Radio Science, Volume 19, Number 5, Pages 1179–1184, September–October 1984

A time domain energy theorem for scattering of plane electromagnetic waves

A. T. de Hoop

Delft University of Technology, Department of Electrical Engineering, Laboratory of Electromagnetic Research, The Netherlands

(Received January 3, 1984; revised March 7, 1984; accepted March 13, 1984.)

A time domain energy theorem for the scattering of plane electromagnetic waves by an obstacle of bounded extent is derived. It is the counterpart in the time domain of the "optical theorem" or the "extinction cross section theorem" in the frequency domain. No assumptions as to the electromagnetic behavior of the obstacle need to be made; so, the obstacle may be electromagnetically nonlinear and/or time variant (a kind of behavior that is excluded in the frequency domain result). As to the wave motion, three different kinds of time behavior are distinguished: (1) transient, (2) periodic, and (3) perpetuating, but with finite mean power flow density. For all three cases the total energy (case 1) or the time-averaged power (cases 2 and 3) that is both absorbed and scattered by the obstacle is related to a certain time interaction integral of the incident plane wave and the spherical-wave amplitude of the scattered wave in the far-field region, when observed in the direction of propagation of the incident wave. The practical implications of the energy theorem are briefly indicated.

1. INTRODUCTION

In the theory of the scattering of electromagnetic waves by an obstacle of bounded extent there are several theorems that interrelate the different quantities associated with this scattering. In the frequency domain analysis of the problem, it must be assumed that the scattering obstacle is linear and time invariant in its electromagnetic behavior. A time domain analysis of the scattering problem reveals the more general conditions under which the relevant theorems may also hold in the time domain. In the present paper, the energy theorem for plane wave scattering is investigated. Its frequency domain counterpart is known as the "optical theorem" or "extinction cross section theorem" [*de Hoop*, 1959; *Van Bladel*, 1964]. For three different kinds of time behavior, namely, (1) transient, (2) periodic, and (3) perpetuating with finite mean power flow density, it is shown that the total energy (case 1) or the time-averaged power (cases 2 and 3), that is both absorbed and scattered by the obstacle, is related to a certain time interaction integral of the incident plane wave and the spherical-wave amplitude of the scattered wave in the far-field region, when observed in the direction of propagation of the incident wave. Our time-domain derivation of the theorem shows that the latter holds for obstacles that may be nonlinear and/or time variant in their electromagnetic behavior. The theorem implies that the total amount of energy (or the total time-averaged power) that is both absorbed and scattered by the obstacle can, in principle, be determined from a measurement at a single position in the far-field region, provided that the incident plane wave is known from a separate measurement.

Paper number 4S0421.
0048-6604/84/004S-0421$08.00

2. FORMULATION OF THE SCATTERING PROBLEM

In three-dimensional space $\mathbf{R}^3$ a scattering object is present. It occupies the bounded domain $\mathscr{D}$. The boundary surface of $\mathscr{D}$ is denoted by $\partial\mathscr{D}$, and the complement of the union of $\mathscr{D}$ and $\partial\mathscr{D}$ in $\mathbf{R}^3$ is denoted by $\mathscr{D}'$. The unit vector along the normal to $\partial\mathscr{D}$, pointing away from $\mathscr{D}$, is denoted by $\mathbf{n}$ (Figure 1). It is assumed that $\partial\mathscr{D}$ is piecewise smooth. The electromagnetic properties of the scattering object remain unspecified; it may show a nonlinear and/or a time variant behavior. The medium occupying the domain $\mathscr{D}'$ is electromagnetically characterized by a scalar, positive, constant permittivity ε and a scalar, positive, constant permeability μ. The speed of electromagnetic waves in this medium is $c = (\varepsilon\mu)^{-1/2}$.

