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TUTORIAL

LARGE-OFFSET APPROXIMATIONS IN THE MODIFIED CAGNIARD METHOD FOR COMPUTING SYNTHETIC SEISMOGRAMS: A SURVEY[1]

A. T. DE HOOP[2]

ABSTRACT

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Large horizontal offset and large vertical offset approximations in the modified Cagniard method for computing synthetic seismograms in a horizontally layered model of the earth are derived. They apply to each of the generalized-ray constituents into which the seismic wave motion is decomposed. For the results applying to large horizontal offset, which are known in the seismological literature, a simplified derivation is given. The results for large vertical offset, which are of particular interest to vertical seismic profiling, are new. The use of the large-offset approximations, both horizontal and vertical, leads to a considerable reduction in computation time for synthetic seismograms compared with the full three-dimensional version of the modified Cagniard method.

1. INTRODUCTION

The modified Cagniard method is one of the methods that can be applied to compute synthetic seismograms. It applies to a lossless, horizontally layered model of the earth. In the method, the total seismic wave motion is decomposed into generalized-ray constituents (Spencer 1960; Cisternas, Betancourt and Leiva 1973) that travel from the source, and after successive reflections and transmissions at the interfaces reach the point of observation. The space–time expressions for these constituents are arrived at by a specific scheme of integral-transform relations, combined with complex-variable analysis. For recent reviews of the method, see Van der Hijden (1987) and De Hoop (1988). Although the numerical evaluation of the

[1] Received June 1987, accepted December 1987.

[2] Laboratory of Electromagnetic Research, Faculty of Electrical Engineering, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands.

resulting integrals and the determination of the relevant modified Cagniard paths consume little time, at least per generalized ray, a considerable simplification still arises for large horizontal and large vertical offsets between source and receiver. The former are often encountered in surface seismics, the latter in vertical seismic profiling. The large horizontal offset approximation has been used by Roever, Vining and Strick (1959), Helmberger (1968) and Wiggins and Helmberger (1974). These authors derived it from the Pekeris representation of the transform-domain seismic wavefield (Pekeris 1955, 1956). Both the large horizontal offset and the large vertical offset approximations can be obtained directly from De Hoop's modification of Cagniard's method (see Cagniard 1939, 1962; De Hoop 1960, 1961; Achenbach 1973; Miklowitz 1978; Aki and Richards 1980) by replacing particular integrals by their appropriate asymptotic representations, without using (modified) Bessel functions in the intermediate steps. Thus, for the large horizontal offset approximation a simplified derivation is obtained, while the results for large vertical offset are new. The use of the large-offset approximations, both horizontal and vertical, leads to a considerable reduction in computation time for synthetic seismograms compared with a full three-dimensional version of the modified Cagniard method. For the large horizontal offset approximation this feature has been investigated by Helmberger and Harkrider (1978). The large vertical offset approximation has been numerically implemented by Van der Hijden (1987).

2. The Generalized-Ray Constituent

By applying the standard scheme of integral transformations of the modified Cagniard method, the time Laplace-transform domain expression of a generalized-ray constituent in a lossless, horizontally layered, structure of isotropic material (model of the earth) is obtained as

$$\hat{w}(\mathbf{x}, s) = (s/2\pi)^2\hat{Q}(s) \int_{\alpha_1=-\infty}^{\infty} \mathrm{d}\alpha_1 \int_{\alpha_2=-\infty}^{\infty} \Pi(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2) \exp\left\{-s\left[\mathrm{i}\alpha_1 x_1 + \mathrm{i}\alpha_2 x_2 + \sum_{\lambda\in\Lambda} \gamma_\lambda(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)h_\lambda\right]\right\} \mathrm{d}\alpha_2 , \tag{1}$$

where s is the real and positive time Laplace-transform parameter, $\mathbf{x}$ is the Cartesian position vector from the source to the point of observation, x_1 and x_2 are the horizontal components of $\mathbf{x}$, h_λ is the (possibly multiple) path that the generalized ray has traversed in the layer with wave speed c_λ (of either P- or S-waves),

$$\gamma_\lambda = (1/c_\lambda^2 + \alpha_1^2 + \alpha_2^2)^{1/2} \tag{2}$$

is the vertical slowness of the wave with wave speed c_λ, Λ is the set of layers in which the generalized ray has propagated, i is the imaginary unit, $\hat{Q}(s)$ is representative of the source signature, and $\Pi = \Pi(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)$ is an s-independent product of the coupling coefficient of the generalized ray to the source, and a number of interface reflection and transmission coefficients. It is assumed that the source has started to act at the instant $t = 0$, where t is the time coordinate. The aim of the modified

