Convergence criterion for the time-domain iterative Born approximation to scattering by an inhomogeneous, dispersive object

Adrianus T. de Hoop*

Schlumberger-Doll Research, Old Quarry Road, P.O. Box 307, Ridgefield, Connecticut 06877-4108

Received June 12, 1990; revised manuscript received March 29, 1991; accepted April 8, 1991

A convergence criterion is derived for the iterative Rayleigh–Gans–Born approximation (or Neumann expansion) applied to the time-domain integral equation for the scattering of transient scalar waves by an inhomogeneous, dispersive object of bounded extent, embedded in a homogeneous, nondispersive medium. The criterion is independent of the size of the object and contains only a bound on the maximum absolute value of the contrast susceptibility of the object with respect to its embedding. Three types of contrast susceptibility relaxation function are considered in more detail: one for an instantaneously reacting (i.e., nondispersive) material, one for a dielectric with a Lorentzian absorption line, and one for a dispersive metal. For the last two cases the convergence proves to be unconditional if the object is embedded in vacuum. The proof makes use of the time Laplace transformation with a real, positive transform parameter and Lerch's theorem on the uniqueness of the one-sided Laplace transformation, which implies that causality of the wave motion plays an essential role.

1. INTRODUCTION

One of the standard procedures used to calculate the wave function associated with the scattering of waves by a penetrable, inhomogeneous object of bounded extent, embedded in a homogeneous medium, is to solve the domain integral equation of the second kind that results from considering the object as the source domain of the scattered wave field. We shall use the source-type integral representation for this wave field, relate the contrast volume source distribution in the object to the total local wave field through the constitutive relation of the material occupying the object, and require field reproduction throughout the object. An analytic procedure used to solve the relevant integral equation is to employ the Neumann expansion (also known as the iterative Rayleigh–Gans–Born approximation), a procedure that is expected to converge for not-too-large contrasts and not-too-large-sized objects. In the frequency-domain analysis of the problem (i.e., for time-harmonic waves) a convergence criterion containing a combined sufficiency bound on the maximum contrast, the size of the object, and the wavelength of the incident radiation can relatively easily be found. For reference, a derivation of this criterion is included.

The major part of the paper is devoted to the time-domain aspects of the problem, which occur when the incident radiation is considered to be transient in nature. It is shown that a sufficient condition for the iterative procedure to converge is that the maximum value of a properly defined relative contrast of the object with respect to its embedding is less than one; the size of the object does not, contrary to what one would expect, occur in the criterion. The proof makes use of the one-sided Laplace transformation with respect to time and Lerch's theorem on the uniqueness of this transformation, so causality of the wave motion plays an essential role.

Arbitrary inhomogeneity and dispersion properties of the object are taken into account. The embedding medium is taken to be homogeneous and dispersion free. Three types of contrast susceptibility relaxation function are considered in more detail: one for an instantaneously reacting (i.e., nondispersive) material, one for a dielectric with a Lorentzian absorption line, and one for a dispersive metal. It is shown that, for the instantaneously reacting material, the iterative Rayleigh–Gans–Born approximation is convergent if the maximum of the absolute value of the contrast of the object with respect to its embedding is less than one, while for the dielectric and for the metal the procedure is unconditionally convergent if the object is embedded in vacuum.

2. FORMULATION OF THE SCATTERING PROBLEM

In three-dimensional space R^3, a scattering object is present that occupies the bounded domain D. The domain exterior to D is denoted by D'. To specify position in the configuration, we shall use the coordinates $\{x, y, z\}$ with respect to an orthogonal, Cartesian reference frame with the origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z\}$ of unit length. In the indicated order the base vectors form a right-handed system. When appropriate, the space coordinates are collectively denoted by the position vector $\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$. The time coordinate is denoted by t. In the configuration, scattering of waves whose physical effects are characterized by the scalar wave function $u = u(\mathbf{r}, t)$ takes place.

