Huygens' Principle 1690–1990: Theory and Applications
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Reciprocity, causality, and Huygens' principle in electromagnetic wave theory

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Abstract

Field reciprocity theorems can be considered to be the most basic relations that exist in the theory of classical fields and waves. In them, two field states occur that could exist in one and the same time-invariant subdomain of configuration space. A specific interaction quantity, when integrated over the boundary of the relevant domain, is then related to an integral over the interior containing the contrast in medium properties and the source distributions associated with the two states. Upon choosing for the two states two physical ones, a number of interesting reciprocity relations can be derived. Another aspect of the field reciprocity theorems is that with the aid of them the different field problems, such as the direct source problem, the inverse source problem, the direct scattering problem, and the inverse scattering problem, can logically be classified, together with their solution schemes.

The time-convolution reciprocity theorem for electromagnetic fields will be discussed, together with a number of its consequences. In particular, Oseen's extinction theorem will receive attention, and its relation to Huygens' principle will be elucidated.

1 Introduction

The present contribution aims to show how Huygens' Principle [1] for electromagnetic wave fields follows from the electromagnetic reciprocity theorem of the time-convolution type and the principle of causality of the generated wave motion. This line of approach enables one to include into the analysis the action of media that are inhomogeneous and anisotropic, and that show arbitrary dispersion (or relaxation) effects as well. Thus, an extension is reached over the classical treatments that apply to homogeneous, isotropic and instantaneously reacting media. For the latter, Baker and Copson [2] is the standard monograph on the subject. The only remaining restriction is that the media in the configuration are linear, time invariant and spatially locally reacting in their electromagnetic behavior. For such media, the time-convolution electromagnetic reciprocity theorem has been extensively discussed by De Hoop in an earlier paper [3]; this paper also discusses the related reciprocity theorem of the time-correlation type and gives a survey of the literature on the subject.

A field reciprocity theorem interrelates, in a specific manner, the field quantities that characterize two admissible states that could occur in one and the same domain in space–time. In electromagnetic theory, Lorentz [4] is commonly credited as the first to derive a reciprocity theorem. Later developments include contributions by Rumsey [5], Welch [6,7], Geurst [8], Ru Shao Cheo [9], Kong [10], and Bojarski [11], while applications to scattering configurations were given by De Hoop [12], to radiation by apertures by Van Bladel [13], to multiport antennas by De Hoop [14], and to inverse scattering by Fisher and Langenberg [15].

For the present analysis we need the time-convolution reciprocity theorem for electromagnetic fields in time-invariant configurations that are linear and locally reacting in their electromagnetic behavior. As regards the space–time geometry in which the two admissible states that occur in the reciprocity theorem are present, this implies that this geometry is the Cartesian product $\mathcal{D} \times \mathcal{R}$ of a time-invariant spatial domain $\mathcal{D} \subset \mathcal{R}^3$ and the real time axis $\mathcal{R}$. Further, the constitutive parameters of the media present in the two states are time invariant and independent of the field values. No further restrictions are imposed. The position of observation in $\mathcal{R}^3$ is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of unit length each. In the indicated order the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors in $\mathcal{R}^3$ is employed and the summation convention applies. The corresponding lowercase Latin subscripts are to be assigned the values {1, 2, 3}. Whenever appropriate, the position vector will be denoted by $\boldsymbol{x} = x_p \boldsymbol{i}_p$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_p denotes differentiation with respect to x_p, while ∂_t is a reserved symbol that denotes differentiation with respect to t. The International System of Units (SI) is used.

The reciprocity theorem is first given for a bounded domain $\mathcal{D}$. In the analysis also the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ occurs, as well as the complement $\mathcal{D}'$ of the union of $\mathcal{D}$ and $\partial\mathcal{D}$ in $\mathcal{R}^3$. The unit vector along the normal to $\partial\mathcal{D}$ is denoted by ν_m; it points away from $\mathcal{D}$. An application of the theorem to unbounded domains (in particular, to the entire $\mathcal{R}^3$) will always be handled as a limiting case where the theorem is first applied to the ball interior to the sphere $\mathcal{S}_\Delta$ of radius Δ and center at the origin of the chosen coordinate system, and next the limit $\Delta \to \infty$ is taken.

By choosing, in the reciprocity theorem, for one of the two states appropriate point-source solutions (Green's functions), source-type integral representations for the electromagnetic field quantities are derived that causally relate the generated field to the action of its sources. Further, through the introduction of appropriate contrast volume source densities of electric and magnetic current, the problem of the (direct) scattering by a contrasting domain of finite extent, present in an unbounded embedding, can be reduced to solving a system of integral equations of the second kind, by invoking the condition of field reproduction in the interior of the scatterer. Next, it is briefly indicated how the reciprocity theorem of the time-convolution type leads, in an elegant manner, to the formulation of inverse source and inverse scattering problems.

Subsequently, the reciprocity theorem is applied to a, bounded or unbounded, subdomain of space and, again, one of the states occurring in it is associated with causal point-source solutions. Then, the theorem gives rise to contributions from the volume sources present in the domain of application and equivalent surface sources on the boundary surface of that domain. For points of observation in the domain of application, the sum of the two types of contributions reproduce the values of the field, while in the exterior domain the two types of contributions cancel each other. The former property lies at the basis of the Kirchhoff theory of diffraction [16] (see also Kottler [17,18] and Stratton and Chu [19]). The latter property is known as Oseen's extinction theorem [20]. Both properties together can be regarded as the mathematical formulation of Huygens' Principle (Baker and Copson [2]).

2 Some properties of the time convolution of space–time functions

In this section we present the properties of the time convolution of space–time functions as far as they are needed in the derivation of the reciprocity theorem. Let $f_1 = f_1(\boldsymbol{x}, t)$ and $f_2 = f_2(\boldsymbol{x}, t)$ be two transient space–time functions. By this we mean that the functions are absolutely integrable on the entire $t \in \mathcal{R}$. Then, time convolution of f_1 and f_2 is defined as

$$\begin{aligned} C(f_1, f_2; \boldsymbol{x}, \tau) &= \int_{t\in\mathcal{R}} f_1(\boldsymbol{x}, t)\, f_2(\boldsymbol{x}, \tau - t)\, \mathrm{d}t \\ &= \int_{t\in\mathcal{R}} f_1(\boldsymbol{x}, \tau - t)\, f_2(\boldsymbol{x}, t)\, \mathrm{d}t \\ &= C(f_2, f_1; \boldsymbol{x}, \tau). \end{aligned} \tag{1}$$

Equation (1) shows that time convolution is symmetrical in f_1 and f_2. For the time derivative of time convolution the rules

$$\partial_\tau C(f_1, f_2; \boldsymbol{x}, \tau) = C(f_1, \partial_t f_2; \boldsymbol{x}, \tau) = C(\partial_t f_1, f_2; \boldsymbol{x}, \tau) \tag{2}$$

apply. For the incorporation of dispersive media in the reciprocity theorem we also need the time convolution of three space–time functions. For this, either of the definitions

$$C(f_1, f_2, f_3; \boldsymbol{x}, \tau) = C(f_1, C(f_2, f_3); \boldsymbol{x}, \tau) = C(C(f_1, f_2), f_3; \boldsymbol{x}, \tau) \tag{3}$$

holds.