Cagniard method is to rewrite the expression

$$\hat{G}(\mathbf{x}, s) = (2\pi)^{-2} \int_{\alpha_1=-\infty}^{\infty} \mathrm{d}\alpha_1 \int_{\alpha_2=-\infty}^{\infty} \Pi(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2) \times \exp\left\{-s\left[\mathrm{i}\alpha_1 x_1 + \mathrm{i}\alpha_2 x_2 + \sum_{\lambda\in\Lambda} \gamma_\lambda(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)h_\lambda\right]\right\} \mathrm{d}\alpha_2 \tag{3}$$

in the form

$$\hat{G}(\mathbf{x}, s) = \int_{\tau=\mathrm{T}}^{\infty} \exp(-s\tau) w^G(\mathbf{x}, \tau)\, \mathrm{d}\tau\,, \tag{4}$$

where τ is a real variable of integration, $T > 0$ and $w^G(\mathbf{x}, \tau)$ is independent of s. If this is accomplished, the uniqueness theorem of the Laplace transformation for real, positive transform parameter (Lerch's theorem; see Widder 1946) ensures that the space–time counterpart $w = w(\mathbf{x}, t)$ of $\hat{w} = \hat{w}(\mathbf{x}, s)$ is given by

$$w(\mathbf{x}, t) = \begin{cases} 0 & \text{for } t < T, \\ \partial_t^2 \int_{\tau=T}^{t} Q(t-\tau) w^G(\mathbf{x}, \tau)\, \mathrm{d}\tau & \text{for } t > T, \end{cases} \tag{5}$$

where additional elementary rules of the Laplace transformation have been used. The function $w^G(\mathbf{x}, \tau)$ is denoted as the (space–time) Green's function of the generalized ray under consideration.

3. The Large Horizontal Offset Approximation

Let the polar-coordinate specification of x_1 and x_2 be given by

$$x_1 = r\cos(\phi), \qquad x_2 = r\sin(\phi), \tag{6}$$

with $r \geq 0$, $0 \leq \phi < 2\pi$. To arrive at the large horizontal offset approximation we replace in (3) the variables of integration $\{\alpha_1, \alpha_2\}$ by $\{\kappa, \psi\}$ via

$$\alpha_1 = \kappa\cos(\psi + \phi), \qquad \alpha_2 = \kappa\sin(\psi + \phi), \tag{7}$$

with $\kappa \geq 0$, $0 \leq \psi < 2\pi$. Then, (3) changes into

$$\hat{G}(\mathbf{x}, s) = (2\pi)^{-2} \int_{\kappa=0}^{\infty} \kappa\, \mathrm{d}\kappa \int_{\psi=0}^{2\pi} \bar{\Pi}(\mathrm{i}\kappa, \psi, \phi) \times \exp\left\{-s\left[\mathrm{i}\kappa r\cos(\psi) + \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda(\mathrm{i}\kappa)h_\lambda\right]\right\} \mathrm{d}\psi, \tag{8}$$

where $\bar{\Pi} = \bar{\Pi}(\mathrm{i}\kappa, \psi, \phi)$ results from $\Pi = \Pi(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)$ and $\bar{\gamma}_\lambda = \bar{\gamma}_\lambda(\mathrm{i}\kappa)$ from $\gamma_\lambda = \gamma_\lambda(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)$ under the substitution (7). From (2) it follows that

$$\bar{\gamma}_\lambda = \bar{\gamma}_\lambda(\mathrm{i}\kappa) = (1/c_\lambda^2 + \kappa^2)^{1/2} \tag{9}$$

is independent of ψ and ϕ. The next step consists of decomposing the integral with respect to ψ as

$$\int_{\psi=0}^{2\pi} \mathrm{d}\psi = \int_{\psi=-\pi/2}^{\pi/2} \mathrm{d}\psi + \int_{\psi=\pi/2}^{3\pi/2} \mathrm{d}\psi, \tag{10}$$

where the periodicity in ψ of the integral has been used. In the last integral we replace the variable of integration ψ by $\psi + \pi$; then the bounds become the same as those of the first integral. However, under the replacement $\psi \to \psi + \pi$, we have $\alpha_1 \to -\alpha_1$ and $\alpha_2 \to -\alpha_2$ as (7) shows. Since the transform-domain seismic wavefield quantities arise from partial differential equations with real-valued coefficients and the wavefield quantities are real-valued, Π has the property $\Pi(-\mathrm{i}\alpha_1, -\mathrm{i}\alpha_2) = \Pi(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)^*$, where * denotes complex conjugate. Since s, κ, r, ϕ and h_λ are real, we have

$$\hat{G}(\mathbf{x}, s) = (2\pi^2)^{-1}\mathrm{Re} \int_{\kappa=0}^{\infty} \kappa \,\mathrm{d}\kappa \int_{\psi=-\pi/2}^{\pi/2} \bar{\Pi}(\mathrm{i}\kappa, \psi, \phi)$$
$$\times \exp\left\{-s\left[\mathrm{i}\kappa r \cos(\psi) + \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda(\mathrm{i}\kappa)h_\lambda\right]\right\} \mathrm{d}\psi. \tag{11}$$

