TRANSIENT ACOUSTIC RADIATION IN A CONTINUOUSLY LAYERED FLUID - AN ANALYSIS BASED ON THE CAGNIARD METHOD

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I. INTRODUCTION

In the present paper, the problem of the transient wave propagation in a continuously layered fluid is addressed directly with the aid of the Cagniard method. The standard integral transformations that are characteristic for this method (Cagniard, 1939, 1962; De Hoop, 1960, 1961, 1988; Aki and Richards, 1980) are applied to the first-order acoustic wave equations of a fluid. The resulting system of differential equations in the depth coordinate is next transformed into a system of integral equations. These integral equations admit a solution by a Neumann iteration. Each higher-order iterate can be physically interpreted as to be generated, through continuous reflection, by the previous one. To show the generality of the method, anisotropy of the fluid in its volume density of mass is included. This type of anisotropy is encountered in the equivalent medium theory of finely discretely layered media (Schoenberg, 1984). The compressibility is a scalar. Next, the transformation back to the space-time domain is performed using the Cagniard method, in which a number of steps can be carried out analytically even for the anisotropic case.

The iterative method is shown to be convergent for any continuous and piecewise continuously differentiable depth profile in the inertia and compressibility properties of the fluid. This is contrary to the frequency-domain analysis of the problem, where the corresponding Neumann series, which is also known as the Bremmer series (Bremmer, 1939, 1949a, 1949b, 1951), can only be shown to be convergent for profiles that vary within certain, frequency dependent, bounds.

The difficulties that are met with an inversion method based on a time Fourier transformation with real frequency variable can be ascribed to the fact that with this transformation causality is lost, while with a time Laplace transformation with a real transform parameter as used by Cagniard (1939, 1962), as a crucial point, causality is automatically taken care of by restricting the transform-domain counterparts of the physical quantities to being bounded functions of the remaining space variables. Also, in the modified Cagniard method the time variable is kept real all the way through, in accordance with its physical meaning. Further, no asymptotics is needed, and only convergent expansions occur. Another aspect of the propagation of transient waves in continuously layered media is covered by the Spectral Theory of

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Transients that has been introduced by Heyman and Felsen (1984, 1987). This theory aims at a complete asymptotic expression for the total wave field in the neighborhood of the wave fronts, rather than an exact expression in terms of successively reflected wave constituents.

II. DESCRIPTION OF THE CONFIGURATION AND FORMULATION OF THE ACOUSTIC WAVE PROBLEM

Small-amplitude acoustic wave motion is considered in an unbounded inhomogeneous fluid, the properties of which vary in a single rectilinear direction in space only. This direction is taken as the vertical one. To specify position in the configuration the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with the origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$, of unit length each, are used; $\mathbf{i}_3$ points vertically downward. The subscript notation for vectors and tensors is used and the summation convention applies. Lowercase Latin subscripts are used for this purpose; they are to be assigned the values $\{1, 2, 3\}$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_m denotes differentiation with respect to x_m, ∂_t is a reserved symbol denoting differentiation with respect to t.

The acoustic properties of the (anisotropic) fluid are characterized by the tensorial volume density of mass ρ_{kr} and the scalar compressibility κ. Both are functions of x_3 only; these functions are assumed to be continuous and piecewise continuously differentiable. For $x_3 \leq x_{3;min}$ and $x_3 \geq x_{3;max}$ the medium is homogeneous, so both functions are constant in these intervals. At any x_3, the tensor ρ_{kr} is assumed to be symmetrical and positive definite and κ is positive. It is advantageous to distinguish, in the vectorial and tensorial quantities, between their horizontal and their vertical components. For the former, lowercase Greek subscripts will be used; for the latter, the subscript 3 will be written explicitly. The position of the point source is indicated by $x_{m;S}$, the receiver position by x'_m.