Position in space is characterized by the position vector $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$, where x, y, and z are the Cartesian coordinates with respect to the orthogonal Cartesian reference frame with origin $\mathcal{O}$ and the three mutually perpendicular base vectors of unit length $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$. In the order indicated, the base vectors

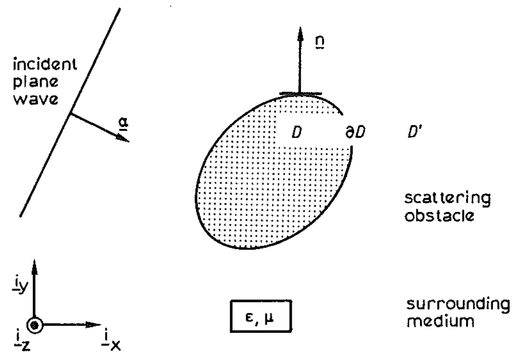

Fig. 1. Scattering configuration with incident plane wave. The speed of electromagnetic waves in the surrounding medium is $c = (\varepsilon\mu)^{-1/2}$.

form a right-handed system. The time coordinate is denoted by t. Partial differentiation is denoted by ∂, and $\nabla = \mathbf{i}_x\partial_x + \mathbf{i}_y\partial_y + \mathbf{i}_z\partial_z$.

The electromagnetic field in the configuration is characterized by the electric field intensity $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and the magnetic field intensity $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$. In $\mathscr{D}'$, where the medium is linear, the total field is written as the sum of the incident field $\{\mathbf{E}^i, \mathbf{H}^i\}$ and the scattered field $\{\mathbf{E}^s, \mathbf{H}^s\}$. Note that, in general, the scattered field is not linearly related to the incident field. The incident field is defined everywhere in $\mathbf{R}^3$ and satisfies in $\mathscr{D}$ the source-free electromagnetic field equations, i.e.,

$$\nabla \times \mathbf{H}^i - \varepsilon\partial_t \mathbf{E}^i = \mathbf{0} \qquad \mathbf{r} \in \mathscr{D} \tag{1}$$

$$\nabla \times \mathbf{E}^i + \mu\partial_t \mathbf{H}^i = \mathbf{0} \qquad \mathbf{r} \in \mathscr{D} \tag{2}$$

The scattered field is defined in $\mathscr{D}'$ and satisfies in this domain the source-free electromagnetic field equations, i.e.,

$$\nabla \times \mathbf{H}^s - \varepsilon\partial_t \mathbf{E}^s = \mathbf{0} \qquad \mathbf{r} \in \mathscr{D}' \tag{3}$$

$$\nabla \times \mathbf{E}^s + \mu\partial_t \mathbf{H}^s = \mathbf{0} \qquad \mathbf{r} \in \mathscr{D}' \tag{4}$$

At large distances from the scattering object the scattered field admits the representation

$$\mathbf{E}^s(\mathbf{r}, t) \sim \mathbf{e}^s(\mathbf{i}_r, t - |\mathbf{r}|/c)/4\pi|\mathbf{r}| \qquad \text{as} \quad |\mathbf{r}| \to \infty \tag{5}$$

$$\mathbf{H}^s(\mathbf{r}, t) \sim \mathbf{h}^s(\mathbf{i}_r, t - |\mathbf{r}|/c)/4\pi|\mathbf{r}| \qquad \text{as} \quad |\mathbf{r}| \to \infty \tag{6}$$

where $\mathbf{i}_r = \mathbf{r}/|\mathbf{r}|$ is the unit vector in the direction of observation. Hence $\mathbf{i}_r \in \Omega$, where Ω is the spherical surface with origin $\mathscr{O}$ and unit radius ($\Omega = \{\mathbf{r}; \mathbf{r} \cdot \mathbf{r} = 1\}$). The right-hand sides of (5) and (6) are the expressions for the field intensities in the far-field region. Between the electric and the magnetic far-field amplitude radiation characteristics $\mathbf{e}^s$ and $\mathbf{h}^s$, the following relations exist:

$$\mathbf{e}^s = Z\mathbf{h}^s \times \mathbf{i}_r \tag{7}$$

$$\mathbf{h}^s = Y\mathbf{i}_r \times \mathbf{e}^s \tag{8}$$

where $Z = (\mu/\varepsilon)^{1/2}$ is the plane wave impedance and $Y = (\varepsilon/\mu)^{1/2}$ is the plane wave admittance in the medium surrounding the obstacle. Equations (7) and (8) imply that $\mathbf{i}_r \cdot \mathbf{e}^s = 0$ and $\mathbf{i}_r \cdot \mathbf{h}^s = 0$, i.e., the scattered spherical wave is asymptotically transverse in its leading term. Note that the time argument of $\mathbf{e}^s$, $\mathbf{h}^s$ is delayed by the travel time from the origin (which is located in the neighborhood of the obstacle) to the point of observation.