The large horizontal offset approximation is found by applying the method of stationary phase to the integral with respect to ψ and replacing it by its asymptotic representation for $r \to \infty$ (Erdelyi 1956, pp. 51–56). The only stationary point of the argument of the exponential function is $\psi = 0$, which is an inner point of the interval of integration. Using around $\psi = 0$ the expansion

$$\cos(\psi) \approx 1 - \psi^2/2, \tag{12}$$

and employing the result

$$\int_{\psi=-\infty}^{\infty} \exp(\mathrm{i}s\kappa r\psi^2/2)\,\mathrm{d}\psi = (2\pi/s\kappa r)^{1/2} \exp(\mathrm{i}\pi/4), \tag{13}$$

we obtain the asymptotic approximation

$$\int_{\psi=-\pi/2}^{\pi/2} \bar{\Pi}(\mathrm{i}\kappa, \psi, \phi) \exp[-\mathrm{i}s\kappa r \cos(\psi)]\,\mathrm{d}\psi$$
$$\sim (2\pi/s\kappa r)^{1/2} \exp(\mathrm{i}\pi/4)\bar{\Pi}(\mathrm{i}\kappa, 0, \phi) \exp(-\mathrm{i}s\kappa r) \quad \text{as} \quad r \to \infty. \tag{14}$$

Finally, we replace in the integration with respect to κ, the variable of integration of κ by $p = \mathrm{i}\kappa$. Then $\kappa^{-1/2} \exp(\mathrm{i}\pi/4) = \mathrm{i}p^{-1/2}$, and $\kappa\,\mathrm{d}\kappa = -p\,\mathrm{d}p$. The resulting expression for $\hat{G}(\mathbf{x}, s)$ is

$$\hat{G}(\mathbf{x}, s) \sim (\pi/s)^{1/2}\hat{g}_{\mathrm{hor}}(\mathbf{x}, s) \quad \text{as} \quad r \to \infty, \tag{15}$$

where

$$\hat{g}_{\mathrm{hor}}(\mathbf{x}; s) = (2/r)^{1/2}(2\pi^2)^{-1}\,\mathrm{Im} \int_{p=0}^{\mathrm{i}\infty} \bar{\Pi}(p, 0, \phi) \exp\left\{-s\left[pr + \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda(p)h_\lambda\right]\right\} p^{1/2}\,\mathrm{d}p. \tag{16}$$

The application of the modified Cagniard method to infer the time-domain counterpart of the right-hand side of (15) will be discussed in Section 5.

4. The Large Vertical Offset Approximation

To arrive at the large vertical offset approximation we replace in (3) the variables of integration $\{\alpha_1, \alpha_2)$ by $\{p, q\}$ via

$$\alpha_1 = -ip\cos(\phi) - q\sin(\phi), \tag{17}$$
$$\alpha_2 = -ip\sin(\phi) + q\cos(\phi),$$

with $p \in I$ and $q \in R$, and where ϕ is the same as in (6). Then, (3) changes into

$$\hat{G}(\mathbf{x}, s) = (4\pi^2 \mathrm{i})^{-1} \int_{p=-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp(-spr)\,\mathrm{d}p \times \int_{q=-\infty}^{\infty} \bar{\Pi}(p, \mathrm{i}q, \phi) \exp\left[-s \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda(p, \mathrm{i}q) h_\lambda\right] \mathrm{d}q, \tag{18}$$

where $\bar{\Pi} = \bar{\Pi}(p, \mathrm{i}q, \phi)$ results from $\Pi = \Pi(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)$ and $\bar{\gamma}_\lambda = \bar{\gamma}_\lambda(p, \mathrm{i}q)$ from $\gamma_\lambda = \gamma_\lambda(\mathrm{i}\alpha_1, \mathrm{i}\alpha_2)$ under the substitution (17). From (2) it follows that

$$\bar{\gamma}_\lambda = \bar{\gamma}_\lambda(p, \mathrm{i}q) = (1/c_\lambda^2 + q^2 - p^2)^{1/2}. \tag{19}$$

