta_{kmpq}\partial_m \tau_{pq} + \rho^s \partial_t v_k = 0, \tag{2a}$$

$$\Delta_{ijmk}\partial_m v_k - S_{ijpq}\partial_t \tau_{pq} = 0, \tag{2b}$$

in which τ_{pq} is the dynamic stress (Pa), v_k is the particle velocity (m/s), ρ^s is the volume density of mass (kg/m^3) and S_{ijpq} is the compliance (Pa^{-1}). Further, $\Delta_{ijpq} = (\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})/2$ is the completely symmetric unit tensor of rank 4 and δ_{ij} is the Kronecker unit tensor of rank 2. The compliance is the inverse of the stiffness C_{ijpq} (Pa), i.e., $S_{ijpq}C_{pqkm} = \Delta_{ijkm}$. For an isotropic solid the stiffness is given by

$$C_{ijpq} = \lambda\delta_{ij}\delta_{pq} + 2\mu\Delta_{ijpq}, \tag{3}$$

where λ and μ (Pa) denote the Lamé coefficients.

Our method of analysis involves the use of a unilateral Laplace transformation with respect to time and a Fourier transformation of the Radon type with respect to the spatial coordinates. For any wavefield quantity Ψ and for any homogeneous subdomain $\mathcal{D}$ of the configuration with a smooth boundary $\partial\mathcal{D}$, we have

$$\hat{\Psi}(\underline{x}, s) = \int_{t=0^-}^{\infty} \exp(-st)\Psi(\underline{x}, t)\, dt \tag{4}$$

and

$$\tilde{\Psi}(\underline{\alpha}, s) = \int_{\underline{x}\in\mathcal{D}} \exp(is\alpha_m x_m)\hat{\Psi}(\underline{x}, s)\, dV, \tag{5}$$

in which s is taken real and positive, and $\underline{\alpha} \in \mathbb{R}^3$ Then, for vanishing initial conditions we have $\partial_t \to s$ and

$$\int_{\underline{x}\in\mathcal{D}} \exp(is\alpha_m x_m)\partial_n\hat{\Psi}\, dV$$

$$= -is\alpha_n\tilde{\Psi} + \int_{\underline{x}\in\partial\mathcal{D}} \hat{\Psi}\exp(is\alpha_m x_m)n_n\, dA, \tag{6}$$

where the unit vector $\underline{n}$ is oriented along the outward normal to $\partial\mathcal{D}$. For $\underline{x} \in \mathcal{D}$ we inversely have

$$\hat{\Psi}(\underline{x}, s) = \left(\frac{s}{2\pi}\right)^3 \int_{\underline{\alpha}\in\mathbb{R}^3} \exp(-is\alpha_m x_m)\tilde{\Psi}(\underline{\alpha}, s)\, dV. \tag{7}$$

THE WAVEFIELD RADIATED INTO THE FORMATION

Upon applying Eq (6) to Eq (2), with $\mathcal{D} = \mathcal{F}$, $\partial\mathcal{D} = \partial\mathcal{B}_S^+ \cup \partial\mathcal{B}_R^+$ and $n_m = -\nu_m$, we obtain

$$is\Delta_{kmpq}\alpha_m\tilde{\tau}_{pq} + \rho^s s\tilde{v}_k = \tilde{f}_k^{\partial\mathcal{B}_S^+} + \tilde{f}_k^{\partial\mathcal{B}_R^+}, \tag{8a}$$

$$-is\Delta_{ijnk}\alpha_n\tilde{v}_k - sS_{ijpq}\tilde{\tau}_{pq} = \tilde{h}_{ij}^{\partial\mathcal{B}_S^+} + \tilde{h}_{ij}^{\partial\mathcal{B}_R^+}. \tag{8b}$$

where the spectral-domain surface force sources and sources of deformation rate are given by

$$\tilde{f}_k^{\partial\mathcal{B}_{S;R}^+} = -\int_{\underline{x}\in\partial\mathcal{B}_{S;R}^+} \exp(is\alpha_m x_m)\Delta_{knpq}\hat{\tau}_{pq}\nu_n\, dA \tag{9}$$