The domain D' is occupied by a homogeneous, dispersion-free medium, which for some applications will be taken to be vacuum. In this medium, the waves propagate with the wave speed c_0. In any source-free subdomain of D', u satisfies the homogeneous three-dimensional wave equation

0740-3232/91/081256-05$05.00

$$\partial_x{}^2 u + \partial_y{}^2 u + \partial_z{}^2 u - c_0{}^{-2}\partial_t{}^2 u = 0, \tag{1}$$

where ∂ denotes partial differentiation. The scattering object shows a contrast in optical properties with respect to its embedding D'. This contrast is characterized by the scalar, causal contrast susceptibility relaxation function $\chi = \chi(\mathbf{r}, t)$. Then, in D, u satisfies the scalar wave equation

$$\partial_x{}^2 u + \partial_y{}^2 u + \partial_z{}^2 u - c_0{}^{-2}\partial_t{}^2 \times \left[u(\mathbf{r}, t) + \int_{t'=0}^{\infty} \chi(\mathbf{r}, t')u(\mathbf{r}, t - t')\mathrm{d}t'\right] = 0. \tag{2}$$

Here, $\chi = \chi(\mathbf{r}, t)$ represents the (causal) time response at position $\mathbf{r}$ in the object if the local wave function were a unit impulse (Dirac delta function) in time. For a homogeneous substance, χ is independent of $\mathbf{r}$. The general analysis will be carried out for arbitrary contrast susceptibility relaxation functions. Three types of contrast susceptibility relaxation function of separate physical importance are considered in more detail.

$\chi = 0$ for $\mathbf{r} \in D'$. In the exterior domain D' sources are present that generate the irradiating incident wave field $u^i = u^i(\mathbf{r}, t)$. The scattered wave field $u^s = u^s(\mathbf{r}, t)$ is defined as

$$u^s = u - u^i. \tag{3}$$

In view of Eqs. (1)–(3) the scattered wave field satisfies the inhomogeneous scalar wave equation

$$\partial_x{}^2 u^s + \partial_y{}^2 u^s + \partial_z{}^2 u^s - c_0{}^{-2}\partial_t{}^2 u^s = -q^s, \tag{4}$$

in which

$$q^s(\mathbf{r}, t) = \left[-c_0{}^{-2}\partial_t{}^2 \int_{t'=0}^{\infty} \chi(\mathbf{r}, t')u(\mathbf{r}, t - t')\mathrm{d}t', 0\right], \quad \mathbf{r} \in \{D, D'\} \tag{5}$$

is the contrast volume source density of the scattered wave. The scattered wave function can be expressed in terms of its contrast volume source density through the retarded potential

$$u^s(\mathbf{r}, t) = \int_{\mathbf{r}' \in D} \frac{q^s(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c_0)}{4\pi|\mathbf{r} - \mathbf{r}'|}\mathrm{d}V. \tag{6}$$

It is now assumed that the sources that generate the wave field start to act at the instant $t = 0$. Then, causality requires that $u(\mathbf{r}, t)$, $u^i(\mathbf{r}, t)$, and $u^s(\mathbf{r}, t)$ vanish at all $\mathbf{r}$ for $t < 0$. Taking a one-sided Laplace transformation according to

$$\hat{u}(\mathbf{r}, s) = \int_{t=0}^{\infty} \exp(-st)u(\mathbf{r}, t)\mathrm{d}t \tag{7}$$

transforms Eqs. (4) and (5) into

$$\partial_x{}^2 \hat{u}^s + \partial_y{}^2 \hat{u}^s + \partial_z{}^2 \hat{u}^s - (s^2/c_0{}^2)\hat{u}^s = -\hat{q}^s, \tag{8}$$

$$\hat{q}^s = [-(s^2/c_0{}^2)\hat{\chi}\hat{u}, 0], \quad \mathbf{r} \in \{D, D'\}, \tag{9}$$

where the rule $\partial_t \to s$ and the product rule for the Laplace transformation of a convolution integral have been used. For our further analysis, s is taken to be real and positive. Then, Lerch's theorem (see Widder[1]) ensures that there is a one-to-one correspondence between a causal time function and its Laplace-transform-domain counterpart, provided that the time function is continuous and is, at most, of exponential growth as $t \to \infty$ and that equality in Eq. (7) is invoked at the real set of points $\{s_n = s_0 + nh; n = 0, 1, 2, \ldots\}$, where s_0 is sufficiently large positive and h is positive. (At these points, the corresponding transform integral is then absolutely convergent.) Now, the physically acceptable wave and relaxation functions are bounded functions of the space and time variables, with, as a limiting case, an impulse (Dirac delta function) time behavior. Under these conditions, their time Laplace-transform-domain counterparts are bounded functions of $\mathbf{r}$ and s for all real $s > 0$.

