Acoustic radiation from an impulsive point source in a continuously layered fluid—An analysis based on the Cagniard method

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The acoustic radiation generated by an impulsive point source in a continuously layered fluid with depth-varying parameters is investigated theoretically with the aid of the modified Cagniard method. Using a one-sided time Laplace transformation with a real positive transform parameter, a Fourier transformation with respect to the horizontal space coordinates and appropriate one-sided Green's functions, the system of transform-domain differential equations in the depth coordinate is rewritten as a system of integral equations that can be solved by a Neumann iteration. The modified Cagniard method leads to space-time expressions for the relevant iterates that physically are representative for the successively reflected waves. This iterative method is shown to be convergent in the time domain for any continuous and piecewise continuously differentiable depth profile in the inertia and compressibility properties of the fluid. To show the generality of the method, the fluid is assumed to show anisotropy in its volume density of mass, which is the kind of anisotropy that shows up in the equivalent medium theory of a finely layered fluid. The continuously refracted waves emitted by the source and the singly, continuously, reflected waves are discussed in detail. With this technique, no difficulties arise with "turning rays," as is the case in the asymptotic ray theory of the (real) frequency-domain analysis of the problem.

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LIST OF SYMBOLS

p	acoustic pressure (Pa)
v_r	particle velocity (m/s)
f_k	volume source density of force (N/m^3)
q	volume source density of injection rate (s^{-1})
$\rho_{k,r}$	tensorial volume density of mass (kg/m^3)
κ	compressibility (Pa^{-1})
$\widetilde{W}_n^-$	spectral-domain wave amplitude of nth order upgoing wave
$\widetilde{W}_n^+$	spectral-domain wave amplitude of nth order downgoing wave
γ^-	spectral-domain vertical wave slowness of upgoing wave
γ^+	spectral-domain vertical wave slowness of downgoing wave
Y	spectral-domain vertical acoustic wave admittance
x_m	space coordinates
t	time coordinate
∂	partial differentiation

INTRODUCTION

The layered fluid with depth-varying properties is one of the canonical model configurations that serves to study the properties of impulsive acoustic or elastic wave motion with regard to its application to ocean acoustics as well as to seismology. In the latter, the model is used when the analysis is concentrated on compressional waves, and interaction with shear waves is neglected. For a vertically inhomogeneous medium that is modeled as a stack of homogeneous plane layers, the author's modification of Cagniard's method,[1–8] also denoted as the generalized-ray method,[9–11] provides an efficient computational method to generate synthetic seismograms, both for two- and three-dimensional wave motion. Recently, the method has also been used to study the influence of anisotropy in the elastic properties of each of the layers.[12] The case of a continuously layered medium has received less attention. Two extensive and thorough papers by Chapman deal with the subject. In the first,[13] the approximation of the continuously layered medium by one consisting of a large number of thin, homogeneous layers is taken as the point of departure. In the second,[14] the case of a continuously layered medium is addressed directly. The transform-domain differential equations are solved by iteratively solving their Volterra integral-equation counterparts, and the different terms in the iteration process are interpreted as generalized rays that have undergone continuous refraction and, single or multiple, continuous reflections in the gradually changing medium. Next, the iterates are transformed back to the time domain with the aid of the Pekeris[15,16] version of the Cagniard technique and the subsequent application of the large horizontal offset approximation used by Roever *et al.*[17] and Wiggins and Helmberger[11] (see, also, Ref. 18). In Chapman's analysis, turning rays are included, while also lateral rays that propagate horizontally over some finite distance, as in the case of discretely layered media (where they are also denoted as head waves), occur.

If the standard Fourier transformation is carried out with respect to the time coordinate, instead of the one-sided

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Laplace transformation according to Cagniard, difficulties arise with the turning points of the corresponding WKBJ iterative method to solve the transform-domain differential equations. These have been overcome by Chapman[19] by using uniformly asymptotic representations of the Langer type (which involve Airy functions) and transforming these back to the time domain. This method is considerably more complicated than the one in the present paper, where only elementary functions and operations are involved. A survey paper by Chapman[20] covers a variety of general aspects of obtaining time-domain expressions for the wave motion in layered media. Recently, the influence of anisotropy has also been studied.[21-23]

In the present paper, the problem of the transient wave propagation in a continuously layered fluid is, just as in Ref. 14, addressed directly. The standard integral transformations that are characteristic for the modified Cagniard method are applied to the first-order acoustic wave equations of a fluid. By the use of appropriate one-sided Green's functions, the resulting system of differential equations in the depth coordinate is next transformed into a system of integral equations. These integral equations admit a solution by a Neumann iteration. Each higher-order iterate can be physically interpreted as to be generated, through continuous reflection, by the previous one. To show the generality of the method, anisotropy of the fluid in its volume density of mass is included (this type of anisotropy is encountered in the equivalent medium theory of finely discretely layered media[24]); the compressibility is a scalar. Next, the transformation back to the space-time domain is discussed, in which a number of steps can be carried out analytically, even for the anisotropic case. The iterative method is shown to be convergent for any continuous and piecewise continuously differentiable depth profile in the inertia and compressibility properties of the fluid. This is contrary to the frequency-domain analysis of the problem, where the corresponding Neumann series (which is also known as the Bremmer[25-28] series) can only be shown to be convergent for profiles that vary within certain, frequency-dependent, bounds. These aspects are discussed by Broer,[29] Sluijter,[30,31] and Broer and Van Vroonhoven,[32] while the corresponding time-domain convergence has been investigated by Gray,[33] for discretely layered media, and by Verheggen *et al.*,[34] for continuously layered media. Finally, a large review paper by Chapman and Orcutt[35] is mentioned.

The difficulties that are met with the inversion method, based on a time Fourier transformation with a real frequency variable, can be ascribed to the fact that, with this transformation, causality is lost, while with a time Laplace transformation with a real transform parameter, as used by Cagniard,[1,2] as a crucial point, causality is automatically taken care of by restricting the transform-domain counterparts of the physical quantities to being bounded functions of the remaining space variables. Also, in the modified Cagniard method, the time variable is kept real all the way through, in accordance with its physical meaning. Furthermore, no asymptotics is needed, and only convergent expansions occur. Another aspect of the propagation of transient waves in continuously layered media is covered by the spectral theory of transients, which has been introduced by Heyman and Felsen[36,37] and has been further applied by Heyman *et al.*,[38,39] and Heyman.[40] This theory aims at a complete asymptotic expression for the total wave field in the neighborhood of the wave fronts, rather than an exact expression in terms of successively reflected wave constituents.

I. DESCRIPTION OF THE CONFIGURATION AND FORMULATION OF THE ACOUSTIC WAVE PROBLEM

Small-amplitude acoustic wave motion is considered in an unbounded inhomogeneous fluid, the properties of which vary in a single rectilinear direction in space only. This direction is taken as the vertical one. To specify position in the configuration, the coordinates $\{x_1, x_2, x_3\}$, with respect to a fixed, orthogonal, Cartesian reference frame, with the origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ of unit length each, are used; $\mathbf{i}_3$ points vertically downward. The subscript notation for vectors and tensors is used and the summation convention applies. Lowercase Latin subscripts are used for this purpose; they are to be assigned the values $\{1,2,3\}$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_m denotes differentiation with respect to x_m; ∂_t is a reserved symbol denoting differentiation with respect to t.

The acoustic properties of the (anisotropic) fluid are characterized by the tensorial volume density of mass $\rho_{k,r}$ and the scalar compressibility κ. Both are functions of x_3 only; these functions are assumed to be continuous and piecewise continuously differentiable. At any x_3, the tensor $\rho_{k,r}$ is assumed to be symmetrical and positive definite (this implies that a nonnegative definite volume density of kinetic energy is associated with the wave motion), and that κ is positive (this implies that a nonnegative definite volume density of deformation energy is associated with the wave motion). In view of the properties stated, the configuration is time invariant as well as shift invariant in the horizontal direction (see Fig. 1). Due to this shift invariance, it is advantageous to distinguish, in the vectorial and tensorial quantities, between their horizontal and their vertical components. For the former, lowercase Greek subscripts will be used; for the latter, the subscript 3 will be written explicitly.

The acoustic wave motion in the configuration is characterized by its acoustic pressure p and its particle velocity v_r. These quantities satisfy the first-order acoustic wave equations

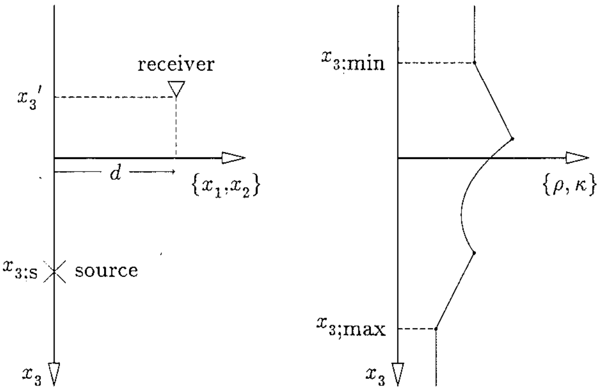

FIG. 1. Acoustic source and receiver in a continuously layered anisotropic fluid with tensorial volume density of mass and scalar compressibility.

$$\partial_k p + \rho_{k,r}\partial_t v_r = f_k, \tag{1}$$

$$\partial_r v_r + \kappa \partial_t p = q, \tag{2}$$

where f_k is the volume source density of force and q is the volume source density of injection rate. For a point source located at $x_m = x_{m;S}$, the volume source densities are given by

$$\{f_k, q\}(x_m, t) = \{F_{k;S}, Q_S\}(t)\delta(x_m - x_{m;S}), \tag{3}$$

where $\delta(x_m)$ is the three-dimensional spatial impulse function (Dirac distribution) operative at the point $x_m = 0$, and $\{F_{k;S}, Q_S\}(t)$ are the source strengths as a function of time. If $F_{k;S} = 0$ and $Q_S \neq 0$, the source is a monopole source; if $F_{k;S} \neq 0$ and $Q_S = 0$, the source is a dipole source. It is assumed that the source starts to act at the instant $t = 0$. The problem is to determine, at an arbitrary point of the configuration, the acoustic wave motion that is causally related to the action of this source.