and

$$\tilde{h}_{ij}^{\partial\mathcal{B}_{S;R}^+} = \int_{\underline{x}\in\partial\mathcal{B}_{S;R}^+} \exp(is\alpha_m x_m)\Delta_{ijnr}\hat{v}_r\nu_n\, dA, \tag{10}$$

respectively. Solving Eq (8) leads to

$$\tilde{v}_r = \tilde{G}_{rk}(\tilde{q}_k^{\partial\mathcal{B}_S^+} + \tilde{q}_k^{\partial\mathcal{B}_R^+}) \tag{11a}$$

$$\tilde{\tau}_{pq} = -C_{pqij}[i\alpha_i\tilde{v}_j + s^{-1}(\tilde{h}_{ij}^{\partial\mathcal{B}_S^+} + \tilde{h}_{ij}^{\partial\mathcal{B}_R^+})], \tag{11b}$$

in which the notional surface sources and Green's tensor are given by

$$\tilde{q}_k^{\partial\mathcal{B}_{S;R}^+} = \frac{s}{\rho^s}(\tilde{f}_k^{\partial\mathcal{B}_{S;R}^+} + C_{mkpq}i\alpha_m\tilde{h}_{pq}^{\partial\mathcal{B}_{S;R}^+}) \tag{12}$$

and

$$\tilde{G}_{rk} = \frac{1}{c_S^2}\tilde{G}_S\delta_{rk} - \alpha_r\alpha_k(\tilde{G}_P - \tilde{G}_S), \tag{13}$$

respectively, with

$$\tilde{G}_{P;S} = s^{-2}\left(\alpha_m\alpha_m + c_{P;S}^{-2}\right)^{-1}, \tag{14}$$

in which c_P and c_S denote the compressional and shear wavespeeds, respectively.

THE RECEIVED ACOUSTIC PRESSURE

To derive an expression for the acoustic pressure on the axis of the receiving hole, we employ the reciprocity theorem for acoustic wavefields in fluid/solid configurations (cf. De Hoop (1990)). In the reciprocity relation two acoustic states A and B occur. The surface interaction integrals relating these two states for the solid and for the fluid are denoted as

$$\hat{\mathcal{I}}^s(A,B) = \Delta_{ijpq}\int_{\underline{x}\in\partial\mathcal{B}_R^+} [-\hat{\tau}_{pq}^A\hat{v}_i^B + \hat{\tau}_{pq}^B\hat{v}_i^A]\nu_j\, dA, \tag{15a}$$

$$\hat{\mathcal{I}}^f(A,B) = \delta_{ij}\int_{\underline{x}\in\partial\mathcal{B}_R^-} [\hat{p}^A\hat{w}_i^B - \hat{p}^B\hat{w}_i^A]\nu_j\, dA, \tag{15b}$$

respectively. As state A we take the total wavefield in the receiving situation, and we neglect multiple scattering between the boreholes. The total wavefield in the formation $\{\hat{\tau}_{pq}^T, \hat{v}_k^T\}$, is written as the linear superposition of an incident wavefield $\{\hat{\tau}_{pq}^{\text{in}}, \hat{v}_k^{\text{in}}\}$ and a scattered wavefield $\{\hat{\tau}_{pq}^{\text{sc}}, \hat{v}_k^{\text{sc}}\}$. The corresponding total wavefield inside the fluid-filled receiving borehole is $\{\hat{p}^T, \hat{w}_k^T\}$. If the receiving borehole were absent, the incident wavefield would be the total wavefield in the formation. As state B, an auxiliary state, we take the acoustic wavefield as it would be generated by a point source of volume injection located on the axis of the receiving borehole where we want to evaluate the received acoustic pressure. The wavefield quantities associated with this auxiliary state at the inside and the outside of the fluid solid boundary boundary $\partial\mathcal{B}_R^-$ are denoted as $\{\hat{p}^R, \hat{w}_k^R\}$ and $\{\hat{v}_k^R, \hat{\tau}_{pq}^R\}$, respectively. In view of the superposition principle Eq (15a) leads to

$$\hat{\mathcal{I}}^s(T,R) = \hat{\mathcal{I}}^s(\text{in},R) + \hat{\mathcal{I}}^s(\text{sc},R). \tag{16}$$