3 Properties of the electromagnetic field in the configuration

In each subdomain of the configuration where the electromagnetic properties vary continuously with position the electromagnetic field vectors are continuously differentiable and satisfy Maxwell's equations

$$-\epsilon_{k,m,p}\partial_m H_p + \partial_t D_k = -J_k, \tag{4}$$

$$\epsilon_{j,m,r}\partial_m E_r + \partial_t B_j = -K_j, \tag{5}$$

where

$$\begin{array}{lcl} E_r & : & \text{electric field strength [Vm}^{-1}], \\ H_p & : & \text{magnetic field strength [Am}^{-1}], \\ D_k & : & \text{electric flux density [Cm}^{-2}], \\ B_j & : & \text{magnetic flux density [T]}, \\ J_k & : & \text{volume source density of electric current [Am}^{-2}], \\ K_j & : & \text{volume source density of magnetic current [Vm}^{-2}], \end{array}$$

$\epsilon_{k,m,p}$ is the Levi-Civita tensor: $\epsilon_{k,m,p} = +1$ if $\{k, m, p\}$ is an even permutation of $\{1,2,3\}$, $\epsilon_{k,m,p} = -1$ if $\{k, m, p\}$ is an odd permutation of $\{1,2,3\}$, and $\epsilon_{k,m,p} = 0$ in all other cases. Equations (4) and (5) are supplemented by the constitutive relations. For a linear, time-invariant, locally reacting medium these are

$$D_k(\boldsymbol{x},t) = \varepsilon_0 \int_{\tau=-\infty}^{\infty} \chi_{k,r}(\boldsymbol{x},\tau)\, E_r(\boldsymbol{x},t-\tau)\,\mathrm{d}\tau, \tag{6}$$

$$B_j(\boldsymbol{x},t) = \mu_0 \int_{\tau=-\infty}^{\infty} \kappa_{j,p}(\boldsymbol{x},\tau)\, H_p(\boldsymbol{x},t-\tau)\,\mathrm{d}\tau, \tag{7}$$

where

$$\begin{array}{lcl} \varepsilon_0 & : & \text{permittivity in vacuum [Fm}^{-1}], \\ \chi_{k,r} & : & \text{electric relaxation function [s}^{-1}], \\ \mu_0 & : & \text{permeability in vacuum [Hm}^{-1}], \\ \kappa_{j,p} & : & \text{magnetic relaxation function [s}^{-1}]. \end{array}$$

In SI, we have $\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1} and $\varepsilon_0 = 1/\mu_0 c_0^2$, with $c_0 = 299792458$ ms^{-1}. Using the notation of eq. (1), eqs. (6) and (7) can be rewritten as

$$D_k(\boldsymbol{x},t) = \varepsilon_0 C(\chi_{k,r}, E_r; \boldsymbol{x}, t), \tag{8}$$

$$B_j(\boldsymbol{x},t) = \mu_0 C(\kappa_{j,p}, H_p; \boldsymbol{x}, t), \tag{9}$$

respectively. In eqs. (8) and (9), inhomogeneity, anisotropy and dispersion of the medium are included.

If $\{\chi_{k,r}, \kappa_{j,p}\}(\boldsymbol{x},\tau) = 0$ when $\tau < 0$, the medium at $\boldsymbol{x}$ is causal. If

$$\varepsilon_0 \chi_{k,r}(\boldsymbol{x},\tau) = \varepsilon_{k,r}(\boldsymbol{x})\,\delta(\tau), \tag{10}$$

$$\mu_0 \kappa_{j,p}(\boldsymbol{x},\tau) = \mu_{j,p}(\boldsymbol{x})\,\delta(\tau), \tag{11}$$

where $\delta(\tau)$ is the unit impulse (Dirac distribution), the medium is instantaneously reacting, and $\varepsilon_{k,r}$ and $\mu_{j,p}$ are its permittivity and permeability, respectively. If $\{\chi_{k,r}, \kappa_{j,p}\}(\boldsymbol{x},\tau) = 0$ when $\tau > 0$, the medium is anti-causal or effectual. From an energy point of view, a medium for which $\{\chi_{k,r}, \kappa_{j,p}\}(\boldsymbol{x},\tau) \neq 0$ when $\tau > 0$ is dissipative, a medium for which eqs. (10) and (11) hold is lossless, and a medium for which $\{\chi_{k,r}, \kappa_{j,p}\}(\boldsymbol{x},\tau) \neq 0$ when $\tau < 0$ is active. A medium that is either dissipative or lossless is also denoted as passive. For the reciprocity theorem itself no specific type of relaxation function is presupposed; in our applications only causal media are considered.

It is assumed that $\chi_{k,r}$ and $\kappa_{j,p}$ are piecewise continuous functions of position. At an interface between two different media they jump by finite amounts. Across such an interface the tangential components of the electric and the magnetic field strengths are continuous. If an impenetrable object is present, either the tangential components of the electric or the tangential components of the magnetic field strength have zero values at its boundary. Through the relevant boundary conditions, the presence of either interfaces or impenetrable objects is accounted for.

The two states that occur in the reciprocity theorem are denoted by the superscripts A and B, respectively. It is noted that the two states can apply to different source distributions and to different media, but they must be present in one and the same domain in space–time.

4 Reciprocity theorem of the time-convolution type

The reciprocity theorem of the time-convolution type follows upon considering the interaction quantity $\epsilon_{m,k,j}[C(E_k^A, H_j^B; \boldsymbol{x}, t) - C(E_k^B, H_j^A; \boldsymbol{x}, t)]$. Using eqs. (4) and (5) for each of the two states, we obtain

$$\begin{aligned}&\epsilon_{m,k,j}\partial_m C(E_k^A, H_j^B; \boldsymbol{x}, t)\\ &\quad = -C(\partial_t B_j^A + K_j^A, H_j^B; \boldsymbol{x}, t) - C(E_k^A, \partial_t D_k^B + J_k^B; \boldsymbol{x}, t),\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\epsilon_{m,k,j}\partial_m C(E_k^B, H_j^A; \boldsymbol{x}, t)\\ &\quad = -C(\partial_t B_j^B + K_j^B, H_j^A; \boldsymbol{x}, t) - C(E_k^B, \partial_t D_k^A + J_k^A; \boldsymbol{x}, t).\end{aligned} \tag{13}$$