For large values of $\sum_{\lambda\in\Lambda} h_\lambda$ the main contribution to the integration with respect to q arises from around $q = 0$. In view of this, the large vertical offset approximation is arrived at by replacing the Laplace-type integral with respect to q by its asymptotic approximation for $\sum_{\lambda\in\Lambda} h_\lambda \to \infty$ (Erdelyi 1956, pp. 36–39). Using the expansion

$$\bar{\gamma}_\lambda \approx \bar{\gamma}_\lambda(p, 0) + q^2/2\bar{\gamma}_\lambda(p, 0) \tag{20}$$

around $q = 0$ and employing the result

$$\int_{q=-\infty}^{\infty} \exp\left[-s(q^2/2) \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda^{-1}(p, 0) h_\lambda\right] \mathrm{d}q = \left[2\pi/s \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda^{-1}(p, 0) h_\lambda\right]^{1/2}, \tag{21}$$

we obtain the asymptotic approximation

$$\int_{q=-\infty}^{\infty} \bar{\Pi}(p, \mathrm{i}q, \phi) \exp\left[-s \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda(p, \mathrm{i}q) h_\lambda\right] \mathrm{d}q \sim \left[2\pi/s \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda^{-1}(p, 0) h_\lambda\right]^{1/2} \bar{\Pi}(p, 0, \phi) \times \exp\left[-s \sum_{\lambda\in\Lambda} \bar{\gamma}_\lambda(p, 0) h_\lambda\right] \quad \text{as} \quad \sum_{\lambda\in\Lambda} h_\lambda \to \infty. \tag{22}$$

Using (22) in (18) we arrive at

$$\hat{G}(\mathbf{x}, s) \sim (\pi/s)^{1/2} \hat{g}_{\mathrm{ver}}(\mathbf{x}; s) \quad \text{as} \quad \sum_{\lambda\in\Lambda} h_\lambda \to \infty, \tag{23}$$

where

$$\hat{g}_{\text{ver}}(\mathbf{x}, s) = (4\pi^2 \mathrm{i})^{-1} 2^{1/2} \int_{p=-\mathrm{i}\infty}^{\mathrm{i}\infty} \bar{\Pi}(p, 0, \phi) \exp\left\{-s\left[pr + \sum_{\lambda \in \Lambda} \bar{\gamma}_\lambda(p, 0) h_\lambda\right]\right\} \times \left[\sum_{\lambda \in \Lambda} \bar{\gamma}_\lambda^{-1}(p, 0) h_\lambda\right]^{-1/2} \mathrm{d}p. \quad (24)$$

The application of the modified Cagniard method to infer the time-domain counterpart of the right-hand side of (24) will be discussed in Section 5.

5. The Modified Cagniard Path

The first step in the modified Cagniard method consists of continuing—in (16) and (24)—the integrand analytically into the complex p-plane, away from the imaginary axis. In this process we encounter the singularities in $\bar{\Pi}(p, 0, \phi)$ and in (cf. (9) and (19))

$$\Gamma_\lambda(p) = (1/c_\lambda^2 - p^2)^{1/2}. \quad (25)$$

These are the branch points of $\Gamma_\lambda(p)$, i.e. $p = \pm 1/c_\lambda$ and the possible poles of $\bar{\Pi}(p, 0, \phi)$. The latter are associated with the possible surface waves of the interface type (non-existent along a fluid/fluid interface, Scholte waves along a fluid/solid interface, Stoneley waves along a solid/solid interface, and Rayleigh waves along the traction-free boundary of a solid). If surface-wave poles do occur in $\bar{\Pi}(p, 0, \phi)$, they are located on the real p-axis in the interval $|\mathrm{Re}\,(p)| > \max_{\lambda \in \Lambda} (1/c_\lambda)$. To make the analytical continuation single-valued, we introduce branch cuts along $\mathrm{Im}(p) = 0$, $|\mathrm{Re}\,(p)| \geq 1/c_\lambda$ for all λ. In the cut p-plane, we then have $\mathrm{Re}\,[\Gamma_\lambda(p)] > 0$.

Next, the integrations in (16) and (24) in the complex p-plane are carried out on a path along which

$$pr + \sum_{\lambda \in \Lambda} \Gamma_\lambda(p) h_\lambda = \tau, \quad (26)$$

where τ is real and positive; such a path is denoted as a modified Cagniard path. Since $r \geq 0$, and $\mathrm{Im}\,(\Gamma_\lambda) < 0$ and > 0 in the upper and lower halves of the complex p-plane, respectively, the modified Cagniard paths are located in the right-half of the p-plane. It is clear that the part of the real p-axis $0 \leq \mathrm{Re}\,(p) < \min_{\lambda \in \Lambda} (1/c_\lambda)$, $\mathrm{Im}\,(p) = 0$, satisfies (26). On this part, $\bar{\Pi}(p, 0, \phi)$ is real-valued, since the transform-domain wave constituents result from the integral transformations of partial differential equations with real-valued coefficients and the wavefield quantities are real-valued. Further, there is a complex part that satisfies (26); it has the asymptotic representation

