R 908 Philips Res. Repts 30, 302*-315*, 1975
Issue in honour of C. J. Bouwkamp

THE N-PORT RECEIVING ANTENNA AND ITS EQUIVALENT ELECTRICAL NETWORK

by A. T. DE HOOP *)

Delft University of Technology
Delft, The Netherlands

(Received January 17, 1975)

Abstract

A rigorous proof is presented of the commonly accepted theorem that an N-port receiving antenna is, in several respects, equivalent to an N-port electrical network containing internal sources. Expressions are derived for the quantities that specify the Thévenin representation of this network. As a basic tool, the reciprocity theorem that relates the electromagnetic fields occurring in the transmitting situation to those occurring in the receiving situation, is used. The most general N-port antenna, if only linear, passive and time-invariant, is investigated; nonreciprocal ones are included as well. The incident radiation consists of an arbitrarily elliptically polarized plane wave. With the aid of the equivalent network, the condition for maximum power transfer from the incident wave to an N-port load is derived.

1. Introduction

The electrical properties of a receiving antenna, for example one that is used in a communication system, are often specified more or less intuitively in terms of an equivalent electrical network with internal sources [1]). The parameters of the relevant network are commonly accepted to follow from related quantities that characterize the same antenna in the transmitting situation, while the strengths of the internal sources will depend on the amplitude, phase and state of polarization of the radiation that is incident upon the antenna in the receiving situation.

The purpose of the present paper is to show how this representation can be justified rigorously. At the same time, expressions are obtained for the network parameters involved as well as for the strengths of the internal sources. As a result, the electrical network that characterizes the properties of the antenna in the receiving situation is completely specified. The conditions under which the representation is shown to hold are:

(a) both in the transmitting and in the receiving situation the antenna is accessible at a finite number of ports at which either the low-frequency voltages and currents replace the general field concept or the single-mode waveguide description (as in a microwave antenna) holds;

*) Dedicated to my friend Dr C. J. Bouwkamp, whose penetrating way of thinking has had a substantial influence on my scientific development.

(b) in the receiving situation the incident electromagnetic radiation consists of a uniform plane wave (with arbitrary amplitude, phase and state of polarization);
(c) the antenna system is linear in its electromagnetic behaviour.

No further restrictions are imposed. In particular, the material of which the antenna is made, may be lossy, inhomogeneous and anisotropic; nonreciprocal antenna configurations are included as well.

The theorem upon which our considerations are based is the reciprocity theorem that relates, for a single antenna, the electromagnetic fields occurring in the transmitting situation to those occurring in the receiving situation. This theorem has been derived by the present author on a previous occasion [2]) and is shortly reconsidered here. Subsequently, it is applied to the configuration under investigation, upon which the desired network representation is obtained.

Once the equivalent network pertaining to the receiving situation is known, several problems related to the further use of the antenna can be solved by employing pure network methods. Among this is the problem of maximum power transfer from the incident wave to a load that is connected with the accessible ports. We show that maximum power transfer occurs if the impedance matrix of the load is the complex conjugate (not the Hermitean conjugate) of the input-impedance matrix of the antenna in the transmitting situation. The load is then "matched" to the antenna.

For recent developments in the network-theoretical aspects of the scattering properties of an N-port receiving antenna, we refer to papers by Mautz and Harrington [3]) and Harrington and Mautz [4]).

Network-theoretical aspects of minimum-scattering antennas are dealt with in papers by Kahn and Kurss [5]) and Wasylkiwskyj and Kahn [6]).

2. Description of the configuration

The antenna system under consideration occupies a bounded domain V in space. Externally, V is bounded by a sufficiently regular closed surface S_0; internally, V is bounded by a sufficiently regular closed surface S_1. The surface S_1 is considered as the termination of the antenna system, and on it a finite number N of ports is defined through which the antenna system is accessible from the "interior" (fig. 1). Parts of S_0 and S_1 may coincide. The region V thus introduced allows us to distinguish the antenna system from the environment into which it radiates or scatters, as well as from the terminals at which it is accessible. The cartesian coordinates of a point in space are denoted by x, y and z; the time variable is denoted by t. The position vector is denoted by $\mathbf{r}$. The electromagnetic fields occurring in the transmitting situation, as well as those occurring in the receiving situation, are assumed to vary sinusoidally in time with the same angular frequency ω. The complex representation of the field vectors is used, and in the formulas, the complex time factor

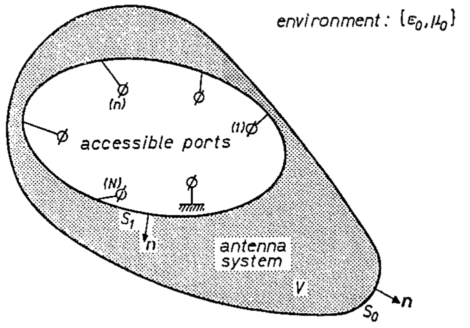

Fig. 1. Antenna configuration with N accessible ports.

exp $(-i\omega t)$, common to all field components, is omitted.

The antenna consists of a medium, the electromagnetic behaviour of which is linear and passive, no further restrictions as to its electromagnetic properties being imposed. The properties of the medium may change abruptly when crossing a (bounded) surface, but, across such a surface of discontinuity in properties, the tangential parts of both the electric- and the magnetic-field vector are continuous. Other parts of the antenna system may consist of conducting surfaces. These surfaces are assumed to be electrically perfectly conducting, and on them the tangential part of the electric-field vector vanishes.

The medium outside S_0 is assumed to be linear, homogeneous, isotropic and lossless, with real scalar permittivity ε_0 and real scalar permeability μ_0; this includes the case of free space.

