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The moving-load problem in soil dynamics—the vertical displacement approximation☆

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Abstract

The moving point load problem in soil dynamics is analyzed in the vertical particle displacement approximation. Prior to its motion, the load is stationary. From the instant at which it is set into motion it moves, with constant speed, along a straight path on the (horizontal) planar surface of a semi-infinite elastic medium. The modified Cagniard method for solving transient wave problems is used to determine closed-form expressions for the vertical component of the particle displacement from the elastodynamic wave equation of which only the vertical component is taken into account. The relevant approximation is standard in soil dynamics. Both the cases of "subsonic" and "supersonic" surface load speeds are considered. Methods to include losses in the model are briefly discussed. The study has been initiated with a view to the application of the results to the analysis of the ground motion generated by high-speed trains traveling on a poorly consolidated soil.

1. Introduction

In this paper, the moving point load problem in soil dynamics is investigated with a view to determine the ground motion generated by a high-speed train traveling on a poorly consolidated soil with low shear wave speed. The foundation on which the load moves is modeled as a homogeneous, isotropic, elastic half-space with a horizontal planar boundary. The load is modeled as a vertical point force that is stationary prior to a certain instant and is then suddenly set into horizontal motion, along a straight trajectory and at constant speed v. As is, in the first instance, usual in soil dynamics, only the vertical component of the particle displacement is taken into account, which component is taken to satisfy the vertical component of the elastodynamic wave equation for a homogeneous, isotropic solid [1,2]. Previous studies on the moving surface load problem include the ones by Cole and Huth [3], Ang [4], Payton [5], Eason [6], Gakenheimer and Miklowitz [7], Freund [8,9], Miklowitz [10], Georgiadis and Barber [11] and Bakker et al. [12].

The mechanical properties of the semi-infinite elastic solid are characterized by its volume density of mass ρ and its Lamé stiffness coefficients λ and μ. The corresponding compressional or P-wave speed is $c_{\mathrm{P}} = [(\lambda+2\mu)/\rho]^{1/2}$ and the corresponding shear or S-wave speed is $c_{\mathrm{S}} = (\mu/\rho)^{1/2}$. Position in the configuration is specified by the

☆ Dedicated to Professor Jan D. Achenbach on the occasion of his 60th birthday.

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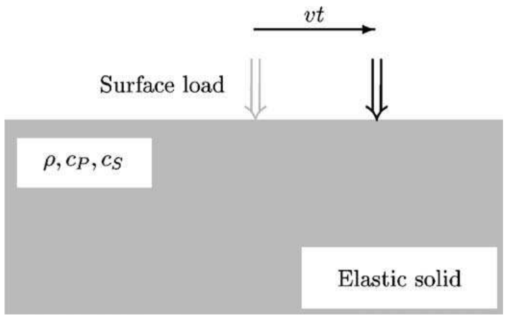

Fig. 1. Surface load on a semi-infinite elastic medium set into motion at $t = 0$ with speed v.

coordinates $\{x, y, z\}$ with respect to an orthogonal, Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_z\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. The reference frame is chosen such that the elastic half-space occupies the domain $\{z > 0\}$ and that the motion of the point force takes place along the trajectory $\{x > 0, y = 0, z = 0\}$. The time coordinate is t. The instant at which the load is set into motion in $t = 0$. Partial differentiation is denoted by ∂. Whenever appropriate, boldface symbols will be used to indicate vectorial quantities. Spatial vectorial differentiation is denoted by the nabla operator $\boldsymbol{\nabla} = \boldsymbol{i}_x\partial_x + \boldsymbol{i}_y\partial_y + \boldsymbol{i}_z\partial_z$.

The configuration is shown in Fig. 1.