$$p \sim \tau \Big/ \left(r \mp \mathrm{i} \sum_{\lambda \in \Lambda} h_\lambda\right) \qquad \text{as} \qquad \tau \to \infty \quad (27)$$

in the upper or lower half of the p-plane. This part is denoted as the body-wave part of the modified Cagniard path; its representation in the first quadrant of the p-plane will be denoted as $p = p^B(\tau)$. Since the left-hand side of (26) satisfies Schwarz's reflection principle, the representation in the fourth quadrant is then $p = p^{B*}(\tau)$, where * denotes complex conjugate. The point of intersection of $p = p^B(\tau)$ and $p = p^{B*}(\tau)$ with the real p-axis follows from the consideration that at that point τ attains its minimum value. Let $p = p_0$ denote the relevant (real) value of p, then by differentiating (26) with respect to p,

$$r - p \sum_{\lambda \in \Lambda} [h_\lambda/\Gamma_\lambda(p)] = 0 \qquad \text{at} \qquad p = p_0. \tag{28}$$

Since $p = p_0$ is necessarily located in between $p = 0$ and each of the branch points $p = 1/c_\lambda$, $\lambda \in \Lambda$ (all Γ_λ with $\lambda \in \Lambda$ must be real at p_0), we can write

$$p_0 = (1/c_\lambda) \sin(\theta_\lambda) \qquad \text{for all} \qquad \lambda \in \Lambda, \tag{29}$$

with $0 \leq \theta_\lambda \leq \pi/2$. Equation (29) reflects Snell's law of refraction. Since

$$\Gamma_\lambda(p) = (1/c_\lambda) \cos(\theta_\lambda) \qquad \text{at} \qquad p = p_0, \tag{30}$$

(28) can be rewritten as

$$r - \sum_{\lambda \in \Lambda} h_\lambda \tan(\theta_\lambda) = 0 \qquad \text{at} \qquad p = p_0. \tag{31}$$

Let $\tau = T^B$ at $p = p_0$, then T^B follows from (26) and (31) as

$$T^B = \sum_{\lambda \in \Lambda} [h_\lambda/c_\lambda \cos(\theta_\lambda)]. \tag{32}$$

Since T^B will be shown to be the arrival time of the body-wave part of the generalized-ray constituent, (32) is in accordance with Fermat's principle.

Firstly we replace the integration in (16) by one along the path $p = p^B(\tau)$ and the integration in (24) by one along the paths $p = p^B(\tau)$ and $p = p^{B*}(\tau)$. In view of Cauchy's theorem, Jordan's lemma and the properties of $p = p^B(\tau)$, this is admissible provided that $\bar{\Pi} = \bar{\Pi}(p; 0, \phi)$ has either no other singularities than $p = 1/c_\lambda$, $\lambda \in \Lambda$ or possible additional branch points (due to reflection against a layer in which the generalized-ray constituent does not propagate) that are outside the range $0 \leq p \leq \min_{\lambda \in \Lambda} (1/c_\lambda)$. The generalized-ray constituent under consideration then contains only a body-wave part. If $\bar{\Pi} = \bar{\Pi}(p, 0, \phi)$ has an additional branch point in the indicated range, the body-wave part of the modified Cagniard path must, depending on the location of the point of observation with respect to the source, be supplemented:

—in (16) by an integral running from this branch point, and just above its branch cut, to $p = p_0$,

—in (24) by a loop integral around the branch cut belonging to this branch point and joining the points $p = p_0 - i0$ and $p = p_0 + i0$, where $p = p^{B*}(\tau)$ and $p = p^B(\tau)$, respectively, were tempted to cross the real p-axis. This part of the modified Cagniard path is the head-wave part, and its contribution to the generalized-ray constituent is denoted as its head-wave contribution. The body-wave and the head-wave contributions to the generalized-ray constituent will be further investigated in Sections 6 and 7, respectively.