Further, in the analysis we need the instantaneous power flow P^i of the incident wave across $\partial\mathscr{D}$ and toward $\mathscr{D}$, i.e.,

$$P^i = -\int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n} \cdot (\mathbf{E}^i \times \mathbf{H}^i)\, dA \tag{9}$$

the instantaneous power flow P^s that the scattered wave carries away from $\partial\mathscr{D}$ toward $\mathscr{D}'$, i.e.,

$$P^s = \int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n} \cdot (\mathbf{E}^s \times \mathbf{H}^s)\, dA \tag{10}$$

and the instantaneous flow P^a of power that is absorbed by the obstacle, i.e.,

$$P^a = -\int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H})\, dA \tag{11}$$

For the incident wave we now take the uniform plane wave propagating in the direction of the unit vector α:

$$\{\mathbf{E}^i, \mathbf{H}^i\} = \{\mathbf{e}^i(t - \alpha \cdot \mathbf{r}/c), \mathbf{h}^i(t - \alpha \cdot \mathbf{r}/c)\} \tag{12}$$

Between $\mathbf{e}^i$ and $\mathbf{h}^i$ the following relations exist:

$$\mathbf{e}^i = Z\mathbf{h}^i \times \alpha \tag{13}$$

$$\mathbf{h}^i = Y\alpha \times \mathbf{e}^i \tag{14}$$

where Z and Y are the same as in (7) and (8). Note that (13) and (14) imply that the wave is transverse.

3. SURFACE SOURCE REPRESENTATION OF THE SCATTERED FIELD

The basic tool in the derivation of the energy theorem is the time domain surface source representation of the scattered field. This representation is the electromagnetic analog of the Kirchhoff representation for scalar wave fields. Let

$$\mathbf{J}_S = \mathbf{n} \times \mathbf{H}^s \qquad \mathbf{r} \in \partial\mathscr{D} \tag{15}$$

and

$$\mathbf{K}_S = \mathbf{E}^s \times \mathbf{n} \qquad \mathbf{r} \in \partial\mathscr{D} \tag{16}$$

denote the scattered-field surface densities of electric and magnetic current, respectively, and let

$$\mathbf{A}^e(\mathbf{r}, t) = \int_{t_0}^{\infty} dt' \int_{\mathbf{r}' \in \partial\mathscr{D}} G(\mathbf{r} - \mathbf{r}', t - t')\mathbf{J}_S(\mathbf{r}', t')\, dA \tag{17}$$

and

$$\mathbf{A}^m(\mathbf{r}, t) = \int_{t_0}^{\infty} dt' \int_{\mathbf{r}' \in \partial\mathscr{D}} G(\mathbf{r} - \mathbf{r}', t - t')\mathbf{K}_S(\mathbf{r}', t')\, dA \tag{18}$$

denote the corresponding vector potentials. In (17) and (18)

$$G(\mathbf{r}, t) = (4\pi|\mathbf{r}|)^{-1}\delta(t - |\mathbf{r}|/c) \tag{19}$$

denotes the free-space Green's function of the three-dimensional scalar wave equation. Then the following integral relation for the scattered field holds:

$$-\mu\partial_t \mathbf{A}^e + \varepsilon^{-1}\nabla\left(\nabla \cdot \int_{t_0}^{t} \mathbf{A}^e dt'\right) - \nabla \times \mathbf{A}^m$$
$$= \{1, \tfrac{1}{2}, 0\}\mathbf{E}^s(\mathbf{r}, t) \qquad \mathbf{r} \in \{\mathscr{D}', \partial\mathscr{D}, \mathscr{D}\} \qquad t \in (t_0, \infty) \tag{20}$$

$$-\varepsilon\partial_t \mathbf{A}^m + \mu^{-1}\nabla\left(\nabla \cdot \int_{t_0}^{t} \mathbf{A}^m dt'\right) + \nabla \times \mathbf{A}^e$$
$$= \{1, \tfrac{1}{2}, 0\}\mathbf{H}^s(\mathbf{r}, t) \qquad \mathbf{r} \in \{\mathscr{D}', \partial\mathscr{D}, \mathscr{D}\} \qquad t \in (t_0, \infty) \tag{21}$$

In (20) and (21) we have taken into account the condition of causality, i.e., we have assumed that the scattered field vanishes everywhere in $\mathscr{D}'$ prior to t_0, where t_0 is the instant at which the incident wave hits the obstacle. A concise derivation of (20) and (21) can be obtained with the aid of a Laplace transform with respect to time and a Fourier transform over $\mathscr{D}'$. From the derivation it follows that in the right-hand sides of (15) and (16) the limiting values upon approaching $\partial\mathscr{D}$ via $\mathscr{D}'$ have to be taken.

