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PHYSICS

VARIATIONAL FORMULATION OF TWO-DIMENSIONAL DIFFRACTION PROBLEMS WITH APPLICATION TO DIFFRACTION BY A SLIT

BY

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1. *Introduction*

In the present paper we deal with the diffraction of a time-harmonic electromagnetic wave by an aperture in a perfectly conducting plane screen of vanishing thickness. The aperture (which may consist of several disjoint parts) is restricted to be cylindrical in the y-direction, i.e., its boundaries are parallel to the y-axis of a rectangular coordinate system. If, moreover, only the y-component of either the electric or the magnetic field of the incident wave is different from zero, the total field is independent of y. In the first case (parallel polarization) the non-zero field components can be derived from E_y, the y-component of the electric field, and in the second case (perpendicular polarization) the non-zero field components can be derived from H_y, the y-component of the magnetic field. Hence, the solution of the diffraction problem is reduced to the determination of two quantities, viz. E_y and H_y, that satisfy: the two-dimensional wave equation, the appropriate boundary conditions at the screen, the edge condition and suitable conditions at infinity.

It will be shown that in the case of plane-wave excitation the complex amplitude of the far-zone diffracted field can be written in a stationary form which is of the well-known Levine-and-Schwinger type [1]. Finally, the variational formulation is applied to the diffraction of a plane wave by an infinite slit of finite width. The aperture distribution, the complex amplitude of the far-zone diffracted field and the transmission coefficient are determined up to and including the terms of relative order $(kb)^4$ (k=wave number, $2b$=width of the slit). For normal incidence our results agree with those of BOUWKAMP [2] and MÜLLER and WESTPFAHL [3].

2. *Integral equation and associated variational principle*

We consider the diffraction of an electromagnetic wave by a cylindrical aperture A in a perfectly conducting plane screen Σ of vanishing thickness, which coincides with the plane $z=0$; the boundaries of A are parallel to the y-axis. The surrounding medium is assumed to be homogeneous and isotropic with inductive capacities ε and μ. The harmonic time-dependence of the form $\exp.(-i\omega t)$ is omitted throughout.

In the first place we deal with the case of parallel polarization. The incident wave, coming from $z<0$, is then given by

$$\mathbf{E}^i=(0,\, u_1^i,\, 0), \quad ik(\mu/\varepsilon)^{\frac{1}{2}}\mathbf{H}^i=(-\partial u_1^i/\partial z,\, 0,\, \partial u_1^i/\partial x), \tag{2.01}$$

in which $k=\omega(\varepsilon\mu)^{\frac{1}{2}}$, with the corresponding total field

$$\mathbf{E}=(0,\, u_1,\, 0), \quad ik(\mu/\varepsilon)^{\frac{1}{2}}\mathbf{H}=(-\partial u_1/\partial z,\, 0,\, \partial u_1/\partial x). \tag{2.02}$$

The wave function u_1 can be written in the following form [4]:

$$\begin{cases} u_1(x,\, z)=u_1^i(x,\, z)-u_1^i(x,\, -z)+\varphi_1(x,\, -z), & \text{when } z \leqq 0, \\ u_1(x,\, z)=\varphi_1(x,\, z), & \text{when } z \geqq 0. \end{cases} \tag{2.03}$$

In (2.03) the function $\varphi_1(x, z)$, defined for $z \geqq 0$, has to satisfy: the two-dimensional wave equation

$$\partial^2\varphi_1/\partial x^2+\partial^2\varphi_1/\partial z^2+k^2\varphi_1=0, \tag{2.04}$$

the boundary condition $\varphi_1(x, 0)=0$ on Σ, Sommerfeld's radiation condition at infinity and the edge condition $\varphi_1(x, z)=O(D^{\frac{1}{2}})$ in the neighbourhood of a sharp edge, where D denotes the distance from the edge. In virtue of the variational principle to be stated later on, we choose for φ_1 a representation which expresses φ_1 in the value of $u_1(x, 0)$ in the aperture, viz.

$$\varphi_1(x,\, z)=-\tfrac{1}{2}i(\partial/\partial z)\int_A \varphi(x')H_0^{(1)}\,[k\{(x-x')^2+z^2\}^{\frac{1}{2}}]dx', \tag{2.05}$$

where $\varphi(x')=u_1(x', 0)$ and $H_0^{(1)}$ denotes the Hankel function of the first kind and order zero. The continuity of $\partial u_1/\partial z$ in the aperture requires

$$[\partial u_1^i/\partial z]_{z=0}=-\tfrac{1}{2}i\lim_{z\to+0}(\partial^2/\partial z^2)\int_A \varphi(x')\, H_0^{(1)}\,[k\{(x-x')^2+z^2\}^{\frac{1}{2}}]\, dx', \tag{2.06}$$