Now, in view of eqs. (8) and (9), we have

$$\begin{aligned}&C(\partial_t B_j^B, H_j^A; \boldsymbol{x}, t) - C(\partial_t B_j^A, H_j^B; \boldsymbol{x}, t)\\ &\quad = \mu_0 \partial_t C(\kappa_{j,p}^B - \kappa_{p,j}^A, H_j^A, H_p^B; \boldsymbol{x}, t)\end{aligned} \tag{14}$$

and

$$C(E_k^B, \partial_t D_k^A; \boldsymbol{x}, t) - C(E_k^A, \partial_t D_k^B; \boldsymbol{x}, t)$$
$$= \varepsilon_0 \partial_t C(\chi_{r,k}^A - \chi_{k,r}^B, E_k^A, E_r^B; \boldsymbol{x}, t), \tag{15}$$

where eq. (2) has been used. Subtracting eq. (13) from eq. (12) and employing eqs. (14) and (15), we arrive at

$$\epsilon_{m,k,j}\partial_m[C(E_k^A, H_j^B; \boldsymbol{x}, t) - C(E_k^B, H_j^A; \boldsymbol{x}, t)]$$
$$= \mu_0 \partial_t C(\kappa_{j,p}^B - \kappa_{p,j}^A, H_j^A, H_p^B; \boldsymbol{x}, t)$$
$$- \varepsilon_0 \partial_t C(\chi_{k,r}^B - \chi_{r,k}^A, E_k^A, E_r^B; \boldsymbol{x}, t)$$
$$- C(K_p^A, H_p^B; \boldsymbol{x}, t) - C(E_k^A, J_k^B; \boldsymbol{x}, t)$$
$$+ C(K_j^B, H_j^A; \boldsymbol{x}, t) + C(E_r^B, J_r^A; \boldsymbol{x}, t). \tag{16}$$

Equation (16) is the local form of the time-convolution reciprocity theorem. The first two terms on the right-hand side are representative of the differences in the properties of the media present in the two states; they vanish at those locations where $\chi_{r,k}^A(\boldsymbol{x}, \tau) = \chi_{k,r}^B(\boldsymbol{x}, \tau)$ and $\kappa_{p,j}^A(\boldsymbol{x}, \tau) = \kappa_{j,p}^B(\boldsymbol{x}, \tau)$ for all $\tau \in \mathcal{R}$. In case the latter conditions hold, the two media are denoted as each other's adjoints. Note in this respect that the adjoint of a causal (effectual) medium is causal (effectual), too. The last four terms on the right-hand side of (16) are associated with the source distributions; they vanish at those locations where no sources are present. Upon integrating eq. (16) over the subdomains of $\mathcal{D}$ where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}} \epsilon_{m,k,j}\nu_m[C(E_k^A, H_j^B; \boldsymbol{x}, t) - C(E_k^B, H_j^A; \boldsymbol{x}, t)]\,\mathrm{d}A$$
$$= \int_{\boldsymbol{x}\in\mathcal{D}} [\mu_0 \partial_t C(\kappa_{j,p}^B - \kappa_{p,j}^A, H_j^A, H_p^B; \boldsymbol{x}, t)$$
$$- \varepsilon_0 \partial_t C(\chi_{k,r}^B - \chi_{r,k}^A, E_k^A, E_r^B; \boldsymbol{x}, t)]\,\mathrm{d}V$$
$$+ \int_{\boldsymbol{x}\in\mathcal{D}} [-C(K_p^A, H_p^B; \boldsymbol{x}, t) - C(E_k^A, J_k^B; \boldsymbol{x}, t)$$
$$+ C(K_j^B, H_j^A; \boldsymbol{x}, t) + C(E_r^B, J_r^A; \boldsymbol{x}, t)]\,\mathrm{d}V. \tag{17}$$

Equation (17) is the global form, for the bounded domain $\mathcal{D}$, of the time-convolution reciprocity theorem. Note that the contributions from interfaces between different media present in $\mathcal{D}$ have cancelled and that the contributions from the boundaries of impenetrable objects present in $\mathcal{D}$ vanish in view of the boundary conditions stated in section 3.

To extend the validity of eq. (17) to an unbounded domain, it is assumed that outside some sphere of finite radius and center at the origin of the chosen Cartesian reference frame the medium is homogeneous, isotropic and lossless. Let, now, $\mathcal{S}_\Delta$ be the sphere of radius Δ and center at the origin of the chosen Cartesian reference frame and let $\mathcal{B}_\Delta$ be the ball interior to $\mathcal{S}_\Delta$. Then, eq. (17) is first applied to the bounded domain $\mathcal{D} \cap \mathcal{B}_\Delta$ and next the limit $\Delta \to \infty$ is taken. From a certain value of Δ onward, then the far-field source-type representations of the field on $\mathcal{S}_\Delta$ pertaining to the homogeneous, isotropic and lossless exterior medium apply, and if the two states A and B are chosen to be causal ones, the contribution from $\partial\mathcal{D} \cap \mathcal{S}_\Delta$ can be shown to vanish in the limit $\Delta \to \infty$.

5 Point-source solutions (Green's functions) and their properties

In this section, the infinite-medium causal point-source solutions to the electromagnetic field equations and their properties are investigated. As in section 4, the medium occupying $\mathcal{R}^3$ is assumed to be homogeneous, isotropic and lossless outside some sphere of finite radius. In this manner, the property of causality can be enforced by prescribing an explicit asymptotic behavior at infinitely large distances from the sources. Point-source solutions are also denoted as Green's functions (after G. Green [21], who introduced them in the theory of electrostatic and magnetostatic fields). Since in conjuction with the introduction of the Green's functions also their reciprocity properties will be investigated, we shall distinguish between the media in the States A and B in which they occur. In all cases, it is assumed that the media in the States A and B are each other's adjoints.

Four point-source solutions will be introduced. The first is the electromagnetic field $\{E_k^{J;A}, H_j^{J;A}\}(\boldsymbol{x}, \boldsymbol{x}', t)$ that is causally related to the action of a point source of electric current that is situated in medium A, at the position $\boldsymbol{x}'$, has the strength a_r^A and a unit impulse (Dirac distribution) time behavior. The associated volume source densities are

$$\{J_r^A, K_p^A\}(\boldsymbol{x}, t) = \{a_r^A, 0\}\, \delta(\boldsymbol{x} - \boldsymbol{x}', t), \tag{18}$$

where δ denotes the four-dimensional unit impulse operative at $\boldsymbol{x} = \boldsymbol{x}'$ and $t = 0$. In view of the linear relationship between the generated field and the source strength a_r^A we write

$$\{E_k^{J;A}, H_j^{J;A}\}(\boldsymbol{x}, \boldsymbol{x}', t) = \{G_{k,r}^{EJ;A}, G_{j,r}^{HJ;A}\}(\boldsymbol{x}, \boldsymbol{x}', t)\, a_r^A. \tag{19}$$