By letting $|\mathbf{r}| \to \infty$ in (17)–(21), we arrive at integral representations for the far-field amplitude radiation characteristics of the scattered wave. In the expression for $G(\mathbf{r} - \mathbf{r}', t - t')$ (see (19)), we employ the relation

$$|\mathbf{r} - \mathbf{r}'| = |\mathbf{r}| - \mathbf{i}_r \cdot \mathbf{r}' + \text{vanishing terms} \qquad \text{as} \quad |\mathbf{r}| \to \infty \tag{22}$$

The use of (22) in (17) and (18) leads to

$$\mathbf{A}^{e,m} \sim \mathbf{a}^{e,m}(\mathbf{i}_r, t - |\mathbf{r}|/c)/4\pi|\mathbf{r}| \qquad \text{as} \quad |\mathbf{r}| \to \infty \tag{23}$$

where

$$\mathbf{a}^e(\mathbf{i}_r, t) = \int_{\mathbf{r}' \in \partial\mathscr{D}} \mathbf{J}_S(\mathbf{r}', t + \mathbf{i}_r \cdot \mathbf{r}'/c)\, dA \tag{24}$$

$$\mathbf{a}^m(\mathbf{i}_r, t) = \int_{\mathbf{r}' \in \partial\mathscr{D}} \mathbf{K}_S(\mathbf{r}', t + \mathbf{i}_r \cdot \mathbf{r}'/c)\, dA \tag{25}$$

The use of (23) in (20) and (21) leads to the asymptotic expressions (5) and (6) with

$$\mathbf{e}^s = -\mu[\partial_t \mathbf{a}^e - \mathbf{i}_r(\mathbf{i}_r \cdot \partial_t \mathbf{a}^e)] + c^{-1}\mathbf{i}_r \times \partial_t \mathbf{a}^m \tag{26}$$

$$\mathbf{h}^s = -\varepsilon[\partial_t \mathbf{a}^m - \mathbf{i}_r(\mathbf{i}_r \cdot \partial_t \mathbf{a}^m)] - c^{-1}\mathbf{i}_r \times \partial_t \mathbf{a}^e \tag{27}$$

It can easily be verified that the right-hand sides of (26) and (27) satisfy (7) and (8).

4. ENERGY THEOREM

The time domain energy theorem takes on different shapes, depending on the type of time behavior of the electromagnetic field. Three cases are considered: (1) transient fields, (2) time-periodic fields, and (3) perpetuating fields with bounded mean value. The three cases will be dealt with separately.

4.1. *Transient fields*

Transient fields vanish prior to a certain instant and go to zero as $t \to \infty$, and these properties hold at any point in space. In our scattering problem the instant t_0 at which the incident wave hits the obstacle marks the onset of the scattering phenomenon. By applying Gauss' divergence theorem to the domain $\mathscr{D}$ and to the vector $\mathbf{E}^i \times \mathbf{H}^i$, and using (1) and (2), it follows that

$$\int_{t_0}^{\infty} P^i\, dt = 0 \tag{28}$$

where P^i is given by (9). This result expresses that the medium with constitutive coefficients ε and μ is lossless. Further, the total energy W^a that is absorbed by the obstacle is

$$W^a = \int_{t_0}^{\infty} P^a\, dt \tag{29}$$

where P^a is given by (11), while the total energy W^s carried by the scattered wave is

$$W^s = \int_{t_0}^{\infty} P^s\, dt \tag{30}$$

where P^s is given by (10). Let us consider now the expression for the sum of the absorbed energy and the scattered energy. With the aid of $\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s$ and $\mathbf{H} = \mathbf{H}^i + \mathbf{H}^s$ the relevant expression can be rewritten as

$$W^a + W^s = -\int_{t_0}^{\infty} dt \int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n}(\mathbf{E}^i \times \mathbf{H}^s + \mathbf{E}^s \times \mathbf{H}^i)\, dA \tag{31}$$