2. Formulation of the problem

In a source-free domain of a homogeneous, isotropic elastic solid, the particle displacement $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ of an elastic wave motion satisfies the elastodynamic wave equation [1,2]

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \rho\partial_t^2\boldsymbol{u} = \boldsymbol{0}. \tag{2.1}$$

Assuming that the vertical point load predominantly only excites the z-component u_z of $\boldsymbol{u}$ and taking into account only the z-component of Eq. (2.1), the *scalar wave equation of soil dynamics* results as

$$c_\mathrm{P}^2\partial_z^2u_z + c_\mathrm{S}^2(\partial_x^2 + \partial_y^2)u_z - \partial_t^2u_z = 0. \tag{2.2}$$

The corresponding vertical traction $\tau_{z,z}$ is related to the vertical particle displacement through

$$\tau_{z,z} = (\lambda + 2\mu)\partial_z u_z. \tag{2.3}$$

The excitation by the point load is modeled via the boundary condition

$$\lim_{z\downarrow 0}\tau_{z,z} = -Mg\delta(x, y)[1 - H(t)] - Mg\delta(x - vt, y)H(t), \tag{2.4}$$

where M is the mass of the load, g the acceleration of free fall and $H(t) = \{0, 1/2, 1\}$ for $\{t < 0, t = 0, t > 0\}$ the Heaviside unit step function. The first term on the right-hand side represents the action of the load while it is stationary, the second term describes the motion from the instant $t = 0$ onwards. Eq. (2.2) indicates that in the horizontal (x, y)direction, where the particle displacement is transverse with respect to the direction of propagation, the wave propagation takes place with the shear wave speed c_S, while in the vertical (z)direction, where the particle displacement is longitudinal with respect to the direction of propagation, the wave propagation takes place with the compressional wave speed c_P. Therefore, the scalar wave equation of soil dynamics exhibits anisotropy in the propagation of its wave fronts.

The calculation of $u_z = u_z(x, y, z, t)$ is carried out as a three-step approach. First, the transient wave motion corresponding to a surface excitation $\lim_{z\downarrow 0}\tau_{z,z} = -Mg\delta(x, y)H(t)$, i.e. for a non-moving suddenly applied load, will be determined. Secondly, by taking in the relevant result the limit $t \to \infty$, the static displacement in the time interval $t < 0$ follows. These two results together yield the displacement field associated with a surface excitation corresponding to the first term on the right-hand side of Eq. (2.4). Thirdly, the wave motion corresponding to the second term in the surface excitation (2.4) is determined. Superposition of the three displacements yields the total particle displacement in the configuration. The two elastodynamic problems involved are solved with the aid of the modified Cagniard method.

3. The non-moving transient point load problem

For the non-moving transient point load problem, to be indicated by the superscript $^\downarrow$, the surface excitation is

$$\lim_{z\downarrow 0}\tau^{\downarrow}_{z,z} = -Mg\delta(x, y)H(t). \tag{3.1}$$

The corresponding particle displacement $u^{\downarrow}_z$ is determined with the aid of the modified Cagniard method [13–20], or rather a variant of it that is particularly adapted to the circumstance that in the moving-load problem the rotational symmetry in the horizontal plane that is present in the the non-moving-load problem is lost. First, the one-sided time Laplace transform is introduced through

$$\hat{u}^{\downarrow}_z(x, y, z, s) = \int_{t=0}^{\infty}\exp(-st)u^{\downarrow}_z(x, y, z, t)\,\mathrm{d}t \quad \text{for } s \in \mathcal{R}, s > 0. \tag{3.2}$$

As indicated, the transform parameter s is taken to be real and positive. Next, the horizontal slowness representation

$$\hat{u}^{\downarrow}_z(x, y, z, s) = \left(\frac{1}{2\pi}\right)^2\int_{\alpha=-\infty}^{\infty}\exp(-\mathrm{i}s\alpha x)\,\mathrm{d}\alpha\int_{\beta=-\infty}^{\infty}\exp(-\mathrm{i}s\beta y)\tilde{u}^{\downarrow}_z(\alpha, \beta, z, s)\,\mathrm{d}\beta \tag{3.3}$$

is introduced. Substitution of this representation in Eq.(2.2) yields

$$c_\mathrm{P}^2\partial_z^2\tilde{u}^{\downarrow}_z - c_\mathrm{S}^2 s^2(\alpha^2 + \beta^2)\tilde{u}^{\downarrow}_z - s^2\tilde{u}^{\downarrow}_z = 0. \tag{3.4}$$