The acoustic wave motion in the configuration is characterized by its acoustic pressure p and its particle velocity v_r. These quantities satisfy the first-order acoustic wave equations

$$\partial_k p + \rho_{kr}\,\partial_t v_r = f_r, \tag{1}$$

$$\partial_r v_r + \kappa\,\partial_t p = q, \tag{2}$$

where f_r is the volume source densitiy of force and q is the volume source density of injection rate. Without loss of generality it is in the further analysis assumed that

$$\{f_r, q\} = \{F_{r;S}(t), Q_S(t)\}\,\delta(x_1, x_2, x_3 - x_{3;S}), \tag{3}$$

i.e., the point source is located at $x_1 = 0, x_2 = 0, x_3 = x_{3;S}$.

III. THE TRANSFORM-DOMAIN ACOUSTIC WAVE EQUATIONS AND THE WAVE-MATRIX FORMALISM

The acoustic wave equations (1) and (2) are subjected to a one-sided Laplace transformation with respect to time with real, positive transform parameter s, and a Fourier transformation with respect to the horizontal space coordinates with real transform parameters $s\alpha_1$ and $s\alpha_2$. For the acoustic pressure the two transformations are

$$\hat{p}(x_m, s) = \int_0^\infty \exp(-st)\,p(x_m, t)\,dt, \tag{4}$$

$$\tilde{p}(i\alpha_\mu, x_3, s) = \int_{x_\mu \in \mathbb{R}^2} \exp(is\alpha_\mu x_\mu)\,\hat{p}(x_m, s)\,dx_1 dx_2, \tag{5}$$

respectively. The extra factor of s in the spatial Fourier-transform parameters has been included for later convenience. In view of this, the transformation inverse to Eq. (5) is given by

$$\hat{p}(x_m, s) = \left(\frac{s}{2\pi}\right)^2 \int_{\alpha_\mu \in \mathbb{R}^2} \exp(-is\alpha_\mu x_\mu)\, \tilde{p}(i\alpha_\mu, x_3, s)\, d\alpha_1 d\alpha_2. \tag{6}$$

Under these transformations Eqs. (1) and (2) transform into

$$-is\alpha_\nu \tilde{p} + s\rho_{\nu r}\tilde{v}_r = \tilde{f}_\nu, \tag{7}$$

$$\partial_3 \tilde{p} + s\rho_{3r}\tilde{v}_r = \tilde{f}_3, \tag{8}$$

$$-is\alpha_\eta \tilde{v}_\eta + \partial_3 \tilde{v}_3 + s\kappa\tilde{p} = \tilde{q}. \tag{9}$$

Upon eliminating the horizontal components $\tilde{v}_\eta$ of the particle velocity from these equations, a system of two ordinary differential equations results with x_3 as independent variable and $\tilde{p}$ and $\tilde{v}_3$ as dependent variables. Let $[\tilde{F}]$ denote the acoustic field matrix, $[\tilde{A}]$ the acoustic system's matrix, and $[\tilde{N}]$ the notional source matrix, then the transform-domain matrix differential equation is

$$\partial_3 \tilde{F}_I + s\tilde{A}_{IJ}\tilde{F}_J = \tilde{N}_I, \tag{10}$$

in which the summation convention again applies, and the elements of the acoustic field matrix are given by

$$\tilde{F}_1 = \tilde{p}, \qquad \tilde{F}_2 = \tilde{v}_3, \tag{11}$$

the elements of the acoustic system's matrix by

$$\begin{aligned} \tilde{A}_{11} &= \rho_{3\eta}\, [\rho_{\parallel}^{-1}]_{\eta\nu}\, i\alpha_\nu, & \tilde{A}_{12} &= \rho_{33} - \rho_{3\eta}\, [\rho_{\parallel}^{-1}]_{\eta\nu}\, \rho_{\nu 3}, \\ \tilde{A}_{21} &= \kappa - i\alpha_\eta\, [\rho_{\parallel}^{-1}]_{\eta\nu}\, i\alpha_\nu, & \tilde{A}_{22} &= i\alpha_\eta\, [\rho_{\parallel}^{-1}]_{\eta\nu}\, \rho_{\nu 3}, \end{aligned} \tag{12}$$

and the elements of the notional source matrix by

$$\tilde{N}_1 = -\rho_{3\eta}\, [\rho_{\parallel}^{-1}]_{\eta\nu}\, \tilde{f}_\nu + \tilde{f}_3, \qquad \tilde{N}_2 = i\alpha_\eta\, [\rho_{\parallel}^{-1}]_{\eta\nu}\, \tilde{f}_\nu + \tilde{q}. \tag{13}$$

Here, $[\rho_{\parallel}^{-1}]_{\eta\nu}$ is the inverse of $\rho_{\nu\eta}$. Note that $[\tilde{A}]$ is independent of s.