II. THE TRANSFORM-DOMAIN ACOUSTIC WAVE EQUATIONS AND THE WAVE-MATRIX FORMALISM

In accordance with the time invariance and the shift invariance of the configuration in the horizontal direction, the acoustic wave equations (1) and (2) are subjected to a sequence of integral transformations that is characteristic for the modified Cagniard method. These are: a one-sided Laplace transformation with respect to time with real, positive transform parameter s, and a Fourier transformation with respect to the horizontal space coordinates with real transform parameters $s\alpha_1$ and $s\alpha_2$. For the acoustic pressure the two transformations are

$$\hat{p}(x_m, s) = \int_{t=0}^{\infty} \exp(-st)p(x_m, t)dt \tag{4}$$

and

$$\tilde{p}(i\alpha_\mu, x_3, s) = \int_{x_\mu \in R^2} \exp(is\alpha_\mu x_\mu)\hat{p}(x_m, s)dx_1\, dx_2, \tag{5}$$

respectively. The extra factor of s in the spatial Fourier-transform parameters has been included for later convenience. In view of this, the transformation inverse to Eq. (5) is given by

$$\hat{p}(x_m, s) = \left(\frac{s}{2\pi}\right)^2 \int_{\alpha_\mu \in R^2} \exp(-is\alpha_\mu x_\mu) \times \tilde{p}(i\alpha_\mu, x_3, s)d\alpha_1\, d\alpha_2. \tag{6}$$

Under these transformations and the condition of causality, the rules $\partial_t \to s$ and $\partial_\mu \to -is\alpha_\mu$ apply, and hence Eqs. (1) and (2) transform into

$$-is\alpha_\kappa \tilde{p} + s\rho_{\kappa,r}\tilde{v}_r = \tilde{f}_\kappa, \tag{7}$$

$$\partial_3 \tilde{p} + s\rho_{3,r}\tilde{v}_r = \tilde{f}_3, \tag{8}$$

$$-is\alpha_\mu \tilde{v}_\mu + \partial_3 \tilde{v}_3 + s\kappa \tilde{p} = \tilde{q}. \tag{9}$$

Upon eliminating the horizontal components $\tilde{v}_\mu$ of the particle velocity from these equations, a system of two ordinary differential equations results with x_3 as the independent variable and $\tilde{p}$ and $\tilde{v}_3$ as dependent variables. For later ease of manipulation, these equations are arranged as a matrix differential equation. The relevant matrices will be denoted by an uppercase symbol enclosed in square brackets when referred to in the text, while in displayed equations the relevant symbol will be supplied with uppercase Latin subscripts to which, again, the summation convention applies; to these subscripts the values 1 and 2 are to be assigned. Let $[\tilde{F}]$ denote the acoustic field matrix, $[\tilde{A}]$ the acoustic system's matrix, and $[\tilde{N}]$ the notional source matrix, then the transform-domain matrix differential equation is

$$\partial_3 \tilde{F}_M + s\tilde{A}_{M,N}\tilde{F}_N = \tilde{N}_M, \tag{10}$$

in which the elements of the acoustic field matrix are given by

$$\tilde{F}_1 = \tilde{p}, \tag{11}$$

$$\tilde{F}_2 = \tilde{v}_3, \tag{12}$$

the elements of the acoustic system's matrix by

$$\tilde{A}_{1,1} = \rho_{3,\rho}(\rho_\parallel^{-1})_{\rho,\kappa} i\alpha_\kappa, \tag{13}$$

$$\tilde{A}_{1,2} = \rho_{3,3} - \rho_{3,\rho}(\rho_\parallel^{-1})_{\rho,\kappa}\rho_{\kappa,3}, \tag{14}$$

$$\tilde{A}_{2,1} = \kappa - i\alpha_\rho(\rho_\parallel^{-1})_{\rho,\kappa} i\alpha_\kappa, \tag{15}$$

$$\tilde{A}_{2,2} = i\alpha_\rho(\rho_\parallel^{-1})_{\rho,\kappa}\rho_{\kappa,3}, \tag{16}$$

and the elements of the notional source matrix by

$$\tilde{N}_1 = -\rho_{3,\rho}(\rho_\parallel^{-1})_{\rho,\kappa}\tilde{f}_\kappa + \tilde{f}_3, \tag{17}$$

$$\tilde{N}_2 = i\alpha_\rho(\rho_\parallel^{-1})_{\rho,\kappa}\tilde{f}_\kappa + \tilde{q}. \tag{18}$$

Here, $(\rho_\parallel^{-1})_{\rho,\kappa}$ is the inverse of $\rho_{\kappa,\rho}$. Note that $[\tilde{A}]$ is independent of s.

Via an appropriate linear transformation to be carried out on the acoustic field matrix, a wave-matrix formalism will be arrived at from which the interaction between up- and downgoing waves in a region of inhomogeneity will be manifest. The same analysis for the isotropic case has been given by Chapman.[13] Because some new features, associated with the anisotropy of the fluid, show up, all steps of the procedure are briefly, but explicitly, indicated. Felsen and Marcuvitz[41] have, for the isotropic case, shown that the decomposition into up- and downgoing waves can also be deduced from a Green's function formalism applied to the second-order differential equation for the transform-domain acoustic pressure. The relevant linear transformation is written as

$$\tilde{F}_N = \tilde{L}_{N,P}\tilde{W}_P, \tag{19}$$

where $[\tilde{W}]$ is the wave matrix, and the matrix $[\tilde{L}]$ is to be chosen appropriately. On the assumption that the inverse $[\tilde{L}]^{-1}$ of $[\tilde{L}]$ exists, substitution of Eq. (19) into Eq. (10) yields

$$\partial_3 \tilde{W}_L + s\tilde{\Lambda}_{L,P}\tilde{W}_P = (\tilde{L}^{-1})_{L,M}\tilde{N}_M - (\tilde{L}^{-1})_{L,M}(\partial_3 \tilde{L}_{M,P})\tilde{W}_P, \tag{20}$$

where

$$\tilde{\Lambda}_{L,P} = (\tilde{L}^{-1})_{L,M}\tilde{A}_{M,N}\tilde{L}_{N,P}. \tag{21}$$

Equation (20) indeed expresses the traveling-wave structure of the up- and downgoing waves, provided that $[\tilde{\Lambda}]$ is a diagonal matrix. From the observation that [cf. Eq. (21)]

$$\tilde{A}_{M,N}\tilde{L}_{N,P} = \tilde{L}_{M,L}\tilde{\Lambda}_{L,P}, \tag{22}$$

it follows that $[\tilde\Lambda]$ is diagonal if $[\tilde L]$ consists of the eigencolumns of $[\tilde A]$; $[\tilde\Lambda]$ then has the eigenvalues of $[\tilde A]$ as its (diagonal) elements. The latter are written as

$$\tilde\Lambda_{I,P} = \gamma^{(P)}\delta_{I,P}, \tag{23}$$

where $\delta_{I,P}$ is the unit matrix.

So far, the normalization of the eigencolumns of $[\tilde A]$, out of which $[\tilde L]$ is composed, is free. This freedom can be exploited to give $[\tilde L]$ a number of additional desirable properties. The most important one of these is that, for the physical interpretation of Eq. (21), one would rather avoid the coupling of some wave occurring in the coupling term on the right-hand side of Eq. (20) to the particular wave that occurs on the left-hand side. This is accomplished if $[\tilde L]^{-1}[\partial_3\tilde L]$ has zero-value diagonal elements. Further, it would be desirable if $[\tilde L]^{-1}$ could be constructed from $[\tilde L]$ without the intervention of a (numerical) matrix inversion procedure. Both properties are realized if each eigencolumn $[\tilde F^{(P)}]$ of $[\tilde A]$ is normalized such that

$$\bar p^{(P)}\tilde v_3^{(P)} + \tilde p^{(P)}\bar v_3^{(P)} = -1, \tag{24}$$

for a wave that is outgoing as $x_3 \to -\infty$, and

$$\bar p^{(P)}\tilde v_3^{(P)} + \tilde p^{(P)}\bar v_3^{(P)} = 1, \tag{25}$$

for a wave that is outgoing as $x_3 \to \infty$. Here, the overbar is used to denote the transform-domain counterpart of a time-reversed space-time quantity; i.e.,

$$\{\bar p,\bar v_3\}(i\alpha_\mu,x_3,s) = \{\tilde p,\tilde v_3\}(i\alpha_\mu,x_3,-s). \tag{26}$$

Under these conditions, $[\tilde L]^{-1}[\partial_3\tilde L]$ indeed has zero-value diagonal elements, while $[\tilde L]^{-1}$ follows, upon interchanging in the transpose of $[\tilde L]$ the pressure and the particle velocity parts, and changing the sign of the row that corresponds to an outgoing wave as $x_3 \to -\infty$, while keeping the sign of the row that corresponds to an outgoing wave as $x_3 \to \infty$. Since Eqs. (24) and (25) are local conditions, the decision whether a wave is outgoing as $x_3 \to -\infty$ or as $x_3 \to \infty$ can be made locally at each depth level by observing the sign of the real parts of the vertical propagation coefficients at that depth level (see Appendix B). The proof of the properties of $[\tilde L]$ is given in Appendix A and is based on a certain reciprocity theorem for the transform-domain acoustic wave field. Related procedures have been employed by Kennett[42] and by Fryer and Frazer.[43] With the normalization thus introduced, the system of differential equations, Eq. (20), truly reflects the physical picture of vertically up- and downgoing waves that travel independently in a homogeneous region (where $[\partial_3\tilde L] = [0]$) and that couple in a region of inhomogeneity (where $[\partial_3\tilde L] \neq [0]$).

In Appendix B, it is shown that the elements of $[\tilde L]$ can be expressed in terms of the vertical acoustic wave admittance

$$Y = (\tilde A_{2,1}/\tilde A_{1,2})^{1/2}, \tag{27}$$

a quantity that, for $\alpha_\mu \in R^2$, is real and positive. (Note that this wave admittance is the same for up- and downgoing waves, even in the case of an anisotropic fluid.) The relevant expressions are

$$\tilde L_{1,1} = (2Y)^{-1/2},\quad \tilde L_{1,2} = (2Y)^{-1/2},$$
$$\tilde L_{2,1} = -(Y/2)^{1/2},\quad \tilde L_{2,2} = (Y/2)^{1/2}. \tag{28}$$

Furthermore, the elements of $[\tilde L]^{-1}$ are, using the procedure discussed in Appendix A, found as

$$(\tilde L^{-1})_{1,1} = (Y/2)^{1/2},\quad (\tilde L^{-1})_{1,2} = -(2Y)^{-1/2},$$
$$(\tilde L^{-1})_{2,1} = (Y/2)^{1/2},\quad (\tilde L^{-1})_{2,2} = (2Y)^{-1/2}. \tag{29}$$

With this, the elements of the coupling matrix $[\tilde L]^{-1}[\partial_3\tilde L]$ are obtained as

$$(\tilde L^{-1}\partial_3\tilde L)_{1,1} = 0,\quad (\tilde L^{-1}\partial_3\tilde L)_{1,2} = -\partial_3 Y/2Y,$$
$$(\tilde L^{-1}\partial_3\tilde L)_{2,1} = -\partial_3 Y/2Y,\quad (\tilde L^{-1}\partial_3\tilde L)_{2,2} = 0. \tag{30}$$

Using these results and writing the elements of the wave matrix $[\tilde W]$ as

$$\tilde W_1 = \tilde W^-,\quad \tilde W_2 = \tilde W^+, \tag{31}$$

where $\tilde W^-$ is the local amplitude of the upgoing wave and $\tilde W^+$ is the local amplitude of the downgoing wave, the system of differential equations, Eq. (20), leads to

$$\partial_3\tilde W^- + s\gamma^-\tilde W^- = \tilde X^- + (\partial_3 Y/2Y)\tilde W^+, \tag{32}$$

$$\partial_3\tilde W^+ + s\gamma^+\tilde W^+ = \tilde X^+ + (\partial_3 Y/2Y)\tilde W^-, \tag{33}$$

where [cf. Eqs. (B7) and (B8)]

$$\gamma^- = (\tilde A_{1,1} + \tilde A_{2,2})/2 - (\tilde A_{1,2}\tilde A_{2,1})^{1/2}, \tag{34}$$

$$\gamma^+ = (\tilde A_{1,1} + \tilde A_{2,2})/2 + (\tilde A_{1,2}\tilde A_{2,1})^{1/2}, \tag{35}$$

and

$$\tilde X^- = (\tilde L^{-1}\tilde N)_1,\quad \tilde X^+ = (\tilde L^{-1}\tilde N)_2. \tag{36}$$

Note that $\mathrm{Re}(\gamma^-) < 0$, $\mathrm{Re}(\gamma^+) > 0$ (cf. Appendix B), and that the right-hand side of Eq. (32) contains, apart from the coupling to the source, only a coupling, due to inhomogeneity, to the downgoing wave, while the right-hand side of Eq. (33) contains, apart from a coupling to the source, only a coupling, due to inhomogeneity, to the upgoing wave. The coupling coefficient is, in both equations, equal to $\partial_3 Y/2Y$, which is half of the local relative change in the vertical acoustic wave admittance.