Both the scattered and the auxiliary wavefields are source-free in the formation and satisfy the causality condition at infinity (outgoing waves). Application of the reciprocity relation relating these two states to the domain outside $\partial\mathcal{B}_R^+$ yields

$$\hat{\mathcal{I}}^s(\text{sc},R) = 0. \tag{17}$$

In view of the reciprocity of the elastostatic fields in the annular region in between $\partial\mathcal{B}_R^-$ and $\partial\mathcal{B}_R^+$, in both states T and R, and the continuity of the radial traction and the radial particle velocity across the interfaces, we further have $\hat{\mathcal{I}}^s(T,R) - \hat{\mathcal{I}}^f(T,R) = 0$. Since $\hat{q}^R = \delta(\underline{x} - \underline{x}^R)$, the reciprocity relation applied to the fluid domain yields

$$\hat{\mathcal{I}}^f(T,R) = \int_{\mathcal{B}_R^-} \hat{p}^T\hat{q}^R\, dV = \hat{p}^T(\underline{x}^R). \tag{18}$$

Upon combining Eqs (16)–(18) we arrive at

$$\hat{p}^T(\underline{x}^R) = \hat{\mathcal{I}}^s(\text{in},R) = \Delta_{ijpq}\int_{\underline{x}\in\partial\mathcal{B}_R^+} [-\hat{\tau}_{pq}^{\text{in}}\hat{v}_i^R + \hat{\tau}_{pq}^R\hat{v}_i^{\text{in}}]\nu_j\, dA. \tag{19}$$

Now, we use Eq (7) to express $\hat{v}_i^{\text{in}}$ and $\hat{\tau}_{pq}^{\text{in}}$ at the wall of the receiving borehole in terms of $\tilde{v}_i^{\text{in}}$ and $\tilde{\tau}_{pq}^{\text{in}}$. Interchanging the order of integration then leads to

$$\hat{p}^T(\underline{x}^R) = \left(\frac{s}{2\pi}\right)^3 \int_{\underline{\alpha}\in\mathbb{R}^3} \exp(-is\alpha_m x_m^R)\tilde{p}^T\, dV. \tag{20}$$

in which we have defined the spectral-domain acoustic pressure at the receiver as

$$\tilde{p}^T(\underline{x}^R) = \int_{\underline{x}\in\partial\mathcal{B}_R^+} \exp[is\alpha_m(x_m^R - x_m)]\Delta_{ijpq} \times [-\tilde{\tau}_{pq}^{\text{in}}\hat{v}_i^R + \hat{\tau}_{pq}^R\tilde{v}_i^{\text{in}}]\nu_j\, dA. \tag{21}$$

Next, we use the shift invariance of the auxiliary state in the vertical direction, viz.,

$$\{\hat{v}_i^R, \hat{\tau}_{pq}^R\}(\underline{x}, \underline{x}^R) = \{\hat{v}_i^R, \hat{\tau}_{pq}^R\}(\underline{x} - \underline{x}^R), \tag{22}$$

and the symmetry properties

$$\Delta_{ijpq}\hat{v}_i^R\nu_j\big|_{x_m - x_m^R} = \Delta_{ijpq}\hat{v}_i^R\nu_j\big|_{x_m^R - x_m}, \tag{23a}$$

$$\Delta_{ijpq}\hat{\tau}_{pq}^R\nu_j\big|_{x_m - x_m^R} = -\Delta_{ijpq}\hat{\tau}_{pq}^R\nu_j\big|_{x_m^R - x_m} \tag{23b}$$

to rewrite Eq (21) as

$$\tilde{p}^T(\underline{x}^R) = (-\tilde{h}_{pq}^{\partial\mathcal{B}^+_R}\tilde{\tau}_{pq}^{\text{in}} + \tilde{f}_k^{\partial\mathcal{B}^+_R}\tilde{v}_k^{\text{in}})\exp(-is\alpha_m x_m^R), \quad (24)$$