In the following sections, **E**, **H**, **D** and **B** denote the space- and frequency-dependent complex representations of the electric-field vector, the magnetic-field vector, the electric-flux density and the magnetic-flux density, respectively. All quantities are expressed in terms of SI units. The superscripts T and R are used to denote the transmitting and the receiving situation, respectively.

3. The antenna in the transmitting situation

In the transmitting situation (fig. 2) the accessible ports of the antenna are fed by an N-port source. Let $I_n{}^T$ denote the electric current fed into the nth port, and let $V_n{}^T$ denote the voltage across the nth port ($n = 1, \ldots, N$). (In the single-mode waveguide description, $I_n{}^T$ and $V_n{}^T$ denote the complex amplitudes, in a chosen transverse reference plane, of the transverse parts of the magnetic and the electric fields pertaining to the waveguide mode under consideration; the relevant transverse modal functions should be properly normalized.) As a consequence of the uniqueness theorem of electromagnetic fields, the voltages $\{V_n{}^T\}$ are linearly related to the currents $\{I_n{}^T\}$ through the relation

$$V_m{}^T = \sum_{n=1}^{N} Z_{m,n}{}^{\text{in}}\, I_n{}^T \qquad (m = 1, \ldots, N), \tag{1}$$

where $Z_{m,n}{}^{\text{in}}$ defines the input-impedance matrix of the radiating antenna system

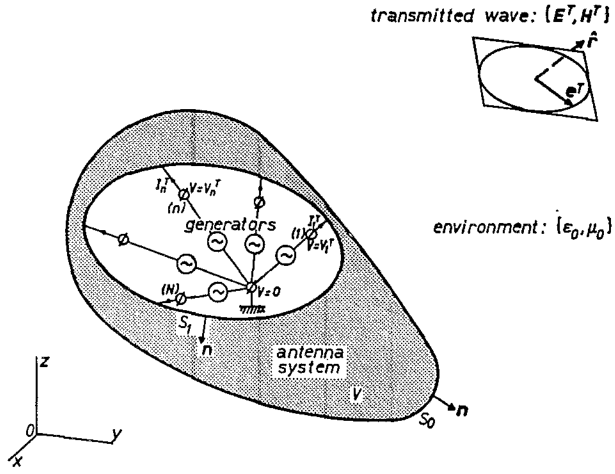

Fig. 2. Antenna in transmitting situation; the accessible ports are fed by a source.

in the transmitting situation. The time-averaged electromagnetic power P^{in} fed into the antenna system in the transmitting situation is then given by

$$P^{\text{in}} = \tfrac{1}{2}\,\text{Re}\left[\sum_{m=1}^{N} V_m{}^T\, I_m{}^{T*}\right], \tag{2}$$

where * denotes the complex conjugate. With the aid of (1), eq. (2) can be rewritten as

$$\begin{aligned} P^{\text{in}} &= \tfrac{1}{2}\,\text{Re}\left[\sum_{m=1}^{N}\sum_{n=1}^{N} Z_{m,n}{}^{\text{in}}\, I_m{}^{T*}\, I_n{}^{T}\right] \\ &= \tfrac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N} (Z_{m,n}{}^{\text{in}} + Z_{n,m}{}^{\text{in}*})\, I_m{}^{T*}\, I_n{}^{T}. \end{aligned} \tag{3}$$

In the domain V, between S_0 and S_1, the electromagnetic-field vectors satisfy the source-free electromagnetic-field equations

$$\text{curl}\,\mathbf{H}^T + i\omega\,\mathbf{D}^T = \mathbf{0}, \tag{4}$$

$$\text{curl}\,\mathbf{E}^T - i\omega\,\mathbf{B}^T = \mathbf{0}, \tag{5}$$

and the constitutive equations which express $\{\mathbf{D}^T, \mathbf{B}^T\}$ linearly in terms of $\{\mathbf{E}^T, \mathbf{H}^T\}$. Owing to the continuity of the tangential parts of $\mathbf{E}^T$ and $\mathbf{H}^T$ across S_1 we can rewrite the expression for P^{in} as

$$P^{\text{in}} = \tfrac{1}{2}\,\text{Re}\left[\iint_{S_1} (\mathbf{E}^T \times \mathbf{H}^{T*}) \cdot \mathbf{n}\,\mathrm{d}A\right], \tag{6}$$

where $\mathbf{n}$ denotes the unit vector along the outward normal.

In the domain outside S_0, $\mathbf{E}^T$ and $\mathbf{H}^T$ satisfy the source-free electromagnetic-field equations

$$\text{curl}\,\mathbf{H}^T + i\omega\,\varepsilon_0\,\mathbf{E}^T = \mathbf{0}, \tag{7}$$

$$\text{curl}\,\mathbf{E}^T - i\omega\,\mu_0\,\mathbf{H}^T = \mathbf{0}. \tag{8}$$

