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PHYSICS

ON INTEGRALS OCCURRING IN THE VARIATIONAL FORMULATION OF DIFFRACTION PROBLEMS

BY

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1. *Introduction*

If the variational formulation of scalar diffraction problems [1] is applied to the diffraction of a plane wave of wavelength $\lambda = 2\pi/k$ by a circular aperture of radius a or an infinite slit of finite width $2b$ in a plane screen, the following integrals are encountered

$$(1.1) \qquad R^{(1)}_{2\mu,2\nu} - iI^{(1)}_{2\mu,2\nu} = \int_0^\infty u^{-2}(u^2-\alpha^2)^{\frac{1}{2}} J_{2\mu}(u)\, J_{2\nu}(u)\, du,$$

in which $\mathrm{Re}(\mu+\nu) > \frac{1}{2}$, and

$$(1.2) \qquad R^{(2)}_{2\mu,2\nu} + iI^{(2)}_{2\mu,2\nu} = \int_0^\infty (u^2-\alpha^2)^{-\frac{1}{2}} J_{2\mu}(u)\, J_{2\nu}(u)\, du,$$

in which $\mathrm{Re}(\mu+\nu) > -\frac{1}{2}$. According to the diffraction problem considered, we have $\alpha = ka$ or $\alpha = kb$, with $\alpha > 0$. The square root in the integrands is defined as $(u^2-\alpha^2)^{\frac{1}{2}} > 0$, if $u > \alpha$ and $(u^2-\alpha^2)^{\frac{1}{2}} = -i(\alpha^2-u^2)^{\frac{1}{2}}$ with $(\alpha^2-u^2)^{\frac{1}{2}} > 0$, if $u < \alpha$. The superscripts 1 and 2 refer to the type of boundary value at the screen (vanishing of the wave function or of its normal derivative respectively).

The case of diffraction by a circular aperture [2] leads to $\mu = m+3/4$, $\nu = n+3/4$ in (1.1) and to $\mu = m+1/4$, $\nu = n+1/4$ in (1.2), with m and n non-negative integers. The case of diffraction by an infinite slit [3] leads to $\mu = m$, $\nu = n$, with m and n positive integers, and $\mu = m+\frac{1}{2}$, $\nu = n+\frac{1}{2}$, with m and n non-negative integers, in (1.1) and to $\mu = m$, $\nu = n$ and $\mu = m+\frac{1}{2}$, $\nu = n+\frac{1}{2}$, with m and n non-negative integers, in (1.2).

The results derived in the present paper are in the form of series in powers of α. In the special case that $\mu+\nu$ is an integer, these series are not power series in the strict sense because their coefficients contain $\log \alpha$.

Separating the right-hand sides of (1.1) and (1.2) into the parts that correspond, for real values of μ and ν (which are of special interest in the physical problems described above), to the real and imaginary parts of the integrals, we have

$$(1.3) \qquad R^{(1)}_{2\mu,2\nu} = \int_\alpha^\infty u^{-2}(u^2-\alpha^2)^{\frac{1}{2}} J_{2\mu}(u)\, J_{2\nu}(u)\, du,$$

$$(1.4)\qquad I^{(1)}_{2\mu,2\nu}=\int_0^\alpha u^{-2}(\alpha^2-u^2)^{\frac{1}{2}}\,J_{2\mu}(u)\,J_{2\nu}(u)\,du,$$

$$(1.5)\qquad R^{(2)}_{2\mu,2\nu}=\int_\alpha^\infty (u^2-\alpha^2)^{-\frac{1}{2}}\,J_{2\mu}(u)\,J_{2\nu}(u)\,du,$$

$$(1.6)\qquad I^{(2)}_{2\mu,2\nu}=\int_0^\alpha (\alpha^2-u^2)^{-\frac{1}{2}}\,J_{2\mu}(u)\,J_{2\nu}(u)\,du.$$

