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The surface line source problem in elastodynamics

by prof. dr. ir. A. T. de Hoop, Delft University of Technology,
Department of Electrical Engineering

Summary: At the plane boundary of a semi-infinite, homogeneous, isotropic, perfectly elastic solid a pulsed line source of normal pressure generates a two-dimensional elastic wave motion in the solid. The displacement vector of this elastic wave motion is calculated at any point inside the solid.

1. Introduction

The surface line source problem in elastodynamics involves the calculation of the displacement vector of the elastic wave motion in a semi-infinite, elastic solid, generated by a pulsed line source of force applied at its plane boundary. Lamb [1], who has been the first to investigate this problem, calculated the displacement vector at the boundary. In the present paper the displacement vector at any point inside the solid is determined by employing the author's modification (de Hoop [2]) of Cagniard's technique for solving seismic pulse problems.

2. Formulation of the problem

The elastic waves under consideration are small-amplitude disturbances travelling in a semi-infinite, homogeneous, isotropic, perfectly elastic solid. The physical properties of the solid are characterized by its mass density ρ and its Lamé constants λ and μ. A right-handed Cartesian co-ordinate system x, y, z is introduced such that the elastic medium occupies the half-space $-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < \infty$. A point inside the solid or at its boundary $z = 0$ is located by either its Cartesian co-ordinates or its cylindrical co-ordinates defined through

$$x = r\sin(\theta); \qquad y = y; \qquad z = r\cos(\theta) \tag{2.1}$$

with $0 \leqq r < \infty$, $-\pi/2 \leqq \theta \leqq \pi/2$. The time co-ordinate is denoted by t.

Along the line $x = 0$, $-\infty < y < \infty$, $z = 0$ a pulsed force is applied to the free surface of the solid. It starts to act at the instant $t = 0$; prior to this instant the medium is assumed to be at rest (Fig. 1).

Let $\vec{u} = \vec{u}(x, y, z, t)$ be the displacement vector of the elastic wave. At any interior point of the solid the stress tensor, with components $\tau_{xx}, \tau_{xy}, \ldots, \tau_{zz}$, is related to the displacement vector through the (linearized) constitutive relations

$$\tau_{xx} = \lambda \operatorname{div} \vec{u} + 2\mu(\partial u_x/\partial x), \text{ etc.} \tag{2.2}$$

$$\tau_{xy} = \tau_{yx} = \mu(\partial u_x/\partial y + \partial u_y/\partial x), \text{ etc.} \tag{2.3}$$

From the (linearized) equation of motion and the constitutive relations it follows that, in the absence of body forces, the displacement vector satisfies the elastodynamic wave equation

$$v_P{}^2 \operatorname{grad} \operatorname{div} \vec{u} - v_S{}^2 \operatorname{rot} \operatorname{rot} \vec{u} - \partial^2\vec{u}/\partial t^2 = \vec{0} \tag{2.4}$$

in which

$$v_P = [(\lambda + 2\mu)/\rho]^{\frac{1}{2}} \tag{2.5}$$

is the velocity of propagation of compressional or P-waves (for which $\operatorname{rot} \vec{u} = \vec{0}$) and

$$v_S = (\mu/\rho)^{\frac{1}{2}} \tag{2.6}$$

is the velocity of propagation of shear or S-waves (for which $\operatorname{div} \vec{u} = 0$).