In view of Eq. (6), the scattered wave function admits the source-type integral representation

$$\hat{u}^s(\mathbf{r}, s) = \int_{\mathbf{r}' \in D} \hat{G}(\mathbf{r} - \mathbf{r}', s)\hat{q}^s(\mathbf{r}', s)\mathrm{d}V, \tag{10}$$

in which

$$\hat{G}(\mathbf{r}, s) = \exp(-s|\mathbf{r}|/c_0)/4\pi|\mathbf{r}|, \quad |\mathbf{r}| > 0 \tag{11}$$

is the Green's function (point-source solution) of the modified Helmholtz equation [Eq. (8)]. Together with

$$\hat{u} = \hat{u}^i + \hat{u}^s, \tag{12}$$

Eqs. (9)–(11) constitute, for points of observation $\mathbf{r} \in D$, an integral equation of the second kind from which $\hat{u} = \hat{u}(\mathbf{r}, s)$ for $\mathbf{r} \in D$ is to be solved. Once this has been done, the result is inserted in Eq. (9) to yield, from Eqs. (10) and (11), the value of $\hat{u}^s$ in all space. Then, the subsequent use of Eq. (12) determines the value of $\hat{u}$ in all space.

3. ITERATIVE SOLUTION TO THE TIME LAPLACE TRANSFORM DOMAIN INTEGRAL EQUATION

To investigate the convergence of the iterative solution to the integral equation for the scattering problem, we write the equation in the operator form

$$\hat{u} = \hat{u}^i + K\hat{u}, \quad \mathbf{r} \in D, \tag{13}$$

where the integral operator K is defined by

$$K\hat{u}(\mathbf{r}, s) = -\left(\frac{s^2}{c_0{}^2}\right) \int_{\mathbf{r}' \in D} \hat{G}(\mathbf{r} - \mathbf{r}', s)\hat{\chi}(\mathbf{r}', s)\hat{u}(\mathbf{r}', s)\mathrm{d}V. \tag{14}$$

The Neumann expansion (also denoted as the iterative Rayleigh–Gans–Born approximation, see Jones[2]) of $\hat{u}$, as it follows from Eq. (13), is defined as the iterative procedure

$$\hat{u}_0 = \hat{u}^i, \tag{15}$$

$$\hat{u}_n = K\hat{u}_{n-1} = K^n\hat{u}_0, \quad n = 1, 2, 3, \ldots. \tag{16}$$

From Eqs. (13), (15), and (16) it follows that if one carries out N steps, the following identity results:

$$\sum_{n=0}^{N} \hat{u}_n = \hat{u} - K^{N+1}\hat{u}. \tag{17}$$

Hence, if $|K^N\hat{u}| \to 0$ as $N \to \infty$, the procedure is convergent, and $\hat{u}$ is given by

$$\hat{u} = \sum_{n=0}^{\infty} \hat{u}_n = \sum_{n=0}^{\infty} K^n \hat{u}^i, \tag{18}$$

where $K^0 = I$ is the identity operator.

To investigate under which conditions Eq. (18) holds, let

$$\hat{X}(s) = \max_{\mathbf{r}' \in D} |\hat{\chi}(\mathbf{r}', s)| \tag{19}$$

be the maximum of the absolute value of the time Laplace-transform-domain contrast susceptibility relaxation function in the scattering object and let

$$\hat{U}(s) = \sup_{\mathbf{r}' \in D} |\hat{u}(\mathbf{r}', s)| \tag{20}$$

be some upper bound of the absolute value of the time Laplace-transform-domain wave function in it; then it follows from Eq. (14) that

$$|K\hat{u}(\mathbf{r}, s)| \leq \left(\frac{s^2}{c_0{}^2}\right) \hat{X}(s) \hat{U}(s) \int_{\mathbf{r}' \in D} \hat{G}(\mathbf{r} - \mathbf{r}', s) \mathrm{d}V, \tag{21}$$

since $\hat{G}$ is real and positive as Eq. (11) shows. (Note that s is real and positive.)