2. *The expansion of* $R^{(1)}_{2\mu,2\nu}-iI^{(1)}_{2\mu,2\nu}$

The simplest way to obtain the series expansion of $I^{(1)}_{2\mu,2\nu}$ is to substitute in (1.4) the power series for the product of two Bessel functions of the first kind [4], viz.

$$(2.1)\quad J_{2\mu}(u)\,J_{2\nu}(u)=\pi^{-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\mu+\nu+\frac{1}{2})\,\Gamma(r+\mu+\nu+1)\,u^{2r+2\mu+2\nu}}{\Gamma(r+1)\,\Gamma(r+2\mu+1)\,\Gamma(r+2\nu+1)\,\Gamma(r+2\mu+2\nu+1)}.$$

Changing the order of summation and integration and evaluating the resulting Eulerian integral of the first kind [5], we obtain

$$(2.2)\qquad I^{(1)}_{2\mu,2\nu}=\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\mu+\nu+\frac{1}{2})\,\Gamma(r+\mu+\nu-\frac{1}{2})\,\alpha^{2r+2\mu+2\nu}}{4\Gamma(r+1)\,\Gamma(r+2\mu+1)\,\Gamma(r+2\nu+1)\,\Gamma(r+2\mu+2\nu+1)}.$$

When $\mu+\nu=q$, where q is a positive integer, we rewrite (2.2), with $p=q+r$, as

$$(2.3)\quad I^{(1)}_{2\mu,2q-2\mu}=\sum_{p=q}^{\infty}\frac{(-)^{p-q}\,\Gamma(p+\frac{1}{2})\,\Gamma(p-\frac{1}{2})\,\alpha^{2p}}{4\Gamma(p-q+1)\,\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)}.$$

To derive the expansion of $R^{(1)}_{2\mu,2\nu}$ we use the Mellin integral [6] for the product of two Bessel functions of the first kind, viz.

$$(2.4)\quad J_{2\mu}(u)\,J_{2\nu}(u)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi^{-\frac{1}{2}}\,\Gamma(-s)\,\Gamma(s+\mu+\nu+\frac{1}{2})\,\Gamma(s+\mu+\nu+1)\,u^{2s+2\mu+2\nu}}{\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)}\,ds,$$

in which $\mathrm{Re}(\mu+\nu)>-\frac{1}{2}$; the path of integration, $\mathrm{Re}\,s=c$, lies in the strip $-\mathrm{Re}(\mu+\nu)-\frac{1}{2}<\mathrm{Re}\,s<0$. Substituting (2.4) in (1.3), changing the order of integration and evaluating the resulting Eulerian integral of the first kind, we obtain

$$(2.5)\quad R^{(1)}_{2\mu,2\nu}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)\,\Gamma(s+\mu+\nu+\frac{1}{2})\,\Gamma(s+\mu+\nu+1)\,\Gamma(-s-\mu-\nu)\,\alpha^{2s+2\mu+2\nu}}{4\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)\,\Gamma(-s-\mu-\nu+3/2)}\,ds,$$

where $\mathrm{Re}\,s=c$ has to satisfy both $-\frac{1}{2}<\mathrm{Re}(s+\mu+\nu)<\mathrm{Re}(\mu+\nu)$ and $\mathrm{Re}(s+\mu+\nu)<0$; the latter condition arises from the integration with respect to u. The poles of the integrand to the right of $\mathrm{Re}\,s=c$ are located at $s=r$ and $s=-\mu-\nu+r$ $(r=0, 1, 2, \ldots)$. If $\mu+\nu$ is not an integer, in both sets the poles are of the first order. Closing the contour towards the right, we learn from Jordan's lemma that the contribution of the large semicircle vanishes if its radius tends to infinity. Application of the theorem of residues gives