Now we investigate the case where the amplitude of the applied force is independent of y; then the boundary conditions as $z \downarrow 0$ are independent of y. Since the elastodynamic wave equation admits y-independent solutions the generated elastic wave motion is independent of y. Hence, $\vec{u} = \vec{u}(x, z, t)$. The y-independent solutions of (2.4) separate into two classes: in

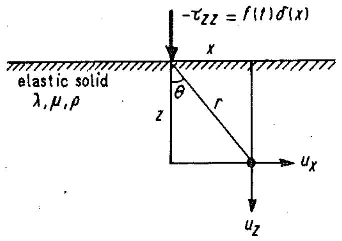

Fig. 1. The configuration to be investigated.

one of them $\vec{u}$ has only a y-component, in the other $\vec{u}$ has only an x- and a z-component. Let the plane of the boundary be 'horizontal', then the former displacement vector corresponds to an SH-wave (horizontally polarized shear wave) while the latter displacement vector corresponds to a superposition of a P-wave and an SV-wave (vertically polarized shear wave) (Bullen [3]). The calculation of the SH-wave is a simple problem in scalar wave propagation and is not considered here. The calculation of the combined P- and SV-wave motion is the subject of our further investigation.

Let $\vec{u}^{(P)}$ be the compressional part and $\vec{u}^{(SV)}$ the shear part of the relevant displacement vector, then $\vec{u} = \vec{u}^{(P)} + \vec{u}^{(SV)}$. From (2.4) it follows that $\vec{u}^{(P)}$ satisfies the two-dimensional wave equation

$$(\partial^2/\partial x^2 + \partial^2/\partial z^2 - v_P^{-2}\partial^2/\partial t^2)\vec{u}^{(P)} = \vec{0} \tag{2.7}$$

together with the auxiliary relation $\vec{i}_y \cdot \mathrm{rot}\, \vec{u}^{(P)} = 0$, i.e.

$$\partial u_x^{(P)}/\partial z - \partial u_z^{(P)}/\partial x = 0, \tag{2.8}$$

while $\vec{u}^{(SV)}$ satisfies the two-dimensional wave equation

$$(\partial^2/\partial x^2 + \partial^2/\partial z^2 - v_S^{-2}\partial^2/\partial t^2)\vec{u}^{(SV)} = \vec{0} \tag{2.9}$$

together with the auxiliary relation $\mathrm{div}\, \vec{u}^{(SV)} = 0$, i.e.

$$\partial u_x^{(SV)}/\partial x + \partial u_z^{(SV)}/\partial z = 0 \tag{2.10}$$

The calculation is further restricted to the case where the force at the boundary is applied normally, then we have

$$\lim_{z\downarrow 0} \tau_{xz} = 0 \text{ and } \lim_{z\downarrow 0} \tau_{zz} = -f(t)\delta(x) \tag{2.11}$$

where $f(t)$ denotes the amplitude of the applied pressure ($f(t) = 0$ when $-\infty < t < 0$) and $\delta(x)$ denotes the one-dimensional delta distribution. In view of the symmetry of the configuration with respect to the plane $x = 0$, u_x is an odd function of x, while u_z is an even function of x, i.e.

$$u_x(x, z, t) = -u_x(-x, z, t) \text{ and } u_z(x, z, t) = u_z(-x, z, t) \tag{2.12}$$

As a consequence, u_x and u_z need only be determined in the quarter-space $0 \leq x < \infty$, $-\infty < y < \infty$, $0 < z < \infty$. Finally, it has to be taken into account that the waves generated by the source travel away from it.

3. Method of solution

We follow the steps as indicated in [2] and subject all time-dependent quantities to a one-sided laplace transform with respect to time. For example:

$$F(s) = \int_0^\infty \exp(-st)f(t)\mathrm{d}t \tag{3.1}$$

in which s is a real, positive number large enough to ensure the convergence of integrals of the type (3.1). (It is assumed that the behaviour of $f(t)$ and $\vec{u}(x, z, t)$ as $t \to \infty$ is such that an appropriate value of s can be found.) Let $\vec{U} = \vec{U}(x; z; s)$ denote the one-sided laplace transform of $\vec{u} = \vec{u}(x, z, t)$ and let, similarly, T_{xx}, etc. be the one-sided laplace transforms of τ_{xx}, etc. Then $\vec{U}^{(P)}$ satisfies the differential equation