To estimate the remaining integral at the right-hand side of formula (21), it is observed that, again since $\hat{G}$ is real and positive,

$$\int_{\mathbf{r}' \in D} \hat{G}(\mathbf{r} - \mathbf{r}', s) \mathrm{d}V \leq \int_{\mathbf{r}' \in B} \hat{G}(\mathbf{r} - \mathbf{r}', s) \mathrm{d}V, \tag{22}$$

where B is the smallest ball in which the scattering object is contained. The integral at the right-hand side of formula (22) can be evaluated analytically. Let R_0 be the radius of B and let r be the distance from the center of B to the point of observation; then the result is

$$\int_{\mathbf{r}' \in B} \hat{G}(\mathbf{r} - \mathbf{r}', s) \mathrm{d}V = \left(\frac{c_0{}^2}{s^2}\right) \left[1 - \left(\frac{1 + sR_0}{c_0}\right) \left(\frac{c_0}{sr}\right) \times \exp\left(\frac{-sR_0}{c_0}\right) \sinh\left(\frac{sr}{c_0}\right)\right], \quad r \leq R_0, \tag{23}$$

$$\int_{\mathbf{r}' \in B} \hat{G}(\mathbf{r} - \mathbf{r}', s) \mathrm{d}V = \left(\frac{c_0{}^2}{s^2}\right) \left(\frac{c_0}{sr}\right) \exp\left(\frac{-sr}{c_0}\right) \times \left[\left(\frac{sR_0}{c_0}\right) \cosh\left(\frac{sR_0}{c_0}\right) - \sinh\left(\frac{sR_0}{c_0}\right)\right], \quad r \geq R_0. \tag{24}$$

Carrying out the necessary differentiations verifies that the right-hand sides of Eqs. (23) and (24) do satisfy the inhomogeneous modified Helmholtz equation with a uniformly distributed volume source density equal to unity throughout B, while the corresponding wave function and its normal (radial) derivative are continuous on crossing the spherical boundary of B. Further, the right-hand sides of Eqs. (23) and (24) are real and positive throughout space, as they should be, while their radial derivative is negative for all $r > 0$. As a consequence, the maximum value of the integrals at the right-hand sides of Eqs. (23) and (24) for fixed s is reached at $r = 0$, where it is found from Eq. (23) to be

$$\max_{\mathbf{r} \in R^3} \int_{\mathbf{r}' \in B} \hat{G}(\mathbf{r} - \mathbf{r}', s) \mathrm{d}V = \left(\frac{c_0{}^2}{s^2}\right) \left[1 - \left(\frac{1 + sR_0}{c_0}\right) \exp\left(\frac{-sR_0}{c_0}\right)\right]. \tag{25}$$

Using formulas (22) and (25) in formula (21) gives, for fixed s,

$$\max_{\mathbf{r} \in R^3} |K\hat{u}(\mathbf{r}, s)| \leq \hat{X}(s) \hat{U}(s) [1 - (1 + sR_0/c_0) \exp(-sR_0/c_0)]. \tag{26}$$

Now, considered as a function of s, the expression in brackets at the right-hand side of Eq. (26) is less than one for any finite, real, positive value of s, while it reaches the limit unity as $s \to \infty$. Hence

$$|K\hat{u}(\mathbf{r}, s)| \leq \hat{X}(s) \hat{U}(s), \tag{27}$$

for all $\mathbf{r} \in R^3$ and all real, positive values of s, and, consequently,

$$|K^N \hat{u}(\mathbf{r}, s)| \leq [\hat{X}(s)]^N \hat{U}(s), \tag{28}$$

for all $\mathbf{r} \in R^3$ and all real, positive values of s. From formula (28) it is concluded that $|K^N \hat{u}| \to 0$ as $N \to \infty$, and hence the Neumann expansion of Eq. (13) is convergent if $\hat{X}(s) < 1$ and $\hat{U}(s)$ is bounded. The latter condition is always satisfied if the behavior of the incident wave field is no more singular in its time dependence than that of an impulse function (Dirac delta function), while the former condition puts a restriction on the maximum absolute value of the time Laplace transform of the contrast susceptibility relaxation function. Note, however, that this restriction is independent of the size of the scattering object, a result that could not be expected from the corresponding frequency-domain convergence analysis (see Section 6). [The shape of the object has been eliminated from the discussion already with the use of formula (22).]