In Sec. III, the coupled wave propagation problem is recast in an integral-equation formulation that is equivalent to Eqs. (32) and (33). In Sec. IV these integral equations are solved iteratively, and, for the transform-domain acoustic pressure and particle velocity, the expansion equations (75) and (76) are obtained. The zero-order term in this expansion is representative for the direct wave generated by the source; the subsequent terms are representative for the waves that are successively reflected at the inhomogeneity levels. The general scheme for the transformation back to the space-time domain is illustrated for a typical generalized-ray constituent. The waves of order zero are discussed in detail in Sec. V, the waves of order one in Sec. VI.

III. INTEGRAL-EQUATION FORMULATION OF THE TRANSFORM-DOMAIN COUPLED WAVE PROBLEM

The integral-equation formulation of the transform-domain coupled wave problem follows from Eqs. (32) and

(33) upon introducing appropriate one-sided Green's functions for the differential operators occurring at the left-hand side of these equations. These Green's functions are defined through the differential equations [note the change in sign in the second term in the left-hand side as compared with Eqs. (32) and (33)]

$$\partial_3 \tilde\Gamma^- - s\gamma^- \tilde\Gamma^- = \delta(x_3 - x_3'), \tag{37}$$

$$\partial_3 \tilde\Gamma^+ - s\gamma^+ \tilde\Gamma^+ = \delta(x_3 - x_3'), \tag{38}$$

where $\delta(x_3)$ denotes the one-dimensional impulse function (Dirac distribution) operative at $x_3 = 0$, together with the condition that $\tilde\Gamma^-$ and $\tilde\Gamma^+$ remain bounded as $|x_3| \to \infty$. The relevant expressions follow as

$$\tilde\Gamma^-(x_3,x_3') = \exp\left(s\int_{\xi=x_3'}^{x_3} \gamma^-(\xi)d\xi\right)H(x_3 - x_3'), \tag{39}$$

$$\tilde\Gamma^+(x_3,x_3') = -\exp\left(s\int_{\xi=x_3'}^{x_3} \gamma^+(\xi)d\xi\right)H(x_3' - x_3), \tag{40}$$

where $H(x_3)$ denotes the unit step function (Heaviside function): $H(x_3) = \{0,\frac{1}{2},1\}$ for $\{x_3 < 0, x_3 = 0, x_3 > 0\}$. In Eqs. (39) and (40), the dependence of $\tilde\Gamma^-$ and $\tilde\Gamma^+$ on α_1, α_2, and s has not been indicated explicitly. Multiplying Eq. (32) by $\tilde\Gamma^-$, Eq. (37) by $\tilde W^-$, adding the results, and integrating this result over all x_3, it is found that

$$[\tilde\Gamma^-(x_3,x_3')\tilde W^-(x_3)]_{x_3=-\infty}^{\infty} = \int_{x_3=-\infty}^{\infty} \tilde\Gamma^-(x_3,x_3')\{\tilde X^-(x_3) + [\partial_3 Y(x_3)/2Y(x_3)]\tilde W^+(x_3)\}dx_3 + \tilde W^-(x_3'). \tag{41}$$

The contribution from $x_3 = -\infty$ to the left-hand side vanishes since $\tilde\Gamma^-(x_3,x_3') = 0$ when $x_3 < x_3'$. At $x_3 = \infty$, the condition $\tilde\Gamma^-(x_3,x_3')\tilde W^-(x_3) \to 0$ is enforced; this condition is certainly satisfied if the fluid is homogeneous in some lower half-space $x_3 > x_{3;\max}$, since, in the latter, $\tilde W^- = 0$ in view of the condition of causality. With this, Eq. (41) leads to

$$\tilde W^-(x_3') = \int_{x_3=x_3'}^{\infty} \tilde G^-(x_3',x_3)\{\tilde X^-(x_3) + [\partial_3 Y(x_3)/2Y(x_3)]\tilde W^+(x_3)\}dx_3, \tag{42}$$

where

$$\tilde G^-(x_3',x_3) = -\tilde\Gamma^-(x_3,x_3') = -\exp\left(-s\int_{\xi=x_3}^{x_3'} \gamma^-(\xi)d\xi\right)H(x_3 - x_3'). \tag{43}$$

Similarly, multiplying Eq. (33) by $\tilde\Gamma^+$, Eq. (38) by $\tilde W^+$, adding the results, and integrating this result over all x_3, it follows that

$$[\tilde\Gamma^+(x_3,x_3')\tilde W^+(x_3)]_{x_3=-\infty}^{\infty} = \int_{x_3=-\infty}^{\infty} \tilde\Gamma^+(x_3,x_3')\{\tilde X^+(x_3) + [\partial_3 Y(x_3)/2Y(x_3)]\tilde W^-(x_3)dx_3\} + \tilde W^+(x_3'). \tag{44}$$

The contribution from $x_3 = \infty$ to the left-hand side vanishes since $\tilde\Gamma^+(x_3,x_3') = 0$ when $x_3 > x_3'$. At $x_3 = -\infty$, the condition $\tilde\Gamma^+(x_3,x_3')\tilde W^+(x_3) \to 0$ is enforced; this condition is certainly satisfied if the fluid is homogeneous in some upper half-space $x_3 < x_{3;\min}$, with $x_{3;\min} < x_{3;\max}$, since, in the latter, $\tilde W^+ = 0$ in view of the condition of causality. With this, Eq. (44) leads to

$$\tilde W^+(x_3') = \int_{x_3=-\infty}^{x_3'} \tilde G^+(x_3',x_3)\{\tilde X^+(x_3) + [\partial_3 Y(x_3)/2Y(x_3)]\tilde W^-(x_3)\}dx_3, \tag{45}$$

where

$$\tilde G^+(x_3',x_3) = -\tilde\Gamma^+(x_3,x_3') = \exp\left(-s\int_{\xi=x_3}^{x_3'} \gamma^+(\xi)d\xi\right)H(x_3' - x_3). \tag{46}$$

Equation (42) expresses $\tilde W^-$ at depth level x_3' in terms of the primary and secondary source distributions, associated with the generating sources and the inhomogeneities, respectively, in the half-space below x_3'. Equation (45) expresses $\tilde W^+$ at depth level x_3' in terms of the primary and secondary source distributions, associated with the generating sources and the inhomogeneities, respectively, in the half-space above x_3'. The integration over the primary source distributions is, at most, extended over the vertical support of the generating sources; the integration over the secondary source distributions is, at most, extended over the vertical range of the region of inhomogeneity, i.e., over $x_{3;\min} < x_3 < x_{3;\max}$. The quantity $\partial_3 Y/2Y$ acts, at each depth level, as a local reflection coefficient, and it is apparently the same for reflection from down- to upgoing and from up- to downgoing waves, even in the case of an anisotropic fluid. For $x_{3;\min} < x_3' < x_{3;\max}$, Eqs. (42) and (45) constitute a system of integral equations of the second kind from which $\tilde W^-$ and $\tilde W^+$ in the region of vertical inhomogeneity can, in principle, be determined. Once this has been achieved, Eqs. (42) and (45) can be reused to determine $\tilde W^-$ and $\tilde W^+$ for any x_3'.

IV. ITERATIVE SOLUTION TO THE SYSTEM OF INTEGRAL EQUATIONS

To elucidate the structure of the system of integral equations and investigate the possibilities for their iterative solution, Eqs. (42) and (45) are written in the operator form

$$\tilde W^- = \tilde W_0^- + K^- \tilde W^+, \tag{47}$$

$$\tilde W^+ = \tilde W_0^+ + K^+ \tilde W^-, \tag{48}$$

where

$$\tilde W_0^-(x_3') = \int_{x_3=x_3'}^{\infty} \tilde G^-(x_3',x_3)\tilde X^-(x_3)dx_3, \tag{49}$$

$$\tilde W_0^+(x_3') = \int_{x_3=-\infty}^{x_3'} \tilde G^+(x_3',x_3)\tilde X^+(x_3)dx_3, \tag{50}$$

and the operators K^- and K^+ are defined through

$$(K^-\tilde{W}^+)(x_3') = \int_{x_3=x_3'}^{\infty} \tilde{G}^-(x_3',x_3) \times [\partial_3 Y(x_3)/2Y(x_3)]\tilde{W}^+(x_3)dx_3, \tag{51}$$

$$(K^+\tilde{W}^-)(x_3') = \int_{x_3=-\infty}^{x_3'} \tilde{G}^+(x_3',x_3) \times [\partial_3 Y(x_3)/2Y(x_3)]\tilde{W}^-(x_3)dx_3, \tag{52}$$

respectively. Equations (47) and (48) suggest the possibility of an iterative solution of the Neumann type by repeated substitution of Eq. (47) in Eq. (48) and vice versa. After carrying out N steps ($N=1,2,\ldots$), the result is

$$\tilde{W}^- = \tilde{W}_0^- + K^-\tilde{W}_0^+ + K^-K^+\tilde{W}_0^- + \cdots + \Delta_N^-, \tag{53}$$

$$\tilde{W}^+ = \tilde{W}_0^+ + K^+\tilde{W}_0^- + K^+K^-\tilde{W}_0^+ + \cdots + \Delta_N^+, \tag{54}$$

where

$$\Delta_N^- = \begin{cases} (K^-K^+)^{N/2}\tilde{W}^-, & \text{if } N \text{ is even,} \\ (K^-K^+)^{(N-1)/2}K^-\tilde{W}^+, & \text{if } N \text{ is odd,} \end{cases} \tag{55}$$

$$\Delta_N^+ = \begin{cases} (K^+K^-)^{N/2}\tilde{W}^+, & \text{if } N \text{ is even,} \\ (K^+K^-)^{(N-1)/2}K^+\tilde{W}^-, & \text{if } N \text{ is odd.} \end{cases} \tag{56}$$