$$(2.6)\quad \begin{cases} R^{(1)}_{2\mu,2\nu} = \sum\limits_{r=0}^{\infty} \dfrac{(-)^r\, \Gamma(r+\mu+\nu+\frac{1}{2})\, \Gamma(r+\mu+\nu-\frac{1}{2}) \cos(\mu+\nu)\pi\, \alpha^{2r+2\mu+2\nu}}{4\Gamma(r+1)\, \Gamma(r+2\mu+1)\, \Gamma(r+2\nu+1)\, \Gamma(r+2\mu+2\nu+1) \sin(\mu+\nu)\pi} + \\ - \sum\limits_{r=0}^{\infty} \dfrac{(-)^r\, \Gamma(r+\frac{1}{2})\, \Gamma(r-\frac{1}{2})\, \alpha^{2r}}{4\Gamma(r-\mu-\nu+1)\, \Gamma(r+\mu-\nu+1)\, \Gamma(r-\mu+\nu+1)\, \Gamma(r+\mu+\nu+1) \sin(\mu+\nu)\pi}. \end{cases}$$

If, however, $\mu+\nu=q$, where q is a positive integer, the poles at $s=-q+p$ $(p=0, 1, 2, \ldots, q-1)$ are of the first order and the poles at $s=r$ $(r=0, 1, 2, \ldots)$ are of the second order. Application of the theorem of residues now leads to an expansion in powers of α, the coefficients of which contain $\log \alpha$. Evaluating the residues, we find

$$(2.7)\quad \begin{cases} R^{(1)}_{2\mu,2q-2\mu} = -\dfrac{1}{4\pi} \sum\limits_{p=0}^{q-1} \dfrac{\Gamma(-p+q)\, \Gamma(p+\frac{1}{2})\, \Gamma(p-\frac{1}{2})\, \alpha^{2p}}{\Gamma(p-q+2\mu+1)\, \Gamma(p+q-2\mu+1)\, \Gamma(p+q+1)} + \\ \quad + \dfrac{1}{4\pi} \sum\limits_{r=0}^{\infty} \dfrac{(-)^r\, \Gamma(r+q+\frac{1}{2})\, \Gamma(r+q-\frac{1}{2})\, \alpha^{2r+2q}}{\Gamma(r+1)\, \Gamma(r+2\mu+1)\, \Gamma(r+2q-2\mu+1)\, \Gamma(r+2q+1)} \cdot \\ \quad \cdot [2 \log \alpha + \psi(r+q+\frac{1}{2}) + \psi(r+q-\frac{1}{2}) - \psi(r+1) + \\ \quad - \psi(r+2\mu+1) - \psi(r+2q-2\mu+1) - \psi(r+2q+1)], \end{cases}$$

where $\psi(z)$ denotes the logarithmic derivative of $\Gamma(z)$. With $p=q+r$ in the second summation in the right-hand side of (2.7), the expression for $R^{(1)}_{2\mu,2q-2\mu}$ can be written as

$$(2.8)\quad \begin{cases} R^{(1)}_{2\mu,2q-2\mu} = -\dfrac{1}{4\pi} \sum\limits_{p=0}^{q-1} \dfrac{\Gamma(-p+q)\, \Gamma(p+\frac{1}{2})\, \Gamma(p-\frac{1}{2})\, \alpha^{2p}}{\Gamma(p-q+2\mu+1)\, \Gamma(p+q-2\mu+1)\, \Gamma(p+q+1)} + \\ \quad + \dfrac{1}{4\pi} \sum\limits_{p=q}^{\infty} \dfrac{(-)^{p-q}\, \Gamma(p+\frac{1}{2})\, \Gamma(p-\frac{1}{2})\, \alpha^{2p}}{\Gamma(p-q+1)\, \Gamma(p-q+2\mu+1)\, \Gamma(p+q-2\mu+1)\, \Gamma(p+q+1)} \cdot \\ \quad \cdot [2 \log \alpha + \psi(p+\frac{1}{2}) + \psi(p-\frac{1}{2}) - \psi(p-q+1) + \\ \quad - \psi(p-q+2\mu+1) - \psi(p+q-2\mu+1) - \psi(p+q+1)]. \end{cases}$$