in which the surface force source and source of deformation rate of the auxiliary state occur that follow from Eqs (9) and (10), respectively. Substituting the expressions for the incident wavefield quantities, which follow from Eq (11) upon setting both $\tilde{q}_k^{\partial\mathcal{B}^+_R} = 0$ and $\tilde{h}_{ij}^{\partial\mathcal{B}^+_R} = 0$ (the single scattering approximation) into Eq (24) leads to

$$\tilde{p}^T = \frac{\rho^s}{s}\tilde{q}_i^{\partial\mathcal{B}^+_R}\tilde{G}_{ij}\tilde{q}_j^{\partial\mathcal{B}^+_S}\exp(-is\alpha_m x_m^R),$$

$$+\frac{1}{s}\tilde{h}_{pq}^{\partial\mathcal{B}^+_R}C_{pqij}\tilde{h}_{ij}^{\partial\mathcal{B}^+_S}\exp(-is\alpha_m x_m^R), \quad (25)$$

in which

$$\tilde{q}_i^{\partial\mathcal{B}^+_R} = \frac{s}{\rho^s}(\tilde{f}_i^{\partial\mathcal{B}^+_R} + C_{pqri}i\alpha_r\tilde{h}_{pq}^{\partial\mathcal{B}^+_R}) \quad (26)$$

is the notional surface source of the auxiliary state. The integral representation for $\hat{p}^T$ given by Eqs (20) and (25) is exact within the framework of the single scattering approximation.

THE EXCITATION AND PROPAGATION OF THE TUBE WAVE

In the low-frequency regime the axisymmetric wave motion in the borehole fluid is dominated by tube waves. As Marzetta and Schoenberg (1985) have demonstrated, the influence of the solid shells and formation on the propagation of the tube wave along a fluid-filled borehole surrounded by a perfectly bonded concentric shell structure may be accounted for by using the plane-strain elastostatic approximation for the field in the solid. To keep the notation transparent, we drop wherever possible, the indices S and R that indicate the source and receiving holes, respectively.

To determine the acoustic wave quantities associated with the tube-wave motion, we need the preliminary result that the Laplace-transform-domain wavefield quantities at the compliant inner borehole wall $\partial\mathcal{B}^-$ are interrelated via the radial wall stiffness η_w at $\partial\mathcal{B}^-$ according to

$$\hat{w}_k\nu_k\big|_{\underline{x}\in\partial\mathcal{B}^-} = \frac{sb^-}{\eta_w}\hat{p}\big|_{\underline{x}\in\partial\mathcal{B}^-} \quad (27)$$

In the quasi-static plane-strain approximation that we consider, the wall stiffness is independent of s.

Upon applying Eq (6) to Eq (1), with $\mathcal{D} = \mathcal{B}^-$, $\partial\mathcal{D} = \partial\mathcal{B}^-$ and $n_m = \nu_m$, we obtain a coupled system of algebraic equations for the spectral wavefield quantities in the fluid given by

$$-is\alpha_k\tilde{p} + \rho^f s\tilde{w}_k = \tilde{f}_k^{\partial\mathcal{B}^-}, \quad (28a)$$

$$-is\alpha_n\tilde{w}_n + \kappa^f s\tilde{p} = \hat{Q}\exp(is\alpha_m x_m^{S;R}) + \tilde{q}^{\partial\mathcal{B}^-}, \quad (28b)$$

where the spectral-domain surface force source and source of volume injection are given by

$$\tilde{f}_k^{\partial\mathcal{B}^-} = -\int_{\underline{x}\in\partial\mathcal{B}^-}\exp(is\alpha_m x_m)\,\hat{p}\,\nu_k\,dA \quad (29)$$

and

$$\tilde{q}^{\partial\mathcal{B}^-} = -\int_{\underline{x}\in\partial\mathcal{B}^-}\exp(is\alpha_m x_m)\,\hat{w}_n\,\nu_n\,dA, \quad (30)$$

respectively. In App A it is shown that in the low-frequency regime, we may replace $\tilde{p}$, $\tilde{f}_k^{\partial\mathcal{B}^-}$ and $\tilde{q}^{\partial\mathcal{B}^-}$ by approximate expressions involving the vertical Fourier transform of the acoustic pressure on the borehole axis only (cf Eq (A.3)). As a consequence, after elimination of $\tilde{w}_n$ from Eq (28) we obtain

$$\tilde{p} = \hat{Q}\rho^f c_B(sc_B)^{-1}(\alpha_3^2 + c_B^{-2})^{-1}\exp(is\alpha_m x_m^{S;R}). \quad (31)$$