To investigate the convergence of the procedure, it is observed that, for $x_{3;\min}\leqslant x_3'\leqslant x_{3;\max}$ (note that s is real and positive),

$$|K^-\tilde{W}|\leqslant A(s)M\int_{x_3=x_3'}^{x_{3;\max}} \exp[-s\gamma(x_3-x_3')]dx_3 = A(s)M\frac{1-\exp[-s\gamma(x_{3;\max}-x_3')]}{s\gamma} \leqslant A(s)M/s\gamma, \tag{57}$$

and

$$|K^+\tilde{W}|\leqslant A(s)M\int_{x_3=x_{3;\min}}^{x_3'} \exp[s\gamma(x_3-x_3')]dx_3 = A(s)M\frac{1-\exp[-s\gamma(x_3'-x_{3;\min})]}{s\gamma} \leqslant A(s)\,M/s\gamma, \tag{58}$$

where

$$A(s) = \max_{x_{3;\min}<x_3<x_{3;\max}}\{|\tilde{W}^-|,|\tilde{W}^+|\}, \tag{59}$$

$$M = \max_{x_{3;\min}<x_3<x_{3;\max}}|\partial_3 Y/2Y|, \tag{60}$$

$$\gamma = \min_{x_{3;\min}<x_3<x_{3;\max}}[\mathrm{Re}(-\gamma^-),\mathrm{Re}(\gamma^+)]>0. \tag{61}$$

On account of Eqs. (57) and (58),

$$|\Delta_N^-|\leqslant A(s)(M/s\gamma)^N, \tag{62}$$

and

$$|\Delta_N^+|\leqslant A(s)(M/s\gamma)^N. \tag{63}$$

To draw further conclusions from Eqs. (53) and (54), the real, positive time Laplace-transform parameter is taken in the semi-infinite interval

$$s>M/\gamma, \tag{64}$$

which can always be done since M is independent of s, positive and bounded, while γ is independent of s, positive and bounded away from zero. Under the condition of Eq. (64), the limit $N\to\infty$ is taken in Eqs. (53) and (54). Since $A(s)$ is independent of N, $|\Delta_N^-|\to 0$ and $|\Delta_N^+|\to 0$ as $N\to\infty$. Upon summing the remaining convergent infinite series, the inequality

$$A(s)\leqslant A_0(s)/[1-(M/s\gamma)] \tag{65}$$

results, where

$$A_0(s) = \max_{x_{3;\min}<x_3<x_{3;\max}}(|\tilde{W}_0^-|,|\tilde{W}_0^+|). \tag{66}$$

Hence, $A(s)$ is a bounded function of s in the interval of Eq. (64) if $A_0(s)$ is so. The latter condition is satisfied if the source is taken to be a point source in space with, at most, a delta function (Dirac distribution) time dependence as Eqs. (49) and (50), with the use of Eqs. (3), (17), (18), and (36), and the property $\{|\tilde{G}^-|,|\tilde{G}^+|\}\leqslant 1$ [cf. Eqs. (43) and (46)], show. This choice does not restrict the applicability of the method, since an extended source in space can be taken care of by an appropriate weighted integration of the point-source result over the spatial support of the source, and different time dependences of the source can be handled by a time convolution of the source signature and the Dirac impulse result. Without loss of generality, it is, therefore, in the further analysis, assumed that

$$\{f_k,q\}(x_m,t) = \{F_{k;S},Q_S\}(t)\delta(x_1,x_2,x_3-x_{3;S}); \tag{67}$$

i.e., the point source is located at $x_1=0$, $x_2=0$, $x_3=x_{3;S}$. Under the conditions indicated, Eqs. (53) and (54) define a convergent iterative process that leads to bounded values of $\tilde{W}^-$ and $\tilde{W}^+$ for values of s in the interval of Eq. (64) and real values of α_1 and α_2. In view of Lerch's theorem,[44] any procedure of transforming back to the time domain after the inverse spatial Fourier transformation of Eq. (6) has been carried out then leads to the unique space-time expressions for $\tilde{W}^-$ and $\tilde{W}^+$. It will be shown that the modified Cagniard method is an appropriate tool in this respect. To apply the method, the expressions for the transform-domain wave amplitudes resulting from Eqs. (53) and (54) are written as

$$\tilde{W}^-(x_3') = \sum_{n=0}^{\infty}\tilde{W}_n^-(x_3'), \tag{68}$$

$$\tilde{W}^+(x_3') = \sum_{n=0}^{\infty}\tilde{W}_n^+(x_3'), \tag{69}$$

with

$$\tilde{W}_0^-(x_3') = \tilde{G}^-(x_3',x_{3;S})\tilde{X}^-, \tag{70}$$

$$\tilde{W}_0^+(x_3') = \tilde{G}^+(x_3',x_{3;S})\tilde{X}^+, \tag{71}$$

$$\tilde{W}_{n+1}^-(x_3') = \left(\int_{x_3=x_3'}^{x_{3;\max}}\tilde{G}^-(x_3',x_3)\tilde{R}(x_3)\tilde{W}_n^+(x_3)dx_3\right) \times H(x_{3;\max}-x_3'), \quad \text{for } n=0,1,2,\ldots, \tag{72}$$

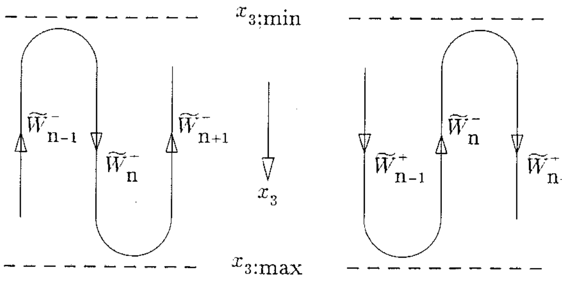

FIG. 2. Successively reflected up- and downgoing waves.

$$\tilde W^+_{n+1}(x_3') = \left(\int_{x_3=x_{3;\min}}^{x_3'} \tilde G^+(x_3',x_3)\tilde R(x_3)\tilde W^-_n(x_3)dx_3\right) \times H(x_3'-x_{3;\min}), \quad \text{for } n=0,1,2,..., \tag{73}$$

where

$$\tilde R = [\partial_3 Y(x_3)/2Y(x_3)] \tag{74}$$

is the local reflection coefficient due to inhomogeneity (see Fig. 2). In Eqs. (68)–(74), the dependence on α_μ and s has not been indicated explicity.

Note that the kind of reasoning employed here cannot be used when a Fourier transformation, with respect to time, with a real angular frequency transform parameter, is carried out and the inversion back to the time domain is based on the Fourier inversion integral. The latter employs, in fact, imaginary values of the time Laplace-transform parameter s, to which values Lerch's theorem does not apply, and for which, most importantly, the estimates of Eqs. (57) and (58) are lost.

Equations (68) and (69) entail, via Eq. (19), the following transform-domain representations for the acoustic pressure and the vertical component of the particle velocity:

$$\tilde p(x_3') = [2Y(x_3')]^{-1/2} \times\left(\sum_{n=0}^{\infty} \tilde W_n^-(x_3') + \sum_{n=0}^{\infty} \tilde W_n^+(x_3')\right), \tag{75}$$

$$\tilde v_3(x_3') = [Y(x_3')/2]^{1/2} \times\left(-\sum_{n=0}^{\infty} \tilde W_n^-(x_3') + \sum_{n=0}^{\infty} \tilde W_n^+(x_3')\right). \tag{76}$$

Furthermore, for the two cases of practical importance of a point source of volume injection (acoustic monopole transducer, explosion source) and a vertical point force (vertical acoustic dipole transducer, vertical mechanical vibrator), the excitation terms are [cf. Eqs. (36) and (67)]

$$\tilde X^- = [Y(x_{3;S})/2]^{1/2}\hat F_{3;S}(s)\delta(x_3-x_{3;S}) - [2Y(x_{3;S})]^{-1/2}\hat Q_S(s)\delta(x_3-x_{3;S}), \tag{77}$$

$$\tilde X^+ = [Y(x_{3;S})/2]^{1/2}\hat F_{3;S}(s)\delta(x_3-x_{3;S}) + [2Y(x_{3;S})]^{-1/2}\hat Q_S(s)\delta(x_3-x_{3;S}). \tag{78}$$

As a consequence, a typical term of order n in the right-hand sides of Eqs. (75) and (76) consists of an n-fold repeated integration in the vertical direction, the limits of which are the successive interaction levels of multiple reflection. In them, the exponential functions, which contain in their arguments additional integrations from the source level to the receiver level are gathered to a single one. The factor that remains in the n-fold integral is the product of one of the source signatures $[\hat F_{3;S}(s)$ or $\hat Q_S(s)]$ that depends only on s, on an s-independent coupling coefficient $[Y(x_{3;S})]^{\pm 1/2}$ that describes the coupling of the source to the wave, on the s-independent reflection coefficients $\tilde R$ at the successive interaction levels of multiple reflection, and on an s-independent coupling coefficient $[Y(x_3')]^{\pm 1/2}$ that describes the coupling of the wave to the receiver (hydrophone for the acoustic pressure, geophone for the vertical component of the particle velocity).

For further analysis, such a typical term is written as

$$\tilde U = \hat S(s)\tilde\Pi(i\alpha_\mu)\exp\left(-s\int_{\xi\in Z^-}\gamma^-(i\alpha_\mu,\xi)d\xi - s\int_{\xi\in Z^+}\gamma^+(i\alpha_\mu,\xi)d\xi\right). \tag{79}$$

Here, $\hat S$ stands for the source signature, $\tilde\Pi$ for the product of coupling coefficients and reflection coefficients, Z^- for the accumulated vertical travel path traversed by the upgoing waves, and Z^+ for the accumulated vertical travel path traversed by the downgoing waves. Both Z^- and Z^+ may, in part or entirely, be multiply covered. In accordance with the property $\mathrm{Re}(\gamma^-)<0$ and $\mathrm{Re}(\gamma^+)>0$, the vertical travel paths can be written as $Z^-=\{\xi\in R; x_3<\xi<x_3^+\}$ for some x_3^+, and $Z^+=\{\xi\in R; x_3^-<\xi<x_3\}$ for some x_3^-, and, consequently, the signed vertical path lengths satisfy the inequalities $\int_{\xi\in Z^-}d\xi\leqslant 0$ and $\int_{\xi\in Z^+}d\xi\geqslant 0$, respectively. Expressions of the type (79) will, just as in the case of discretely layered media, be denoted as generalized-ray constituents.[9,11] Their transformation back to the space-time domain with the aid of the modified Cagniard method is discussed in Appendix C. The final result is

$$U=\partial_t^2\int_{\tau=T_0}^{t} S(t-\tau)g(\mathbf{x},\tau)d\tau, \tag{80}$$

where

$$g = -(2\pi^2)^{-1}\int_{\psi=\Psi_1(\mathbf{x},\tau)}^{\Psi_2(\mathbf{x},\tau)} \mathrm{Re}\left(\tilde\Pi(p,\psi)p\frac{\partial p}{\partial\tau}\right)d\psi \times H(\tau-T_0), \tag{81}$$

in which T_0 is the arrival time of the generalized ray constituent, $H(\tau)$ is the Heaviside unit step function, and p is related to ψ and τ via the equation of the modified Cagniard path

$$pd\cos(\psi)+\int_{\xi\in Z^-}\gamma^-(p,\psi,\xi)d\xi + \int_{\xi\in Z^+}\gamma^+(p,\psi,\xi)d\xi=\tau, \quad \text{with } \mathrm{Im}(\tau)=0. \tag{82}$$

In going from Eq. (79) to Eq. (80), the x_1 axis of the Cartesian reference frame is chosen along the horizontal line joining the source and the receiver, and d is the horizontal offset.