In addition, the transmitted field satisfies the radiation conditions [7,8,9])

$$\hat{\mathbf{r}}\times\mathbf{H}^T + (\varepsilon_0/\mu_0)^{1/2}\,\mathbf{E}^T = O(r^{-2}) \qquad \text{as} \qquad r\to\infty, \tag{9}$$

$$\hat{\mathbf{r}}\times\mathbf{E}^T - (\mu_0/\varepsilon_0)^{1/2}\,\mathbf{H}^T = O(r^{-2}) \qquad \text{as} \qquad r\to\infty, \tag{10}$$

in which $\hat{\mathbf{r}}(=\mathbf{r}/r)$ denotes the unit vector in the radial direction and r is the distance from the origin to a point in space. As a consequence of eqs (7)–(10), the following representation holds:

$$\{\mathbf{E}^T(\mathbf{r}_P), \mathbf{H}^T(\mathbf{r}_P)\} \sim \{\mathbf{e}^T(\hat{\mathbf{r}}_P), \mathbf{h}^T(\hat{\mathbf{r}}_P)\}\exp(i\,k_0\,r_P)/4\,\pi\,r_P \qquad \text{as} \qquad r_P\to\infty, \tag{11}$$

in which

$$k_0 = \omega\,(\varepsilon_0\,\mu_0)^{1/2} = 2\pi/\lambda_0, \tag{12}$$

λ_0 being the wavelength in the medium outside S_0. P denotes the point of observation with position vector $\mathbf{r}_P$. Between $\mathbf{e}^T$ and $\mathbf{h}^T$ the following relations exist:

$$\mathbf{e}^T = (\mu_0/\varepsilon_0)^{1/2}\,(\mathbf{h}^T\times\hat{\mathbf{r}}_P), \tag{13}$$

$$\mathbf{h}^T = (\varepsilon_0/\mu_0)^{1/2}\,(\hat{\mathbf{r}}_P\times\mathbf{e}^T). \tag{14}$$

These relations, together with eq. (11), are in accordance with eqs (9) and (10). We shall denote $\mathbf{e}^T = \mathbf{e}^T(\hat{\mathbf{r}}_P)$ as the electric-field and $\mathbf{h}^T = \mathbf{h}^T(\hat{\mathbf{r}}_P)$ as the magnetic-field amplitude radiation characteristic of the antenna system. For a given N-port antenna, they only depend on the direction of observation and on the way in which the N accessible ports are fed. Both amplitude radiation characteristics are transverse with respect to the direction of propagation of the expanding spherical wave generated by the antenna, i.e. $\hat{\mathbf{r}}_P\,.\,\mathbf{e}^T = 0$ and $\hat{\mathbf{r}}_P\,.\,\mathbf{h}^T = 0$. We exhibit the dependence of $\mathbf{e}^T$ and $\mathbf{h}^T$ on the way in which the N accessible ports are fed, by writing

$$\{\mathbf{e}^T, \mathbf{h}^T\} = \sum_{n=1}^{N} \{\mathbf{e}_n{}^T, \mathbf{h}_n{}^T\}\, I_n{}^T. \tag{15}$$

The time-averaged electromagnetic power P^T radiated by the antenna is given by

$$P^T = \tfrac{1}{2}\,\mathrm{Re}\left[\iint_{S_0} (\mathbf{E}^T\times\mathbf{H}^{T*})\,.\,\mathbf{n}\,\mathrm{d}A\right], \tag{16}$$

where $\mathbf{n}$ denotes the unit vector along the direction of the outward normal. Since the medium outside S_0 is lossless, we can replace S_0 in eq. (16) by a sphere whose radius is taken to be so large that the representation (11) holds. Then, we can rewrite eq. (16) as

$$P^T = \frac{1}{32\,\pi^2}\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} \iint_{\Omega} \mathbf{e}^T\,.\,\mathbf{e}^{T*}\,\mathrm{d}\Omega, \tag{17}$$

where Ω denotes the sphere of unit radius. Incidentally, eq. (17) proves that $P^T > 0$ for any nonidentically vanishing $\mathbf{e}^T$. With the aid of (15), eq. (17) can be rewritten as

$$P^T = \sum_{m=1}^{N} \sum_{n=1}^{N} I_m{}^{T*} I_n{}^{T} \frac{1}{32\,\pi^2} \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} \iint_\Omega \mathbf{e}_m{}^{T*} \cdot \mathbf{e}_n{}^T \,\mathrm{d}\Omega. \tag{18}$$

Now, for a *lossless antenna* we have $P^T = P^{\mathrm{in}}$. In this case the right-hand sides of (3) and (18) should be equal, irrespective of the values of $\{I_n{}^T\}$. This condition leads to the relation

$$\tfrac{1}{4}\,(Z_{m,n}{}^{\mathrm{in}} + Z_{n,m}{}^{\mathrm{in}*}) = \frac{1}{32\,\pi^2} \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} \iint_\Omega \mathbf{e}_m{}^{T*} \cdot \mathbf{e}_n{}^T \,\mathrm{d}\Omega. \tag{19}$$

A relation similar to (19), has been derived by Van Bladel [10]).

It is known that $\mathbf{e}^T = \mathbf{e}^T(\hat{\mathbf{r}}_P)$ and $\mathbf{h}^T = \mathbf{h}^T(\hat{\mathbf{r}}_P)$ can be expressed in terms of the values that the tangential parts of $\mathbf{E}^T$ and $\mathbf{H}^T$ admit on S_0; the relevant expression for $\mathbf{e}^T$ is [2])

$$\mathbf{e}^T = i\,k_0\,\hat{\mathbf{r}}_P \times \iint_{S_0} \mathbf{n} \times \mathbf{E}^T(\mathbf{r}) \exp(-i\,k_0\,\hat{\mathbf{r}}_P \cdot \mathbf{r})\,\mathrm{d}A\, +$$
$$-\,i\,k_0\,(\mu_0/\varepsilon_0)^{1/2}\,\hat{\mathbf{r}}_P \times \left[\hat{\mathbf{r}}_P \times \iint_{S_0} \mathbf{n} \times \mathbf{H}^T(\mathbf{r}) \exp(-i\,k_0\,\hat{\mathbf{r}}_P \cdot \mathbf{r})\,\mathrm{d}A\right]. \tag{20}$$

Our proof of the reciprocity relation is in part based on this expression.