The second point-source solution is the electromagnetic field $\{E_k^{J;B}, H_j^{J;B}\}$ $(\boldsymbol{x}, \boldsymbol{x}'', t)$ that is causally related to the action of a point source of electric current that is situated in medium B, at the position $\boldsymbol{x}''$, has the strength a_r^B and a unit impulse time behavior. The associated volume source densities are

$$\{J_r^B, K_p^B\}(\boldsymbol{x}, t) = \{a_r^B, 0\}\, \delta(\boldsymbol{x} - \boldsymbol{x}'', t). \tag{20}$$

In view of the linear relationship between the generated field and the source strength a_r^B we write

$$\{E_k^{J;B}, H_j^{J;B}\}(\boldsymbol{x}, \boldsymbol{x}'', t) = \{G_{k,r}^{EJ;B}, G_{j,r}^{HJ;B}\}(\boldsymbol{x}, \boldsymbol{x}'', t)\, a_r^B. \tag{21}$$

The third point-source solution is the electromagnetic field $\{E_k^{K;A}, H_j^{K;A}\}$ $(\boldsymbol{x}, \boldsymbol{x}', t)$ that is causally related to the action of a point source of magnetic current that is situated in medium A, at the position $\boldsymbol{x}'$, has the source strength b_p^A and a unit impulse time behavior. The associated volume source densities are

$$\{J_r^A, K_p^A\}(\boldsymbol{x}, t) = \{0, b_p^A\}\, \delta(\boldsymbol{x} - \boldsymbol{x}', t). \tag{22}$$

In view of the linear relationship between the generated field and the source strength b_p^A we write

$$\{E_k^{K;A}, H_j^{K;A}\}(\boldsymbol{x}, \boldsymbol{x}', t) = \{G_{k,p}^{EK;A}, G_{j,p}^{HK;A}\}(\boldsymbol{x}, \boldsymbol{x}', t)\, b_p^A. \tag{23}$$

The fourth point-source solution is the electromagnetic field $\{E_k^{K;B}, H_j^{K;B}\}$ $(\boldsymbol{x}, \boldsymbol{x}'', t)$ that is causally related to the action of a point source of magnetic current that is situated in medium B, at the position $\boldsymbol{x}''$, has the source strength b_p^B and a unit impulse time behavior. The associated volume source densities are

$$\{J_r^B, K_p^B\}(\boldsymbol{x}, t) = \{0, b_p^B\}\, \delta(\boldsymbol{x} - \boldsymbol{x}'', t). \tag{24}$$

In view of the linear relationship between the generated field and the source strength b_p^B we write

$$\{E_k^{K;B}, H_j^{K;B}\}(\boldsymbol{x}, \boldsymbol{x}'', t) = \{G_{k,j}^{EK;B}, G_{j,p}^{HK;B}\}(\boldsymbol{x}, \boldsymbol{x}'', t)\, b_p^B. \tag{25}$$

The four Green's tensors $G_{k,r}^{EJ}$, $G_{j,r}^{HJ}$, $G_{k,p}^{EK}$ and $G_{j,p}^{HK}$ fully characterize the generation and propagation of electromagnetic waves in the medium under consideration. Between them, certain reciprocity relations exist. They follow upon applying the reciprocity relation of eq. (17) to the entire $\mathcal{R}^3$. In view of the causality condition, there is no contribution from the "sphere at infinity". Further, we take $\boldsymbol{x}' \neq \boldsymbol{x}''$, while the media are each other's adjoints. For the combination of the fields according to eqs. (19) and (21), eq. (17) leads to

$$a_k^B G_{k,r}^{EJ;A}(\boldsymbol{x}'',\boldsymbol{x}',t)\, a_r^A = a_k^A G_{k,r}^{EJ;B}(\boldsymbol{x}',\boldsymbol{x}'',t)\, a_r^B . \tag{26}$$

Since a_r^A and a_k^B are arbitrary, it follows from this result that

$$G_{k,r}^{EJ;A}(\boldsymbol{x}'',\boldsymbol{x}',t) = G_{r,k}^{EJ;B}(\boldsymbol{x}',\boldsymbol{x}'',t) \quad \text{for } \boldsymbol{x}' \neq \boldsymbol{x}''. \tag{27}$$

For the combination of the fields according to eqs. (23) and (25), eq. (17) leads to

$$b_j^B G_{j,p}^{HK;A}(\boldsymbol{x}'',\boldsymbol{x}',t)\, b_p^A = b_j^A G_{j,p}^{HK;B}(\boldsymbol{x}',\boldsymbol{x}'',t)\, b_p^B . \tag{28}$$

Since b_p^A and b_j^B are arbitrary, it follows from this result that

$$G_{j,p}^{HK;A}(\boldsymbol{x}'',\boldsymbol{x}',t) = G_{p,j}^{HK;B}(\boldsymbol{x}',\boldsymbol{x}'',t) \quad \text{for } \boldsymbol{x}' \neq \boldsymbol{x}''. \tag{29}$$

For the combination of the fields according to eqs. (19) and (25), eq. (17) leads to

$$b_j^B G_{j,r}^{HJ;A}(\boldsymbol{x}'',\boldsymbol{x}',t)\, a_r^A = -a_k^A G_{k,p}^{EK;B}(\boldsymbol{x}',\boldsymbol{x}'',t)\, b_p^B . \tag{30}$$

Since a_r^A and b_j^B are arbitrary, it follows from this result that

$$G_{j,r}^{HJ;A}(\boldsymbol{x}'',\boldsymbol{x}',t) = -G_{r,j}^{EK;B}(\boldsymbol{x}',\boldsymbol{x}'',t) \quad \text{for } \boldsymbol{x}' \neq \boldsymbol{x}''. \tag{31}$$

For the combination of the fields according to eqs. (21) and (23), eq. (17) leads to

$$a_k^B G_{k,p}^{EK;A}(\boldsymbol{x}'',\boldsymbol{x}',t)\, b_p^A = -b_j^A G_{j,r}^{HJ;B}(\boldsymbol{x}',\boldsymbol{x}'',t)\, a_r^B . \tag{32}$$

Since b_p^A and a_k^B are arbitrary, it follows from this result that

$$G_{k,p}^{EK;A}(\boldsymbol{x}'',\boldsymbol{x}',t) = -G_{p,k}^{HJ;B}(\boldsymbol{x}',\boldsymbol{x}'',t) \quad \text{for } \boldsymbol{x}' \neq \boldsymbol{x}''. \tag{33}$$

Note that eq. (33) is in accordance with eq. (31).