It is to be noted that for obtaining the correct early time asymptotic expressions for the total wave amplitude at and immediately behind the wave front, all contributions that travel in a particular direction (up or down) must be added, since they all arrive at the same instant as the direct wave, be it with decreasing initial amplitudes. This kind of asymptotics is investigated in Refs. 21 and 36. Our decomposition yields expressions for the successively reflected waves *at all times*; we have not yet been able to sum up these contributions analytically, for example, at the wave front.

In the two subsequent sections, the waves of orders zero and one are discussed in more detail for the case of the acoustic pressure due to a point source of volume injection. The expressions for the particle velocity, and the expressions for the acoustic pressure and the particle velocity of the waves transmitted by a force source follow in a similar manner.

V. THE WAVES OF ORDER ZERO

In this section, the acoustic pressure of the waves of order zero transmitted by a point source of volume injection is discussed in more detail (see Fig. 3). For this case, Eq. (75) yields

$$\tilde{p} = \tilde{p}_0^- + \tilde{p}_0^+, \tag{83}$$

where, in view of Eqs. (43), (46), (70), (71), (77), and (78),

$$\tilde{p}_0^-(x_3') = \hat{Q}_S(s)\,[4Y(x_3')Y(x_{3;S})]^{-1/2} \times \exp\left(-s\int_{\xi=x_{3;S}}^{x_3'} \gamma^-(\xi)d\xi\right)H(x_{3;S}-x_3'), \tag{84}$$

$$\tilde{p}_0^+(x_3') = \hat{Q}_S(s)\,[4Y(x_3')Y(x_{3;S})]^{-1/2} \times \exp\left(-s\int_{\xi=x_{3;S}}^{x_3'} \gamma^+(\xi)d\xi\right)H(x_3'-x_{3;S}). \tag{85}$$

The theory of Appendix C leads to the following time-domain results:

$$p_0^-(\mathbf{x}',t) = \left(\partial_t^2\int_{\tau=T_0}^{t} Q_S(t-\tau)g_0^-(\mathbf{x}',\tau)d\tau\right) \times H(x_{3;S}-x_3'), \tag{86}$$

where

$$g_0^- = -(4\pi^2)^{-1}\int_{\psi=\Psi_1(\mathbf{x},\tau)}^{\Psi_2(\mathbf{x},\tau)} \mathrm{Re}\{[Y(p,\psi,x_3') \times Y(p,\psi,x_{3;S})]^{-1/2}p\left(\frac{\partial p}{\partial\tau}\right)d\psi, \tag{87}$$

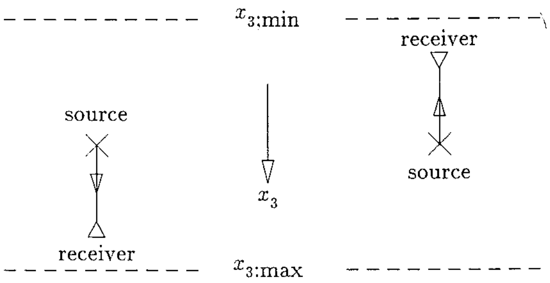

FIG. 3. The waves of order zero.

in which p is related to ψ and τ via the equation of the modified Cagniard path

$$pd\cos(\psi) + \int_{\xi=x_{3;S}}^{x_3'} \gamma^-(p,\psi,\xi)d\xi = \tau, \quad \text{with } \mathrm{Im}(\tau)=0 \tag{88}$$

and

$$p_0^+(\mathbf{x}',t) = \left(\partial_t^2\int_{\tau=T_0}^{t} Q_S(t-\tau)g_0^+(\mathbf{x}',\tau)d\tau\right)H(x_3'-x_{3;S}), \tag{89}$$

where

$$g_0^+ = -(4\pi^2)^{-1}\int_{\psi=\Psi_1(\mathbf{x},\tau)}^{\Psi_2(\mathbf{x},\tau)} \mathrm{Re}\left\{[Y(p,\psi,x_3') \times Y(p,\psi,x_{3;S})]^{-1/2}p\left(\frac{\partial p}{\partial\tau}\right)\right\}d\psi, \tag{90}$$

in which p is related to ψ and τ via the equation of the modified Cagniard path

$$pd\cos(\psi) + \int_{\xi=x_{3;S}}^{x_3'} \gamma^+(p,\psi,\xi)d\xi = \tau, \quad \text{with } \mathrm{Im}(\tau)=0. \tag{91}$$

Equations (84) and (85) have some resemblance to the corresponding WKBJ-result in the frequency-angular-wave-number analysis. Note, however, that, in the present analysis, they are the first step in a convergent iterative process, and that Eqs. (86) and (87) and (89) and (90) hold exactly for all observation times after the arrival of the wave and not just asymptotically behind the wave front, as the seismograms of Singh and Chapman[21] and Garmany[22,23] do. Furthermore, observe that no difficulties with turning points occur, since the latter simply to not arise in the s-domain treatment of the problem.

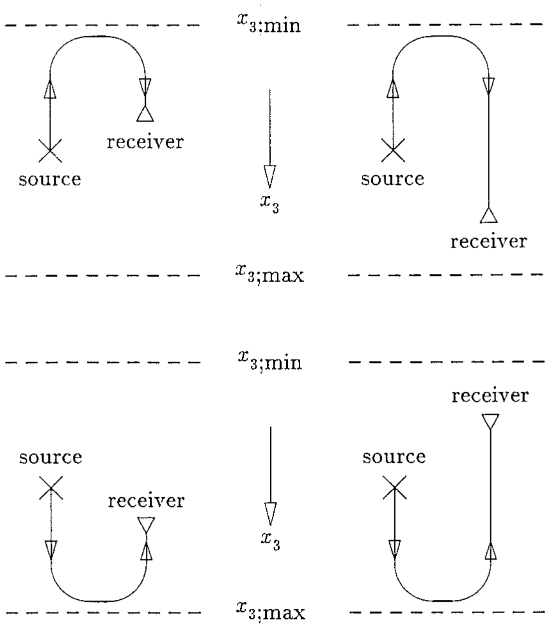

FIG. 4. The waves of order one.

VI. THE WAVES OF ORDER ONE

In this section, the acoustic pressure of the waves of order one, i.e., the waves that have undergone a single continuous reflection associated with the transmission by a point source of volume injection, is discussed in more detail (see Fig. 4).

For this case, Eq. (75) yields

$$\tilde{p} = \tilde{p}_1^- + \tilde{p}_1^+, \tag{92}$$

where, in view of Eqs. (43), (46), (70)–(73), (77), and (78),

$$\tilde{p}_1^-(x_3') = \hat{Q}_S(s)\,[4Y(x_3')Y(x_{3;S})]^{-1/2} \times \left[\int_{x_3=\max(x_3',x_{3;S})}^{x_{3;\max}} \tilde{R}(x_3)\exp\left(-s\int_{\xi=x_3}^{x_3'}\gamma^-(\xi)d\xi - s\int_{\xi=x_{3;S}}^{x_3}\gamma^+(\xi)d\xi\right)dx_3\right] \times H(x_{3;\max}-x_3')H(x_{3;\max}-x_{3;S}), \tag{93}$$

$$\tilde{p}_1^+(x_3') = \hat{Q}_S(s)\,[4Y(x_3')Y(x_{3;S})]^{-1/2}\left[\int_{x_3=x_{3;\min}}^{\min(x_3',x_{3;S})} \tilde{R}(x_3)\exp\left(-s\int_{\xi=x_3}^{x_3'}\gamma^+(\xi)d\xi - s\int_{\xi=x_{3;S}}^{x_3}\gamma^-(\xi)d\xi\right)\right] \times H(x_3'-x_{3;\min})H(x_{3;S}-x_{3;\min}). \tag{94}$$

The theory of Appendix C leads to the following time-domain results:

$$p_1^-(\mathbf{x}',t) = \left(\partial_t^2\int_{\tau=T_0}^{t} Q_S(t-\tau)d\tau \times \int_{x_3=\max(x_3',x_{3;S})}^{x_{3;\max}} g_1^-(\mathbf{x},\tau)dx_3\right) \times H(x_{3;\max}-x_3')H(x_{3;\max}-x_{3;S}), \tag{95}$$

where

$$g_1^- = -(4\pi^2)^{-1} \times \int_{\psi=-\Psi_1(\mathbf{x},\tau)}^{\Psi_1(\mathbf{x},\tau)} \mathrm{Re}\left[[Y(p,\psi,x_3')Y(p,\psi,x_{3;S}]^{-1/2} \times \tilde{R}(p,\psi,x_3)p\left(\frac{\partial p}{\partial \tau}\right)\right]d\psi, \tag{96}$$

in which p is related to ψ and τ via the equation of the modified Cagniard path

$$pd\cos(\psi) + \int_{\xi=x_3}^{x_3'}\gamma^-(p,\psi,\xi)d\xi + \int_{\xi=x_{3;S}}^{x_3}\gamma^+(p,\psi,\xi)d\xi = \tau, \quad \text{with } \mathrm{Im}(\tau)=0 \tag{97}$$

and

$$p_1^+(\mathbf{x}',t) = \left(\partial_t^2\int_{\tau=T_0}^{t} Q_S(t-\tau)d\tau \int_{x_3=x_{3;\min}}^{\min(x_3',x_{3;S})} g_1^+(\mathbf{x},\tau)dx_3\right) \times H(x_3'-x_{3;\min})H(x_{3;S}-x_{3;\min}), \tag{98}$$

where

$$g_1^+ = -(4\pi^2)^{-1} \times \int_{\psi=-\Psi_1(\mathbf{x},\tau)}^{\Psi_1(\mathbf{x},\tau)} \mathrm{Re}\left[[Y(p,\psi,x_3')Y(p,\psi,x_{3;S})]^{-1/2} \times \tilde{R}(p,\psi,x_3)p\left(\frac{\partial p}{\partial \tau}\right)\right]d\psi, \tag{99}$$

in which p is related to ψ and τ via the equation of the modified Cagniard path

$$pd\cos(\psi) + \int_{\xi=x_3}^{x_3'}\gamma^+(p,\psi,\xi)d\xi + \int_{\xi=x_{3;S}}^{x_3}\gamma^+(p,\psi,\xi)d\xi = \tau, \quad \text{with } \mathrm{Im}(\tau)=0. \tag{100}$$

VII. THE CASE OF AN ISOTROPIC FLUID

For an isotropic fluid, $\rho_{k,r} = \rho\delta_{k,r}$, where ρ is the scalar volume density of mass. In this case,

$$\gamma^- = -\gamma \text{ and } \gamma^+ = \gamma, \tag{101}$$

where

$$\gamma = (c^{-2} + \alpha_p\alpha_p)^{1/2}, \tag{102}$$

in which

$$c = (\rho\kappa)^{-1/2} \tag{103}$$

in the acoustic wave speed. Furthermore, the vertical acoustic wave admittance becomes

$$Y = \gamma/\rho. \tag{104}$$

For the isotropic case, the version of the modified Cagniard method, where the variable of integration $i\alpha_1$ is replaced by the complex variable p and $\alpha_2 = q$ is kept real, seems most appropriate (see Ref. 3, and Ref. 12, Secs. 4.4, 5.4.2).

