1959

Appl. sci. Res. Section B, Vol. 8

A RECIPROCITY THEOREM FOR THE ELECTROMAGNETIC FIELD SCATTERED BY AN OBSTACLE

by A. T. DE HOOP

Laboratorium voor Theoretische Elektrotechniek en Elektromagnetische Straling, Technische Hogeschool, Delft, Netherlands

Summary

When a time-harmonic plane electromagnetic wave is incident upon a scattering obstacle of finite dimensions, the far-zone scattered field satisfies a reciprocity relation. This reciprocity relation is derived with the aid of H. A. Lorentz's theorem. The result is valid under rather general assumptions as far as the electromagnetic properties of the obstacle are concerned. As a special case, the result for a perfectly conducting obstacle is obtained.

§ 1. *Introduction.* In several branches of electromagnetic theory some kind of reciprocity theorem holds. Restricting ourselves to those branches where the field concept plays an essential role, we mention the relations between the transmitting and receiving properties of antennas [1][2]) and the symmetry of the impedance (or admittance) matrix characterizing the properties of a waveguide junction [3][4]). The cited literature shows that the proof of the reciprocity relations under consideration is based upon a theorem due to H. A. Lorentz [5]).

When the scattering of a plane electromagnetic wave by an obstacle of finite dimensions is considered, it can be shown that a reciprocity relation exists for the far-zone scattered field. The special case of perfectly conducting obstacles has been investigated by Levine and Schwinger [6]) (scattering by a plane obstacle of vanishing thickness) and by Storer and Sevick [7]) (scattering by an obstacle of arbitrary shape). The proof given by these authors is based upon the integral equation to be satisfied by the surface-current density at the boundary of the obstacle.

In the present paper it is shown that the relevant reciprocity relation holds in case the obstacle has rather general electromagnetic properties (for the precise conditions, see § 3) and is surrounded by a homogeneous, isotropic, non-conducting medium. The proof is based upon Lorentz's theorem.

Reciprocity relations in electromagnetic scattering problems have also been studied by Saxon [8]). In studying Saxon's paper, the present author encountered some difficulties in understanding the physical meaning of his separation of the field at infinity into incident and outgoing spherical waves. In any case, the proof in the present paper is mathematically rigorous and the theorem applies to an idealization (plane-wave excitation) of a physically realizable situation. Our results agree with Saxon's reciprocity relation for plane-wave scattering.

§ 2. *The field outside the obstacle.* A time-harmonic, elliptically polarized, plane electromagnetic wave is incident upon an obstacle of finite dimensions. The boundary of the obstacle is a sufficiently regular closed surface S. The electric and magnetic properties of the obstacle are assumed to be linear; they will be specified in § 3. The medium in the domain outside S is assumed to be homogeneous, isotropic and non-conducting (which includes the case of free space), with permittivity ε_0 and permeability μ_0. In the exterior domain, the electric field vector $\boldsymbol{E}$ and the magnetic field vector $\boldsymbol{H}$ are written as the sum of the incident field $\boldsymbol{E}^i$, $\boldsymbol{H}^i$ and the scattered field $\boldsymbol{E}^s$, $\boldsymbol{H}^s$:

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s, \qquad (2.1)$$

$$\boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s. \qquad (2.2)$$

Both the incident and the scattered field satisfy Maxwell's equations

$$\operatorname{curl} \boldsymbol{H} = - i\omega\varepsilon_0 \boldsymbol{E}, \qquad (2.3)$$

$$\operatorname{curl} \boldsymbol{E} = i\omega\mu_0 \boldsymbol{H}, \qquad (2.4)$$

where ω is the angular frequency of the exponential time dependence of the form $\exp(- i\omega t)$. This factor, which is common to all field components, has been omitted throughout. In addition, the scattered field shall satisfy the radiation condition [9]) *)

$$\int_{S_R} |\boldsymbol{E}^s - (\mu_0/\varepsilon_0)^{\frac{1}{2}} (\boldsymbol{H}^s \times \boldsymbol{i}_R)|^2 \, \mathrm{d}S = o(1) \quad (R \to \infty), \qquad (2.5)$$

*) For a vector $\boldsymbol{A}$ whose components are complex numbers, we have $|\boldsymbol{A}|^2 = \boldsymbol{A}\cdot\boldsymbol{A}^*$, where $\boldsymbol{A}^*$ denotes the complex conjugate to $\boldsymbol{A}$.

where S_R is the surface of a sphere of radius R around some point of observation and $\boldsymbol{i}_R$ is the unit vector in the direction of the outward normal to S_R.