6 The direct source problem

In the direct (or forward) source problem, given source distributions $\{J_r^T, K_p^T\}$ with a bounded support $\mathcal{D}^T$ generate in a medium with known constitutive coefficients an electromagnetic field $\{E_k^T, H_j^T\}$ the values of which are to be expressed, in all space and in a causal manner, in terms of the relevant source distributions. It is expected that the point-source solutions introduced in section 5 will, through the principle of superposition, be instrumental in this respect. In the present section, it will be shown how the reciprocity relation of section 4 leads to the desired representation. To this end, the field in state A is identified with the actual field generated by the source distribution, i.e.,

$$\{E_k^A, H_j^A\} = \{E_k^T, H_j^T\}(\boldsymbol{x}, t), \tag{34}$$

and

$$\{J_r^A, K_p^A\} = \{J_r^T, K_p^T\}(\boldsymbol{x}, t). \tag{35}$$

Further, state B is first identified with the field generated by a point source of electric current situated in the medium adjoint to the actual one, i.e.,

$$\{E_k^B, H_j^B\} = \{E_k^{J;B}, H_j^{J;B}\}(\boldsymbol{x}, \boldsymbol{x}', t) \tag{36}$$

and

$$\{J_r^B, K_p^B\} = \{a_r^B, 0\}\,\delta(\boldsymbol{x} - \boldsymbol{x}', t). \tag{37}$$

Now, eq. (17) is applied to the entire $\mathcal{R}^3$. Using eq. (21) and invoking the condition that a_r^B is arbitrary, the representation of the electric field strength follows as

$$\begin{aligned} &E_k^T(\boldsymbol{x}', t) \\ &= \int_{\boldsymbol{x}\in\mathcal{D}^T} \{C[G_{k,r}^{EJ}(\boldsymbol{x}', \boldsymbol{x}), J_r^T(\boldsymbol{x}), t] + C[G_{k,p}^{EK}(\boldsymbol{x}', \boldsymbol{x}), K_p^T(\boldsymbol{x}), t]\}\,\mathrm{d}V \\ &\text{for all } \boldsymbol{x}' \in \mathcal{R}^3, \end{aligned} \tag{38}$$

where eqs. (27) and (33) have been used to rewrite the Green's tensors in terms of the ones applying to the medium in which the actual radiation takes place. Secondly, state B is identified with the field generated by a point source of magnetic current situated in the medium adjoint to the actual one, i.e.,

$$\{E_k^B, H_j^B\} = \{E_k^{K;B}, H_j^{K;B}\}(\boldsymbol{x}, \boldsymbol{x}', t) \tag{39}$$

and

$$\{J_r^B, K_p^B\} = \{0, b_p^B\}\,\delta(\boldsymbol{x} - \boldsymbol{x}', t). \tag{40}$$

Again eq. (17) is applied to the entire $\mathcal{R}^3$. Using eq. (25) and invoking the condition that b_p^B is arbitrary, the representation of the magnetic field strength follows as

$$H_j^T(\boldsymbol{x}', t) = \int_{\boldsymbol{x}\in\mathcal{D}^T} \{C[G_{j,r}^{HJ}(\boldsymbol{x}', \boldsymbol{x}), J_r^T(\boldsymbol{x}), t] + C[G_{j,p}^{HK}(\boldsymbol{x}', \boldsymbol{x}), K_p^T(\boldsymbol{x}), t]\}\,\mathrm{d}V \quad \text{for all } \boldsymbol{x}' \in \mathcal{R}^3, \tag{41}$$

where eqs. (31) and (29) have been used to rewrite the Green's tensors in terms of the ones applying to the medium in which the actual radiation takes place.

Equations (38) and (41) express the field values at any point in space as a superposition of the fields radiated by the elementary volume sources out of which the distributed sources can be envisaged to be composed. In view of the singularity of the Green's tensors at $\boldsymbol{x} = \boldsymbol{x}'$, the integrals at the right-hand sides have, for $\boldsymbol{x}' \in \mathcal{D}^T$ and $\boldsymbol{x}' \in \partial\mathcal{D}^T$, to be interpreted as their Cauchy principal values, i.e., they are, when necessary, calculated by a limiting procedure that excludes the singularity of the integrand by the ball $0 \le |\boldsymbol{x} - \boldsymbol{x}'| < \delta$ of radius δ about the singular point $\boldsymbol{x}'$, after which the limit $\delta \downarrow 0$ is taken. The same procedure applies to points of observation on an interface of discontinuity in electromagnetic medium properties, where the right-hand sides yield, at each point where the interface is smooth, half the sum of the limiting values of the relevant field quantity on either side of the interface.

7 The direct scattering problem

In the direct (or forward) scattering problem, the situation is considered where in some bounded domain $\mathcal{D}^s$ in space the electromagnetic medium properties differ by a given amount from the ones of a given background medium ("embedding"), the Green's tensors for the embedding being assumed to be known. The "scattering domain" $\mathcal{D}^s$ is irradiated by some known electromagnetic field and the problem is to calculate the resulting total electromagnetic field in all space. Let $\{\chi_{k,r}, \kappa_{j,p}\}$ be the medium parameters of the embedding and let $\{\Delta\chi_{k,r}, \Delta\kappa_{j,p}\}$ be the contrasts in electric and magnetic properties, respectively, that the scatterer shows with the embedding. Let, further, $\{E_k^i, H_j^i\}$ denote the incident field (i.e., the field in the absence of the scatterer) and let $\{E_k, H_j\}$ be the total field. Then, the scattered field $\{E_k^s, H_j^s\}$ is introduced as

$$\{E_k^s, H_j^s\} = \{E_k - E_k^i, H_j - H_j^i\} \quad \text{for all } \boldsymbol{x} \in \mathcal{R}^3. \tag{42}$$

It is assumed that the sources that generate the field are located in $\mathcal{D}^{s\prime}$, where $\mathcal{D}^{s\prime}$ is the complement of the closure of $\mathcal{D}^s$ in $\mathcal{R}^3$. Then, the total field is source-free in $\mathcal{D}^s$ and we have from eqs. (4), (5), (8) and (9)

$$\begin{aligned} &-\epsilon_{k,m,p}\partial_m H_p + \partial_t C(\chi_{k,r}, E_r, \boldsymbol{x}, t) \\ &\quad = -\,\partial_t C(\Delta\chi_{k,r}, E_r, \boldsymbol{x}, t) \quad \text{for } \boldsymbol{x} \in \mathcal{D}^s, \end{aligned} \tag{43}$$

$$\begin{aligned} &\epsilon_{j,m,r}\partial_m E_r + \partial_t C(\kappa_{j,p}, H_p, \boldsymbol{x}, t) \\ &\quad = -\partial_t C(\Delta\kappa_{j,p}, H_p, \boldsymbol{x}, t) \quad \text{for } \boldsymbol{x} \in \mathcal{D}^s. \end{aligned} \tag{44}$$

Since the incident wave is sourcefree in $\mathcal{D}^s$ as well and travels in the embedding, we have

$$-\epsilon_{k,m,p}\partial_m H_p^i + \partial_t C(\chi_{k,r}, E_r^i, \boldsymbol{x}, t) = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}^s, \tag{45}$$

$$\epsilon_{j,m,r}\partial_m E_r^i + \partial_t C(\kappa_{j,p}, H_p^i, \boldsymbol{x}, t) = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}^s. \tag{46}$$