If the procedure is convergent, it follows from Eqs. (17) and (18) that the bound on the error after one carries out N steps is given by

$$|\hat{u} - \hat{u}_N| = \left|\sum_{n=N+1}^{\infty} K^n \hat{u}^i\right| \leq \left\{\sum_{n=N+1}^{\infty} [\hat{X}(s)]^n\right\} \hat{U}^i = \frac{[\hat{X}(s)]^{N+1}}{1 - \hat{X}(s)} \hat{U}^i, \tag{29}$$

where

$$\hat{U}^i(s) = \max_{\mathbf{r} \in D} |\hat{u}^i(\mathbf{r}, s)| \tag{30}$$

is the maximum of $|\hat{u}^i|$ over the scattering object.

4. TIME-DOMAIN ITERATIVE RAYLEIGH–GANS–BORN APPROXIMATION

In view of Lerch's theorem all results of Section 3 directly carry over to the time domain. Hence the time-domain Rayleigh–Gans–Born approximation is convergent provided that the time-domain counterpart of the convergence condition

$$\max_{\mathbf{r}' \in D} |\hat{\chi}(\mathbf{r}', s)| < 1 \tag{31}$$

is satisfied at the real set of points $\{s_n; n = 0, 1, 2, \ldots\}$, where

$$s_n = s_0 + nh, \qquad s_0 > 0, h > 0. \tag{32}$$

Note that in this condition s_0 can still freely be chosen, provided that it is positive and, in view of applications below, the freedom can be exploited to take it as large as needed. Of course, the larger the value of s_0 that must be chosen in order for the procedure to converge, the more accurately the early time behavior of the incident wave function must be known. Also, the closer the contrast properties of the object are to the convergence bound, the slower the rate of convergence will be. Some specific cases of the contrast susceptibility relaxation function will be considered in more detail in Section 5.

The numerical implementation of the method requires, for the first-order approximation, an integration over the scattering domain. For the next and higher orders, the relevant space–time integral equation has to be solved numerically. In the corresponding frequency-domain analysis, recent research in this respect includes the generalized overrelaxation method discussed by Kleinman *et al.*[3] and the discrete-dipole approximation discussed by Flatau and Stephens.[4] Also of importance to the numerics of the problem is the convergence analysis of the scattering-order formulation of the coupled-dipole method discussed by Singham and Bohren.[5]

For some results on the time-domain far-field scattering of plane waves in the first-order approximation by objects of various shapes, see de Hoop.[6]

5. THE CONTRAST SUSCEPTIBILITY RELAXATION FUNCTIONS CONSIDERED

Three types of contrast susceptibility relaxation function of direct importance to optics will be considered in more detail below.

Instantaneously Reacting (Nondispersive) Material

For an instantaneously reacting (i.e., nondispersive) material the contrast susceptibility relaxation function is of the form

$$\chi(\mathbf{r}, t) = \chi_1(\mathbf{r})\delta(t), \tag{33}$$

where $\delta(t)$ is the temporal unit impulse (Dirac delta function). Then,

$$\hat{\chi}(\mathbf{r}, s) = \chi_1(\mathbf{r}) \tag{34}$$

is independent of s. For this class of materials the condition for the convergence of the iterative Rayleigh–Gans–Born approximation is [cf., formula (31)]

$$\max_{\mathbf{r}\in D}|\chi_1(\mathbf{r})| < 1. \tag{35}$$

Dielectric Object with a Lorentzian Absorption Line Embedded in Vacuum

The susceptibility relaxation function of a dielectric with a single absorption line, according to the classical Lorentz model, is given by (Born and Wolf[7] and Jones[8])

$$\chi = \chi_d = (\omega_p^2/\Omega)\exp(-\Gamma t)\sin(\Omega t)H(t), \tag{36}$$

with

$$\omega_p = (Nq^2/m\epsilon_0)^{1/2}, \tag{37}$$

$$\Omega = (\omega_0^2 + \omega_p^2/3 - \Gamma^2)^{1/2}. \tag{38}$$

In Eqs. (36)–(38) the symbols have the following meaning:

N, number density;
q, absolute value of electric charge;
m, mass;
ϵ_0, permittivity *in vacuo*;
ω_0, angular resonance frequency associated with the restoring force (Coulomb force);
ω_p, angular plasma frequency;
Γ, phenomenological damping coefficient;
Ω, natural angular frequency;

all associated with the atom's moving electric charge distribution. $H(t)$ is the Heaviside unit step function: $H(t) = \{1, 1/2, 1\}$ for $\{t < 0, t = 0, t > 0\}$. In the model, the spherical-cavity Lorentz correction has been taken into account. The s-domain counterpart of Eq. (36) is

$$\hat{\chi}_d = \frac{\omega_p^2}{s^2 + 2s\Gamma + \omega_0^2 + \omega_p^2/3}. \tag{39}$$

Equation (39) shows that $\hat{\chi}_d$ is real and positive for all real, positive s, and is monotonically decreasing toward zero as $s \to \infty$. Hence, from a certain value of s onward, the condition for convergence is always satisfied. Taking s_0 greater than or equal to this value, it is concluded that, for a scattering object of bounded extent, embedded in vacuum and consisting of a collection of polarizable atoms within Lorentz's classical theory of electrons, the time-domain iterative Rayleigh–Gans–Born approximation is unconditionally convergent.

Dispersive Metal Object Embedded in Vacuum

The susceptibility relaxation function of a dispersive metal is given by (Born and Wolf[9])

$$\chi = \chi_m = (\omega_p^2/\nu_c)[1 - \exp(-\nu_c t)]H(t), \tag{40}$$

in which ω_p is again the angular plasma frequency given by Eq. (37) and ν_c is the collision frequency of the conduction electrons with the atomic lattice. The s-domain counterpart of Eq. (40) is

$$\hat{\chi}_m = (\omega_p^2)/s(s + \nu_c). \tag{41}$$

Equation (41) shows that $\hat{\chi}_m$ is real and positive for all real, positive s, and is monotonically decreasing toward zero as $s \to \infty$. Hence, from a certain value of s onward, the condition for convergence is always satisfied. Taking s_0 greater than or equal to this value results in the conclusion that, for a scattering object of bounded extent, embedded in vacuum and consisting of a collection of electrically charged particles with inertia properties and subject to collisions with an atomic lattice within Lorentz's classical theory of electrons, the time-domain iterative Rayleigh–Gans–Born approximation is unconditionally convergent.

6. FREQUENCY-DOMAIN ITERATIVE RAYLEIGH–GANS–BORN APPROXIMATION

For completeness, the frequency-domain bound on the convergence of the iterative Rayleigh–Gans–Born approximation will be derived in this section. In the frequency domain the estimates of Section 3 have to be redone for $s = -i\omega$, where $\omega > 0$ is the real angular frequency of the oscillations and the complex time factor $\exp(-i\omega t)$ is understood. For $s = -i\omega$, formula (22) is replaced by [note that $|\exp(-i\omega|\mathbf{r} - \mathbf{r}'|/c_0)| = 1$]

$$\int_{\mathbf{r}'\in D} \hat{G}(\mathbf{r} - \mathbf{r}', -i\omega)\mathrm{d}V \leq \int_{\mathbf{r}'\in B} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\mathrm{d}V, \qquad (42)$$

where again B is the smallest ball in which the scattering object is contained. The integral at the right-hand side of formula (42) can be evaluated analytically; also, its value directly follows from Eqs. (23) and (24) by putting $s = 0$. The result is

$$\int_{\mathbf{r}'\in B} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\mathrm{d}V = R_0^2/2 - r^2/6, \qquad r \leq R_0, \qquad (43)$$

$$\int_{\mathbf{r}'\in B} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\mathrm{d}V = R_0^3/3r, \qquad r \geq R_0. \qquad (44)$$

From this result it follows that

$$\max_{\mathbf{r}\in R^3} \int_{\mathbf{r}'\in B} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\mathrm{d}V = \frac{R_0^2}{2}. \qquad (45)$$