VIII. CONCLUSION

A convergent time-domain iterative scheme has been developed for computing exact transient responses from impulsive acoustic sources present in a continuously layered fluid. It employs the modified Cagniard method. The numerical effort involved for a Dirac impulse source signature consists of carrying out certain integrations in the depth coordinate, the determination of the modified Cagniard paths for given positions of a source, receiver, and reflection levels, and a final integral over a finite range. For other source signatures, the impulse response has to be convolved with the relevant source signature. To demonstrate the generality of the method, the fluid has been taken to be anisotropic by introducing a tensorial volume density of mass. The time Laplace-transform parameter is kept strictly real and positive (as Cagniard did). Thus causality is ensured, while it gives rise to a new result on the convergence of the associated time-domain "Bremmer series."

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APPENDIX A: THE TRANSFORM-DOMAIN TIME-CORRELATION TYPE RECIPROCITY RELATION

In this Appendix the transform-domain time-correlation type of reciprocity relation pertaining to an anisotropic continuously layered fluid is derived and some of its consequences are discussed. A reciprocity relation interrelates, in a specific manner, two states of a field that can occur in one and the same domain in the space where the relevant field is defined. The reciprocity relation that is needed in the present analysis is the one pertaining to the transform-domain acoustic pressure and vertical component of the particle velocity; general space-time reciprocity relations are discussed in Refs. 45 and 46. Let $\{\tilde{p}^A,\tilde{v}_3^A\}$ and $\{\tilde{p}^B,\tilde{v}_3^B\}$ denote the two states; then, the interaction quantity to be considered is

$$\partial_3(\bar{p}^B\tilde{v}_3^A+\tilde{p}^A\bar{v}_3^B)=(\partial_3\bar{p}^B)\tilde{v}_3^A+\bar{p}^B(\partial_3\tilde{v}_3^A)+(\partial_3\tilde{p}^A)\bar{v}_3^B+\tilde{p}^A(\partial_3\bar{v}_3^B), \quad (A1)$$

where

$$\{\bar{p}^B,\bar{v}_3^B\}(i\alpha_\mu,x_3,s)=\{\tilde{p}^B,\tilde{v}_3^B\}(i\alpha_\mu,x_3,-s). \quad (A2)$$

Using Eqs. (7)–(9) on the right-hand side of Eq. (A1), it is found that

$$\begin{aligned}\partial_3(\bar{p}^B\tilde{v}_3^A+\tilde{p}^A\bar{v}_3^B)&=s(\rho^B_{r,k}-\rho^A_{k,r})\bar{v}_k^B\tilde{v}_r^A+s(\kappa^B-\kappa^A)\bar{p}^B\tilde{p}^A\\&\quad+\tilde{v}_k^A\bar{f}_k^B+\bar{v}_k^B\tilde{f}_k^A+\tilde{q}^A\bar{p}^B+\bar{q}^B\tilde{p}^A.\end{aligned} \quad (A3)$$

Equation (A3) is the desired reciprocity relation. To derive from it the relations for the acoustic wave field that are needed in the main text, it is assumed that $\rho^B_{r,k}=\rho^A_{k,r}$ and $\kappa^B=\kappa^A$. If these relations hold, the fluids in the two states are denoted as each others' adjoints. If the relations hold for one and the same fluid, such a fluid is denoted as self-adjoint. In the main text, a self-adjoint fluid is considered, but the analysis of this Appendix is kept more general. Under the conditions indicated, the terms associated with the fluid properties on the right-hand side of Eq. (A3) vanish and only the terms associated with the sources remain. In particular, the right-hand side of Eq. (A3) then vanishes at a source-free level in the layered fluid. In terms of the acoustic field matrix [cf. Eqs. (11) and (12)], the interaction quantity is expressed as

$$\bar{p}^B\tilde{v}_3^A+\tilde{p}^A\bar{v}_3^B=\bar{F}^B_L C_{L,P}\tilde{F}^A_P, \quad (A4)$$

in which the elements of the matrix $[C]$ are given by

$$C_{1,1}=0,\quad C_{1,2}=1,\quad C_{2,1}=1,\quad C_{2,2}=0. \quad (A5)$$

Substituting Eq. (A4) in Eq. (A3), and applying this equation to a source-free level, the following relation is obtained:

$$\partial_3(\bar{F}^B_L C_{L,P}\tilde{F}^A_P)=(\partial_3\bar{F}^B_L)C_{L,Q}\tilde{F}^A_Q+\bar{F}^B_K C_{K,P}(\partial_3\tilde{F}^A_P)=0. \quad (A6)$$

Subsequent use of the source-free versions of Eq. (10) for the two states leads to

$$s\bar{F}^B_K(\bar{A}^B_{L,K}C_{L,Q}-C_{K,P}\tilde{A}^A_{P,Q})\tilde{F}^A_Q=0. \quad (A7)$$

Since Eq. (A7) has to hold for any two linearly independent choices of both $[\tilde{F}^A]$ and $[\bar{F}^B]$, it follows that

$$\bar{A}^B_{L,K}C_{L,Q}=C_{K,P}\tilde{A}^A_{P,Q}, \quad (A8)$$

a relation that also follows by direct inspection of Eqs. (13)–(16). (Note that $[\tilde{A}]$ is independent of s.) Equation (A8) is equivalent to

$$\bar{A}^B_{L,K}=C_{K,P}\tilde{A}^A_{P,Q}C^{-1}_{Q,L}. \quad (A9)$$

Since under a transposition and a similarity transformation the eigenvalues of a matrix are invariant, the eigenvalues of $[\bar{A}^B]^T$, where "T" means "transpose," are the same as those of $[\tilde{A}^A]$. For the rest of the analysis, they will be numbered in the same order. With [cf. Eq. (22)]

$$\bar{A}^B_{L,K}=\bar{L}^B_{L,M}\gamma^{(M)}(\bar{L}^B)^{-1}_{M,K} \quad (A10)$$

and

$$\tilde{A}^A_{P,Q}=\tilde{L}^A_{P,N}\gamma^{(N)}(\tilde{L}^A)^{-1}_{N,Q}, \quad (A11)$$

Eq. (A9) leads to

$$\bar{L}^B_{L,M}\gamma^{(M)}C_{L,Q}\tilde{L}^A_{Q,N}=\bar{L}^B_{K,M}C_{K,P}\tilde{L}^A_{P,N}\gamma^{(N)}. \quad (A12)$$

Assuming that all eigenvalues of $[\tilde{A}]$ are different, Eq. (A12) implies that $[\bar{L}^B]^T[C][\tilde{L}^A]$ is a diagonal matrix, the values of the elements of which depend on the normalization of the eigencolumns of $[\tilde{A}]$. For waves that are outgoing as $x_3\to-\infty$, the normalization is taken such that

$$\bar{p}^{B(N)}\tilde{v}_3^{A(N)}+\tilde{p}^{A(N)}\bar{v}_3^{B(N)}=-1, \quad (A13)$$

while, for waves that are outgoing as $x_3\to\infty$, the normalization is taken such that

$$\bar{p}^{B(N)}\tilde{v}_3^{A(N)}+\tilde{p}^{A(N)}\bar{v}_3^{B(N)}=1. \quad (A14)$$

With this, the elements of $[\bar{L}^B]^T[C][\tilde{L}^A]$ have the value -1 on the diagonal at those positions that correspond to waves that are outgoing as $x_3\to-\infty$ and the value 1 at those positions that correspond to waves that are outgoing as $x_3\to\infty$. Owing to the structure of $[\tilde{A}]$, $[\bar{L}^B]^T[C]$ has as its rows the columns of $[\tilde{L}^A]$, but with the acoustic pressure and particle velocity parts in the adjoint fluid interchanged. Hence, the inverse of $[\tilde{L}]$ for a particular fluid is found by interchanging in its transpose, applying to the adjoint fluid, the acoustic pressure and particle velocity parts, and changing the sign in the rows that apply to outgoing waves as $x_3\to-\infty$, but keeping the sign in the rows that correspond to outgoing waves as $x\to\infty$. Matrix inversion is, therefore, not needed for this. Observe that, in this whole procedure, it has been used, and that $[\tilde{A}]$, and therefore $[\tilde{L}]$, are independent of s. With the normalization thus carried out, it also follows that

$$\partial_3(\overline{L}^B_{K,M}C_{K,P}\widetilde{L}^A_{P,N}) = (\partial_3\overline{L}^B_{K,M})C_{K,P}\widetilde{L}^A_{P,N} + \overline{L}^B_{K,M}C_{K,P}(\partial_3\widetilde{L}^A_{P,N}) = 0. \tag{A15}$$

Application of Eq. (A15) to a self-adjoint fluid and taking $M=N$, the two terms in the middle of the result are equal and, hence,

$$\widetilde{L}_{K,M}C_{K,P}\partial_3\widetilde{L}_{P,N} = 0, \quad \text{for } M=N, \tag{A16}$$

which implies that, for a self-adjoint fluid, $[\widetilde{L}]^{-1}[\partial_3\widetilde{L}]$ has zero values on its diagonal. These properties have been used in the main text.