6. The Body-Wave Contribution

The body-wave contributions to the generalized-ray Green's function in the large-offset approximations follow from (16), (24) and (26) as

$$\hat{g}^{B}_{\text{hor}}(\mathbf{x}, s) = (2/r)^{1/2}(2\pi^2)^{-1} \int_{\tau=T^B}^{\infty} \exp(-s\tau) \operatorname{Im}[\bar{\Pi}(p^B, 0, \phi)(p^B)^{1/2}(\partial p^B/\partial\tau)]\, d\tau, \quad (33)$$

and

$$\hat{g}^{B}_{\text{ver}}(\mathbf{x}, s) = (2\pi^2)^{-1}2^{1/2} \int_{\tau=T^B}^{\infty} \exp(-s\tau) \times \operatorname{Im}\left\{\bar{\Pi}(p^B, 0, \phi)\left[\sum_{\lambda\in\Lambda} \Gamma_\lambda^{-1}(p^B)h_\lambda\right]^{-1/2}(\partial p^B/\partial\tau)\right\} d\tau. \quad (34)$$

In (34), the parts along $p = p^B(\tau)$ and $p = p^{B*}(\tau)$ have been taken together and Schwarz's reflection principle has been used. The Jacobian of the transformation from p^B to τ follows from (26) as

$$\partial p^B/\partial\tau = \left[r - p^B \sum_{\lambda\in\Lambda} \Gamma_\lambda^{-1}(p^B)h_\lambda\right]^{-1}. \quad (35)$$

Equations (33) and (34) are the shape of a Laplace transformation over the range (T^B, ∞). The uniqueness theorem of the Laplace transformation with real, positive transform parameter then yields the space–time expressions

$$\hat{g}^{B}_{\text{hor}}(\mathbf{x}, t) = \begin{cases} 0 & \text{for} \quad t < T^B, \\ (2/r)^{1/2}(2\pi^2)^{-1} \operatorname{Im}[\bar{\Pi}(p^B, 0, \phi)(p^B)^{1/2}(\partial p^B/\partial\tau)] & \text{for} \quad t > T^B, \end{cases} \quad (36)$$

and

$$\hat{g}^{B}_{\text{ver}}(\mathbf{x}, t) = \begin{cases} 0 & \text{for} \quad t < T^B, \\ (2\pi^2)^{-1}2^{1/2} \operatorname{Im}\{\bar{\Pi}(p^B, 0, \phi) \left[\sum_{\lambda\in\Lambda} \Gamma_\lambda^{-1}(p^B)h_\lambda\right]^{-1/2}(\partial p^B/\partial\tau)\} & \text{for} \quad t > T^B. \end{cases} \quad (37)$$

Further, $(\pi/s)^{1/2}$ corresponds in the time domain to the function $t^{-1/2}$ for $t > 0$. Then the body-wave contribution to the generalized-ray Green's function follows in the large horizontal offset approximation from (15) as

$$w^{G;B}(\mathbf{x}, t) \sim \begin{cases} 0 & \text{for} \quad t < T^B, \\ \int_{t'=T^B}^{t} (t - t')^{-1/2} g^{B}_{\text{hor}}(\mathbf{x}, t')\, dt' & \text{for} \quad t > T^B, \end{cases} \quad (38)$$

as $r \to \infty$, and in the large vertical offset approximation from (23) as

$$w^{G;B}(\mathbf{x}, t) \sim \begin{cases} 0 & \text{for} \quad t < T^B, \\ \displaystyle\int_{t'=T^B}^{t} (t - t')^{-1/2} g^B_{\text{ver}}(\mathbf{x}, t')\, \mathrm{d}t' & \text{for} \quad T > T^B, \end{cases} \tag{39}$$

as $\sum_{\lambda \in \Lambda} h_\lambda \to \infty$.

7. The Head-wave Contribution

Let us consider the head-wave contribution that is due to the occurrence of Γ_μ in $\bar{\Pi}(p, 0, \phi)$, where $\mu \notin \Lambda$ and $1/c_\mu < \min_{\lambda \in \Lambda} (1/c_\lambda)$. Then the body-wave part of the modified Cagniard path must be supplemented by the head-wave part (discussed at the end of Section 5). Along this part too, the parametrization (26) has to be carried out. The relevant values of p in the upper and lower halves of the p-plane will be denoted by $p = p^H(\tau)$ and $p = p^{H*}(\tau)$, respectively. Let $\tau = T^H$ denote the value of τ at $p = 1/c_\mu$, then (26) leads to

$$T^H = r/c_\mu + \sum_{\lambda \in \Lambda} (1/c_\lambda^2 - 1/c_\mu^2)^{1/2} h_\lambda. \tag{40}$$

For a head-wave contribution to occur we must have $1/c_\mu < p_0$, or, using (29),

$$1/c_\mu < (1/c_\lambda) \sin(\theta_\lambda) \qquad \text{for all} \qquad \lambda \in \Lambda. \tag{41}$$