4. The antenna in the receiving situation

In the receiving situation a time-harmonic, uniform, plane electromagnetic wave is incident upon the antenna system, while the accessible ports are connected to an N-port load (fig. 3). As incident field $\{\mathbf{E}^i, \mathbf{H}^i\}$ we take

$$\mathbf{E}^i = \mathbf{A} \exp(-i\,k_0\,\boldsymbol{\alpha} \cdot \mathbf{r}), \tag{21}$$

$$\mathbf{H}^i = (\varepsilon_0/\mu_0)^{1/2}\,(\mathbf{A} \times \boldsymbol{\alpha}) \exp(-i\,k_0\,\boldsymbol{\alpha} \cdot \mathbf{r}), \tag{22}$$

where $\mathbf{A}$ is a constant, complex vector that specifies the amplitude and the phase of the plane wave at the origin, as well as its state of polarization, and $-\boldsymbol{\alpha}$ denotes the unit vector in the direction of propagation. (We call $\boldsymbol{\alpha}$ the direction of incidence.) The state of polarization is, in general, elliptic, but is linear if $\mathbf{A} \times \mathbf{A}^* = \mathbf{0}$ and circular if $\mathbf{A} \cdot \mathbf{A} = 0$. Since the wave is transverse, we have $\boldsymbol{\alpha} \cdot \mathbf{A} = 0$. In the domain outside S_0, the scattered field $\{\mathbf{E}^s, \mathbf{H}^s\}$ is introduced as the difference between the actual (total) field $\{\mathbf{E}^R, \mathbf{H}^R\}$ and the field of the incident wave $\{\mathbf{E}^i, \mathbf{H}^i\}$:

$$\mathbf{E}^s \stackrel{\mathrm{def}}{=} \mathbf{E}^R - \mathbf{E}^i, \qquad \mathbf{H}^s \stackrel{\mathrm{def}}{=} \mathbf{H}^R - \mathbf{H}^i. \tag{23}$$

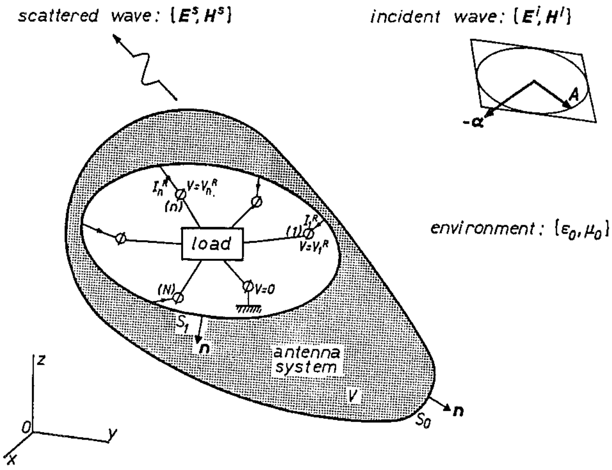

Fig. 3. Antenna in receiving situation: a uniform, plane, electromagnetic wave is incident on it and the accessible ports are connected to a load.

In the domain outside S_0, both the incident and the total fields, and hence the scattered field, satisfy the source-free electromagnetic-field equations

$$\text{curl}\,\mathbf{H}^{i,s} + i\omega\,\varepsilon_0\,\mathbf{E}^{i,s} = \mathbf{0}, \tag{24}$$

$$\text{curl}\,\mathbf{E}^{i,s} - i\omega\,\mu_0\,\mathbf{H}^{i,s} = \mathbf{0}. \tag{25}$$

In addition, the scattered field satisfies the radiation conditions

$$\hat{\mathbf{r}}\times\mathbf{H}^s + (\varepsilon_0/\mu_0)^{1/2}\,\mathbf{E}^s = O(r^{-2}) \quad \text{as} \quad r\to\infty, \tag{26}$$

$$\hat{\mathbf{r}}\times\mathbf{E}^s - (\mu_0/\varepsilon_0)^{1/2}\,\mathbf{H}^s = O(r^{-2}) \quad \text{as} \quad r\to\infty. \tag{27}$$

As a consequence of eqs (24)–(27) the following representation holds:

$$\{\mathbf{E}^s(\mathbf{r}_P), \mathbf{H}^s(\mathbf{r}_P)\} \sim \{\mathbf{e}^s(\hat{\mathbf{r}}_P), \mathbf{h}^s(\hat{\mathbf{r}}_P)\}\exp(i\,k_0\,r_P)/4\,\pi\,r_P \quad \text{as} \quad r_P\to\infty. \tag{28}$$

Between $\mathbf{e}^s$ and $\mathbf{h}^s$, relations of the type (13)–(14) exist, and for $\mathbf{e}^s$, a representation similar to (20) can be obtained.

The time-averaged power $P^R = P^R(\boldsymbol{\alpha})$, received by the antenna system, is given by

$$P^R = -\tfrac{1}{2}\,\text{Re}\left[\iint_{S_0} (\mathbf{E}^R\times\mathbf{H}^{R*})\cdot\mathbf{n}\,\mathrm{d}A\right]. \tag{29}$$

Further, the scattered power $P^s = P^s(\boldsymbol{\alpha})$ is defined as

$$P^s \overset{\text{def}}{=} \tfrac{1}{2}\,\text{Re}\left[\iint_{S_0} (\mathbf{E}^s\times\mathbf{H}^{s*})\cdot\mathbf{n}\,\mathrm{d}A\right]. \tag{30}$$

Since the medium outside S_0 is lossless, we can, on account of (28), rewrite (30) as

$$P^s = \frac{1}{32\,\pi^2}\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\iint_{\Omega} \mathbf{e}^s\cdot\mathbf{e}^{s*}\,\mathrm{d}\Omega. \tag{31}$$

Next, we add (29) and (30), subsequently use (23), (21), (22) and the representation of $\mathbf{e}^s$ similar to (20), and observe that the incident wave would in the absence of the antenna dissipate no power when travelling in the domain inside S_0. This procedure leads to

$$P^R + P^s = \tfrac{1}{2} \operatorname{Re} \left[(i\omega\, \mu_0)^{-1}\, \mathbf{A}^* \cdot \mathbf{e}^s(-\boldsymbol{\alpha})\right]. \tag{32}$$

Equation (32) is directly related to the "cross-section theorem" in electromagnetic scattering [11,12].