if $x \in A$. This leads, with (2.04), to the following differential-integral equation

$$[\partial u_1^i/\partial z]_{z=0}=2(k^2+\partial^2/\partial x^2)\int_A \varphi(x')\, G(x,\, x')dx', \tag{2.07}$$

if $x \in A$, where

$$G(x,\, x')=(i/4)H_0^{(1)}(k|x-x'|). \tag{2.08}$$

Using the condition $\varphi(x')=0$ on the rim of the aperture, integration by parts of the second term in the right-hand side of (2.07) gives the equivalent form

$$[\partial u_1^i/\partial z]_{z=0}=2k^2\int_A \varphi(x')\, G(x,\, x')\, dx'+2(\partial/\partial x)\int_A (\partial\varphi/\partial x')\, G(x,\, x')\, dx', \tag{2.09}$$

if $x \in A$. For plane-wave excitation,

$$u_1^i(x,\, z)=\exp\,[ik(x\sin\theta_0+z\cos\theta_0)], \tag{2.10}$$

the left-hand sides of (2.07) and (2.09) read

$$[\partial u_1^i/\partial z]_{z=0}=ik\cos\theta_0\exp\,(ikx\sin\theta_0). \tag{2.11}$$

To obtain the far-zone diffracted field, let (R,θ) be the polar coordinates of the point of observation at a large distance from the aperture. Inserting in (2.05) the asymptotic expansion of the Hankel function, we have

$$\varphi_1(x,z) \sim A_1(\theta,\theta_0)(8\pi kR)^{-\frac{1}{2}} \exp(ikR+i\pi/4), \tag{2.12}$$

with

$$A_1(\theta,\theta_0) = -2ik\cos\theta \int_A \varphi(\theta_0;x')\exp(-ikx'\sin\theta)dx', \tag{2.13}$$

in which $\varphi(\theta_0;x')$ denotes the aperture distribution due to an incident plane wave travelling in the direction θ_0. Multiplying through in (2.07) or (2.09) with $\varphi(\theta;x)$, using (2.11) and integrating over the aperture, we obtain

$$\left\{\begin{aligned} &ik\cos\theta_0 \int_A \varphi(\theta;x)\exp(ikx\sin\theta_0)dx = \\ &= 2\int_A dx \int_A [k^2\varphi(\theta_0;x')\varphi(\theta;x) - \{\partial\varphi(\theta_0;x')/\partial x'\}\{\partial\varphi(\theta;x)/\partial x\}]G(x,x')dx', \end{aligned}\right. \tag{2.14}$$

where in the second integral of the right-hand side we have used the condition $\varphi=0$ at the edge of the aperture. Owing to the symmetry in rand θ_0 of the right-hand side of (2.14) we find, with (2.13), the reciprocity θelation

$$A_1(\pi+\theta_0,\theta) = A_1(\pi+\theta,\theta_0), \tag{2.15}$$

which also follows directly from a theorem due to H. A. Lorentz. Dividing (2.14) by $A_1(\pi+\theta_0,\theta)A_1(\pi+\theta,\theta_0)$ and inverting, we find

$$\left\{\begin{aligned} &A_1(\pi+\theta_0,\theta) = A_1(\pi+\theta,\theta_0) = \\ &= -\frac{k^2\cos\theta\cos\theta_0 \int_A \varphi(\theta;x)\exp(ikx\sin\theta_0)\,dx \int_A \varphi(\theta_0;x')\exp(ikx'\sin\theta)\,dx'}{\int_A dx \int_A [k^2\varphi(\theta;x)\,\varphi(\theta_0;x') - \{\partial\varphi(\theta;x)/\partial x\}\,\{\partial\varphi(\theta_0;x')/\partial x'\}]\,G(x,x')\,dx'}. \end{aligned}\right. \tag{2.16}$$

It can be shown that the right-hand side of (2.16) is stationary with respect to independent variations of $\varphi(\theta;x)$ and $\varphi(\theta_0;x')$ about their correct values following from the differential-integral equation (2.07) (or its equivalent (2.09)); only those variations which satisfy the condition $\varphi=0$ at the edge of the aperture are admissible. For the analogous expression in three-dimensional scalar diffraction theory, see Bouwkamp [5].