The poles in Eq (31) represent up- and downgoing tube waves propagating with a tube-wave speed

$$c_B = c_f\left(1 + \frac{2c_f^2\rho^f}{\eta_w}\right)^{-1/2} \quad (32)$$

where $c_f = (\rho^f\kappa^f)^{-1/2}$ denotes the acoustic wavespeed in the fluid.

THE LOW-FREQUENCY-REGIME EXPRESSION FOR THE RECEIVED ACOUSTIC PRESSURE

In the low-frequency regime the radial stiffness at the inner borehole wall and the transfer of the acoustic wavefield quantities across the shell structure can be determined using a quasi-static plane-strain recurrence scheme (cf De Hoop et al. (1993)). Both the resulting traction and the particle velocity at $\partial\mathcal{B}^+$ are oriented along $\underline{\nu}$ and are given by

$$\left\{\begin{array}{l}\Delta_{mjpq}\nu_j\hat{\tau}_{pq} = \hat{\tau}\delta_{m\mu}\nu_\mu \\ \hat{v}_k = \delta_{k\kappa}\hat{v}_\kappa = -\dfrac{sb^+}{2\mu}\hat{\tau}\delta_{k\kappa}\nu_\kappa\end{array}\right. \quad \text{at } \partial\mathcal{B}^+, \quad (33)$$

in which $\hat{\tau}$ denotes the elastodynamic normal traction at the outer wall. Further, the amplitude of the traction is transferred from the inner to the outer borehole wall according to

$$\hat{\tau} = -T\hat{p}, \quad (34)$$

where T is the elastostatic traction transfer coefficient.

In App A it is shown that the spectral-domain surface sources of deformation rate and the notional surface sources may be approximated by

$$\tilde{h}_{ij}^{\partial\mathcal{B}^+_{S;R}} = T_{S;R}\frac{\Omega^+_{S;R}}{\Omega^-_{S;R}}\tilde{p}^{S;R}\frac{s}{2\mu}(\delta_{ij} - \delta_{i3}\delta_{j3}), \quad (35a)$$

$$\tilde{q}_i^{\partial\mathcal{B}^+_{S;R}} = T_{S;R}\frac{\Omega^+_{S;R}}{\Omega^-_{S;R}}\frac{s^2}{\rho^s}\tilde{p}^{S;R}(\frac{c_P^2}{c_S^2}\delta_{ik} - 2\delta_{i3}\delta_{k3})i\alpha_k, \quad (35b)$$

in which $\tilde{p}^S$ and $\tilde{p}^R$, follow from Eq (31), with $\hat{Q}_S = \hat{Q}$ and $\hat{Q}_R = 1$.

With Eq (35), Eq (25) is now, in principle, amenable to transformation back to the space-time domain. Before doing this, we cast the expression in a more transparent form. Considering the generic case in which c_{BS} and c_{BR} are different, we substitute Eqs (13), (35a) and (35b) into Eq (25) and contract over the repeated subscripts. The resulting expression can be simplified with the aid of several straightforward algebraic manipulations. This leads to

$$\tilde{p}^T = \hat{K}\left\{\frac{2c_{BS}^2c_{BR}^2}{c_{BS}+c_{BR}}\left[\frac{\tilde{h}_P(c_{BS}) - \tilde{h}_P(c_{BR})}{c_{BS}-c_{BR}}\right]\right.$$

$$+\frac{2c_{BS}^2c_{BR}^2}{c_{BS}+c_{BR}}\left[\frac{\tilde{h}_S(c_{BS}) - \tilde{h}_S(c_{BR})}{c_{BS}-c_{BR}}\right]$$