In the domain V, between S_0 and S_1, the electromagnetic-field vectors satisfy the source-free electromagnetic-field equations

$$\operatorname{curl} \mathbf{H}^R + i\omega\, \mathbf{D}^R = \mathbf{0}, \tag{33}$$

$$\operatorname{curl} \mathbf{E}^R - i\omega\, \mathbf{B}^R = \mathbf{0}, \tag{34}$$

and the constitutive equations which express $\{\mathbf{D}^R, \mathbf{B}^R\}$ linearly in terms of $\{\mathbf{E}^R, \mathbf{H}^R\}$. The time-averaged electromagnetic power $P^L = P^L(\boldsymbol{\alpha})$ dissipated in the load is given by

$$P^L = -\tfrac{1}{2} \operatorname{Re} \left[\iint_{S_1} (\mathbf{E}^R \times \mathbf{H}^{R*}) \cdot \mathbf{n}\, \mathrm{d}A \right]. \tag{35}$$

Owing to the continuity of the tangential parts of $\mathbf{E}^R$ and $\mathbf{H}^R$ across S_1, we can express P^L also in terms of the electric currents $\{I_n^R\}$ flowing into the load and the voltages $\{V_n^R\}$ across the ports of the load. The result is

$$P^L = \tfrac{1}{2} \operatorname{Re} \left[\sum_{m=1}^{N} V_m^R\, I_m^{R*} \right]. \tag{36}$$

Since the electromagnetic properties of the N-port load are assumed to be linear, the voltages $\{V_n^R\}$ are linearly related to the currents $\{I_n^R\}$ through the relation

$$V_m^R = \sum_{n=1}^{N} Z_{m,n}^L\, I_n^R \qquad (m = 1, \ldots, N), \tag{37}$$

where $Z_{m,n}^L$ defines the impedance matrix of the load. With the aid of (37), eq. (36) can be rewritten as

$$P^L = \tfrac{1}{2} \operatorname{Re} \left[\sum_{m=1}^{N} \sum_{n=1}^{N} Z_{m,n}^L\, I_m^{R*}\, I_n^R \right]$$

$$= \tfrac{1}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} (Z_{m,n}^L + Z_{n,m}^{L*})\, I_m^{R*}\, I_n^R. \tag{38}$$

Now, for a *lossless antenna* we have $P^R = P^L$. In this case we obtain from (31), (32) and (38)

$$\tfrac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N}(Z_{m,n}{}^{L}+Z_{n,m}{}^{L*})\,I_m{}^{R*}\,I_n{}^{R}+\frac{1}{32\,\pi^2}\left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}\iint_{\Omega}\mathbf{e}^s\,.\,\mathbf{e}^{s*}\,\mathrm{d}\Omega=$$

$$=\tfrac{1}{2}\,\mathrm{Re}\,[(i\omega\,\mu_0)^{-1}\,\mathbf{A}^*\,.\,\mathbf{e}^s(-\boldsymbol{\alpha})]. \tag{39}$$

5. The reciprocity relation

The starting point for the derivation of the reciprocity relation is Lorentz's reciprocity theorem for electromagnetic fields [13]). This theorem can be applied, provided that the electromagnetic properties of the medium present in the transmitting situation and those of the medium present in the receiving situation are interrelated in such a way that, at all points in space, the relation

$$\mathbf{E}^T\,.\,\mathbf{D}^R-\mathbf{E}^R\,.\,\mathbf{D}^T-\mathbf{H}^T\,.\,\mathbf{B}^R+\mathbf{H}^R\,.\,\mathbf{B}^T=0 \tag{40}$$

holds. In the domain outside S_0 this is obviously the case as the constitutive equations here are simply $\mathbf{D}=\varepsilon_0\,\mathbf{E}$ and $\mathbf{B}=\mu_0\,\mathbf{H}$, both in the transmitting and in the receiving situation. In the domain V, between S_0 and S_1, the situation may be more complicated. Here, eq. (40) holds without change of properties of the medium when the medium is reciprocal. In all other cases, the medium is nonreciprocal, and the appropriate change in properties has to be made when switching from transmission to reception and vice versa. It is noted that, in the general condition (40), nonreciprocal media, including those showing the magnetoelectric effect [14]), are included. If (40) is satisfied, Lorentz's theorem states that

$$\iint_S(\mathbf{E}^T\times\mathbf{H}^R-\mathbf{E}^R\times\mathbf{H}^T)\,.\,\mathbf{n}\,\mathrm{d}A=0 \tag{41}$$

for any sufficiently regular, bounded, closed surface S, provided that the domain bounded by S is free from electromagnetic sources. If (41) is applied to a domain outside S_0, we may, on account of eqs (23)–(25), also replace $\{\mathbf{E}^R,\mathbf{H}^R\}$ in (40) and (41) by either $\{\mathbf{E}^s,\mathbf{H}^s\}$ or $\{\mathbf{E}^i,\mathbf{H}^i\}$.