Let $\boldsymbol{r} = (x, y, z)$ be the radius vector from a fixed origin to the point of observation. The origin is located at some finite distance from the obstacle. Further, the unit vector $\boldsymbol{\theta}$ in the direction of observation is introduced. Then, $\boldsymbol{r} = r\boldsymbol{\theta}$. If, now, the scattered field satisfies (2.3), (2.4) and (2.5), the following expansion holds[10]:

$$\boldsymbol{E}^s(\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{\theta}) \frac{e^{ikr}}{ikr} + O(r^{-2}) \quad (r \to \infty), \tag{2.6}$$

$$(\mu_0/\varepsilon_0)^{\frac{1}{2}} \boldsymbol{H}^s(\boldsymbol{r}) = [\boldsymbol{\theta} \times \boldsymbol{F}(\boldsymbol{\theta})] \frac{e^{ikr}}{ikr} + O(r^{-2}) \quad (r \to \infty), \tag{2.7}$$

where the (complex) factor $\boldsymbol{F}(\boldsymbol{\theta})$ is given by

$$4\pi\boldsymbol{F}(\boldsymbol{\theta}) = -\, k^2\boldsymbol{\theta} \times \int_S [\boldsymbol{n} \times \boldsymbol{E}^s(\boldsymbol{\varrho})] \exp(-\, ik\boldsymbol{\theta}\cdot\boldsymbol{\varrho})\, dS +$$
$$+ (\mu_0/\varepsilon_0)^{\frac{1}{2}} k^2\boldsymbol{\theta} \times \{\boldsymbol{\theta} \times \int_S [\boldsymbol{n} \times \boldsymbol{H}^s(\boldsymbol{\varrho})] \exp(-ik\boldsymbol{\theta}\cdot\boldsymbol{\varrho})\, dS\}, \tag{2.8}$$

in which $\boldsymbol{\varrho} = (\xi, \eta, \zeta)$ is the radius vector to the point of integration, and

$$k = \omega(\varepsilon_0\mu_0)^{\frac{1}{2}} = 2\pi/\lambda, \tag{2.9}$$

λ being the wavelength in the medium outside the obstacle. The first term of the right-hand side of (2.6) and (2.7) is called the "far-zone approximation". Although S in (2.8) could be any sufficiently regular bounded closed surface completely surrounding the obstacle, it will be convenient to take S to be the boundary of the obstacle.

§ 3. *The field inside the obstacle.* The total field inside the obstacle, too, will be denoted by $\boldsymbol{E}, \boldsymbol{H}$. The electromagnetic properties of the obstacle are characterized by its tensor permittivity ε_{ij}, its tensor conductivity σ_{ij} and its tensor permeability μ_{ij} $(i, j = 1, 2, 3)$. The field vectors $\boldsymbol{E} = (E_1, E_2, E_3)$ and $\boldsymbol{H} = (H_1, H_2, H_3)$ satisfy Maxwell's equations, which, in subscript notation, are

$$(\text{curl}\, \boldsymbol{H})_i = \sum_{j=1}^{3} (\sigma_{ij} - i\omega\varepsilon_{ij})E_j \quad (i = 1, 2, 3), \tag{3.1}$$

$$(\text{curl}\, \boldsymbol{E})_i = i\omega \sum_{j=1}^{3} \mu_{ij}H_j \quad (i = 1, 2, 3). \tag{3.2}$$

It is assumed that ε_{ij}, σ_{ij} and μ_{ij} are symmetric tensors, i.e. $\varepsilon_{ij} = \varepsilon_{ji}$, $\sigma_{ij} = \sigma_{ji}$, $\mu_{ij} = \mu_{ji}$. Further, we impose the restriction that ε_{ij}, σ_{ij} and μ_{ij} are independent of the electric and magnetic field vector (thus excluding non-linear effects) and are continuous functions of position with the possible exception of a finite number of sufficiently regular bounded surfaces across which they may jump by finite amounts. If these conditions are satisfied, Lorentz's theorem holds[11]).