Via an appropriate linear transformation to be carried out on the acoustic field matrix, a wave-matrix formalism will be arrived at from which the interaction between up- and down-going waves in a region of inhomogeneity will be manifest. The relevant linear transformation is written as (cf. Chapman (1974) for the isotropic case)

$$\tilde{F}_I = \tilde{L}_{IJ}\tilde{W}_J, \tag{14}$$

where $[\tilde{W}]$ is the wave matrix and the matrix $[\tilde{L}]$ is to be chosen appropriately. On the assumption that the inverse $[\tilde{L}^{-1}]$ of $[\tilde{L}]$ exists, substitution of Eq. (14) into Eq. (10) yields

$$\partial_3 \tilde{W}_I + s\tilde{\Lambda}_{IJ}\tilde{W}_J = [\tilde{L}^{-1}]_{IJ}\tilde{N}_J - [\tilde{L}^{-1}]_{IJ}\, [\partial_3 \tilde{L}]_{JK}\, \tilde{W}_K, \tag{15}$$

where $[\tilde{\Lambda}] = [\tilde{L}^{-1}][\tilde{A}][\tilde{L}]$. Equation (15) indeed expresses the traveling-wave structure of the up- and downgoing waves provided that $[\tilde{\Lambda}]$ is a diagonal matrix. From the observation that $[\tilde{A}][\tilde{L}] = [\tilde{L}][\tilde{\Lambda}]$ it follows that $[\tilde{\Lambda}]$ is diagonal if $[\tilde{L}]$ consists of the eigencolumns of $[\tilde{A}]$; $[\tilde{\Lambda}]$ then has the eigenvalues of $[\tilde{A}]$ as its (diagonal) elements.

The elements of $[\tilde{L}]$ can be expressed in terms of the vertical acoustic wave admittance $Y = (\tilde{A}_{21}/\tilde{A}_{12})^{1/2}$, a quantity that, for $\alpha_\mu \in \mathbb{R}^2$, is real and positive. Note that this wave admittance is the same for up- and downgoing waves even in the case of an anisotropic fluid. The relevant expressions are

$$\tilde{L}_{11} = (2Y)^{-1/2}, \quad \tilde{L}_{12} = (2Y)^{-1/2}, \quad \tilde{L}_{21} = -(Y/2)^{1/2}, \quad \tilde{L}_{22} = (Y/2)^{1/2}. \tag{16}$$

With this, the coupling matrix becomes $[\tilde{L}^{-1}][\partial_3\tilde{L}] = R\,[C]$, where $C_{11} = C_{22} = 0$, $C_{12} = C_{21} = -1$ and $R = \partial_3 Y/2Y$ is the local reflection coefficient. Using these results and writing the elements of the wave matrix $[\tilde{W}]$ as $\tilde{W}_1 = \tilde{W}^-$ and $\tilde{W}_2 = \tilde{W}^+$, where $\tilde{W}^-$ is the local amplitude of the upgoing wave and $\tilde{W}^+$ is the local amplitude of the downgoing wave, the system of differential equations Eq. (15) leads to

$$\partial_3\tilde{W}^- + s\gamma^-\tilde{W}^- = \tilde{X}^- + R\,\tilde{W}^+, \tag{17}$$

$$\partial_3\tilde{W}^+ + s\gamma^+\tilde{W}^+ = \tilde{X}^+ + R\,\tilde{W}^-, \tag{18}$$

where

$$\gamma^- = \tfrac{1}{2}\,(\tilde{A}_{11} + \tilde{A}_{22}) - (\tilde{A}_{12}\tilde{A}_{21})^{1/2}, \qquad \gamma^+ = \tfrac{1}{2}\,(\tilde{A}_{11} + \tilde{A}_{22}) + (\tilde{A}_{12}\tilde{A}_{21})^{1/2}, \tag{19}$$

are the vertical slownesses of the up- and downgoing waves, respectively, while

$$\tilde{X}^- = [\tilde{L}^{-1}\tilde{N}]_1, \qquad \tilde{X}^+ = [\tilde{L}^{-1}\tilde{N}]_2. \tag{20}$$