Let S_r denote the sphere with radius r and centre at the origin, where r is chosen so large that S_r completely surrounds S_0. Since the fields $\{\mathbf{E}^T,\mathbf{H}^T\}$ and $\{\mathbf{E}^s,\mathbf{H}^s\}$ both satisfy the radiation conditions (eqs (9) and (10) and (26) and (27), respectively), we have

$$\lim_{r\to\infty}\iint_{S_r}(\mathbf{E}^T\times\mathbf{H}^s-\mathbf{E}^s\times\mathbf{H}^T)\,.\,\mathbf{n}\,\mathrm{d}A=0. \tag{42}$$

Consequently, the application of Lorentz's theorem (41) to the domain bounded internally by S_0 and externally by S_r, and to the fields $\{\mathbf{E}^T,\mathbf{H}^T\}$ and $\{\mathbf{E}^s,\mathbf{H}^s\}$ leads in the limit $r\to\infty$ to

$$\iint_{S_0}(\mathbf{E}^T\times\mathbf{H}^s-\mathbf{E}^s\times\mathbf{H}^T)\,.\,\mathbf{n}\,\mathrm{d}A=0. \tag{43}$$

Next, we observe that, from (20), (21) and (22), it follows that

$$\iint_{S_0} (\mathbf{E}^T \times \mathbf{H}^i - \mathbf{E}^i \times \mathbf{H}^T) \cdot \mathbf{n}\, \mathrm{d}A = (i\omega\, \mu_0)^{-1}\, \mathbf{A} \cdot \mathbf{e}^T(\boldsymbol{\alpha}). \tag{44}$$

On adding eqs (43) and (44), and using (23), we arrive at

$$\iint_{S_0} (\mathbf{E}^T \times \mathbf{H}^R - \mathbf{E}^R \times \mathbf{H}^T) \cdot \mathbf{n}\, \mathrm{d}A = (i\omega\, \mu_0)^{-1}\, \mathbf{A} \cdot \mathbf{e}^T(\boldsymbol{\alpha}). \tag{45}$$

We proceed with the application of Lorentz's theorem (41) to the domain V, bounded internally by S_1 and externally by S_0 and to the fields $\{\mathbf{E}^T, \mathbf{H}^T\}$ and $\{\mathbf{E}^R, \mathbf{H}^R\}$, which gives

$$\iint_{S_1} (\mathbf{E}^T \times \mathbf{H}^R - \mathbf{E}^R \times \mathbf{H}^T) \cdot \mathbf{n}\, \mathrm{d}A = \iint_{S_0} (\mathbf{E}^T \times \mathbf{H}^R - \mathbf{E}^R \times \mathbf{H}^T) \cdot \mathbf{n}\, \mathrm{d}A. \tag{46}$$

However, on S_1 the field description in terms of the N accessible ports holds. Taking into account the direction of $\mathbf{n}$ and the chosen directions of the currents $\{I_n^T\}$ and $\{I_n^R\}$, we can rewrite the left-hand side of eq. (46) as

$$\iint_{S_1} (\mathbf{E}^T \times \mathbf{H}^R - \mathbf{E}^R \times \mathbf{H}^T) \cdot \mathbf{n}\, \mathrm{d}A = -\sum_{m=1}^{N} (V_m^T\, I_m^R + V_m^R\, I_m^T). \tag{47}$$

Combining eqs (45), (46) and (47), we arrive at the amplitude reciprocity relation (cf. ref. 15)

$$\sum_{m=1}^{N} (V_m^T\, I_m^R + V_m^R\, I_m^T) = -(i\omega\, \mu_0)^{-1}\, \mathbf{A} \cdot \mathbf{e}^T(\boldsymbol{\alpha}). \tag{48}$$

This relation will serve as the starting point for the derivation of the network representation of the antenna in the receiving situation.

6. The equivalent network for an N-port receiving antenna

In the reciprocity relation (48) we substitute eq. (1) in the left-hand side and use (15) in the right-hand side. Rearranging the result we obtain

$$\sum_{m=1}^{N} \left(\sum_{n=1}^{N} Z_{n,m}^{\text{in}}\, I_n^R + V_m^R \right) I_m^T = -(i\omega\, \mu_0)^{-1} \sum_{m=1}^{N} [\mathbf{A} \cdot \mathbf{e}_m^T(\boldsymbol{\alpha})]\, I_m^T. \tag{49}$$

This equation should hold irrespective of the values of $\{I_m^T\}$. As a consequence, it follows that

$$\sum_{n=1}^{N} Z_{n,m}^{\text{in}}\, I_n^R + V_m^R = \mathsf{E}_m^R \qquad (m = 1, \ldots, N), \tag{50}$$

in which the "equivalent electromotive force" $E_m^R = E_m^R(\boldsymbol{\alpha})$ is given by

$$E_m^R = -(i\omega\,\mu_0)^{-1}\,\mathbf{A}\,.\,\mathbf{e}_m^T(\boldsymbol{\alpha}) \qquad (m = 1, \ldots, N). \tag{51}$$

Equation (50) describes the properties of an N-port electrical network with internal voltage sources (Thévenin representation). The N-port network whose network equations are (50), is the equivalent electrical network for the antenna in the receiving situation. Equation (51) shows how E_m^R depends on the amplitude, the phase and the state of polarization of the incident wave and, through the geometrical factor $\mathbf{e}_m^T(\boldsymbol{\alpha})$, on the direction of incidence. Note that the internal-impedance matrix of the network is the transpose of the input-impedance matrix of the antenna in the transmitting situation.