In the second place we consider the case of perpendicular polarization. The incident wave is then given by

$$(\mu/\varepsilon)^{\frac{1}{2}}\mathbf{H}^i = (0,u_2^i,0), \quad ik\mathbf{E}^i = (\partial u_2^i/\partial z, 0, -\partial u_2^i/\partial x), \tag{2.17}$$

with the corresponding total field

$$(\mu/\varepsilon)^{\frac{1}{2}}\mathbf{H} = (0,u_2,0), \quad ik\mathbf{E} = (\partial u_2/\partial z, 0, -\partial u_2/\partial x). \tag{2.18}$$

The wave function u_2 can be written in the following form [4]:

$$\left\{\begin{aligned} &u_2(x,z) = u_2^i(x,z)+u_2^i(x,-z)-\varphi_2(x,-z), \quad \text{when } z<0, \\ &u_2(x,z) = \varphi_2(x,z), \quad \text{when } z>0. \end{aligned}\right. \tag{2.19}$$

In (2.19) the function $\varphi_2(x, z)$, defined for $z>0$, has to satisfy: the two-dimensional wave equation

$$\partial^2\varphi_2/\partial x^2+\partial^2\varphi_2/\partial z^2+k^2\varphi_2=0, \tag{2.20}$$

the boundary condition $\partial\varphi_2/\partial z=0$ on Σ, Sommerfeld's radiation condition at infinity and the edge condition $\varphi_2(x, z)=O(D^{-\frac{1}{2}})$ in the neighbourhood of a sharp edge, if D denotes the distance from the edge. In virtue of the variational principle for the polarization under consideration, we choose for φ_2 a representation which expresses φ_2 in the value of $\partial u_2/\partial z$ in the aperture, viz.

$$\varphi_2(x, z)=-\tfrac{1}{2}i\int_A \psi(x')H_0^{(1)}[k\{(x-x')^2+z^2\}^{\frac{1}{2}}]dx', \tag{2.21}$$

where $\psi(x')=[\partial u_2(x', z')/\partial z']_{z'=0}$. The continuity of u_2 in the aperture leads to the following (pure) integral equation

$$u_2^i(x, 0)=-2\int_A \psi(x')G(x, x')dx', \tag{2.22}$$

if $x\in A$, in which $G(x, x')$ is given by (2.08). For plane-wave excitation,

$$u_2^i(x, z)=\exp\,[ik(x\sin\theta_0+z\cos\theta_0)], \tag{2.23}$$

the left-hand side of (2.22) reads

$$u_2^i(x, 0)=\exp\,(ikx\sin\theta_0). \tag{2.24}$$

Inserting in (2.21) the asymptotic expansion of the Hankel function, we obtain for the far-zone diffracted field

$$\varphi_2(x, z)\sim A_2(\theta, \theta_0)(8\pi kR)^{-\frac{1}{2}}\exp\,(ikR+i\pi/4), \tag{2.25}$$

with

$$A_2(\theta, \theta_0)=-2\int_A \psi(\theta_0; x')\exp\,(-ikx'\sin\theta)dx', \tag{2.26}$$

where $\psi(\theta_0; x')$ denotes the aperture distribution due to an incident plane wave travelling in the direction θ_0. Multiplying through in (2.22) with $\psi(\theta; x)$, using (2.24) and integrating over the aperture, we obtain

$$\int_A \psi(\theta; x)\exp\,(ikx\sin\theta_0)dx=-2\int_A dx\int_A \psi(\theta; x)\,G(x, x')\,\psi(\theta_0; x')dx'. \tag{2.27}$$

In this case, too, we have the reciprocity relation

$$A_2(\pi+\theta_0, \theta)=A_2(\pi+\theta, \theta_0). \tag{2.28}$$

Dividing (2.27) by $A_2(\pi+\theta_0, \theta)A_2(\pi+\theta, \theta_0)$ and inverting, we find

$$\left\{\begin{aligned} &A_2(\pi+\theta_0, \theta)=A_2(\pi+\theta, \theta_0)= \\ &\quad=\frac{\int_A \psi(\theta; x)\exp\,(ikx\sin\theta_0)dx\int_A \psi(\theta_0; x')\exp\,(ikx'\sin\theta)\,dx'}{\int_A dx\int_A \psi(\theta; x)\,G(x, x')\,\psi(\theta_0; x')\,dx'}.\end{aligned}\right. \tag{2.29}$$

It can be shown that the right-hand side of (2.29) is stationary with respect to independent variations of $\psi(\theta; x)$ and $\psi(\theta_0; x')$ about their correct values following from the integral equation (2.22); in this case the variations are not restricted by a condition at the edge of the aperture. For the analogous expression in three-dimensional scalar diffraction theory, see BOUWKAMP [6].