In Section IV, the coupled wave propagation problem is recast in an integral-equation formulation that is equivalent to Eqs. (17) and (18). Next, these integral equations are solved iteratively and for the transform-domain acoustic pressure and particle velocity series expansions are obtained. The zero-order term in this expansion is representative for the direct wave generated by the source; the subsequent terms are representative for the waves that are successively reflected at the inhomogeneity levels.

IV. INTEGRAL-EQUATION FORMULATION AND ITERATIVE SOLUTION OF THE TRANSFORM- DOMAIN COUPLED WAVE PROBLEM

The integral-equation formulation of the transform-domain coupled wave problem follows from Eqs. (17) and (18) upon introducing appropriate one- sided Green's functions for the differential operators occurring at the left-hand side of these equations. The result can be written as

$$\tilde{W}^- = \tilde{W}_0^- + K^-\tilde{W}^+, \qquad \tilde{W}^+ = \tilde{W}_0^+ + K^+\tilde{W}^-, \tag{21}$$

where

$$\tilde{W}_0^-(x_3') = \int_{x_3'}^{\infty} \tilde{G}^-(x_3', x_3)\,\tilde{X}^-(x_3)\,dx_3, \tag{22}$$

$$\tilde{W}_0^+(x_3') = \int_{-\infty}^{x_3'} \tilde{G}^+(x_3', x_3)\,\tilde{X}^+(x_3)\,dx_3, \tag{23}$$

$$[K^-\tilde{W}^+](x_3') = \int_{x_3'}^{\infty} \tilde{G}^-(x_3', x_3)\,R(x_3)\,\tilde{W}^+(x_3)\,dx_3, \tag{24}$$

$$[K^+\tilde{W}^-](x_3') = \int_{-\infty}^{x_3'} \tilde{G}^+(x_3', x_3)\,R(x_3)\,\tilde{W}^-(x_3)\,dx_3, \tag{25}$$

$$\tilde{G}^-(x_3', x_3) = -H(x_3 - x_3')\,\exp\left[-s\int_{x_3}^{x_3'} \gamma^-(\varsigma)\,d\varsigma\right], \tag{26}$$

$$\tilde{G}^+(x_3', x_3) = H(x_3' - x_3)\,\exp\left[-s\int_{x_3}^{x_3'} \gamma^+(\varsigma)\,d\varsigma\right]. \tag{27}$$

The equations in (21) suggest the possibility of an iterative solution of the Neumann type by repeated substitution of the first equation in the second and vice versa. Since s is real and positive,

$$|K^-\tilde{W}| \leq A(s)\,M\int_{x_3'}^{x_{3;max}} \exp[-s\gamma(x_3 - x_3')]\,dx_3 \leq \frac{A(s)\,M}{s\gamma}, \tag{28}$$

$$|K^+\tilde{W}| \leq A(s)\,M\int_{x_{3;min}}^{x_3'} \exp[-s\gamma(x_3 - x_3')]\,dx_3 \leq \frac{A(s)\,M}{s\gamma}, \tag{29}$$

where

$$\left.\begin{array}{lcl} A(s) & = & \max\{|\tilde{W}^-|, |\tilde{W}^+|\}, \\ M & = & \max\{|R|\}, \\ \gamma & = & \max\{\mathrm{Re}(-\gamma^-), \mathrm{Re}(\gamma^+)\} > 0, \end{array}\right\} x_3 \in [x_{3;min}, x_{3;max}]. \tag{30}$$