Combining (2.2) with (2.6) and (2.3) with (2.8), we obtain as final results

$$(2.9)\quad \begin{cases} R^{(1)}_{2\mu,2\nu} - iI^{(1)}_{2\mu,2\nu} = \\ = \sum\limits_{r=0}^{\infty} \dfrac{(-)^r\, \Gamma(r+\mu+\nu+\frac{1}{2})\, \Gamma(r+\mu+\nu-\frac{1}{2}) \exp[-i(\mu+\nu)\pi]\, \alpha^{2r+2\mu+2\nu}}{4\Gamma(r+1)\, \Gamma(r+2\mu+1)\, \Gamma(r+2\nu+1)\, \Gamma(r+2\mu+2\nu+1) \sin(\mu+\nu)\pi} + \\ - \sum\limits_{r=0}^{\infty} \dfrac{(-)^r\, \Gamma(r+\frac{1}{2})\, \Gamma(r-\frac{1}{2})\, \alpha^{2r}}{4\Gamma(r-\mu-\nu+1)\, \Gamma(r+\mu-\nu+1)\, \Gamma(r-\mu+\nu+1)\, \Gamma(r+\mu+\nu+1) \sin(\mu+\nu)\pi}, \end{cases}$$

which holds if $\mathrm{Re}(\mu+\nu)>\frac{1}{2}$ and $\mu+\nu$ is not an integer, while

$$(2.10)\quad \begin{cases} R^{(1)}_{2\mu,2q-2\mu} - iI^{(1)}_{2\mu,2q-2\mu} = -\dfrac{1}{4\pi} \sum\limits_{p=0}^{q-1} \dfrac{\Gamma(-p+q)\, \Gamma(p+\frac{1}{2})\, \Gamma(p-\frac{1}{2})\, \alpha^{2p}}{\Gamma(p-q+2\mu+1)\, \Gamma(p+q-2\mu+1)\, \Gamma(p+q+1)} + \\ \quad + \dfrac{1}{4\pi} \sum\limits_{p=q}^{\infty} \dfrac{(-)^{p-q}\, \Gamma(p+\frac{1}{2})\, \Gamma(p-\frac{1}{2})\, \alpha^{2p}}{\Gamma(p-q+1)\, \Gamma(p-q+2\mu+1)\, \Gamma(p+q-2\mu+1)\, \Gamma(p+q+1)} \cdot \\ \quad \cdot [2 \log \alpha - \pi i + \psi(p+\frac{1}{2}) + \psi(p-\frac{1}{2}) - \psi(p-q+1) + \\ \quad - \psi(p-q+2\mu+1) - \psi(p+q-2\mu+1) - \psi(p+q+1)], \end{cases}$$

which holds if q is a positive integer.

3. *The expansion of* $R^{(2)}_{2\mu,2\nu}+iI^{(2)}_{2\mu,2\nu}$

To derive the expansion of $I^{(2)}_{2\mu,2\nu}$ we substitute (2.1) in (1.6). Changing the order of summation and integration, evaluating the resulting Eulerian integral of the first kind, we are led to

$$(3.1)\qquad I^{(2)}_{2\mu,2\nu}=\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\mu+\nu+\frac{1}{2})\,\Gamma(r+\mu+\nu+\frac{1}{2})\,\alpha^{2r+2\mu+2\nu}}{2\Gamma(r+1)\,\Gamma(r+2\mu+1)\,\Gamma(r+2\nu+1)\,\Gamma(r+2\mu+2\nu+1)}.$$