Let the 'critical angles' $\{\theta_\lambda^\mu; \lambda \in \Lambda\}$, with $0 \leq \theta_\lambda^\mu \leq \pi/2$, be introduced through

$$\sin(\theta_\lambda^\mu) = c_\lambda/c_\mu. \tag{42}$$

Then, (41) implies

$$\sin(\theta_\lambda) > \sin(\theta_\lambda^\mu), \qquad \text{for all} \qquad \lambda \in \Lambda, \tag{43}$$

which in turn implies that

$$\tan(\theta_\lambda) > \tan(\theta_\lambda^\mu) \qquad \text{for all} \qquad \lambda \in \Lambda. \tag{44}$$

Using (44) in combination with (31), the condition $1/c_\mu < p_0$ is then equivalent to

$$r > \sum_{\lambda \in \Lambda} h_\lambda \tan(\theta_\lambda^\mu), \tag{45}$$

or

$$r > \sum_{\lambda \in \Lambda} h_\lambda \frac{c_\lambda/c_\mu}{(1 - c_\lambda^2/c_\mu^2)^{1/2}}. \tag{46}$$

Equation (46) is the condition for 'total reflection' against an interface of a medium with wave speed c_μ, in accordance with Snell's law of refraction at the interfaces of the media with wave speed $c_\lambda (\lambda \in \Lambda)$.

The head-wave contributions to the generalized-ray Green's functions in the large-offset approximations follow from (16), (24) and (26) as

$$\hat{g}^H_{\text{hor}}(\mathbf{x}, s) = (2/r)^{1/2}(2\pi^2)^{-1} \int_{\tau=T^H}^{T^B} \exp(-s\tau)\, \text{Im}\, [\bar{\Pi}(p^H, 0, \phi)(p^H)^{1/2}(\partial p^H/\partial\tau)]\, \text{d}\tau, \tag{47}$$

and

$$\hat{g}^H_{\text{ver}}(\mathbf{x}, s) = (2\pi^2)^{-1}2^{1/2} \int_{\tau=T^H}^{T^B} \exp(-s\tau) \times \text{Im}\, \{\bar{\Pi}(p^H, 0, \phi)\Big[\sum_{\lambda\in\Lambda} \Gamma_\lambda^{-1}(p^H)h_\lambda\Big]^{-1/2}(\partial p^H/\partial\tau)\}\, \text{d}\tau. \tag{48}$$

In (48), the parts along $p = p^H(\tau)$ and $p = p^{H*}(\tau)$ have been taken together, and Schwarz's reflection principle has been used. The Jacobian of the transformation from p^H to τ follows from (26) as

$$\partial p^H/\partial\tau = \Big[r - p^H \sum_{\lambda\in\Lambda} \Gamma_\lambda^{-1}(p^H)h_\lambda\Big]^{-1}. \tag{49}$$

Now, (47) and (48) are the shape of a Laplace transformation over the range (T^H, T^B). As in Section 6 we then obtain the space–time expressions

$$\hat{g}^H_{\text{hor}}(\mathbf{x}, t) = \begin{cases} 0 & \text{for} \quad t < T^H \quad \text{and} \quad t > T^B, \\ (2/r)^{1/2}(2\pi^2)^{-1} & \\ \quad \times \text{Im}\, [\bar{\Pi}(p^H, 0, \phi)(p^H)^{1/2}(\partial p^H/\partial\tau)] & \text{for} \quad T^H < t < T^B, \end{cases} \tag{50}$$

and

$$\hat{g}^H_{\text{ver}}(\mathbf{x}, t) = \begin{cases} 0 & \text{for} \quad t < T^H \quad \text{and} \quad t > T^B, \\ (2\pi^2)^{-1}2^{1/2}\, \text{Im} \Big\{\bar{\Pi}(p^H, 0, \phi) & \\ \quad \times \Big[\sum_{\lambda\in\Lambda} \Gamma_\lambda^{-1}(p^H)h_\lambda\Big]^{-1/2}(\partial p^H/\partial\tau)\Big\} & \text{for} \quad T^H < t < T^B. \end{cases} \tag{51}$$

The head-wave contribution to the generalized-ray Green's function then follows in the large horizontal offset approximation from (15) as

$$w^{G;H}(\mathbf{x}, t) \sim \begin{cases} 0 & \text{for} \quad t < T^H, \\ \displaystyle\int_{t'=T^H}^{t} (t - t')^{-1/2} g^H_{\text{hor}}(\mathbf{x}, t')\, \text{d}t' & \text{for} \quad T^H < t < T^B, \\ \displaystyle\int_{t'=T^H}^{T^B} (t - t')^{-1/2} g^H_{\text{hor}}(\mathbf{x}, t')\, \text{d}t' & \text{for} \quad t > T^B, \end{cases} \tag{52}$$