The substitution of (12) and the use of (15) and (16) yield

$$W^a + W^s = \int_{t_0}^{\infty} dt \int_{\mathbf{r}\in\partial\mathscr{D}} [\mathbf{e}^i(t - \alpha \cdot \mathbf{r}/c) \cdot \mathbf{J}_S(\mathbf{r}, t) + \mathbf{h}^i(t - \alpha \cdot \mathbf{r}/c) \cdot \mathbf{K}_S(\mathbf{r}, t)]\, dA \tag{32}$$

We now introduce the instant t^i at which the incident wave reaches the origin of our chosen coordinate system. Then we have $\mathbf{e}^i(t) = \mathbf{0}$ and $\mathbf{h}^i(t) = \mathbf{0}$ when $-\infty < t < t^i$, and the right-hand side of (32) can, upon shifting the time integration, be rewritten as

$$W^a + W^s = \int_{t^i}^{\infty} dt \int_{\mathbf{r}\in\partial\mathscr{D}} [\mathbf{e}^i(t) \cdot \mathbf{J}_S(\mathbf{r}, t + \alpha \cdot \mathbf{r}/c) + \mathbf{h}^i(t) \cdot \mathbf{K}_S(\mathbf{r}, t + \alpha \cdot \mathbf{r}/c)]\, dA \tag{33}$$

Obviously, the relation between t_0 and t^i is given by

$$t_0 = t^i + \min_{\mathbf{r}\in\partial\mathscr{D}}(\alpha \cdot \mathbf{r}/c) \tag{34}$$

After comparing the right-hand side of (33) with the expressions for $\mathbf{e}^s(\alpha, t)$ and $\mathbf{h}^s(\alpha, t)$ that result from (24)–(27), and taking into account that $\alpha \cdot \mathbf{e}^i = 0$ and $\alpha \cdot \mathbf{h}^i = 0$, we arrive at

$$W^a + W^s = -\mu^{-1} \int_{t^i}^{\infty} \mathbf{e}^i(t) \cdot \left[\int_{t^i}^{t} \mathbf{e}^s(\alpha, t')dt' \right] dt \tag{35}$$

and

$$W^a + W^s = -\varepsilon^{-1} \int_{t^i}^{\infty} \mathbf{h}^i(t) \cdot \left[\int_{t^i}^{t} \mathbf{h}^s(\alpha, t')dt' \right] dt \tag{36}$$

Equations (35) and (36) constitute the energy theorem for transient plane wave scattering. In the right-hand sides the scattered-field, spherical-wave amplitudes in the far-field region occur in the direction of observation α, i.e., in the direction of propagation of the incident wave.

4.2. *Time-periodic fields*

For time-periodic fields, with period T, we introduce the time-averaged values, over a period, of the different power flows. Let $\langle\ \rangle_T$ denote the time average over a period, i.e.,

$$\langle\ \rangle_T = T^{-1} \int_{t_0}^{t_0+T} dt \tag{37}$$

Then the counterpart of (28) is

$$\langle P^i \rangle_T = 0 \tag{38}$$

Further, the counterparts of (29) and (30) are

$$\langle P^a \rangle_T = T^{-1} \int_{t_0}^{t_0+T} P^a\, dt \tag{39}$$

and

$$\langle P^s \rangle_T = T^{-1} \int_{t_0}^{t_0+T} P^s\, dt \tag{40}$$

respectively. Again we consider the expression for the sum of the (time-averaged) absorbed power and the (time-averaged) scattered power. This can be rewritten as (see (31))

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\left\langle \int_{\mathbf{r}\in\partial\mathscr{D}} \mathbf{n} \cdot (\mathbf{E}^i \times \mathbf{H}^s + \mathbf{E}^s \times \mathbf{H}^i)\, dA \right\rangle_T \tag{41}$$

The substitution of (12) and the use of (15) and (16) yield

$$\langle P^a \rangle_T + \langle P^s \rangle_T = \left\langle \int_{\mathbf{r}\in\partial\mathscr{D}} [\mathbf{e}^i(t - \alpha \cdot \mathbf{r}/c) \cdot \mathbf{J}_S(\mathbf{r}, t) + \mathbf{h}^i(t - \alpha \cdot \mathbf{r}/c) \cdot \mathbf{K}_S(\mathbf{r}, t)]\, dA \right\rangle_T \tag{42}$$