The procedure is convergent if the real, positive time Laplace-transform parameter is taken in the semi-infinite interval $s > M/\gamma$, which can always be done since M is independent of s, positive and bounded, while γ is independent of s, positive and bounded away from zero. Upon summing the remaining convergent infinite series

$$\tilde{W}^-(x_3') = \sum_{n=0}^{\infty} \tilde{W}_n^-(x_3'), \qquad \tilde{W}^+(x_3') = \sum_{n=0}^{\infty} \tilde{W}_n^+(x_3'), \tag{31}$$

with

$$\begin{array}{lll} \tilde{W}_0^- = \tilde{G}^-(x_3', x_{3;S})\,\tilde{X}^-, & \tilde{W}_0^+ = \tilde{G}^+(x_3', x_{3;S})\,\tilde{X}^+, & \\ \tilde{W}_{n+1}^- = [K^-\tilde{W}_n^+](x_3'), & \tilde{W}_{n+1}^+ = [K^+\tilde{W}_n^-](x_3'), & (n = 0, 1, 2, \ldots), \end{array} \tag{32}$$

the inequality

$$A(s) \le \frac{A_0(s)}{1 - M/s\gamma} \tag{33}$$

results. In the latter, $A_0(s) = \max\{|\tilde{W}_0^-|, |\tilde{W}_0^+|\}$ on $[x_{3;min}, x_{3;max}]$. Hence, $A(s)$ is a bounded function in the interval $s > M/\gamma$ if $A(s)$ is so. The latter condition is satisfied if the source is taken to have at most a delta function (Dirac distribution) time dependence as Eqs. (13), (20), (32), and the property $\{|\tilde{G}^-|, |\tilde{G}^+|\} \le 1$ (cf. Eqs. (26) and (27)), show.

Note that the kind of reasoning here employed cannot be used when a Fourier transformation with respect to time, with real angular frequency transform parameter, is carried out and the inversion back to the time domain is based on the Fourier inversion integral. The latter employs, in fact, imaginary values of the time Laplace-transform parameter s, to which values Lerch's theorem does not apply, and for which, most importantly, the estimates of Eqs. (28) and (29) are lost.

The equations under (31) entail, via Eq. (14), the following transform-domain representations for the acoustic pressure and the vertical component of the particle velocity:

$$\tilde{p}(x_3') = [2Y(x_3')]^{-1/2} \left[\sum_{n=0}^{\infty} \tilde{W}_n^-(x_3') + \sum_{n=0}^{\infty} \tilde{W}_n^+(x_3'),\right] \tag{34}$$

$$\tilde{v}_3(x_3') = [Y(x_3')/2]^{1/2} \left[-\sum_{n=0}^{\infty} \tilde{W}_n^-(x_3') + \sum_{n=0}^{\infty} \tilde{W}_n^+(x_3'),\right] \tag{35}$$

A typical term of order n in the right-hand sides of Eqs. (34) and (35) now consists of an n-fold repeated integration in the vertical direction, the limits in which are the successive interaction levels of multiple reflection. In them, the exponential functions, that contain in their arguments additional integrations from the source level to the receiver level, are gathered to a single one. The factor that remains in the n-fold integral is the product of one of the source signatures that only depend on s, an s-independent coupling coefficient that describes the coupling of the source to the wave, the s-independent reflection coefficients R at the successive interaction levels of multiple reflection, and an s-independent coupling coefficient that describes the coupling of the wave to the receiver. For our further analysis, such a typical term is written as

$$\tilde{U} = \hat{S}(s)\,\tilde{\Pi}(i\alpha_\mu)\,\exp\left[-s\int_{Z^-} \gamma^-(i\alpha_\mu, \varsigma)\,d\varsigma - s\int_{Z^+} \gamma^+(i\alpha_\mu, \varsigma)\,d\varsigma\right]. \tag{36}$$

Here, $\hat{S}$ stands for the source signature, $\tilde{\Pi}$ for the product of coupling coefficients and reflection coefficients, Z^- for the accumulated vertical travel path traversed by the upgoing waves and

Z^+ for the accumulated vertical travel path traversed by the downgoing waves. Both Z^- and Z^+ may, in part or entirely, be multiply covered. In accordance with the property $\mathrm{Re}\{\gamma^-\} < 0$ and $\mathrm{Re}\{\gamma^+\} > 0$, the vertical travel paths can be written as $Z^- = \{\varsigma \in \mathbb{R} | x_3 < \varsigma < x_3^+\}$ for some x_3^+ and $Z^+ = \{\varsigma \in \mathbb{R} | x_3^- < \varsigma < x_3\}$ for some x_3^-, and consequently the signed vertical path lengths satisfy the inequalities $\int_{Z^-} d\varsigma \leq 0$ and $\int_{Z^+} d\varsigma \geq 0$, respectively. Expressions of the type (36) will, just as in the case of discretely layered media, be denoted as generalized-ray constituents (Wiggins and Helmberger, 1974). Their transformation back to the space-time domain with the aid of the modified Cagniard method will be described in section V.