3. *The transmission cross-section*

The transmission cross-section is defined as the ratio of the average power transmitted through the aperture to the average incident power per unit area normal to the direction of propagation of the incident wave. The average transmitted power for parallel polarization is given by

$$P_1 = \tfrac{1}{2}\,\mathrm{Re}(i/\omega\mu) \int_A u_1[\partial u_1^*/\partial z']_{z'=0}\, dx', \tag{3.01}$$

where the asterisk denotes the complex conjugate quantity. However, from (2.03), we have

$$[\partial u_1/\partial z']_{z'=0} = [\partial u_1^i/\partial z']_{z'=0} = ik \cos\theta_0 \exp(ikx' \sin\theta_0), \tag{3.02}$$

if $x' \in A$. Hence, eq. (3.01) can be rewritten as

$$P_1 = -(4\omega\mu)^{-1}\,\mathrm{Im}\, A_1(\theta_0, \theta_0). \tag{3.03}$$

In the case of perpendicular polarization the average transmitted power is

$$P_2 = \tfrac{1}{2}\, Re(i/\omega\mu) \int_A u_2^*[\partial u_2/\partial z']_{z'=0}\, dx'. \tag{3.04}$$

In this case, from (2.19), we have

$$u_2(x', 0) = u_2^i(x', 0) = \exp(ikx' \sin\theta_0), \tag{3.05}$$

if $x' \in A$. Hence, eq. (3.04) can be rewritten as

$$P_2 = -(4\omega\mu)^{-1}\,\mathrm{Im}\, A_2(\theta_0, \theta_0). \tag{3.06}$$

For both polarizations the average incident power per unit area normal to the direction of propagation of the incident plane wave is

$$P^i = \tfrac{1}{2}(\varepsilon/\mu)^{\frac{1}{2}}. \tag{3.07}$$

From (3.03), (3.06) and (3.07) the transmission cross-section turns out to be

$$\sigma_{1,2}(\theta_0) = -\frac{1}{2k}\,\mathrm{Im}\, A_{1,2}(\theta_0, \theta_0). \tag{3.08}$$

A stationary expression for $\sigma_{1,2}$ can be obtained by using the stationary expression for $A_{1,2}(\theta_0, \theta_0)$ which follows from either (2.16) or (2.29). The relation (3.08) is analogous to the one in three-dimensional scalar diffraction theory; see LEVINE and SCHWINGER [7] and BOUWKAMP [8].

4. *Diffraction by a slit. Polarization parallel to the edge of the slit*

The variational formulation will now be applied to the diffraction of a plane wave by an infinite slit located at $z=0$, $-b<x<b$. For parallel polarization, the relevant aperture distribution $\varphi(x) = u_1(x, 0)$ will be expanded in terms of the eigenfunctions of Laplace's equation

in the coordinates of the elliptic cylinder, viz. cos $[n \arcsin (x/b)]$ and sin $[n \arcsin (x/b)]$. This expansion is adapted to diffraction by narrow slits, as Laplace's equation is the limiting form of the wave equation for $kb \to 0$. In view of the boundary condition at the screen there results

$$\text{(4.01)} \qquad \begin{cases} \varphi(\theta_0; x) = \sum_{n=0}^{\infty} \{a_{2n+1} \cos [(2n+1) \arcsin (x/b)] + \\ \qquad\qquad + i a_{2n} \sin [2n \arcsin (x/b)]\}, \end{cases}$$

where the coefficients a_n are independent of x and the factor i in the second summation is added for convenience. It may be remarked that each term of the expansion (4.01) satisfies the edge condition. As the variational formulation will only be used to obtain the exact solution, it will be sufficient to require the expression for, e.g., $A_1(\theta_0, \theta_0)$ to be stationary. From (2.16) it is clear that for this special choice the aperture distributions $\varphi(\theta_0; x)$ and $\varphi(\pi+\theta_0; x)$ are needed; however, in virtue of the symmetry of the configuration, we have: $\varphi(\pi+\theta_0; x) = \varphi(\theta_0; -x)$. The stationary expression for $A_1(\theta_0, \theta_0)$ now reads

$$\text{(4.02)} \qquad \begin{cases} A_1(\theta_0, \theta_0) = \\ = \dfrac{k^2 \cos^2 \theta_0 \int_{-b}^{b} \varphi(\theta_0; x) \exp(-ikx \sin \theta_0)\, dx \int_{-b}^{b} \varphi(\theta_0; -x') \exp(ikx' \sin \theta_0)\, dx'}{\int_{-b}^{b} dx \int_{-b}^{b} [k^2 \varphi(\theta_0; x)\, \varphi(\theta_0; -x') - \{\partial\varphi(\theta_0; x)/\partial x\}\{\partial\varphi(\theta_0; -x')/\partial x'\}]\, G(x, x') dx'}. \end{cases}$$

Using the expansion (4.01) and a well-known integral representation of the Bessel function of the first kind, we obtain

$$\text{(4.03)} \qquad \int_{-b}^{b} \varphi(\theta_0; x) \exp(-ikx \sin \theta_0)\, dx = (\pi/k \sin \theta_0) \sum_{n=1}^{\infty} n a_n J_n(kb \sin \theta_0).$$