After interchanging the time integration with the one over $\partial\mathscr{D}$ and shifting the variable in the resulting time integration, we obtain

$$\langle P^a \rangle_T + \langle P^s \rangle_T = \int_{\mathbf{r}\in\partial\mathscr{D}} \langle \mathbf{e}^i(t) \cdot \mathbf{J}_S(\mathbf{r}, t + \alpha \cdot \mathbf{r}/c) + \mathbf{h}^i(t) \cdot \mathbf{K}_S(\mathbf{r}, t + \alpha \cdot \mathbf{r}/c) \rangle_T\, dA \tag{43}$$

Upon comparing the right-hand side with the expressions for $\mathbf{e}^s(\alpha, t)$ and $\mathbf{h}^s(\alpha, t)$ that result from (24)–(27), and taking into account that $\alpha \cdot \mathbf{e}^i = 0$ and $\alpha \cdot \mathbf{h}^i = 0$, we arrive at

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\mu^{-1} \left\langle \mathbf{e}^i(t) \cdot \int_{t_0}^{t} \mathbf{e}^s(\alpha, t')\, dt' \right\rangle_T \tag{44}$$

and

$$\langle P^a\rangle_T + \langle P^s\rangle_T = -\varepsilon^{-1}\left\langle \mathbf{h}^i(t)\cdot\int_{t_0}^{t}\mathbf{h}^s(\alpha, t')\,dt'\right\rangle_T \quad (45)$$

Equations (44) and (45) constitute the energy theorem for time-periodic plane wave scattering. Note that in the time integration of the scattered-field, spherical-wave amplitudes the properties

$$\langle \mathbf{e}^s(\alpha, t')\rangle_T = 0 \quad (46)$$

and

$$\langle \mathbf{h}^s(\alpha, t')\rangle_T = 0 \quad (47)$$

hold in view of (26) and (27).

Obviously, it has been assumed here that the incident field and the scattered field are both time periodic with the same period T. Now, with regard to the scattering object, this implies that a possibly time varying behavior has to comply with this assumption, i.e., the electromagnetic properties of the scattering object must at most be time periodic with the same period T, also.

4.3. *Perpetuating fields*

For perpetuating fields we assume that the time-averaged values of the different power flow densities exist. Let $\langle\ \rangle_\infty$ denote the relevant time averages; then

$$\langle\ \rangle_\infty = \lim_{T_1\to-\infty,\, T_2\to\infty}(T_2 - T_1)^{-1}\int_{T_1}^{T_2} dt \quad (48)$$

In accordance with this, the fields are assumed to have bounded values as $t\to-\infty$ and as $t\to\infty$. Then with (1) and (2), it follows that

$$\langle P^i\rangle_\infty = 0 \quad (49)$$

As in the case of transient fields we consider the expression for the sum of the time-averaged absorbed power and the time-averaged scattered power. This can be written as (see (41))

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = -\left\langle\int_{\mathbf{r}\in\partial\mathcal{D}}\mathbf{n}\cdot(\mathbf{E}^i\times\mathbf{H}^s + \mathbf{E}^s\times\mathbf{H}^i)\,dA\right\rangle_\infty \quad (50)$$

The substitution of (12) and the use of (15) and (16) yield

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = \left\langle\int_{\mathbf{r}\in\partial\mathcal{D}}[\mathbf{e}^i(t-\alpha\cdot\mathbf{r}/c)\cdot\mathbf{J}_S(\mathbf{r}, t) + \mathbf{h}^i(t-\alpha\cdot\mathbf{r}/c)\cdot\mathbf{K}_S(\mathbf{r}, t)]\,dA\right\rangle_\infty \quad (51)$$

After interchanging the time integration with the one over $\partial\mathcal{D}$ and shifting the variable in the resulting time integration, we obtain

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = \int_{\mathbf{r}\in\partial\mathcal{D}}\left\langle\mathbf{e}^i(t)\cdot\mathbf{J}_S(\mathbf{r}, t+\alpha\cdot\mathbf{r}/c) + \mathbf{h}^i(t)\cdot\mathbf{K}_S(\mathbf{r}, t+\alpha\cdot\mathbf{r}/c)\right\rangle_\infty dA \quad (52)$$