Let $\boldsymbol{E}_a$, $\boldsymbol{H}_a$ and $\boldsymbol{E}_b$, $\boldsymbol{H}_b$ denote two vector fields which satisfy Maxwell's equations (3.1) and (3.2) in a certain domain, bounded by a sufficiently regular closed surface S. If $\boldsymbol{n}$ is the unit vector in the direction of the outward normal to S, we have

$$\int_S (\boldsymbol{E}_a \times \boldsymbol{H}_b - \boldsymbol{E}_b \times \boldsymbol{H}_a)\cdot\boldsymbol{n}\,\mathrm{d}S = 0. \tag{3.3}$$

Equation (3.3) is called Lorentz's theorem. The proof is easily obtained by making use of Maxwell's equations and Green's divergence theorem. Although S in (3.3) could be any sufficiently regular bounded closed surface, it will be convenient to take S to be the boundary of the obstacle.

§ 4. *Proof of the reciprocity theorem.* The vector fields $\boldsymbol{E}_a$, $\boldsymbol{H}_a$ and $\boldsymbol{E}_b$, $\boldsymbol{H}_b$ are chosen as follows. The field $\boldsymbol{E}_a$, $\boldsymbol{H}_a$ is the total field due to an incident plane wave of the form

$$\boldsymbol{E}_a{}^i(\boldsymbol{r}) = \boldsymbol{A}\exp(-\,ik\boldsymbol{\alpha}\cdot\boldsymbol{r}), \tag{4.1}$$

$$\boldsymbol{H}_a{}^i(\boldsymbol{r}) = (\varepsilon_0/\mu_0)^{\frac{1}{2}}\,(\boldsymbol{A}\times\boldsymbol{\alpha})\exp(-\,ik\boldsymbol{\alpha}\cdot\boldsymbol{r}), \tag{4.2}$$

where $\boldsymbol{A}$ specifies the polarization of the wave (in general, elliptic) and $\boldsymbol{\alpha}$ is the unit vector pointing *towards* the source at infinity. Similarly $\boldsymbol{E}_b$, $\boldsymbol{H}_b$ is the total field due to an incident plane wave of the form

$$\boldsymbol{E}_b{}^i(\boldsymbol{r}) = \boldsymbol{B}\exp(-ik\boldsymbol{\beta}\cdot\boldsymbol{r}), \tag{4.3}$$

$$\boldsymbol{H}_b{}^i(\boldsymbol{r}) = (\varepsilon_0/\mu_0)^{\frac{1}{2}}(\boldsymbol{B}\times\boldsymbol{\beta})\exp(-ik\boldsymbol{\beta}\cdot\boldsymbol{r}). \tag{4.4}$$

Since the waves are transverse, we have $\boldsymbol{A}\cdot\boldsymbol{\alpha} = 0$ and $\boldsymbol{B}\cdot\boldsymbol{\beta} = 0$. From Lorentz's theorem (3.3) it follows that

$$\int_S (\boldsymbol{E}_a \times \boldsymbol{H}_b - \boldsymbol{E}_b \times \boldsymbol{H}_a)\cdot\boldsymbol{n}\,\mathrm{d}S = 0, \tag{4.5}$$

where S is the boundary of the obstacle. Since $\boldsymbol{n}\times\boldsymbol{E}_{a,b}$ and

$\boldsymbol{n} \times \boldsymbol{H}_{a,b}$ are continuous across S, we need not indicate whether S is approached from the inside or the outside, respectively, when evaluating the integrand of (4.5).