Subtracting eq. (45) from eq. (43) and eq. (46) from eq. (44), and using eq. (42), it follows that

$$-\epsilon_{k,m,p}\partial_m H_p^s + \partial_t C(\chi_{k,r}, E_r^s, \boldsymbol{x}, t) = -J_k^s \quad \text{for } \boldsymbol{x} \in \mathcal{D}^s, \tag{47}$$

$$\epsilon_{j,m,r}\partial_m E_r^s + \partial_t C(\kappa_{j,p}, H_p^s, \boldsymbol{x}, t) = -K_j^s \quad \text{for } \boldsymbol{x} \in \mathcal{D}^s, \tag{48}$$

where

$$\begin{aligned} &\{J_k^s, K_j^s\} = \{\partial_t C(\Delta\chi_{k,r}, E_r, \boldsymbol{x}, t), \partial_t C(\Delta\kappa_{j,p}, H_p, \boldsymbol{x}, t)\} \\ &\text{for } \boldsymbol{x} \in \mathcal{D}^s \end{aligned} \tag{49}$$

are the contrast volume source densities of the scattered field; they have the support $\mathcal{D}^s$. Since the sources that generate the field are held fixed, the scattered field is sourcefree outside D^s and hence

$$-\epsilon_{k,m,p}\partial_m H_p^s + \partial_t C(\chi_{k,r}, E_r^s, \boldsymbol{x}, t) = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}^{s\prime}, \tag{50}$$

$$\epsilon_{j,m,r}\partial_m E_r^s + \partial_t C(\kappa_{j,p}, H_p^s, \boldsymbol{x}, t) = 0 \quad \text{for } \boldsymbol{x} \in \mathcal{D}^{s\prime}. \tag{51}$$

If $\{J_k^s, K_j^s\}$ were known, eqs. (47) – (51) would constitute a direct source problem of the type discussed in section 6; as yet they are, however, unknown. To construct a system of equations through which the problem can be solved, we use the relevant source-type integral representations, viz. (cf. eqs. (38) and (41))

$$E_k^s(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \{C[G_{k,r}^{EJ}(\boldsymbol{x}',\boldsymbol{x}), J_r^s(\boldsymbol{x}),t] + C[G_{k,p}^{EK}(\boldsymbol{x}',\boldsymbol{x}), K_p^s(\boldsymbol{x}),t]\}\,\mathrm{d}V \quad \text{for all } \boldsymbol{x}' \in \mathcal{R}^3, \tag{52}$$

$$H_j^s(\boldsymbol{x}',t) = \int_{\boldsymbol{x}\in\mathcal{D}^s} \{C[G_{j,r}^{HJ}(\boldsymbol{x}',\boldsymbol{x}), J_r^s(\boldsymbol{x}),t] + C[G_{j,p}^{HK}(\boldsymbol{x}',\boldsymbol{x}), K_p^s(\boldsymbol{x}),t]\}\,\mathrm{d}V \quad \text{for all } \boldsymbol{x}' \in \mathcal{R}^3. \tag{53}$$

Using the known incident field values, combining these for $\boldsymbol{x}' \in D^s$ with the values of the scattered field as represented by eqs. (52)–(53) and requiring field reproduction, a system of linear integral equations of the second kind is obtained from which $\{J_k^s, K_j^s\}$ can be solved. Using, subsequently, these values in the right-hand sides of eqs. (52)–(53), the scattered field in all space can be determined and, hence, the total field in all space is known. The relevant integral equations are usually solved by numerical procedures; the first step in their iterative Neumann solution is known as the (first) Rayleigh–Gans–Born approximation. (For applications of the latter, see Quak and De Hoop [22].)

8 The inverse source problem

In the inverse source problem, unknown volume source distributions $\{J_r^T, K_p^T\}$ with a bounded support $\mathcal{D}^T$ that is either known or guessed, radiate an electromagnetic field $\{E_k^T, H_j^T\}$ into a known embedding. In some bounded domain $\mathcal{D}^\Omega$ of observation the radiated field is accessible to measurement. In general, the domains $\mathcal{D}^T$ and $\mathcal{D}^\Omega$ are disjoint. The source distributions in $\mathcal{D}^T$ are now to be reconstructed from appropriate measurement operations carried out in $\mathcal{D}^\Omega$. Since this requires an interaction of the "remote-sensing type", Lorentz's reciprocity theorem provides a basis for this interaction.

In eq. (17) we take

$$\{J_r^A, K_p^A\} = \{J_r^T, K_p^T\}, \tag{54}$$

and correspondingly,

$$\{E_k^A, H_j^A\} = \{E_k^T, H_j^T\}. \tag{55}$$

Further, we take the medium in state B to be the adjoint of the known embedding and for the electromagnetic field in state B a "computational" or "observational" one with volume source distributions

$$\{J_r^B, K_p^B\} = \{J_r^\Omega, K_p^\Omega\}, \tag{56}$$

with support $\mathcal{D}^\Omega$, and field values

$$\{E_k^B, H_j^B\} = \{E_k^\Omega, H_j^\Omega\}, \tag{57}$$

that are causally related to them. The two states are substituted in eq. (17) and the reciprocity relation is applied to the entire $\mathcal{R}^3$. The result is

$$\int_{\boldsymbol{x}\in\mathcal{D}^T}[C(E_r^\Omega, J_r^T, \boldsymbol{x}, t) - C(H_p^\Omega, K_p^T, \boldsymbol{x}, t)]\,\mathrm{d}V$$
$$= \int_{\boldsymbol{x}\in\mathcal{D}^\Omega}[C(E_k^T, J_k^\Omega, \boldsymbol{x}, t) - C(H_j^T, K_j^\Omega, \boldsymbol{x}, t)]\,\mathrm{d}V. \tag{58}$$

In eq. (58), the left-hand side contains the unknown quantities, while the right-hand side contains known quantities. As far as the "computational" or "observational" state Ω is concerned, one can say that it is representative for the processing of the measured data. As eq. (58) shows, it makes, for the data processing represented by this equation, no sense to make the support of $\{J_r^\Omega, K_p^\Omega\}$ larger than $\mathcal{D}^\Omega$, since no information on the radiated field is available outside $\mathcal{D}^\Omega$.

Commonly, a solution to the inverse source problem is constructed by discretizing in eq. (58) both $\mathcal{D}^T$ and $\mathcal{D}^\Omega$ as well as the source and field distributions, and solving the corresponding system of linear algebraic equations for the coefficients in the expansions of the unknown source distributions. A discussion of the details of this procedure, which is in the category of ill-posed problems, is beyond the scope of the present paper.