as $r \to \infty$, and in the large vertical offset approximation from (23) as

$$w^{G;H}(\mathbf{x}, t) \sim \begin{cases} 0 & \text{for} \quad t < T^H, \\ \displaystyle\int_{t'=T^H}^{t} (t - t')^{-1/2} g^H_{\text{ver}}(\mathbf{x}, t')\, dt' & \text{for} \quad T^H < t < T^B, \\ \displaystyle\int_{t'=T^H}^{T^B} (t - t')^{-1/2} g^H_{\text{ver}}(\mathbf{x}, t')\, dt' & \text{for} \quad t > T^B, \end{cases} \tag{53}$$

as $\sum_{\lambda \in \Lambda} h_\lambda \to \infty$. Note that for (52) and (53) to occur, (46) must be satisfied, i.e. the horizontal offset between the source and the point of observation must be large enough.

8. Numerical Implementation

Except for the simplest case—in the summation in the exponential function in (26) only a single term is present (see Appendix)—the modified Cagniard path must be determined with the aid of numerical methods. First, for each given r and $\{h_\lambda;\ \lambda \in \Lambda\}$, (28) is solved for p_0. T^B follows using (26). Next, for each given r and $\{h_\lambda;\ \lambda \in \Lambda\}$, (26) is solved for p^B in the range $\tau > T^B$, and this value is used in the body-wave contributions. If head-wave contributions are present, T^H follows from (40). Subsequently, (26) is solved for p^H in the range $T^H < \tau < T^B$ for each r and $\{h_\lambda;\ \lambda \in \Lambda\}$ that are subject to the condition (46), and this value is used in the head-wave contributions. For the evaluation of the generalized-ray Green's functions, the integrations occurring in (38), (39), (52) and (53) have to be carried out numerically. This must be done carefully because of the inverse square root singularities in $(t - t')^{-1/2}$ near $t' = t$ and in $\partial p^B/\partial \tau$ and $\partial p^H/\partial t$ near $t = T^B$. The application of a local stretching procedure circumvents this difficulty. The final evaluation of the convolution integrals (5) usually presents no difficulties. Numerical results obtained along these lines can be found in Roever *et al.* (1959), Helmberger (1968) and Wiggins and Helmberger (1974) for the large horizontal offset approximations, and Van der Hijden (1987) for both the large horizontal offset and the large vertical offset approximations. In the last reference, the application to anisotropic layered media is also discussed.

Appendix

The Modified Cagniard Path for a Generalized Ray that Propagates in a Single Medium Only

For a generalized-ray constituent that propagates in a single medium only, the summation in (26) contains a single term. The modified Cagniard path then follows from an expression of the form

$$pr + (1/c^2 - p^2)^{1/2} h = \tau. \tag{A1}$$

The value of p_0 follows from (cf. (28))

$$r - p(1/c^2 - p^2)^{-1/2}h = 0 \qquad \text{at} \qquad p = p_0. \tag{A2}$$

Equation (A2) leads to

$$p_0 = r/c(r^2 + h^2)^{1/2}. \tag{A3}$$

Substitution of (A3) in (A1) yields

$$T^B = (r^2 + h^2)^{1/2}/c. \tag{A4}$$

Solving p from (A1), we obtain

$$p^B = \frac{\tau r + ih[\tau^2 - (T^B)^2]^{1/2}}{r^2 + h^2}, \tag{A5}$$

from which the Jacobian of the transformation from p^B to τ is found as

$$\partial p^B/\partial\tau = \frac{r + ih\tau[\tau^2 - (T^B)^2]^{-1/2}}{r^2 + h^2}. \tag{A6}$$

Further, let the presence of $\gamma_\mu = \bar{\gamma}_\mu(p, 0)$ in one of the reflection coefficients be responsible for the occurrence of a head wave. Then (cf. (40)),

$$T^H = r/c_\mu + (1/c^2 - 1/c_\mu^2)^{1/2}h. \tag{A7}$$

Solving p from (A1), we obtain

$$p^H = \frac{\tau r - h[(T^B)^2 - \tau^2]^{1/2}}{r^2 + h^2}, \tag{A8}$$

from which the Jacobian of the transformation from p^H to τ is found:

$$\partial p^H/\partial\tau = \frac{r + h\tau[(T^B)^2 - \tau^2]^{-1/2}}{r^2 + h^2}. \tag{A9}$$

The head-wave contribution is present if $c_\mu > c$, and (cf. (46))

$$r > h\frac{c/c_\mu}{(1 - c^2/c_\mu^2)^{1/2}}. \tag{A10}$$

From (A6) and (A9) the inverse square root singularities in $\partial p^B/\partial\tau$ and $\partial p^H/\partial\tau$ in the neighbourhood of $\tau = T^B$ are obvious.

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