$$(\partial^2/\partial x^2 + \partial^2/\partial z^2 - s^2/v_P^2)\vec{U}^{(P)} = \vec{0} \tag{3.2}$$

together with the auxiliary condition

$$\partial U_x^{(P)}/\partial z - \partial U_z^{(P)}/\partial x = 0 \tag{3.3}$$

while $\vec{U}^{(SV)}$ satisfies the differential equation

$$(\partial^2/\partial x^2 + \partial^2/\partial z^2 - s^2/v_S^2)\vec{U}^{(SV)} = \vec{0} \tag{3.4}$$

together with the auxiliary condition

$$\partial U_x^{(SV)}/\partial x + \partial U_z^{(SV)}/\partial z = 0 \tag{3.5}$$

Further the constitutive relations (2.2) and (2.3) are replaced by corresponding relations involving the laplace-transformed quantities, whereas the boundary conditions (2.11) are replaced by

$$\lim_{z\downarrow 0} T_{xz} = 0 \text{ and } \lim_{z\downarrow 0} T_{zz} = -F(s)\delta(x) \tag{3.6}$$

Next we introduce the integral representations

$$\vec{U}^{(P)} = (F(s)/2\pi\mathrm{i}) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (p\vec{i}_x + \gamma_P\vec{i}_z)A(p) \times \times \exp[-s(px + \gamma_P z)]\mathrm{d}p \tag{3.7}$$

and

$$\vec{U}^{(SV)} = (F(s)/2\pi\mathrm{i}) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (\gamma_S\vec{i}_x - p\vec{i}_z)B(p) \times \times \exp[-s(px + \gamma_S z)]\mathrm{d}p \tag{3.8}$$

in which

$$\gamma_{P,S} = (1/v_{P,S}^2 - p^2)^{\frac{1}{2}} \text{ with } \mathrm{Re}(\gamma_{P,S}) \geq 0 \tag{3.9}$$

The right-hand side of (3.7) satisfies (3.2) and (3.3), the right-hand side of (3.8) satisfies (3.4) and (3.5); the choice of the square root in (3.9) will lead to waves travelling away from the origin. Substituting (3.7) and (3.8) in the boundary conditions (3.6) and using the representation

$$\delta(x) = (s/2\pi\mathrm{i}) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp(-spx)\mathrm{d}p \tag{3.10}$$

we obtain two linear, algebraic equations for $A(p)$ and $B(p)$. The solution of these equations is obtained as

$$A(p) = (1/2v_S^2 - p^2)/2\mu R(p);\; B(p) = p\gamma_P/2\mu R(p) \tag{3.11}$$

in which

$$R(p) = (1/2v_S^2 - p^2)^2 + p^2\gamma_P\gamma_S \tag{3.12}$$

At this point we observe that $A(p)$ and $B(p)$ have the following singularities in the complex p-plane: (a) branch points at $p = \pm 1/v_P$, (b) branch points at $p = \pm 1/v_S$, (c) simple poles at $p = \pm 1/v_R$ where $p = 1/v_R$ and $p = -1/v_R$ (v_R = Rayleigh wave velocity) are the simple zeros of the right-hand side of (3.12).

Now that $\vec{U}^{(P)}$ and $\vec{U}^{(SV)}$ have been determined, the transformation back to the time domain follows by changing the path of integration in (3.7) and (3.8) to a curve along which

$$\mathrm{Re}(px + \gamma_{P,S}z) = \tau > 0 \quad \text{and} \quad \mathrm{Im}(px + \gamma_{P,S}z) = 0 \tag{3.13}$$