Another consequence of the normalization is that

$$\partial_3[\overline{W}^B_M J_{M,N}\widetilde{W}^A_N] = 0 \tag{A17}$$

in any source-free subregion of the configuration. Here, $[J]$ is the matrix with elements

$$J_{1,1} = -1, \quad J_{1,2} = 0, \quad J_{2,1} = 0, \quad J_{2,2} = 1. \tag{A18}$$

Equation (A17) follows from Eqs. (A6) and (19) and the properties of $[\widetilde{L}]$ derived thus far. Furthermore, it is understood that $\widetilde{W}_1$ applies to upgoing waves and $\widetilde{W}_2$ to downgoing waves. As a result, $[\overline{W}^B]^T[J][\widetilde{W}^A]$ is a propagation invariant in source-free regions; its value can most profitably be calculated in the homogeneous outer regions of the configuration.

APPENDIX B: THE VERTICAL PROPAGATION COEFFICIENTS AND THE VERTICAL ACOUSTIC WAVE ADMITTANCE

The columns of $[\widetilde{L}]$, normalized as in Appendix A, for a self-adjoint fluid can conveniently be expressed in terms of the acoustic wave admittance of the up- and downgoing waves. For a column that refers to a wave that is outgoing as $x_3 \to -\infty$ (upgoing wave), the vertical acoustic wave admittance is introduced through

$$\tilde{v}_3^- = -Y^-\tilde{p}^-, \tag{B1}$$

and for a column that refers to a wave that is outgoing as $x_3 \to \infty$ (downgoing wave) through

$$\tilde{v}_3^+ = Y^+\tilde{p}^+. \tag{B2}$$

The orthogonality property

$$\overline{p}^+\tilde{v}_3^- + \tilde{p}^-\overline{v}_3^+ = 0 \tag{B3}$$

that follows from Eq. (A3) leads to

$$Y^- = Y^+ = Y, \tag{B4}$$

where Y is the common vertical acoustic wave admittance of up- and downgoing waves. The normalization conditions Eqs. (A13) and (A14) next lead to

$$\tilde{p}^+ = \tilde{p}^- = (2Y)^{-1/2}. \tag{B5}$$

To express Y in terms of the fluid properties and the transform parameters, it is observed that the vertical propagation coefficients γ^- and γ^+ of the up- and downgoing waves, respectively, are the eigenvalues of the matrix $[\widetilde{A}]$. Hence they satisfy the quadratic equation

$$\gamma^2 - (\widetilde{A}_{1,1} + \widetilde{A}_{2,2})\gamma + \widetilde{A}_{1,1}\widetilde{A}_{2,2} - \widetilde{A}_{1,2}\widetilde{A}_{2,1} = 0. \tag{B6}$$

Now, for the self-adjoint fluid under consideration, $\widetilde{A}_{1,1} = \widetilde{A}_{2,2}$, as Eqs. (13) and (16) show, while the positive definiteness of the tensorial volume density of mass ensures that $\widetilde{A}_{1,2} > 0$ and $\widetilde{A}_{2,1} > 0$. [For the former, this follows upon taking in $v_k\rho_{k,r}v_r > 0$, $v_\rho = -(\rho_\|^{-1})_{\rho,\sigma}\rho_{\sigma,3}v_3$; for the latter, it follows from the observation that $\alpha_\rho(\rho_\|^{-1})_{\rho,\kappa}\alpha_\kappa > 0$. The two solutions of Eq. (B6) are given by

$$\gamma^- = (\widetilde{A}_{1,1} + \widetilde{A}_{2,2})/2 - (\widetilde{A}_{1,2}\widetilde{A}_{2,1})^{1/2}, \tag{B7}$$

$$\gamma^+ = (\widetilde{A}_{1,1} + \widetilde{A}_{2,2})/2 + (\widetilde{A}_{1,2}\widetilde{A}_{2,1})^{1/2}. \tag{B8}$$

Obviously, γ^- and γ^+ are complex with the same imaginary parts, while $\mathrm{Re}(\gamma^-) < 0$ and $\mathrm{Re}(\gamma^+) > 0$; furthermore, γ^- and γ^+ are independent of s. Next, it is observed that the first equation of Eq. (10) in a source-free domain yields

$$-s\gamma^-\tilde{p}^- + s(\widetilde{A}_{1,1}\tilde{p}^- + \widetilde{A}_{1,2}\tilde{v}_3^-) = 0. \tag{B9}$$

Using Eqs. (B1), (B2), (B4), and (B7), it follows that

$$Y = (\widetilde{A}_{2,1}/\widetilde{A}_{1,2})^{1/2}, \tag{B10}$$

where the property $\widetilde{A}_{1,1} = \widetilde{A}_{2,2}$ has been reused. Note that Y is real and positive, and independent of s.

APPENDIX C: THE MODIFIED CAGNIARD METHOD APPLIED TO A GENERALIZED-RAY CONSTITUENT

In this Appendix, the transformation back to the space-time domain of the generalized-ray constituent Eq. (79) with the aid of the modified Cagniard method is discussed. Using Eq. (6), the s-domain expression corresponding to Eq. (79) is given by

$$\hat{U} = \left(\frac{s}{2\pi}\right)^2 \hat{S}(s)\int_{\alpha_\mu\in R^2} \widetilde{\Pi}(i\alpha_\mu) \times\exp\left[-s\left(i\alpha_\mu x_\mu + \int_{\zeta\in Z^-}\gamma^-(i\alpha_\mu,\zeta)d\zeta + \int_{\zeta\in Z^+}\gamma^+(i\alpha_\mu,\zeta)d\zeta\right)\right]d\alpha_\mu. \tag{C1}$$

So far, the orientation of the horizontal axes of the chosen Cartesian reference frame is arbitrary. This arbitrariness is now exploited to simplify the further analysis. A convenient choice is to take one of the axes, the x_1 axis, for example, along the straight line joining the projections of the source and the receiver, respectively, on the horizontal plane. Accordingly,

$$x_1 = d, \quad x_2 = 0, \tag{C2}$$

where $d > 0$ is the horizontal offset between source and receiver. (Note that, as a consequence, the components of the different tensors and vectors, with respect to this reference frame, occur in the wave-field expressions.) For the present case of an anisotropic medium, the most appropriate version of the Cagniard method seems to be one where the variables of integration $i\alpha_\mu$ in Eq. (C1) are replaced by

$$i\alpha_1 = p\cos(\psi), \quad i\alpha_2 = p\sin(\psi), \tag{C3}$$

where p is positive imaginary and ψ is real in the interval $(0,2\pi)$. With this, Eq. (C1) changes into

$$\hat{U} = -\left(\frac{s}{2\pi}\right)^2 \hat{S}(s)\int_{\psi=0}^{2\pi}d\psi\int_{p=0}^{i\infty}\widetilde{\Pi}(p,\psi)$$

$$\times \exp\Big[-s\Big(pd\cos(\psi)+\int_{\zeta\in Z^-}\gamma^-(p,\psi,\zeta)d\zeta$$

$$+\int_{\zeta\in Z^+}\gamma^+(p,\psi,\zeta)d\zeta\Big)\Big]pdp, \qquad \text{(C4)}$$

where $\tilde{\Pi}(p,\psi)$ stands for $\tilde{\Pi}[p\cos(\psi),\ p\sin(\psi)]$ and $\gamma^{\pm}(p,\psi,\zeta)$ for $\gamma^{\pm}[p\cos(\psi),p\sin(\psi),\zeta]$. Next, the periodicity of the integrand in ψ is used to replace the interval $(0,2\pi)$ of integration by $(-\pi/2, 3\pi/2)$, the integrals over the intervals $(-\pi/2, \pi/2)$ and $(\pi/2, 3\pi/2)$ are taken together, and the property is used, such that, for imaginary values of p, we have $\tilde{\Pi}(p,\psi+\pi)=\tilde{\Pi}(-p,\psi)=\tilde{\Pi}(p,\psi)^*$ and $\gamma^{\pm}(p,\psi+\pi,\zeta)=\gamma^{\pm}(-p,\psi,\zeta)=\gamma^{\pm}(p,\psi,\zeta)^*$, where $*$ denotes the complex conjugate. This procedure leads to

$$\hat{U}=-\left(\frac{s^2}{2\pi^2}\right)\hat{S}(s)\int_{\psi=-\pi/2}^{\pi/2}d\psi\,\mathrm{Re}\Big\{\int_{p=0}^{i\infty}\tilde{\Pi}(p,\psi)$$

$$\times\exp\Big[-s\Big(pd\cos(\psi)+\int_{\zeta\in Z^-}\gamma^-(p,\psi,\zeta)d\zeta$$

$$+\int_{\zeta\in Z^+}\gamma^+(p,\psi,\zeta)d\zeta\Big)\Big]pdp\Big\}. \qquad \text{(C5)}$$

The essential feature of the modified Cagniard method consists of replacing the integration with respect to p along the positive imaginary axis, through continuous deformation, with one along a modified Cagniard path that follows from

$$pd\cos(\psi)+\int_{\zeta\in Z^-}\gamma^-(p,\psi,\zeta)d\zeta$$

$$+\int_{\zeta\in Z^+}\gamma^+(p,\psi,\zeta)d\zeta=\tau, \qquad \text{(C6)}$$

with τ real and positive. The admissibility of the contour deformation rests on the applicability of Cauchy's theorem and of Jordan's lemma, according to which the contributions from joining circular arcs at infinity vanish, provided that the contour deformation takes place into the right half of the p plane. The only singularities of the integrand are the branch points due to the occurrence of $\tilde{A}_{2,1}^{1/2}$ in the expressions for $\tilde{\Pi}$ and $\gamma^{\pm}$, i.e., the zeros of $\tilde{A}_{2,1}$. These zeros can easily be proved to reside on the real p axis. (This would not be the case if the de Hoop[3] version of the modified Cagniard method had been used; see also Ref. 12.) From Eq. (C6), it follows that the part of the real axis from the origin to the branch point nearest to the origin, as well as the complex path that satisfies the equation and has a straight asymptote as $\tau\to\infty$, is a candidate for a modified Cagniard path. As to the complex part of the modified Cagniard path, two possibilities exist: (a) It intersects the real p axis at a regular point of the left-hand side of Eq. (C6), in which case τ reaches, at that point, a minimum that follows from equating to zero the expressions for $\partial\tau/\partial p$ that follows from Eq. (C6); (b) the modified Cagniard path touches the real p axis at the branch point of $\tilde{A}_{2,1}^{1/2}$ nearest to the origin [i.e., a singular point of the left-hand side of Eq. (C6)]. The two cases are shown in Figs. C1 and C2. Which of the two cases applies depends on the vertical profiles of the constitutive parameters, the mutual positions of source and point of observation, and the reflection levels of the generalized ray. Note that, in the latter case, the modified Cagniard path must, in the immediate neighborhood of the branch point under consideration, be replaced by a circular arc around that branch point since the branch point of $\tilde{A}_{2,1}^{1/2}$ is a simple pole of $\tilde{R}=\partial_3 Y/2Y$. If the latter situation occurs, one can therefore expect a relatively large contribution from such a generalized ray to the total wave motion. A generalized ray of this type is the time-domain counterpart of a turning ray in the asymptotic frequency-domain treatment of the problem. Along the modified Cagniard path, τ is introduced as the variable of integration, and the integrations with respect to τ and ψ are interchanged. The final result can be written as

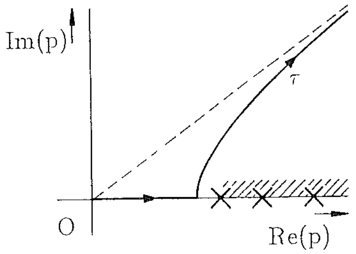

FIG. C1. Complex part of modified Cagniard path where τ reaches a minimum at a regular point.

$$\hat{U}=s^2\hat{S}(s)\hat{G}(\mathbf{x},s), \qquad \text{(C7)}$$

in which

$$\hat{G}(\mathbf{x},s)=\int_{\tau=T_0}^{\infty}\exp(-s\tau)g(\mathbf{x},\tau)d\tau, \qquad \text{(C8)}$$

with

$$g(\mathbf{x},\tau)=-(2\pi^2)^{-1}\int_{\psi=\Psi_1(\mathbf{x},\tau)}^{\Psi_2(\mathbf{x},\tau)}\mathrm{Re}\Big[\tilde{\Pi}(p,\psi)p\Big(\frac{\partial p}{\partial\tau}\Big)\Big]d\psi. \qquad \text{(C9)}$$

For details of the procedure, in particular of the composition of the ψ-integral in the different τ-intervals, we refer to van der Hijden,[12] Secs. 4.6, 6.4, and 7.4.