$$\left.+4s^{-1}(\tilde{G}_P - \tilde{G}_S)\exp(is\alpha_m x_m^S)\right\}, \quad (36)$$

in which

$$\hat{K} = s^2 \hat{Q}_S T_S T_R \frac{\Omega_S^+ \Omega_R^+}{\Omega_S^- \Omega_R^-} \frac{\rho_S^f \rho_R^f}{\rho^s}, \tag{37}$$

while

$$\hat{h}_{P;S}(c_B) = \frac{2A_{P;S}}{sc_B} \tilde{G}_{P;S}(\alpha_3^2 + c_B^2)^{-1} \exp(is\alpha_m x_m^S) \tag{38}$$

denote compressional and shear constituents and

$$A_P = c_B^{-3}\left(\frac{c_B^2}{2c_S^2} - 1\right)^2 \text{ and } A_S = c_B^{-3}\left(\frac{c_B^2}{c_S^2} - 1\right) \tag{39}$$

denote their respective amplitudes.

THE SPACE-TIME-DOMAIN ACOUSTIC WAVE MOTION

In this section, we perform the inverse Fourier transformation on $\tilde{p}^T$ (cf. Eq (20)) by inspection. From Eq (14) we infer that the Laplace-transform-domain counterpart of $s^{-1}\tilde{G}_{P;S}\exp(is\alpha_m x_m^S)$ is given by

$$s^{-1}\hat{G}_{P;S}(d,0,z,s) = s^{-1}\frac{\exp(-sR/c_{P;S})}{4\pi R}, \tag{40}$$

Its space-time-domain counterpart is

$$I_t G_{P;S}(d,0,z,t) = \frac{H(t - t_{P;S})}{4\pi R}, \tag{41}$$

where H denotes the Heaviside step function and $t_P = R/c_P$ and $t_S = R/c_S$ are the traveltimes of the compressional and shear waves, respectively.

In view of Eqs (38) and (40) we write $\hat{h}_{P;S}$ as a convolution along a borehole axis of a vertical tube-wave propagation term and a spherical cross-hole propagation term according to

$$\begin{aligned}\hat{h}_{P;S}(c_B) &= A_{P;S}\int_{\zeta=-\infty}^{\infty} \hat{G}_{P;S}(d,0,z-\zeta)\exp(-s|\zeta|/c_B)\,d\zeta \\ &= A_{P;S}[\hat{\chi}_{P;S}(z,c_B) + \hat{\chi}_{P;S}(-z,c_B)],\end{aligned} \tag{42}$$

in which

$$\hat{\chi}_{P;S}(\pm z, c_B) = \int_{\zeta=0}^{\infty} \hat{G}_{P;S}(d,0,\pm z-\zeta)\exp(-s\zeta/c_B)\,d\zeta \tag{43}$$

form the elementary building blocks that describe the kinematic aspects of the compressional and shear coupling of an up- and downgoing tube wave in one hole to the acoustic wave motion at a single point in the other hole, or vice versa.

Below, we shall determine the space-time-domain expression for the acoustic pressure for the general situation in which c_{BS} and c_{BR} could exceed both c_S and c_P, which entails that up to four (different) angles of conical refraction can occur. To this end, we rewrite Eq (43) as

$$\hat{\chi}_{P;S}(\pm z, c_B) = \int_{\zeta=0}^{\infty} \frac{\exp[-s\tau(\zeta)]}{4\pi[d^2 + (\pm z - \zeta)^2]^{1/2}}\,d\zeta, \tag{44}$$

in which

$$\tau(\zeta) = \zeta/c_B + [d^2 + (\pm z - \zeta)^2]^{1/2}/c_{P;S}. \tag{45}$$

Now, we cast Eq (44) in such a form that it can be recognized as a one-sided Laplace transformation of a function of time (cf Eq (4)), which — according to Lerch's theorem — can uniquely be identified with the space-time-domain constituent $\chi_{P;S}$. To do this, we partition the interval of integration such that τ is monotonic in each subinterval and subsequently take τ as the new variable of integration. We refer to De Hoop et al. (1993) for a detailed analysis, which leads to