Since this process requires the integrands to be single-valued, branch cuts are introduced in accordance with (3.9), i.e. along $\mathrm{Re}(\gamma_{P,S}) = 0$. Further, neither singularities nor branch cuts may be passed. The modified path of integration in (3.7) is then obtained as $p = \omega_P(x, z, \tau)$, together with its image $p = \omega_P^*$

with respect to the real axis, where ω_P is given by

$$\omega_P = (\tau/r)\sin(\theta) + i(\tau^2/r^2 - 1/v_P^2)^{\frac{1}{2}}\cos(\theta) \qquad (3.14)$$

with $T_P < \tau < \infty$ and $T_P = r/v_P$

The modified path of integration in (3.8) in general consists of the two parts $p = \omega_{PS}(x, z, \tau)$ and $p = \omega_S(x, z, \tau)$, together with their images $p = \omega_{PS}^*$ and $p = \omega_S^*$ with respect to the real axis, where ω_{PS} is given by

$$\omega_{PS} = (\tau/r)\sin(\theta) - (1/v_S^2 - \tau^2/r^2)^{\frac{1}{2}}\cos(\theta) + i0 \qquad (3.15)$$

with $T_{PS} < \tau < T_S$ and

$$T_{PS} = T_P \sin(\theta) + T_S(1 - v_S^2/v_P^2)^{\frac{1}{2}}\cos(\theta),$$

$$T_S = r/v_S$$

and ω_S by

$$\omega_S = (\tau/r)\sin(\theta) + i(\tau^2/r^2 - 1/v_S^2)^{\frac{1}{2}}\cos(\theta) \qquad (3.16)$$

with $T_S < \tau < \infty$

The part $p = \omega_{PS}$, together with its image $p = \omega_{PS}^*$ with respect to the real axis, is only present for points of observation in the domain $v_S/v_P < |\sin(\theta)| \leqq 1$ and is due to the presence of the branch points $p = \pm 1/v_P$ (Fig. 2). Introducing in (3.7) and (3.8) τ as variable of integration and taking into account the symmetry of the paths of integration with respect to the real axis, we obtain expressions for $\vec{U}^{(P)}$ and $\vec{U}^{(SV)}$ of the form

$$\vec{U}^{(P)} = F(s)\vec{G}^{(P)}(x, z; s) \text{ and } \vec{U}^{(SV)} = F(s)\vec{G}^{(SV)}(x, z; s) \quad (3.17)$$

where

$$\vec{G}^{(P)} = \int_{T_P}^{\infty} \exp(-s\tau)\,\vec{g}^{(P)}(x, z, \tau)\,d\tau \qquad (3.18)$$

and

$$\vec{G}^{(SV)} = \int_{T_{PS}}^{T_S} \exp(-s\tau)\vec{g}^{(PS)}(x, z, \tau)\,d\tau +$$

$$+ \int_{T_S}^{\infty} \exp(-s\tau)\,\vec{g}^{(S)}(x, z, \tau)\,d\tau \qquad (3.19)$$

with

$$\vec{g}^{(P)} = (1/\pi)\,\mathrm{Im}[(\omega_P\vec{i}_x + \gamma_P(\omega_P)\vec{i}_z)A(\omega_P)(\partial\omega_P/\partial\tau)] \qquad (3.20)$$

$$\vec{g}^{(PS)} = (1/\pi)\,\mathrm{Im}[(\gamma_S(\omega_{PS})\vec{i}_x - \omega_{PS}\vec{i}_z)B(\omega_{PS})(\partial\omega_{PS}/\partial\tau)] \qquad (3.21)$$

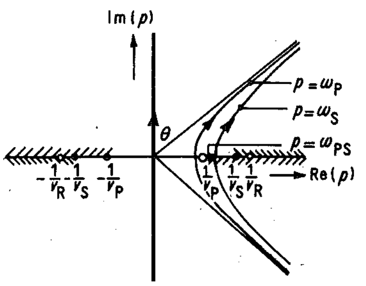

Fig. 2. The modified paths of integration in the complex p-plane.