When $\mu+\nu=q$, where q is a non-negative integer, we rewrite (3.1), with $p=q+r$, so as to obtain

$$(3.2)\qquad I^{(2)}_{2\mu,2q-2\mu}=\sum_{p=q}^{\infty}\frac{(-)^{p-q}\,\Gamma(p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})\,\alpha^{2p}}{2\Gamma(p-q+1)\,\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)}.$$

To obtain the Mellin integral for $R^{(2)}_{2\mu,2\nu}$ we substitute (2.4) in (1.5), change the order of integration and evaluate the resulting Eulerian integral of the first kind. This procedure leads to

$$(3.3)\qquad R^{(2)}_{2\mu,2\nu}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)\,\Gamma(s+\mu+\nu+\frac{1}{2})\,\Gamma(s+\mu+\nu+1)\,\Gamma(-s-\mu-\nu)\,\alpha^{2s+2\mu+2\nu}}{2\Gamma(s+2\mu+1)\,\Gamma(s+2\nu+1)\,\Gamma(s+2\mu+2\nu+1)\,\Gamma(-s-\mu-\nu+\frac{1}{2})}\,ds,$$

where $\operatorname{Re} s=c$ lies in the common part of the strip $-\frac{1}{2}<\operatorname{Re}(s+\mu+\nu)<$ $<\operatorname{Re}(\mu+\nu)$ and the half-plane $\operatorname{Re}(s+\mu+\nu)<0$; the latter condition arises from the integration with respect to u. The poles of the integrand to the right of $\operatorname{Re} s=c$ are located at $s=r$ and at $s=-\mu-\nu+r$ $(r=0, 1, 2, \ldots)$. If $\mu+\nu$ is not an integer, in both sets the poles are of the first order. Upon closing the contour towards the right, the contribution of the large semicircle vanishes in virtue of Jordan's lemma. Application of the theorem of residues gives

$$(3.4)\qquad \begin{cases} R^{(2)}_{2\mu,2\nu}=-\displaystyle\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\mu+\nu+\frac{1}{2})\,\Gamma(r+\mu+\nu+\frac{1}{2})\cos(\mu+\nu)\pi\,\alpha^{2r+2\mu+2\nu}}{2\Gamma(r+1)\,\Gamma(r+2\mu+1)\,\Gamma(r+2\nu+1)\,\Gamma(r+2\mu+2\nu+1)\sin(\mu+\nu)\pi}+\\ \quad+\displaystyle\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\frac{1}{2})\,\Gamma(r+\frac{1}{2})\,\alpha^{2r}}{2\Gamma(r-\mu-\nu+1)\,\Gamma(r+\mu-\nu+1)\,\Gamma(r-\mu+\nu+1)\,\Gamma(r+\mu+\nu+1)\sin(\mu+\nu)\pi}.\end{cases}$$

If, however, $\mu+\nu=q$, where q is a non-negative integer, the poles at $s=-q+p$ $(p=0, 1, 2, \ldots, q-1)$ are of the first order and the poles at $s=r$ $(r=0, 1, 2, \ldots)$ are of the second order. Application of the theorem of residues now leads to

$$(3.5)\qquad \begin{cases} R^{(2)}_{2\mu,2q-2\mu}=\displaystyle\frac{1}{2\pi}\sum_{p=0}^{q-1}\frac{\Gamma(-p+q)\,\Gamma(p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})\,\alpha^{2p}}{\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)}+\\ \quad-\displaystyle\frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+q+\frac{1}{2})\,\Gamma(r+q+\frac{1}{2})\,\alpha^{2r+2q}}{\Gamma(r+1)\,\Gamma(r+2\mu+1)\,\Gamma(r+2q-2\mu+1)\,\Gamma(r+2q+1)}\cdot\\ \quad\cdot[2\log\alpha+2\psi(r+q+\frac{1}{2})-\psi(r+1)-\psi(r+2\mu+1)+\\ \qquad-\psi(r+2q-2\mu+1)-\psi(r+2q+1)].\end{cases}$$