With the use of formulas (42) and (45) in the frequency-domain counterpart of formula (21), the result

$$|K\hat{u}(\mathbf{r}, -i\omega)| \leq (\omega^2/c_0^2)|\hat{X}(-i\omega)||\hat{U}(-i\omega)|R_0^2/2, \qquad (46)$$

for all $\mathbf{r} \in R^3$ and all real, positive values of ω, is obtained and consequently,

$$|K^N\hat{u}(\mathbf{r}, -i\omega)| \leq [(\omega^2R_0^2/2c_0^2)|\hat{X}(-i\omega)|]^N|\hat{U}(-i\omega)|, \qquad (47)$$

for all $\mathbf{r}\epsilon R^3$ and all real, positive values of ω, which implies that the Neumann expansion of the frequency-domain counterpart of Eq. (13) is convergent if

$$(\omega^2R_0^2/2c_0^2)|\hat{X}(-i\omega)| < 1. \qquad (48)$$

In terms of the free-space wavelength $\lambda_0 = 2\pi c_0/\omega$ in the embedding, formula (48) can be rewritten as

$$(2\pi^2R_0^2/\lambda_0^2)|\hat{X}(-i\omega)| < 1. \qquad (49)$$

As formula (49) shows, the convergence condition for the frequency-domain iterative Rayleigh–Gans–Born approximation is dependent on frequency (or the wavelength) and on the size and the maximum of the absolute value of the contrast susceptibility of the scattering object.

7. CONCLUSION

The convergence criterion for the time-domain iterative Rayleigh–Gans–Born approximation to the scattering of scalar waves by an object of bounded extent embedded in a homogeneous, dispersion-free medium is investigated. The object may be arbitrarily inhomogeneous and may have arbitrary dispersion properties. The convergence condition contains only the maximum absolute value of the contrast susceptibility relaxation function of the material out of which the object is composed and is independent of the size of the object. It is shown that, for an instantaneously reacting material, the iterative Rayleigh–Gans–Born approximation is convergent if the maximum of the absolute value of the contrast of the object with respect to its embedding is less than one, while for dielectric and metal objects the procedure is unconditionally convergent if the object is embedded in vacuum.

For reference, also the frequency-domain convergence criterion for the iterative Rayleigh–Gans–Born approximation is derived. The relevant convergence criterion depends, in addition to the maximum absolute value of the contrast susceptibility, on the frequency (or the wavelength) of the incident radiation and on the size of the scattering object. From the frequency-domain result the time-domain convergence condition could, as far as the author can see, in no way be deduced.

*Permanent address, Laboratory of Electromagnetic Research, Faculty of Electrical Engineering, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands.

REFERENCES

1. D. V. Widder, *The Laplace Transform* (Princeton U. Press, Princeton, N.J., 1946), pp. 61–63.
2. D. S. Jones, *The Theory of Electromagnetism* (Pergamon, Oxford, 1964), pp. 513–515.
3. R. E. Kleinman, G. F. Roach, and P. M. van den Berg, "Convergent Born series for large refractive indices," J. Opt. Soc. Am. A **7,** 890–897 (1990).
4. P. J. Flatau and G. L. Stephens, "Light scattering by rectangular solids in the discrete-dipole approximation: a new algorithm exploiting the Block–Toeplitz structure," J. Opt. Soc. Am. A **7,** 589–600 (1990).
5. S. B. Singham and C. F. Bohren, "Light scattering by an arbitrary particle: the scattering-order formulation of the coupled dipole method," J. Opt. Soc. Am. A **5,** 1867–1872 (1988).
6. A. T. de Hoop, "Time-domain far-field scattering of plane scalar waves in the Born approximation," J. Opt. Soc. Am. A **2,** 1961–1964 (1985).
7. M. Born and E. Wolf, *The Principles of Optics,* 6th ed. (Pergamon, Oxford, 1986), pp. 84–94.
8. D. S. Jones, *The Theory of Electromagnetism* (Pergamon, Oxford, 1964), pp. 709–712.
9. M. Born and E. Wolf, *The Principles of Optics,* 6th ed. (Pergamon, Oxford, 1986), pp. 624–627.