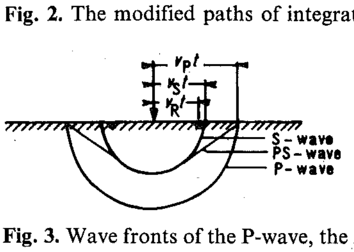

Fig. 3. Wave fronts of the P-wave, the PS-wave and the S-wave.

$$\vec{g}^{(S)} = (1/\pi)\,\mathrm{Im}[(\gamma_S(\omega_S)\vec{i}_x - \omega_S\vec{i}_z)B(\omega_S)(\partial\omega_S/\partial\tau)] \qquad (3.22)$$

Expressions for $\vec{u}^{(P)}(x, z, t)$ and $\vec{u}^{(SV)}(x, z, t)$ are now obtained by applying the convolution theorem to (3.17) and using the uniqueness of the inverse laplace transform. The results are presented in Table 1 and Table 2.

Table 1. P-wave as a function of time.

t	$\vec{u}^{(P)}(x, z, t)$
$-\infty < t < T_P$	$\vec{0}$
$T_P < t < \infty$	$\int_{T_P}^{t} f(t-\tau)\,\vec{g}^{(P)}(x, z, \tau)\,d\tau$

Table 2. SV-wave as a function of time (the term $\vec{g}^{(PS)}$ is only present in the domain $v_S/v_P < |\sin(\theta)| \leqq 1$).

t	$\vec{u}^{(SV)}(x, z, t)$
$-\infty < t < T_{PS}$	$\vec{0}$
$T_{PS} < t < T_S$	$\int_{T_{PS}}^{t} f(t-\tau)\,\vec{g}^{(PS)}(x, z, \tau)\,d\tau$
$T_S < t < \infty$	$\int_{T_{PS}}^{T_S} f(t-\tau)\,\vec{g}^{(PS)}(x, z, \tau)\,d\tau +$
	$+ \int_{T_S}^{t} f(t-\tau)\,\vec{g}^{(S)}(x, z, \tau)\,d\tau$

4. Discussion of the results

The total elastodynamic wave motion $\vec{u}$ generated by the line source is given by $\vec{u} = \vec{u}^{(P)} + \vec{u}^{(SV)}$, where $\vec{u}^{(P)}$ follows from Table 1 and $\vec{u}^{(SV)}$ follows from Table 2 (see Fig. 3). The P-wave is a cylindrical wave arriving at $t = T_P$, where T_P is the travel time for P-waves travelling along a straight line from the line source to the point of observation. The SV-wave consists of a PS-conversion wave arriving at $t = T_{PS}$ and a cylindrical SV-wave arriving at $t = T_S$. The PS-conversion wave is only present in the domain $v_S/v_P < |\sin(\theta)| \leqq 1$ and precedes the cylindrical SV-wave. T_{PS} is the travel time for SV-waves travelling from the line source a distance $x - z\,(v_S/v_P)\,(1 - v_S^2/v_P^2)^{-\frac{1}{2}}$ along the boundary with P-wave velocity and then along a straight line to the point of observation with S-wave velocity; T_S is the travel time for S-waves travelling along a straight line from the line source to the point of observation. At the boundary of the solid there is a singularity in the displacement vector travelling with velocity v_R; this singularity is due to the presence of the pole at $p = \pm 1/v_R$.

The features that are characteristic for the configuration occur in $\vec{g}^{(P)}$, $\vec{g}^{(PS)}$ and $\vec{g}^{(S)}$. The numerical computation of these quantities involves only algebraic operations of the type complex.

References

[1] Lamb, H.: On the propagation of tremors over the surface of an elastic solid. Phil. Trans. Roy. Soc. (London) **A 203** (1904) 1-42.

[2] De Hoop, A. T.: A modification of Cagniard's method for solving seismic pulse problems. Appl. Sci. Res. **B 8** (1960) 349-356.

[3] Bullen, K. E.: An introduction to the theory of seismology. Cambridge University Press. 3rd edition, 1963, p. 75.