In the first place we observe that

$$\int_{S_r} (\boldsymbol{E}_a{}^s \times \boldsymbol{H}_b{}^s - \boldsymbol{E}_b{}^s \times \boldsymbol{H}_a{}^s) \cdot \boldsymbol{n} \, \mathrm{d}S = O(r^{-1}) \quad (r \to \infty), \tag{4.6}$$

where S_r is a sphere of radius r around the origin. This relation is proved by substituting (2.6) and (2.7) in the left-hand side of (4.6). Since Lorentz's theorem also applies to the domain bounded internally by S and externally by S_r, (4.6) implies that

$$\int_S (\boldsymbol{E}_a{}^s \times \boldsymbol{H}_b{}^s - \boldsymbol{E}_b{}^s \times \boldsymbol{H}_a{}^s) \cdot \boldsymbol{n} \, \mathrm{d}S = 0. \tag{4.7}$$

Secondly, it can be shown that

$$\int_S (\boldsymbol{E}_a{}^i \times \boldsymbol{H}_b{}^i - \boldsymbol{E}_b{}^i \times \boldsymbol{H}_a{}^i) \cdot \boldsymbol{n} \, \mathrm{d}S = 0. \tag{4.8}$$

Consequently, we obtain from (4.5), (4.7) and (4.8)

$$\int_S (\boldsymbol{E}_a{}^s \times \boldsymbol{H}_b{}^i - \boldsymbol{E}_b{}^i \times \boldsymbol{H}_a{}^s) \cdot \boldsymbol{n} \, \mathrm{d}S = = \int_S (\boldsymbol{E}_b{}^s \times \boldsymbol{H}_a{}^i - \boldsymbol{E}_a{}^i \times \boldsymbol{H}_b{}^s) \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{4.9}$$

Now, we have from (2.8), using (4.3) and (4.4),

$$4\pi \boldsymbol{B} \cdot \boldsymbol{F}_a(\beta) = -(\mu_0/\varepsilon_0)^{\frac{1}{2}} k^2 \int_S (\boldsymbol{E}_a{}^s \times \boldsymbol{H}_b{}^i - \boldsymbol{E}_b{}^i \times \boldsymbol{H}_a{}^s) \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{4.10}$$

Application of the identity (4.9) to the right-hand side of (4.10) yields the result

$$\boldsymbol{B} \cdot \boldsymbol{F}_a(\beta) = \boldsymbol{A} \cdot \boldsymbol{F}_b(\alpha). \tag{4.11}$$

Equation (4.11) is the reciprocity theorem to be proved.

The proof given above also applies to the case of a perfectly conducting obstacle. For, in this case each term of the left-hand side of (4.5) vanishes by virtue of the boundary condition: $\boldsymbol{n} \times \boldsymbol{E}_{a,b} = \boldsymbol{0}$ on S.

Received 19th June, 1959.

REFERENCES

1) Stevenson, A. F., Quart. Appl. Math. **5** (1948) 369.
2) King, R. W. P., Electromagnetic Engineering, Vol. I, McGraw-Hill Book Company, Inc., New York, 1945; p. 311.
3) Montgomery, C. G. et al., Principles of Microwave Circuits, M. I. T. Radiation Laboratory Series, Vol. 8, McGraw-Hill Book Company, Inc., New York, 1948; p. 141.
4) Flügge, S., Handbuch der Physik, Bd. 16, Elektrische Felder und Wellen, Springer, Berlin, 1958. Article by F. E. Borgnis and C. H. Papas; p. 399.
5) Lorentz, H. A., Versl. Kon. Akad. Wetensch. Amsterdam **4** (1895) 176.
6) Levine, H. and J. Schwinger, Comm. Pure Appl. Math. **3** (1950) 355.
7) Storer, J. E. and J. Sevick, J. Appl. Phys. **25** (1954) 369.
8) Saxon, D. S., Phys. Rev. **100** (1955) 1771.
9) Wilcox, C. H., Comm. Pure Appl. Math. **9** (1956) 115.
10) Hoop, A. T. de, Appl. sci. Res. **B 7** (1959) 463.
11) Stevenson, A. F., Ref. [1]), Appendix.