One application of eqs (50) and (51) is the experimental technique for determining the electric-field amplitude radiation characteristics $\{\mathbf{e}_m^T(\boldsymbol{\alpha})\}$ of a given N-port antenna, in the transmitting situation, by using the antenna as a receiving antenna and measuring its reaction on an incident plane wave. A simple procedure performing this, runs as follows:

(a) a specific value of $\boldsymbol{\alpha}$ and two different values of $\mathbf{A}$, corresponding to two independent states of polarization, are chosen (the values of $\mathbf{A}$ can be determined experimentally by removing the antenna and measuring $\mathbf{E}^i$ at the origin of the coordinate system, the latter point serving as a reference point for the phase of the fields);

(b) all ports are left open, i.e. $I_n^R = 0$ for all n, and V_m^R is measured for all m (in this case V_m^R equals E_m^R as (50) shows);

(c) from eq. (51) we calculate the two complex components of $\mathbf{e}_m^T$ at the selected value of $\boldsymbol{\alpha}$ (note that both $\mathbf{A}$ and $\mathbf{e}_m^T$ are transverse with respect to $\boldsymbol{\alpha}$);

(d) new values of $\boldsymbol{\alpha}$ and $\mathbf{A}$ are selected, and the procedure is repeated until enough values of $\{\mathbf{e}_m^T(\boldsymbol{\alpha})\}$ have been obtained.

7. The condition for maximum power transfer from the incident wave to the load

Let now an N-port load be connected to the accessible ports. Then, the relations (37) hold. For any given load, substitution of (37) in (50) leads to a system of N linear, algebraic equations, from which the currents $\{I_n^R\}$ can be determined. Once this has been done, the power P^L dissipated in the load can be calculated from (38).

One of the problems associated with the use of a receiving antenna is to design such a load that, for a given antenna system, maximum power is transferred from the incident wave to the load. On account of the relations (37) and (50), this is a pure network-theoretical problem, and it has a unique solution. The answer is that the impedance matrix of the load should be the Hermitean conjugate of the internal-impedance matrix of the electric network under consideration. For reference, a proof of this statement is included in an appendix. As the internal-impedance matrix of the equivalent network for the antenna in the receiving situation is the transpose of the input-impedance matrix of the antenna in the transmitting situation, maximum power is dissipated in the load if

$$Z_{m,n}{}^{L} = Z_{m,n}{}^{\mathrm{in}*} \qquad (m,n = 1, \ldots, N). \tag{52}$$

Hence, for maximum power transfer, the impedance matrix of the load should be equal to the complex conjugate (not the Hermitean conjugate) of the input-impedance matrix of the antenna in the transmitting situation. If (52) holds, the load is called "matched" to the antenna. Apparently, the condition for matching is independent of the direction of incidence $\boldsymbol{\alpha}$.

Appendix

The maximum-power-transfer theorem for an N-port network with prescribed internal sources

We consider a linear, passive, time-invariant electric network with prescribed internal sources and N accessible ports ($N \geqslant 1$). The network is operating in the sinusoidal steady state. The equations corresponding to the Thévenin representation of the network are given by

$$\sum_{n=1}^{N} Z_{m,n} I_n + V_m = E_m \qquad (m = 1, \ldots, N), \tag{A.1}$$

where $\{I_n\}$ are the currents flowing out of the ports, $\{V_m\}$ are the voltages across the ports, $Z_{m,n}$ defines the internal-impedance matrix and $\{E_m\}$ are the equivalent electromotive forces of the internal sources. The time-averaged power P^L dissipated in an N-port load is given by

$$P^L = \tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N} V_m I_m{}^*\right]. \tag{A.2}$$

We now define the "optimum state" of the network as the one that maximizes P^L. In the optimum state, let $\{V_m{}^{\mathrm{opt}}\}$ be the voltages, $\{I_n{}^{\mathrm{opt}}\}$ the currents and $P^{L,\mathrm{opt}}$ the power dissipated in the load. Then

$$P^{L,\mathrm{opt}} = \tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N} V_m{}^{\mathrm{opt}} I_m{}^{\mathrm{opt}*}\right]. \tag{A.3}$$

Let us now consider arbitrary (not necessarily small) variations *) $\{\delta V_m\}$ and $\{\delta I_n\}$ around $\{V_m{}^{\mathrm{opt}}\}$ and $\{I_n{}^{\mathrm{opt}}\}$, respectively, and take

$$V_m = V_m{}^{\mathrm{opt}} + \delta V_m \qquad (m = 1, \ldots, N) \tag{A.4}$$

and

$$I_n = I_n{}^{\mathrm{opt}} + \delta I_n \qquad (n = 1, \ldots, N). \tag{A.5}$$

---

*) While preparing the manuscript, the author's attention was called to a paper by Desoer [16]), where a similar line of reasoning is followed and where a more sophisticated treatment is given.

While varying the voltages and the currents, we keep $\{Z_{m,n}\}$ and $\{E_m\}$ fixed. Consequently, $\{\delta V_m\}$ and $\{\delta I_n\}$ are interrelated through

$$\sum_{n=1}^{N} Z_{m,n}\,\delta I_n + \delta V_m = 0 \qquad (m = 1, \ldots, N), \tag{A.6}$$

as follows from (A.1). Substitution of (A.3), (A.4) and (A.5) in (A.2) yields

$$P^L = P^{L,\mathrm{opt}} + \tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N} (\delta V_m\, I_m{}^{\mathrm{opt}*} + V_m{}^{\mathrm{opt}}\,\delta I_m{}^* + \delta V_m\,\delta I_m{}^*)\right]. \tag{A.7}$$