$$\chi_{P;S}(z,c_B) = \begin{cases} \dfrac{2H(t-t_C) - H(t-t_{P;S})}{4\pi\left[(t - z/c_B)^2 - d^2\left(c_{P;S}^{-2} - c_B^{-2}\right)\right]^{1/2}} & \text{for } \begin{cases} c_B > c_{P;S} \text{ and} \\ z/d > \cot(\theta_C), \end{cases} \\[2ex] \dfrac{H(t-t_{P;S})}{4\pi\left[(t - z/c_B)^2 - d^2\left(c_{P;S}^{-2} - c_B^{-2}\right)\right]^{1/2}} & \text{otherwise,} \end{cases} \tag{46}$$

in which $\theta_C = \arccos(c_{P;S}/c_B)$ for $c_B > c_{P;S}$ is related to the angle of conical refraction α_C via $\theta_C = \pi/2 - \alpha_C$. In Eq (42), the constituents $\chi_{P;S}(z,c_B)$ and $\chi_{P;S}(-z,c_B)$ occur in pairs, and therefore the offset is pre-critical for $|z/d| < \cot(\theta_C)$ and post-critical for $|z/d| > \cot(\theta_C)$. For pre-critical offsets conical waves are absent, whereas for post-critical offsets a conical wave precedes the pertaining body wave. The traveltime of a conical wave is found to be

$$t_C = |z|/c_B + d(c_{P;S}^{-2} - c_B^{-2})^{1/2} \quad \begin{cases} c_B > c_{P;S} \text{ and} \\ |z/d| > \cot(\theta_C). \end{cases} \tag{47}$$

Having determined $G_{P;S}$ and $\chi_{P;S}$, the time-domain representation of the acoustic pressure at the receiver is obtained by replacing s by ∂_t; thus yielding

$$p^T = \partial_t^2 Q(t) \overset{t}{*} g_{SR}(t), \tag{48}$$

in which $\overset{t}{*}$ denotes a convolution with respect to time, while the cross-well acoustic-pressure/volume-injection-source Green's function is given by

$$\begin{aligned} g_{SR}(t) = T_S T_R \frac{\rho_S^f \rho_R^f}{\rho^s} \frac{\Omega_S^+ \Omega_R^+}{\Omega_S^- \Omega_R^-} &\times \left\{ \frac{2c_{BS}^2 c_{BR}^2}{c_{BS} + c_{BR}}\left[\frac{h_P(c_{BS}) - h_P(c_{BR})}{c_{BS} - c_{BR}}\right] + 4I_t G_P \right. \\ &\left. + \frac{2c_{BS}^2 c_{BR}^2}{c_{BS} + c_{BR}}\left[\frac{h_S(c_{BS}) - h_S(c_{BR})}{c_{BS} - c_{BR}}\right] - 4I_t G_S \right\}, \end{aligned} \tag{49}$$

where the compressional and shear constituents are given by

$$h_{P;S}(c_B) = A_{P;S}[\chi_{P;S}(z,c_B) + \chi_{P;S}(-z,c_B)], \tag{50}$$

The particular case in which the tube-wave speeds in the source and receiving boreholes are equal can be treated by considering the limit of $\tilde{p}^T$ for $c_{BS} \to c_{BR}$ (cf. De Hoop et al. (1993)).

SUMMARY

We have presented closed-form time-domain expressions for the acoustic pressure on the axis of a receiving borehole, due to the action of a point source of volume injection on the axis of a source borehole, on the assumption that multiple scattering effects may be omitted and that the traveltimes of the elastic waves in the formation over distances of the order of the borehole diameters can be neglected.

We have carried out a major part of the analysis leading to this result in the spectral domain, so as to streamline the mathematical framework presented in our previous paper (De Hoop et al. (1993)).