9 The inverse scattering problem

In the inverse scattering problem, a contrasting domain of bounded support $\mathcal{D}^s$, which is either known or guessed, is present in a known embedding with constitutive parameters $\{\chi_{k,r}, \kappa_{j,p}\}$. The contrasts in medium parameters $\{\Delta\chi_{k,r}, \Delta\kappa_{j,p}\}$ have the support $\mathcal{D}^s$, but are otherwise unknown. The contrasting domain is probed by irradiating it with an incident electromagnetic field $\{E_k^i, H_j^i\}$ that propagates in the embedding. Due to the presence of the contrasting domain, scattering takes place which manifests itself through the presence of a scattered field $\{E_k^s, H_j^s\}$. In some bounded domain $\mathcal{D}^\Omega$ of observation, the total field (and hence the scattered field) is accessible to measurement. In general, the domains $\mathcal{D}^s$ and $\mathcal{D}^\Omega$ are disjoint. The contrast medium parameters in $\mathcal{D}^s$ are now to be reconstructed from the measured data in $\mathcal{D}^\Omega$. As in section 8, the Lorentz reciprocity theorem provides a basis for this "remote-sensing type" of the problem. In eq. (17) we take

$$\{J_r^A, K_p^A\} = \{J_r^s, K_p^s\}, \tag{59}$$

where $\{J_r^s, K_p^s\}$ is given by eq. (49), and correspondingly,

$$\{E_k^A, H_j^A\} = \{E_k^s, H_j^s\}. \tag{60}$$

Further, we take the medium in state B to be the adjoint of the known embedding and for the electromagnetic field in state B a "computational" or "observational " one, with known volume source distributions

$$\{J_r^B, K_p^B\} = \{J_r^\Omega, K_p^\Omega\}, \tag{61}$$

with support $\mathcal{D}^\Omega$, and field values

$$\{E_k^B, H_j^B\} = \{E_k^\Omega, H_j^\Omega\}, \tag{62}$$

that are causally related to them. The two states are substituted in eq. (17) and the reciprocity relation is applied to the entire $\mathcal{R}^3$. The result is

$$\int_{\boldsymbol{x}\in\mathcal{D}^s} [C(E_r^\Omega, J_r^s, \boldsymbol{x}, t) - C(H_p^\Omega, K_p^s, \boldsymbol{x}, t)]\,\mathrm{d}V$$
$$= \int_{\boldsymbol{x}\in\mathcal{D}^\Omega} [C(E_k^s, J_k^\Omega, \boldsymbol{x}, t) - C(H_j^s, K_j^\Omega, \boldsymbol{x}, t)]\,\mathrm{d}V. \tag{63}$$

In eq. (63), the left-hand side contains the unknown quantities, while the right-hand side contains known quantities. As far as the "computational" or "observational" state is concerned, one can say that it is representative for the processing of the measured data. As eq. (63) shows, it makes, for the data processing represented by this equation, no sense to make the support of $\{J_r^\Omega, K_p^\Omega\}$ larger than $\mathcal{D}^\Omega$, since no information on the scattered field is available outside $\mathcal{D}^\Omega$.

Commonly, a solution to the inverse source problem is constructed by discretizing in eq. (63) both $\mathcal{D}^s$ and $\mathcal{D}^\Omega$ as well as the source and field distributions and contrast medium parameters and solving the unknown expansion coefficients from the resulting equations that are nonlinear because in (cf. eq. (49))

$$\{J_k^s, K_j^s\} = \{\partial_t C(\Delta\chi_{k,r}, E_r, \boldsymbol{x}, t), \partial_t C(\Delta\kappa_{j,p}, H_p, \boldsymbol{x}, t)\}$$
$$\text{for } \boldsymbol{x} \in \mathcal{D}^s, \tag{64}$$

both the contrast medium parameters and the total field values are unknown. A discussion of the details of the relevant procedures is beyond the scope of the present paper.

10 Electromagnetic field representations in a subdomain of space. Equivalent surface sources. Huygens' principle and Oseen's extinction theorem

As has been shown in section 6, source-type electromagnetic field representations can be obtained that express the electric and the magnetic field strengths at all points in space in terms of the volume source distributions that generate the field, provided that the relevant Green's tensors can be constructed. In a number of cases, however, one is only interested in the field values in some, bounded or unbounded, subdomain of $\mathcal{R}^3$, and a field representation for the relevant domain would suffice. In the present section it is shown how the Lorentz reciprocity theorem of section 4 leads to such a representation, be it that now, in addition to the volume integrals containing the volume source distributions in the domain of interest, also surface integrals over the boundary of this domain occur. Let $\mathcal{D}$ be the relevant subdomain of $\mathcal{R}^3$ and let $\partial\mathcal{D}$ be the boundary surface of $\mathcal{D}$. Further, the complement of $\mathcal{D} \cup \partial\mathcal{D}$ in $\mathcal{R}^3$ is denoted by $\mathcal{D}'$. To arrive at the relevant representation, we apply eq. (17) to the domain $\mathcal{D}$ and take state A to be the actual electromagnetic field

$$\{E_k^A, H_j^A\} = \{E_k, H_j\} \tag{65}$$

present in $\mathcal{D}$. The corresponding volume source distributions are identified with the actual ones insofar these are present in $\mathcal{D}$, i.e.,

$$J_r^A = \{J_r, 0\}, \quad K_p^A = \{K_p, 0\} \quad \text{for } \boldsymbol{x} \in \{\mathcal{D}, \mathcal{D}'\}. \tag{66}$$

Further, state B is first identified with the field generated by, and causally related to, a point source of electric current situated in a medium adjoint to the actual one, i.e.,

$$\{E_k^B, H_j^B\} = \{E_k^{J;B}, H_j^{J;B}\}(\boldsymbol{x}, \boldsymbol{x}', t) \tag{67}$$

and

$$\{J_r^B, K_p^B\} = \{J_r^B, K_p^B\}(\boldsymbol{x}, \boldsymbol{x}', t) = \{a_r^B, 0\}\delta(\boldsymbol{x} - \boldsymbol{x}', t)$$
$$\text{for } \boldsymbol{x}' \in \mathcal{R}^3. \tag{68}$$

Now, eq. (17) is applied to the domain $\mathcal{D}$, while it is taken into account that

$$\int_{\boldsymbol{x}\in\mathcal{D}} C[E_k^A(\boldsymbol{x}), J_k^B(\boldsymbol{x}, \boldsymbol{x}'), t]\,\mathrm{d}V$$
$$= \int_{\boldsymbol{x}\in\mathcal{D}} E_k(\boldsymbol{x}, t) a_k^B \delta(\boldsymbol{x} - \boldsymbol{x}')\,\mathrm{d}V = \chi_{\mathcal{D}}(\boldsymbol{x}')\, E_k(\boldsymbol{x}', t)\, a_k^B, \tag{69}$$

where

$$\chi_{\mathcal{D}}(\boldsymbol{x}') = \{1, 1/2, 0\} \quad \text{for } \boldsymbol{x}' \in \{\mathcal{D}, \partial\mathcal{D}, \mathcal{D}'\} \tag{70}$$

is the characteristic function of the set $\mathcal{D}$. Using eq. (69) and invoking the condition that a_k^B is arbitrary, we end up with