After comparing the right-hand side with the expressions for $\mathbf{e}^s(\alpha, t)$ and $\mathbf{h}^s(\alpha, t)$ that result from (24)–(27), and taking into account that $\alpha\cdot\mathbf{e}^i = 0$ and $\alpha\cdot\mathbf{h}^i = 0$, we arrive at

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = -\mu^{-1}\left\langle\mathbf{e}^i(t)\cdot\int_{-\infty}^{t}\mathbf{e}^s(\alpha, t')\,dt'\right\rangle_\infty \quad (53)$$

and

$$\langle P^a\rangle_\infty + \langle P^s\rangle_\infty = -\varepsilon^{-1}\left\langle\mathbf{h}^i(t)\cdot\int_{-\infty}^{t}\mathbf{h}^s(\alpha, t')\,dt'\right\rangle_\infty \quad (54)$$

Equations (53) and (54) constitute the energy theorem for the scattering of perpetuating plane waves. Note that in the time integration of the scattered-field, spherical-wave amplitudes the properties

$$\langle\mathbf{e}^s(\alpha, t')\rangle_\infty = 0 \quad (55)$$

and

$$\langle\mathbf{h}^s(\alpha, t')\rangle_\infty = 0 \quad (56)$$

hold. In comparison with the case of time-periodic fields, no restrictions are, in this case, laid upon the possible time behavior of the electromagnetic properties of the scattering object.

5. USAGE OF THE ENERGY THEOREM

The energy theorem derived in the previous section relates the quantity $W^a + W^s$ to the far-field amplitude radiation characteristic of the scattered field observed in the direction of propagation of the illuminating plane wave. As such, it can serve as a check on the different approximations (of an analytical or a computational nature) that are made in the practical analysis of scattering problems. The analysis of the scattering by a nonlinear optical grating by *Reinisch and Nevière* [1983] yields an example where approximations of an analytical nature are involved. Approximations of a computational nature are induced by the discretization, both in time and in space, used to solve a scattering problem numerically. In this respect it is noted that the energy theorem contains only integrals over finite domains which can be evaluated by standard numerical procedures. They run over the surface of the scattering object (see (24) and (25)) and over a certain time interval. With regard to the time integrations, the following observations are made. For transient fields, the right-hand sides of (35) and (36) formally require an integration over the interval (t_i, ∞); in practice, this integration needs only to be carried out over the interval during which the incident pulse significantly differs from zero. For time-periodic fields, the right-hand sides of (44) and (45) require, as expected, only integrations over a single time period. For perpetuating fields, the right-hand sides of (53) and (54) fall in the category of time correlation integrals for which appropriate routines should be employed.

Suppose now that the quantity $W^a + W^s$ has been computed, independently, from the field quantities in the domain occupied by the scattering object. Then the energy theorem shows up to what accuracy the computations are consistent. If, on the other hand, one trusts the computed field values, one need not compute the quantity $W^a + W^s$ (which requires extra integrations), but directly obtains it by use of the energy theorem.

6. CONCLUSION

For the scattering of plane electromagnetic waves by an object of bounded extent embedded in a homogeneous, isotropic, linear, lossless medium, an energy theorem has been derived. This theorem is the time domain counterpart of the frequency domain "optical theorem." It relates the energy that is both absorbed and scattered by the object to the spherical-wave amplitude of the scattered field in the far-field region, when observed in the direction of propagation of the incident plane wave. Depending on the time behavior of the incident wave (transient, periodic, perpetuating), the energy theorem takes on slightly different forms. An important consequence of the full time domain analysis is that the electromagnetic properties of the scattering object hardly need any specification, i.e., unlike the case of frequency domain analysis, time-variant and/or nonlinear behavior is included. The practical implications of the energy theorem are briefly indicated.

REFERENCES

de Hoop, A. T., On the plane-wave extinction cross-section of an obstacle, *Appl. Sci. Res.*, *B7*, 463–469, 1959.

Reinisch, R., and M. Neviére, Electromagnetic theory of gratings ruled on nonlinear materials and surface-enhanced nonlinear optical effects, paper presented at International Symposium on Electromagnetic Theory, URSI, Santiago de Compostela, Spain, 1983.

Van Bladel, J., *Electromagnetic Fields*, pp. 254–258, McGraw-Hill, New York, 1964.

A. T. de Hoop, Delft University of Technology, Department of Electrical Engineering, Laboratory of Electromagnetic Research, P.O. Box 5031, 2600 GA Delft, The Netherlands.