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APPENDIX A: APPROXIMATE EXPRESSIONS FOR SPECTRAL QUANTITIES

In the low-frequency regime we express the acoustic pressure inside a borehole in terms of its Taylor expansion in a cross-sectional plane about its axis according to

$$\begin{aligned}\hat{p}(\underline{x}, s) &= \hat{p}_0 + O[s^2(x_\mu - x_\mu^{S;R})(x_\mu - x_\mu^{S;R})/c_f^2] \\ &\approx \hat{p}_0 = \hat{p}(x_\mu^{S;R}, x_3, s). \end{aligned} \tag{A.1}$$

To approximate the Fourier transform of $\hat{p}$ and the surface integrals representing the spectral-domain surface sources given by Eqs (9), (10), (29) and (30), we replace the Fourier kernel by the first two terms of its Taylor expansion about the borehole axis, according to

$$\begin{aligned}\exp(is\alpha_m x_m) &\approx \exp(is\alpha_m x_m^{S;R})\exp[is\alpha_3(x_3 - x_3^{S;R})] \\ &\times [1 + is\alpha_\mu(x_\mu - x_\mu^{S;R})] \end{aligned} \tag{A.2}$$

As the radius of a borehole is much smaller than the distance between the boreholes, this approximation amounts to neglecting the traveltime across the borehole cross-sections.

Now, we substitute Eqs (A.1) and (A.2) into Eq (5) with $\hat{\Psi} = \hat{p}$ and employ the identities $\int dA = \Omega^-$ and $\int \alpha_\mu(x_\mu - x_\mu^{S;R})\, dA = 0$, where the integrations are carried out in a cross-sectional plane bounded by $\partial\mathcal{B}^-$. As a consequence, we obtain an approximate expression for $\tilde{p}$ in terms of a vertical Fourier transform, viz.

$$\tilde{p} = \Omega^- \int_{\underline{x}\in\mathcal{L}} \exp(is\alpha_m x_m)\hat{p}_0(\underline{x}, s)\, d\ell. \tag{A.3}$$

Next, we employ Eqs (33)–(34) to rewrite Eqs (9)–(10) according to

$$\tilde{f}_k^{\partial\mathcal{B}^+} = T \int_{\underline{x}\in\partial\mathcal{B}^+} \exp(is\alpha_m x_m)\hat{p}\nu_k\, dA \tag{A.4}$$

and

$$\tilde{h}_{ij}^{\partial\mathcal{B}^+} = T \int_{\underline{x}\in\partial\mathcal{B}^+} \exp(is\alpha_m x_m)\hat{p}\nu_i\nu_j\, dA, \tag{A.5}$$

respectively. Finally, we substitute Eq (27) into Eq (30) and Eqs (A.1) and (A.2) into Eqs (29), (30), (A.4) and (A.5) after which we employ the identities $\oint \nu_k\, d\ell = 0$, $\oint \nu_m\nu_n\, d\ell = \pi b^\pm(\delta_{mn} - \delta_{m3}\delta_{n3})$ and $\oint \nu_k\nu_m\nu_n\, d\ell = 0$, where the integrations are carried out along a cross-sectional circular boundary. As a consequence, the expressions for the spectral surface sources associated with the wave motion inside the fluid reduce to

$$\tilde{f}_k^{\partial\mathcal{B}^-} = \tilde{p}(\delta_{kn} - \delta_{k3}\delta_{n3})is\alpha_n, \tag{A.6}$$

$$\tilde{q}^{\partial\mathcal{B}^-} = 2\tilde{p}s\eta_w^{-1}, \tag{A.7}$$

respectively, while the spectral surface sources associated with the wave motion inside the solid formation reduce to

$$\tilde{f}_k^{\partial\mathcal{B}^+} = T\frac{\Omega^+}{\Omega^-}\tilde{p}(\delta_{kn} - \delta_{k3}\delta_{n3})is\alpha_n, \tag{A.8}$$

$$\tilde{h}_{ij}^{\partial\mathcal{B}^+} = T\frac{\Omega^+}{\Omega^-}\tilde{p}\frac{s}{2\mu}(\delta_{ij} - \delta_{i3}\delta_{j3}), \tag{A.9}$$

respectively. Upon substituting Eqs (A.8)–(A.9) into Eq (12) and into Eq (26) we finally obtain the expression for the spectral notional surface sources, given by Eq (35b).

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