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}} \{C[G_{k,r}^{EJ}(\boldsymbol{x}', \boldsymbol{x}), \partial J_r(\boldsymbol{x}), t] + C[G_{k,p}^{EK}(\boldsymbol{x}', \boldsymbol{x}), \partial K_p(\boldsymbol{x}), t]\}\,\mathrm{d}A$$
$$+ \int_{\boldsymbol{x}\in\mathcal{D}} \{C[G_{k,r}^{EJ}(\boldsymbol{x}', \boldsymbol{x}), J_r(\boldsymbol{x}), t] + C[G_{k,p}^{EK}(\boldsymbol{x}', \boldsymbol{x}), K_p(\boldsymbol{x}), t]\}\,\mathrm{d}V$$
$$= \chi_{\mathcal{D}}(\boldsymbol{x}') E_k(\boldsymbol{x}', t) \quad \text{for } \boldsymbol{x}' \in \mathcal{R}^3, \tag{71}$$

where

$$\partial J_r = -\epsilon_{r,m,n}\nu_m H_n \quad \text{for } \boldsymbol{x} \in \partial\mathcal{D} \tag{72}$$

is the equivalent surface density of electric current on $\partial\mathcal{D}$, and

$$\partial K_p = \epsilon_{p,m,n}\nu_m E_n \quad \text{for } \boldsymbol{x} \in \partial\mathcal{D} \tag{73}$$

is the equivalent surface density of magnetic current on $\partial\mathcal{D}$, while ν_m is the unit vector along the normal to $\partial\mathcal{D}$ pointing away from $\mathcal{D}$.

Secondly, state B is identified with the field generated by, and causally related to, a point source of magnetic current situated in a medium adjoint to the actual one, i.e.,

$$\{E_k^B, H_j^B\} = \{E_k^{K;B}, H_j^{K;B}\}(\boldsymbol{x}, \boldsymbol{x}', t) \tag{74}$$

and

$$\{J_r^B, K_p^B\} = \{J_r^B, K_p^B\}(\boldsymbol{x}, \boldsymbol{x}', t) = \{0, b_p^B\}\,\delta(\boldsymbol{x} - \boldsymbol{x}', t). \tag{75}$$

Now, eq. (17) is applied to the domain $\mathcal{D}$, while it is taken into account that

$$\int_{\boldsymbol{x}\in\mathcal{D}} C[H_j^A(\boldsymbol{x}), K_j^B(\boldsymbol{x},\boldsymbol{x}'), t]\,\mathrm{d}V$$
$$= \int_{\boldsymbol{x}\in\mathcal{D}} H_j(\boldsymbol{x},t)\, b_j^B\, \delta(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}V = \chi_{\mathcal{D}}(\boldsymbol{x}')\, H_j(\boldsymbol{x}',t)\, b_j^B, \tag{76}$$

where $\chi_{\mathcal{D}}$ is given by eq. (70). Using eq. (76) and invoking the condition that b_j^B is arbitrary, we end up with

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}} \{C[G_{j,r}^{HJ}(\boldsymbol{x}',\boldsymbol{x}), \partial J_r(\boldsymbol{x}), t] + C[G_{j,p}^{HK}(\boldsymbol{x}',\boldsymbol{x}), \partial K_p(\boldsymbol{x}), t]\}\,\mathrm{d}A$$
$$+ \int_{\boldsymbol{x}\in\mathcal{D}} \{C[G_{j,r}^{HJ}(\boldsymbol{x}',\boldsymbol{x}), J_r(\boldsymbol{x}), t] + C[G_{j,p}^{HK}(\boldsymbol{x}',\boldsymbol{x}), K_p(\boldsymbol{x}), t]\}\,\mathrm{d}V$$
$$= \chi_{\mathcal{D}}(\boldsymbol{x}') H_j(\boldsymbol{x}',t) \quad \text{for } \boldsymbol{x}' \in \mathcal{R}^3, \tag{77}$$

where ∂J_r and ∂K_p are given by eqs. (72) and (73), respectively.

The results for $\boldsymbol{x}' \in \mathcal{D}$ and $\boldsymbol{x}' \in \mathcal{D}'$ in eqs. (71) and (77) are obvious; the ones for $\boldsymbol{x}' \in \partial\mathcal{D}$ hold at points where ∂D has a unique tangent plane, and follow from a detailed analysis. The latter also reveals that the integrals in the left-hand side have to be interpreted as their Cauchy principal values, i.e., the integrals are, when necessary, calculated by a limiting procedure that excludes the singularity in the integrand by a ball $0 \le |\boldsymbol{x}-\boldsymbol{x}'| < \delta$ of radius $\delta > 0$ about the singular point $\boldsymbol{x}'$, after which the limit $\delta \downarrow 0$ is taken.

When $\boldsymbol{x}' \in \mathcal{D}$, eqs. (71) and (77) express the value of the electric and the magnetic field strengths in some point of $\mathcal{D}$ as the sum of the contributions from the volume source densities of electric and magnetic current as far as these are present in $\mathcal{D}$, and the equivalent surface source densities of electric and magnetic current on $\partial\mathcal{D}$. Evidently, the latter yield, in the interior of $\mathcal{D}$, the contributions to the field values insofar these arise from sources located in $\mathcal{D}'$ (i.e., outside $\mathcal{D}$).

Another property of eqs. (71) and (77) is that the field emitted by the volume sources in $\mathcal{D}$ and the field emitted by the surface sources on $\partial\mathcal{D}$ cancel each other for $\boldsymbol{x}' \in \mathcal{D}'$. This property is known as Oseen's extinction theorem [20].

When eqs. (71) and (77) are used in a domain in which no volume source distributions are present, it expresses Huygens' principle [1] which states that an electromagnetic field due to sources "behind" some surface that divides space into two disjoint regions, can in front of the relevant surface be represented as due to equivalent surface sources located at that surface, while that representation yields the value zero "behind" that surface.

Equations (71) and (77) have been the basis for the analysis of numerous diffraction problems. This development has started with the Kirchhoff theory of the diffraction of scalar light waves by apertures in "black" screens [16] (see also Baker and Copson [2]) and was further extended by Kottler [17, 18] to the diffraction of electromagnetic waves by apertures in "black" screens (see also Baker and Copson [2]), while Stratton and Chu [19] derived equivalent formulas as general solutions to the electromagnetic radiation problem (see also Stratton [23]).

11 Conclusion

It has been shown how the time-domain electromagnetic reciprocity theorem of the time-convolution type leads to source-type representations for the electromagnetic field quantities that can be used to analyze electromagnetic direct source, inverse source, direct scattering, and inverse scattering problems. Further, the source-type representations for a bounded domain are shown to be mathematical formulations of Huygens' principle and Oseen's extinction theorem.

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