It is to be noted that for obtaining the correct early time asymptotic expressions for the total wave amplitude at and immediately behind the wave front, all contributons that travel in a particular direction (up or down) must be added, since they all arrive at the same instant as the direct wave, be it with decreasing initial amplitudes. This kind of asymptotics is investigated in Singh and Chapman (1988) and Heyman and Felsen (1984). Our decomposition yields expressions for the successively reflected waves *at all times*; we have not been able yet to sum these contributions analytically, for example, at the wave front.

V. THE MODIFIED CAGNIARD METHOD APPLIED TO A GENERALIZED-RAY CONSTITUENT

The transformation back to the space-time domain of the generalized-ray constituent in Eq. (36) will now be discussed. Using Eq. (6), the s-domain expression corresponding to Eq. (36) is given by

$$\hat{U} = \left(\frac{s}{2\pi}\right)^2 \hat{S}(s) \int_{\alpha_\mu \in \mathbb{R}^2} \tilde{\Pi}(i\alpha_\mu) \exp\left[-s\left(i\alpha_\mu x_\mu + \int_{Z^-} \gamma^- d\varsigma + \int_{Z^+} \gamma^+ d\varsigma\right)\right] d\alpha_1 d\alpha_2. \quad (37)$$

For the present case of an anisotropic medium the most appropriate version of the Cagniard method seems to be the one where the variables of integration $i\alpha_\mu$ are replaced by

$$i\alpha_1 = p\cos(\theta + \psi), \qquad i\alpha_2 = p\sin(\theta + \psi), \quad (38)$$

where $\theta = \arctan(x_2'/x_1')$, p is positive imaginary and $0 \leq \psi \leq 2\pi$. Using the relevant symmetry properties, these transformations lead to

$$\begin{aligned} \hat{U} \quad = \quad & -\left(\frac{s}{2\pi}\right)^2 \hat{S}(s) \int_{-\pi/2}^{\pi/2} d\psi \, \mathrm{Re}\left\{\int_0^{i\infty} \tilde{\Pi}(p,\psi)\right. \\ & \left.\exp\left[-s\left(pd\cos\psi + \int_{Z^-} \gamma^- d\varsigma + \int_{Z^+} \gamma^+ d\varsigma\right)\right] p\,dp\right\}. \end{aligned} \quad (39)$$

The essential feature of the modified Cagniard method consists of replacing the integration with respect to p along the positive imaginary axis, through continuous deformation, by one along a modified Cagniard contour that follows from

$$\tau = pd\cos\psi + \int_{Z^-} \gamma^-(p,\psi,\varsigma)\, d\varsigma + \int_{Z^+} \gamma^+(p,\psi,\varsigma)\, d\varsigma = \text{Real}. \quad (40)$$

The admissibility of the contour deformation rests on the applicability of Cauchy's theorem and Jordan's lemma. The latter only allows for a deformation into the right half of the complex p-plane. The only singularities of the integrand are the branch points due to the occurrence of $\tilde{A}_{21}^{1/2}$ in the expressions for $\tilde{\Pi}$, γ^+ and γ^-, i.e., the zeros of $\tilde{A}_{21}$. These zeros can easily be proved to reside on the real p-axis. From Eq. (40) it follows that the part of the real p-axis from the origin to the branch point nearest to the origin, as well as the complex path that satisfies the equation for $\tau \to \infty$, are candidates for modified Cagniard contours. As to the complex part of the modified Cagniard contours two possibilities exist: (a) it intersects the real p-axis at a regular point of the left-hand side of Eq. (40), in which case $\partial\tau/\partial p = 0$ at that point; (b) the modified Cagniard contour touches the real p-axis at the branch point nearest to the origin. These two cases are shown in Fig. 1.