In a similar way the denominator of (4.02) is reduced to

$$\text{(4.04)} \qquad \begin{cases} \int_{-b}^{b} dx \int_{-b}^{b} [k^2 \varphi(\theta_0; x)\, \varphi(\theta_0; -x') - \{\partial\varphi(\theta_0; x)/\partial x\}\{\partial\varphi(\theta_0; -x')/\partial x'\}]\, G(x, x')\, dx' = \\ \qquad = -(\pi/4) \int_{-\infty}^{\infty} \lambda^{-2}(\lambda^2-k^2)^{\frac{1}{2}} [\sum_{n=1}^{\infty} n a_n J_n(\lambda b)]^2\, d\lambda, \end{cases}$$

where we have used the integral representation

$$\text{(4.05)} \qquad H_0^{(1)}(k|x-x'|) = (\pi i)^{-1} \int_{-\infty}^{\infty} (\lambda^2-k^2)^{-\frac{1}{2}} \exp[-i\lambda(x-x')]\, d\lambda.$$

The square roots in (4.04) and (4.05) are defined as $(\lambda^2-k^2)^{\frac{1}{2}} > 0$ if $|\lambda| > k$ and $(\lambda^2-k^2)^{\frac{1}{2}} = -i(k^2-\lambda^2)^{\frac{1}{2}}$, with $(k^2-\lambda^2)^{\frac{1}{2}} > 0$, if $|\lambda| < k$, (λ real). If we substitute

$$\text{(4.06)} \qquad c_n = n a_n,$$

and

$$\text{(4.07)} \qquad d_{m,n} = d_{n,m} = -\tfrac{1}{2} \int_{-\infty}^{\infty} \lambda^{-2}(\lambda^2-k^2)^{\frac{1}{2}} J_m(\lambda b)\, J_n(\lambda b)\, d\lambda,$$

eq. (4.02) can be rewritten as

$$A_1(\theta_0, \theta_0) = \frac{\pi \cot^2 \theta_0 \, [\sum_{n=1}^{\infty} c_n J_n(kb \sin \theta_0)]^2}{\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} d_{m,n} c_m c_n}. \tag{4.08}$$

Let us now apply independent first-order variations δc_n to the correct values of c_n. In view of the stationary character of (4.08), the condition $\delta A_1 = 0$ leads to the following set of linear equations in c_n:

$$A_1(\theta_0, \theta_0) \sum_{n=1}^{\infty} d_{m,n} c_n = 2\pi \cot^2 \theta_0 \, J_m(kb \sin \theta_0) \sum_{n=1}^{\infty} c_n J_n(kb \sin \theta_0), \quad (m = 1, 2, 3, \ldots). \tag{4.09}$$

On the other hand, from the non-stationary expression (2.13), we have

$$A_1(\theta_0, \theta_0) = -\, 2\pi i \cot \theta_0 \sum_{n=1}^{\infty} c_n J_n(kb \sin \theta_0). \tag{4.10}$$

Combining (4.09) and (4.10), we obtain

$$\sum_{n=1}^{\infty} d_{m,n} c_n = i \cot \theta_0 \, J_m(kb \sin \theta_0), \qquad (m = 1, 2, 3, \ldots). \tag{4.11}$$

Now, from (4.07), it is clear that $d_{m,n} = 0$ if $m + n$ is odd; hence, the system of equations (4.11) breaks down into two independent systems of equations, viz.

$$\sum_{n=0}^{\infty} d_{2m+1,2n+1} \, c_{2n+1} = i \cot \theta_0 \, J_{2m+1}(kb \sin \theta_0), \qquad (m = 0, 1, 2, \ldots), \tag{4.12}$$

$$\sum_{n=1}^{\infty} d_{2m,2n} \, c_{2n} = i \cot \theta_0 \, J_{2m}(kb \sin \theta_0), \qquad (m = 1, 2, 3, \ldots). \tag{4.13}$$

These systems of equations show properties which are very similar to those of the analogous system of equations encountered in the variational formulation of the scalar diffraction by a circular aperture [9]. Using the power-series expansion of the Bessel function of the first kind and expanding [10] $d_{m,n}$ in series of powers of kb the coefficients of which contain log kb, we arrive at the following results:

$$\left\{\begin{aligned}
a_1 = c_1 &= -\, ikb \cos \theta_0 \Big[1 - \tfrac{1}{4}\Big(p - \tfrac{3}{4} + \tfrac{1}{2} \sin^2 \theta_0\Big) (kb)^2 + \\
&\quad + \tfrac{1}{16}\Big(p^2 - \tfrac{5}{4}\, p + \tfrac{1}{2}\, p \sin^2 \theta_0 + \tfrac{7}{16} - \tfrac{1}{3} \sin^2 \theta_0 + \\
&\quad + \tfrac{1}{12} \sin^4 \theta_0\Big) (kb)^4 + \ldots\Big], \\
a_3 = \tfrac{1}{3}\, c_3 &= -\, \tfrac{1}{48}\, i(kb)^3 \cos \theta_0 \Big[1 + 2 \sin^2 \theta_0 + \\
&\quad - \tfrac{1}{8}\Big(3p - \tfrac{15}{8} + \tfrac{1}{2} \sin^2 \theta_0 + \sin^4 \theta_0\Big) (kb)^2 + \ldots\Big], \\
a_5 = \tfrac{1}{5}\, c_5 &= -\, \tfrac{1}{5120}\, i(kb)^5 \cos \theta_0 \Big[1 + \tfrac{4}{3} \sin^2 \theta_0 + \tfrac{8}{3} \sin^4 \theta_0 + \ldots\Big],
\end{aligned}\right. \tag{4.14}$$

and

$$(4.15)\quad \begin{cases} a_2 = \frac{1}{2}c_2 = -\frac{1}{4}i(kb)^2 \cos\theta_0 \sin\theta_0 \left[1 + \frac{1}{12}(1-\sin^2\theta_0)(kb)^2 + \ldots\right], \\ a_4 = \frac{1}{4}c_4 = -\frac{1}{384}i(kb)^4 \cos\theta_0 \sin\theta_0 [1 + 2\sin^2\theta_0 + \ldots], \end{cases}$$

in which $p = \log(\gamma kb/4) - \pi i/2$ and $\log\gamma = 0.577215\ldots$ (Euler's constant). All other coefficients a_n contain only powers of kb higher than the fifth. To the same order of accuracy the complex amplitude of the far-zone diffracted field is given by

$$(4.16)\quad \begin{cases} A_1(\theta, \theta_0) = \\ \quad = -\pi(kb)^2 \cos\theta_0 \cos\theta \left[1 - \frac{1}{8}\left\{2p - \frac{3}{2} + \sin^2\theta_0 - \sin\theta_0 \sin\theta + \right.\right. \\ \quad + \left.\sin^2\theta\right\}(kb)^2 + \frac{1}{16}\left\{p^2 - \frac{5}{4}p + \frac{1}{2}p\sin^2\theta_0 + \frac{7}{16} + \right. \\ \quad - \frac{1}{3}\sin^2\theta_0 + \frac{1}{12}\sin^4\theta_0 + \frac{1}{6}(1-\sin^2\theta_0)\sin\theta_0\sin\theta + \\ \quad + \frac{1}{2}\left(p - \frac{2}{3} + \frac{2}{3}\sin^2\theta_0\right)\sin^2\theta - \frac{1}{6}\sin\theta_0\sin^3\theta + \\ \quad + \left.\left.\frac{1}{12}\sin^4\theta\right\}(kb)^4 + \ldots\right]. \end{cases}$$

The corresponding transmission coefficient τ_1, which is defined as the ratio of the average power transmitted through the aperture to the average power transmitted through the aperture in the sense of geometrical optics, can be determined from

$$(4.17)\qquad \tau_1 = \frac{1}{16\,\pi kb \cos\theta_0} \int_{-\pi/2}^{\pi/2} |A_1(\theta, \theta_0)|^2\, d\theta.$$

As a check on the calculations the relation

$$(4.18)\qquad \tau_1 = -\frac{1}{4\,kb\cos\theta_0}\,\mathrm{Im}\, A_1(\theta_0, \theta_0),$$

which follows from (3.08), may be used. The result is found to be

$$(4.19)\quad \begin{cases} \tau_1 = \frac{\pi^2(kb)^3\cos\theta_0}{32}\left[1 + \frac{1}{2}\left\{-\log(\gamma kb/4) + \frac{5}{8} - \frac{1}{2}\sin^2\theta_0\right\}(kb)^2 + \right. \\ \quad + \frac{1}{16}\left\{3\log^2(\gamma kb/4) - \frac{7}{2}\log(\gamma kb/4) + 2\sin^2\theta_0 \log(\gamma kb/4) + \right. \\ \quad + \left.\left.\frac{109}{96} - \frac{\pi^2}{4} - \frac{17}{16}\sin^2\theta_0 + \frac{5}{12}\sin^4\theta_0\right\}(kb)^4 + \ldots\right]. \end{cases}$$

For normal incidence, $\theta_0 = 0$, our results (4.14), (4.15), (4.16) and (4.19) agree with those of Bouwkamp [2] and Müller and Westpfahl [3].