Since

$$\tfrac{1}{2}\,\mathrm{Re}\,(\delta V_m\, I_m{}^{\mathrm{opt}*}) = \tfrac{1}{2}\,\mathrm{Re}\,(\delta V_m{}^*\, I_m{}^{\mathrm{opt}}), \tag{A.8}$$

the first-order terms on the right-hand side of (A.7) cancel if

$$\mathrm{Re}\left[\sum_{m=1}^{N} (\delta V_m{}^*\, I_m{}^{\mathrm{opt}} + V_m{}^{\mathrm{opt}}\,\delta I_m{}^*)\right] = 0. \tag{A.9}$$

On account of (A.6), this can be rewritten as

$$\mathrm{Re}\left[\sum_{m=1}^{N} \left(-\sum_{n=1}^{N} Z_{n,m}{}^*\, I_n{}^{\mathrm{opt}} + V_m{}^{\mathrm{opt}}\right)\delta I_m{}^*\right] = 0. \tag{A.10}$$

Equation (A.10) holds for arbitrary complex $\{\delta I_m{}^*\}$, provided that

$$V_m{}^{\mathrm{opt}} = \sum_{n=1}^{N} Z_{n,m}{}^*\, I_n{}^{\mathrm{opt}} \qquad (m = 1, \ldots, N). \tag{A.11}$$

Introducing the impedance matrix of the load through

$$V_m = \sum_{n=1}^{N} Z_{m,n}{}^L\, I_n \qquad (m = 1, \ldots, N), \tag{A.12}$$

eq. (A.11) implies that in the optimum state we have

$$Z_{m,n}{}^{L,\mathrm{opt}} = Z_{n,m}{}^* \qquad (m, n = 1, \ldots, N). \tag{A.13}$$

Equation (A.13) states that the optimum N-port load has an impedance matrix which is the Hermitean conjugate of the internal-impedance matrix of the network. It now remains to be shown that the condition (A.9) maximizes P^L. To this aim we observe that if, apart from the sources, the given N-port network is passive, we have

$$\tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N}\sum_{n=1}^{N} \delta I_m{}^*\, Z_{m,n}\,\delta I_n\right] \geqslant 0 \tag{A.14}$$

for any sequence $\{\delta I_n\}$. Using (A.6), (A.14) and (A.9) in (A.7) we arrive at

$$P^L \leqslant P^{L,\mathrm{opt}} \tag{A.15}$$

for any sequence $\{\delta I_n\}$. This result completes the proof. If the given N-port network is dissipative, the left-hand side of (A.14) is positive for any non-identically vanishing sequence $\{\delta I_n\}$ and hence $P^L < P^{L,\mathrm{opt}}$ for any non-identically vanishing sequence $\{\delta I_n\}$.

The time-averaged power P_i that is dissipated internally in the network is given by

$$P_i = \tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N}\sum_{n=1}^{N} I_m^*\, Z_{m,n}\, I_n\right] = \tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N}\sum_{n=1}^{N} I_n\, Z_{n,m}^*\, I_m^*\right]. \tag{A.16}$$

In the optimum state this reduces, on account of (A.13), to

$$P_i^{\mathrm{opt}} = P^{L,\mathrm{opt}}. \tag{A.17}$$

The time-averaged power P_s that is delivered by the sources is given by

$$P_s = \tfrac{1}{2}\,\mathrm{Re}\left[\sum_{m=1}^{N} \mathsf{E}_m\, I_m^*\right]. \tag{A.18}$$

Using (A.1), (A.2) and (A.16), this becomes

$$P_s = P_i + P^L. \tag{A.19}$$

In the optimum state we therefore have, by virtue of (A.17) and (A.19)

$$P_i^{\mathrm{opt}} = P^{L,\mathrm{opt}} = \tfrac{1}{2}\, P_s^{\mathrm{opt}}. \tag{A.20}$$

REFERENCES

1) R. E. Collin and F. J. Zucker, Antenna theory, Part 1, McGraw-Hill Book Company, New York, 1969, Chapter 4.
2) A. T. de Hoop, Appl. sci. Res. **19**, 90-96, 1968.
3) J. R. Mautz and R. F. Harrington, IEEE Trans. Ant. Prop. **AP-21**, 188-199, 1973.
4) R. F. Harrington and J. R. Mautz, IEEE Trans. Ant. Prop. **AP-22**, 184-190, 1974.
5) W. K. Kahn and H. Kurss, IEEE Trans. Ant. Prop. **AP-13**, 671-675, 1965.
6) W. Wasylkiwskyj and W. K. Kahn, IEEE Trans. Ant. Prop. **AP-18**, 204-216, 1970.
7) S. Silver, Microwave antenna theory and design, McGraw-Hill Book Company, New York, 1949, p. 85.
8) C. Müller, Foundations of the mathematical theory of electromagnetic waves, Springer-Verlag, Berlin, 1969, p. 136.
9) J. van Bladel, Electromagnetic fields, McGraw-Hill Book Company, New York, 1964, p. 204.
10) J. van Bladel, Arch. elektr. Uebertr. **20**, 447-450, 1966.
11) A. T. de Hoop, Appl. sci. Res. **B7**, 463-469, 1959.
12) J. van Bladel, Electromagnetic fields, McGraw-Hill Book Company, New York, 1964, p. 258.
13) R. E. Collin and F. J. Zucker, Antenna theory, Part 1, McGraw-Hill Book Company, New York, 1969, p. 24.
14) T. H. O'Dell, The electrodynamics of magneto-electric media, North-Holland Publishing Company, Amsterdam, 1970, p. 22.
15) A. T. de Hoop and G. de Jong, Proc. IEE **121**, 1051-1056, 1974.
16) C. A. Desoer, IEEE Trans. Circuit Theory **CT-20**, 328-330, 1973.