Which of the two cases applies depends on the vertical profiles of the constitutive parameters and the mutual positions of the source, the reflection levels and the receiver. In the latter case the modified Cagniard contour must be supplemented by a circular arc around the relevant branch point since this branch point can be a simple pole of R (in this case the corresponding generalized ray is the time-domain conterpart of a turning ray in the asymptotic frequency-domain approach). Along the modified Cagniard contour τ is introduced as the variable of integration and the integrations with respect to τ and ψ are interchanged. The final result can be written as

$$\hat{U} = s^2\,\hat{S}(s)\,\hat{G}(Z^-, Z^+, s), \tag{41}$$

in which

$$\hat{G}(Z^-, Z^+, s) = \frac{-1}{2\pi^2}\int_{T_0}^{\infty} d\tau\, \exp(-s\tau) \int_{\Psi_1(Z^-,Z^+,\tau)}^{\Psi_2(Z^-,Z^+,\tau)} \mathrm{Re}\{\tilde{\Pi}(p,\psi)\, p\, \frac{\partial p}{\partial \tau}\}\, d\psi. \tag{42}$$

For details of the procedure we refer to Van der Hijden (1987), Sections 4.6, 6.4 and 7.4. The time-domain counterpart of Eq. (41) is

$$U = \partial_t^2 \int_{T_0}^{t} S(t-\tau)\, G(Z^-, Z^+, \tau)\, d\tau, \tag{43}$$

where $G(Z^-, Z^+, t)$ can be recognized form Eq. (42) as

$$G(Z^-, Z^+, t) = \frac{-1}{2\pi^2} H(t - T_0) \int_{\Psi_1(Z^-,Z^+,t)}^{\Psi_2(Z^-,Z^+,t)} \mathrm{Re}\{\tilde{\Pi}(p,\psi)\, p\, \frac{\partial p}{\partial \tau}\}\, d\psi. \tag{44}$$

Obviously, T_0 is the arrival time of the generalized ray constituent under consideration.

VI. THE CASE OF AN ISOTROPIC FLUID

For an isotropic fluid, $\rho_{kr} = \rho\delta_{kr}$, where ρ is the scalar volume density of mass. In this case $\gamma^- = -\gamma$ and $\gamma^+ = \gamma$, where $\gamma = (c^{-2} + \alpha_\eta\alpha_\eta)^{1/2}$. In the latter expression $c = (\rho\kappa)^{-1/2}$ is the acoustic wave speed. The vertical acoustic wave admittance becomes $Y = \gamma/\rho$.

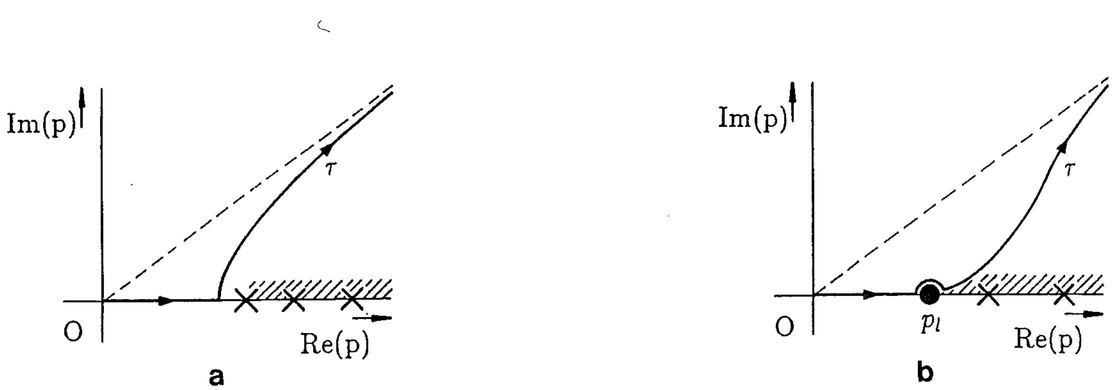

Fig.1. (a) Modified Cagniard countour with complex part intersecting the real p-axis in a point where $\partial\tau/\partial p = 0$; (b) Modified Cagniard contour with complex part touching the real p-axis in the leftmost branch point.

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