5. *Diffraction by a slit. Polarization perpendicular to the edge of the slit*

For perpendicular polarization the expansion of the relevant aperture distribution $\psi(x)$, which follows from separation of variables in Laplace's equation in the coordinates of the elliptic cylinder and the boundary condition at the screen, reads

$$(5.01)\quad \begin{cases} \psi(\theta_0; x) = \sum_{n=0}^{\infty} \Big\{ C_{2n} \dfrac{\cos[2n \arcsin(x/b)]}{(1-x^2/b^2)^{\frac{1}{2}}} + \\ \qquad + iC_{2n+1} \dfrac{\sin[(2n+1)\arcsin(x/b)]}{(1-x^2/b^2)^{\frac{1}{2}}} \Big\}. \end{cases}$$

In this case, too, each term of the expansion (5.01) satisfies the edge condition. The stationary expression for $A_2(\theta_0, \theta_0)$ becomes

$$(5.02)\quad A_2(\theta_0, \theta_0) = \frac{\int_{-b}^{b} \psi(\theta_0; x) \exp(-ikx \sin\theta_0)\, dx \int_{-b}^{b} \psi(\theta_0; -x') \exp(ikx' \sin\theta_0)\, dx'}{\int_{-b}^{b} dx \int_{-b}^{b} \psi(\theta_0; x)\, G(x, x')\, \psi(\theta_0; -x')\, dx'},$$

where we used the relation $\psi(\pi+\theta_0; x) = \psi(\theta_0; -x)$. With the expansion (5.01) we obtain

$$(5.03)\quad \int_{-b}^{b} \psi(\theta_0; x) \exp(-ikx \sin\theta_0)\, dx = \pi b \sum_{n=0}^{\infty} C_n J_n(kb \sin\theta_0).$$

In a similar way the denominator of (5.02) is reduced to

$$(5.04)\quad \begin{cases} \int_{-b}^{b} dx \int_{-b}^{b} \psi(\theta_0; x)\, G(x, x')\, \psi(\theta_0; -x')\, dx' = \\ \qquad = (\pi b^2/4) \int_{-\infty}^{\infty} (\lambda^2 - k^2)^{-\frac{1}{2}} \Big[\sum_{n=0}^{\infty} C_n J_n(\lambda b)\Big]^2 d\lambda, \end{cases}$$

where we have used the integral representation (4.05). The square root in the right-hand side of (5.04) is defined in the same way as in section 4. If we make the substitution

$$(5.05)\quad D_{m,n} = D_{n,m} = -\tfrac{1}{2} \int_{-\infty}^{\infty} (\lambda^2 - k^2)^{-\frac{1}{2}} J_m(\lambda b)\, J_n(\lambda b)\, d\lambda,$$

eq. (5.02) can be rewritten as

$$(5.06)\quad A_2(\theta_0, \theta_0) = -\frac{2\pi \Big[\sum_{n=0}^{\infty} C_n J_n(kb \sin\theta_0)\Big]^2}{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{m,n} C_m C_n}.$$

Applying independent first-order variations δC_n to the correct values of C_n, the condition $\delta A_2 = 0$ leads to the following system of linear equations in C_n:

$$(5.07)\quad A_2(\theta_0, \theta_0) \sum_{n=0}^{\infty} D_{m,n} C_n = -2\pi J_m(kb \sin\theta_0) \sum_{n=0}^{\infty} C_n J_n(kb \sin\theta_0), \qquad (m = 0, 1, 2, \ldots).$$

On the other hand, from the non-stationary expression (2.26), we have

$$(5.08)\quad A_2(\theta_0, \theta_0) = -2\pi b \sum_{n=0}^{\infty} C_n J_n(kb \sin\theta_0).$$

Combining (5.07) and (5.08), we obtain

$$\sum_{n=0}^{\infty} D_{m,n} C_n = \frac{1}{b} J_m(kb \sin\theta_0), \qquad (m = 0, 1, 2, \ldots). \tag{5.09}$$

From (5.05) it is clear that $D_{m,n}=0$ if $m+n$ is odd; hence, eq. (5.09) breaks down into two independent systems of linear equations, viz.

$$\sum_{n=0}^{\infty} D_{2m,2n} C_{2n} = \frac{1}{b} J_{2m}(kb \sin\theta_0), \qquad (m = 0, 1, 2, \ldots), \tag{5.10}$$

$$\sum_{n=0}^{\infty} D_{2m+1,2n+1} C_{2n+1} = \frac{1}{b} J_{2m+1}(kb \sin\theta_0), \qquad (m = 0, 1, 2, \ldots). \tag{5.11}$$