The time-domain counterpart of Eq. (C7) is

$$U=\partial_t^2\int_{\tau=T_0}^{t}S(t-\tau)g(\mathbf{x},\tau)d\tau, \qquad \text{(C10)}$$

in which, in view of Lerch's theorem,[44] applied to Eq. (C8)

$$g=-(2\pi^2)^{-1}$$

$$\times\Big\{\int_{\psi=\Psi_1(\mathbf{x},\tau)}^{\Psi_2(\mathbf{x},\tau)}\mathrm{Re}\Big[\tilde{\Pi}(p,\psi)p\Big(\frac{\partial p}{\partial\tau}\Big)d\psi\Big\}$$

$$\times H(\tau-T_0), \qquad \text{(C11)}$$

where $H(\tau)$ denotes the Heaviside unit step function: $H(\tau)=\{0,\frac{1}{2},1\}$ for $\{\tau<0,\tau=0,\tau>0\}$. Obviously, T_0 is the

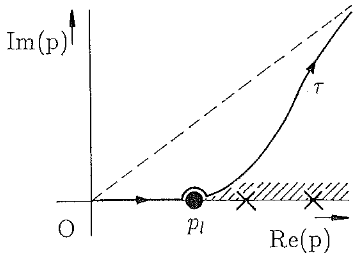

FIG. C2. Complex part of modified Cagniard path where τ reaches a smallest value at the leftmost branch point of $\tilde{A}_{2,1}^{1/2}$.

arrival time of the generalized ray constituent under consideration.

[1] L. Cagniard, *Reflection et refraction des ondes seismiques progressives* (Gauthier-Villars, Paris, 1939).

[2] L. Cagniard, *Reflection and Refraction of Progressive Seismic Waves* (McGraw-Hill, New York, 1962); translation and revision of Cagniard[1] by E. A. Flinn and C. H. Dix.

[3] A. T. de Hoop, "A modification of Cagniard's method for solving seismic pulse problems," Appl. Sci. Res. **B8**, 349–356 (1960).

[4] A. T. de Hoop, "Theoretical determination of the surface motion of a uniform elastic half-space produced by a dilatational, impulsive point source," in *Proceedings Colloque International du C.N.R.S. no. 111, Marseille*, 21–32 (1961) (in English).

[5] A. T. de Hoop, "Acoustic radiation from impulsive sources in a layered fluid," Nieuw Archief voor Wiskunde **6**, 111–129 (1988).

[6] J. D. Achenbach, *Wave Propagation in Elastic Solids* (North-Holland, Amsterdam, 1973), p. 187.

[7] J. Miklowitz, *The Theory of Elastic Waves and Waveguides* (North-Holland, Amsterdam, 1978), p. 302.

[8] K. Aki and P. G. Richards, *Quantitative Seismology* (Freeman, San Francisco, 1980), p. 224.

[9] T. Spencer, "The method of generalized reflection and transmission coefficients," Geophysics **25**, 625–641 (1960).

[10] D. V. Helmberger, "The crust-mantle transition in the Bering Sea," Bull. Seismol. Soc. Am. **58**, 179–214 (1968).

[11] R. A. Wiggins and D. V. Helmberger, "Synthetic seismogram computation by expansion in generalized rays," Geophys. J. R. Astron. Soc. **37**, 73–90 (1974).

[12] J. H. M. T. van der Hijden, *Propagation of Transient Elastic Waves in Stratified Anisotropic Media* (North-Holland, Amsterdam, 1988).

[13] C. H. Chapman, "Generalized ray theory for an inhomogeneous medium," Geophys. J. R. Astron. Soc. **36**, 673–704 (1974).

[14] C. H. Chapman, "Exact and approximate generalized ray theory in vertically inhomogeneous media," Geophys. J. R. Astron. Soc. **46**, 201–233 (1976).

[15] C. L. Pekeris, "The seismic surface pulse," Proc. Nat. Acad. Sci. U.S.A. **41**, 469–480 (1955).

[16] C. L. Pekeris, "The seismic buried pulse," Proc. Nat. Acad. Sci. U.S.A. **41**, 629–639 (1955).

[17] W. L. Roever, T. F. Vining, and E. Strick, "Propagation of elastic wave motion from an impulsive source along a fluid/solid interface," Philos. Trans. R. S. London, Ser. A **251**, 455–523 (1959).

[18] A. T. de Hoop, "Large-offset approximations in the modified Cagniard method for computing synthetic seismograms: a survey," Geophys. Prospect. **36**, 465–477 (1988).

[19] C. H. Chapman, "Long-period corrections to body waves: theory," Geophys. J. R. Astron. Soc. **64**, 321–372 (1981).

[20] C. H. Chapman, "A new method for computing synthetic seismograms," Geophys. J. R. Astron. Soc. **54**, 481–518 (1978).

[21] S. C. Singh and C. H. Chapman, "WKBJ seismogram theory in anisotropic media," J. Acoust. Soc. Am. **84**, 732–741 (1988).

[22] J. Garmany, "Seismograms in stratified anisotropic media—I. WKBJ theory," Geophys. J. **92** (3), 365–377 (1988).

[23] J. Garmany, "Seismograms in stratified anisotropic media—II. Uniformly asymptotic approximations," Geophys. J. **92**, 379–389 (1988).

[24] M. Schoenberg, "Wave propagation in alternating solid and fluid layers," Wave Motion **6**, 303–320 (1984).

[25] H. Bremmer, "Geometrisch-optische benadering van de golfvergelijking," Handel. Ned. Nat. en Geneeskd. Congr. 88–91 (1939) (in Dutch).

[26] H. Bremmer, "The propagation of electromagnetic waves through a stratified medium and its WKB approximation for oblique incidence," Physica **15**, 593–608 (1949).

[27] H. Bremmer, "Some remarks on the ionospheric double refraction. Part II: Reduction of Maxwell's equations; WKB approximation," Philips Res. Rep. **4**, 189–205 (1949).

[28] H. Bremmer, "The WKB approximation as the first term of a geometric-optical series," Commun. Pure Appl. Math. **4**, 105–115 (1951); also in *The Theory of Electromagnetic Waves*, edited by M. Kline (Interscience, New York, 1951), pp. 169–179.

[29] L. J. F. Broer, "Note on approximate solutions of the wave equation," Appl. Sci. Res. **B10**, 110–118 (1962).

[30] F. W. Sluijter, "Generalizations of the Bremmer series based on physical concepts," J. Math. Anal. Appl. **27**, 282–302 (1969).

[31] F. W. Sluijter, "Arbitrariness of dividing the total field in an optically inhomogeneous medium into direct and reversed waves," J. Opt. Soc. Am. **60**, 8–10 (1970).

[32] L. J. F. Broer and J. B. van Vroonhoven, "On some solutions of the wave equation," Physica **52**, 441–454 (1971).

[33] S. H. Gray, "On the convergence of the time-domain Bremmer series," Wave Motion **5**, 249–255 (1983).

[34] M. M. Verheggen, L. J. F. Broer, and F. W. Sluijter, "Nonmonochromatic solutions of the spatially inhomogeneous wave equation based on the WKB approximation," J. Math. Anal. Appl. **62**, 512–524 (1978).

[35] C. H. Chapman and J. A. Orcutt, "The computation of body wave synthetic seismograms in laterally homogeneous media," Rev. Geophys. **23**, 105–163 (1985).

[36] E. Heyman and L. B. Felsen, "Non-dispersive approximations for transient ray fields in an inhomogeneous medium," in *Hybrid Formulation of Wave Propagation and Scattering*, edited by L. B. Felsen (Martinus Nijhoff, Dordrecht, The Netherlands, 1984), pp. 269–284.

[37] E. Heyman and L. B. Felsen, "Real and complex spectra—a generalization of WKBJ seismograms," Geophys. J. R. Astron. Soc. **91**, 1087–1126 (1987).

[38] E. Heyman, B. Z. Steinberg, and L. B. Felsen, "Spectral analysis of focus wave modes," J. Opt. Soc. Am. **A4**, 2081–2091 (1987).

[39] E. Heyman, R. Ianconescu, and L. B. Felsen, "Pulsed beam interaction with propagation environments; canonical example of reflection and diffraction," Proceedings of the SPIE/OE Symposium of Microwave Particle and Beam Propagation, held in Los Angeles, CA, 18–20 January 1989.

[40] E. Heyman, "Complex source pulsed beam representation of transient radiation," Wave Motion **11**, 337–349 (1989).

[41] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves* (Prentice-Hall, Englewood Cliffs, NJ, 1973), Secs. 3.5a and 3.5c.

[42] B. L. N. Kennett, *Seismic Wave Propagation in Stratified Media* (Cambridge U. P., London, 1984), pp. 39, 40 and 50, 51.

[43] G. J. Fryer and F. N. Frazer, "Seismic waves in stratified anisotropic media," Geophys. J. R. Astron. Soc. **78**, 691–710 (1984).

[44] D. V. Widder, *The Laplace Transform* (Princeton U. P., Princeton, NJ, 1946), p. 63.

[45] A. T. de Hoop, "Time-domain reciprocity theorems for acoustic wave fields in fluids with relaxation," J. Acoust. Soc. Am. **84**, 1877–1882 (1988).

[46] A. T. de Hoop and H. J. Stam, "Time-domain reciprocity theorems for elastodynamic wave fields in solids with relaxation and their application to inverse problems," Wave Motion **10**, 479–489 (1988).