With $p=q+r$ in the second summation in the right-hand side of (3.5), the expression for $R^{(2)}_{2\mu,2q-2\mu}$ can be rewritten as

$$(3.6)\quad \left\{\begin{aligned} R^{(2)}_{2\mu,2q-2\mu} = {} & \frac{1}{2\pi}\sum_{p=0}^{q-1}\frac{\Gamma(-p+q)\,\Gamma(p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})\,\alpha^{2p}}{\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)} + \\ & -\frac{1}{2\pi}\sum_{p=q}^{\infty}\frac{(-)^{p-q}\,\Gamma(p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})\,\alpha^{2p}}{\Gamma(p-q+1)\,\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)}\cdot \\ & \cdot[2\log\alpha+2\psi(p+\tfrac{1}{2})-\psi(p-q+1)-\psi(p-q+2\mu+1)+ \\ & -\psi(p+q-2\mu+1)-\psi(p+q+1)]. \end{aligned}\right.$$

Combining (3.1) with (3.4) and (3.2) with (3.6), we obtain as final results

$$(3.7)\quad \left\{\begin{aligned} & R^{(2)}_{2\mu,2\nu}+iI^{(2)}_{2\mu,2\nu} = \\ & = -\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\mu+\nu+\frac{1}{2})\,\Gamma(r+\mu+\nu+\frac{1}{2})\exp[-i(\mu+\nu)\pi]\,\alpha^{2r+2\mu+2\nu}}{2\Gamma(r+1)\,\Gamma(r+2\mu+1)\,\Gamma(r+2\nu+1)\,\Gamma(r+2\mu+2\nu+1)\sin(\mu+\nu)\pi} + \\ & +\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma(r+\frac{1}{2})\,\Gamma(r+\frac{1}{2})\,\alpha^{2r}}{2\Gamma(r-\mu-\nu+1)\,\Gamma(r+\mu-\nu+1)\,\Gamma(r-\mu+\nu+1)\,\Gamma(r+\mu+\nu+1)\sin(\mu+\nu)\pi}, \end{aligned}\right.$$

which holds if $\mathrm{Re}(\mu+\nu)>-\frac{1}{2}$ and $\mu+\nu$ is not an integer, while

$$(3.8)\quad \left\{\begin{aligned} R^{(2)}_{2\mu,2q-2\mu}+iI^{(2)}_{2\mu,2q-2\mu} = {} & \frac{1}{2\pi}\sum_{p=0}^{q-1}\frac{\Gamma(-p+q)\,\Gamma(p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})\,\alpha^{2p}}{\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)} + \\ & -\frac{1}{2\pi}\sum_{p=q}^{\infty}\frac{(-)^{p-q}\,\Gamma(p+\frac{1}{2})\,\Gamma(p+\frac{1}{2})\,\alpha^{2p}}{\Gamma(p-q+1)\,\Gamma(p-q+2\mu+1)\,\Gamma(p+q-2\mu+1)\,\Gamma(p+q+1)}\cdot \\ & \cdot[2\log\alpha-\pi i+2\psi(p+\tfrac{1}{2})-\psi(p-q+1)-\psi(p-q+2\mu+1)+ \\ & -\psi(p+q-2\mu+1)-\psi(p+q+1)], \end{aligned}\right.$$

which holds if q is a non-negative integer. Equation (3.7) has been obtained by Bouwkamp [7].

4. *Concluding remarks*

Finally, it may be remarked that the integrals (1.1) and (1.2) are analytic functions of μ and ν in the domain of convergence. Hence, the results (2.10) and (3.8) follow from (2.9) and (3.7) respectively by analytic continuation. This procedure is most easily carried out by differentiating (2.9) and (3.7) with respect to $\mu+\nu$, after having multiplied both sides of these equations with $\sin(\mu+\nu)\pi$. Taking the limit for $\mu+\nu=q$, where q is an integer in the domain of convergence of the integrals, we obtain the desired results.

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