The properties of these systems of equations are similar to those of the corresponding systems of equations of section 4. Using the power-series expansion of the Bessel function of the first kind and expanding [10] $D_{m,n}$ in series of powers of kb the coefficients of which contain $\log kb$, we arrive at the following results:

$$\left\{\begin{aligned} bC_0 &= \frac{1}{p} + \frac{1}{4p}\left(1-\sin^2\theta_0\right)(kb)^2 + \frac{1}{64}\left\{\frac{1}{2p^2} + \frac{1}{p}\left(\frac{3}{4} - 3\sin^2\theta_0 + \right.\right. \\ &\qquad \left.\left. + \sin^4\theta_0\right) - (1-2\sin^2\theta_0)\right\}(kb)^4 + \ldots, \\ bC_2 &= \frac{1}{4}\left(\frac{1}{2p} + 1 - 2\sin^2\theta_0\right)(kb)^2 + \left\{\frac{1}{32p}(1-\sin^2\theta_0) + \right. \\ &\qquad \left. - \frac{1}{48}(1+\sin^2\theta_0 - 2\sin^4\theta_0)\right\}(kb)^4 + \ldots, \\ bC_4 &= \frac{1}{512}\left\{\frac{1}{p} + \frac{4}{3}(1 + 4\sin^2\theta_0 - 8\sin^4\theta_0)\right\}(kb)^4 + \ldots, \end{aligned}\right. \tag{5.12}$$

and

$$\left\{\begin{aligned} bC_1 &= -\,kb\sin\theta_0\left[1 + \frac{1}{4}\left\{p + \frac{1}{4} - \frac{1}{2}\sin^2\theta_0\right\}(kb)^2 + \ldots\right], \\ bC_3 &= \frac{1}{16}(kb)^3\sin\theta_0\,[1 - 2\sin^2\theta_0 + \ldots], \end{aligned}\right. \tag{5.13}$$

in which p has the same meaning as in section 4. All other coefficients C_n contain only powers of kb higher than the fourth. To the same order of accuracy the complex amplitude of the far-zone diffracted field is given by

$$\left\{\begin{aligned} &A_2(\theta,\theta_0) = \\ &\quad = -\,2\pi\left[\frac{1}{p} + \left\{\frac{1}{4p}(1-\sin^2\theta_0) - \frac{1}{2}\sin\theta_0\sin\theta - \frac{1}{4p}\sin^2\theta\right\}(kb)^2 + \right. \\ &\qquad + \left\{\frac{1}{128p^2} + \frac{1}{64p}\left(\frac{3}{4} - 3\sin^2\theta_0 + \sin^4\theta_0\right) + \right. \\ &\qquad - \frac{1}{64}(1-2\sin^2\theta_0) - \frac{1}{8}\sin\theta_0\left(p + \frac{1}{4} - \frac{1}{2}\sin^2\theta_0\right)\sin\theta + \\ &\qquad + \left(-\frac{3}{64p} + \frac{1}{16p}\sin^2\theta_0 + \frac{1}{32} - \frac{1}{16}\sin^2\theta_0\right)\sin^2\theta + \\ &\qquad \left.\left. + \frac{1}{16}\sin\theta_0\sin^3\theta + \frac{1}{64p}\sin^4\theta\right\}(kb)^4 + \ldots\right]. \end{aligned}\right. \tag{5.14}$$

The corresponding transmission coefficient τ_2 can be determined from

$$\tau_2 = \frac{1}{16\pi kb \cos\theta_0} \int_{-\pi/2}^{\pi/2} |A_2(\theta, \theta_0)|^2 \, d\theta. \tag{5.15}$$

As a check on the calculations the relation

$$\tau_2 = -\frac{1}{4kb \cos\theta_0} \operatorname{Im} A_2(\theta_0, \theta_0), \tag{5.16}$$

which follows from (3.08), may be used. The result is found to be

$$\left\{\begin{aligned} \tau_2 = \frac{\pi^2}{4|p|^2 kb\cos\theta_0} \Big[1 + \left\{\frac{1}{4} - \frac{1}{2}\sin^2\theta_0\right\} (kb)^2 + \\ + \left\{\frac{3}{256} + \frac{1}{64} \operatorname{Re}\frac{1}{p} + \left(\frac{1}{8}|p|^2 - \frac{3}{32}\right)\sin^2\theta_0 + \right. \\ \left. + \frac{3}{32}\sin^4\theta_0\right\} (kb)^4 + \dots \Big], \end{aligned}\right. \tag{5.17}$$

with $|p|^2 = \log^2(\gamma kb/4) + \pi^2/4$ and $\operatorname{Re}(1/p) = \log(\gamma kb/4)[\log^2(\gamma kb/4) + \pi^2/4]^{-1}$. For normal incidence our results (5.12), (5.13), (5.14) and (5.17) agree with those of BOUWKAMP [2] and MÜLLER and WESTPFAHL [3].

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