The solution of Eq. (3.4) that is bounded as $z \to \infty$ is

$$\tilde{u}^{\downarrow}_z = \tilde{A}^{\downarrow}\exp(-s\gamma z) \tag{3.5}$$

in which

$$\gamma(\alpha, \beta) = \frac{c_\mathrm{S}}{c_\mathrm{P}}\left(\frac{1}{c_\mathrm{S}^2} + \alpha^2 + \beta^2\right)^{1/2} \quad \text{with } (\dots)^{1/2} > 0. \tag{3.6}$$

Substitution of this expression in the boundary condition

$$\tilde{\tau}^{\downarrow}_{z,z} = -Mgs \tag{3.7}$$

that follows from Eq. (3.1) and the horizontal slowness representation $\tilde{\delta}(\alpha, \beta, s) = s^2$ of $\delta(x, y)$, together with s^{-1} for $\mathrm{Re}(s) > 0$ as the Laplace transform of $H(t)$, yields

$$\tilde{A} = \frac{Mg}{\lambda + 2\mu}\frac{1}{\gamma}. \tag{3.8}$$

The first step towards arriving at the corresponding time-domain result consists of rewriting the integration with respect to β in Eq. (3.3) as an integration with respect to Ω via the change of variables

$$\mathrm{i}\beta = (c_\mathrm{S}^{-2} + \alpha^2)^{1/2}\Omega. \tag{3.9}$$

With this

$$\frac{1}{2\pi}\int_{\beta=-\infty}^{\infty}\frac{1}{\gamma}\exp(-\mathrm{i}s\beta y - s\gamma z)\,\mathrm{d}\beta$$
$$= \frac{1}{2\pi \mathrm{i}}\int_{\Omega=-\mathrm{i}\infty}^{\infty}\exp\left\{-s\left[\Omega y + \left(\frac{c_{\mathrm{S}}}{c_{\mathrm{P}}}\right)(1-\Omega^2)^{1/2}z\right](c_{\mathrm{S}}^{-2}+\alpha^2)^{1/2}\right\}, \qquad \frac{1}{(c_{\mathrm{S}}/c_{\mathrm{P}})(1-\Omega^2)^{1/2}}\,\mathrm{d}\Omega. \tag{3.10}$$

Through an application of Cauchy's theorem and Jordan's lemma, the integration in the right-hand side is replaced with one along the modified Cagniard path:

$$\Omega y + \left(\frac{c_{\mathrm{S}}}{c_{\mathrm{P}}}\right)(1-\Omega^2)^{1/2}z = q \tag{3.11}$$

with q real and positive. This path is the hyperbolic arc $\{\Omega = \Omega_\perp\} \cup \{\Omega = \Omega_\perp^*\}$, where

$$\Omega_\perp = \frac{y}{r_\perp^2}q + \mathrm{i}\frac{(c_{\mathrm{S}}/c_{\mathrm{P}})z}{r_\perp^2}(q^2 - r_\perp^2)^{1/2} \quad \text{for } r_\perp < q < \infty \tag{3.12}$$

in which

$$r_\perp = \left[y^2 + \left(\frac{c_{\mathrm{S}}}{c_{\mathrm{P}}}\right)^2 z^2\right]^{1/2} > 0 \tag{3.13}$$

and * denotes complex conjugate. Taking together the contributions from $\Omega = \Omega_\perp$ and $\Omega = \Omega_\perp^*$, using Schwarz's reflection principle of complex function theory and employing the relation

$$\frac{1}{(1-\Omega_\perp^2)^{1/2}}\frac{\partial \Omega_\perp}{\partial q} = \frac{\mathrm{i}}{(q^2 - r_\perp^2)^{1/2}}, \tag{3.14}$$

the following result is obtained:

$$\frac{1}{2\pi}\int_{\beta=-\infty}^{\infty}\frac{1}{\gamma}\exp(-\mathrm{i}s\beta y - s\gamma z)\,\mathrm{d}\beta = \frac{1}{\pi}\int_{q=r_\perp}^{\infty}\exp[-sq(c_{\mathrm{S}}^{-2}+\alpha^2)^{1/2}]\frac{1}{(c_{\mathrm{S}}/c_{\mathrm{P}})(q^2-r_\perp^2)^{1/2}}\,\mathrm{d}q. \tag{3.15}$$

Using this in Eq. (3.3), the next step consists of interchanging the order of the integrations and replacing in the integration with respect to α the variable $\mathrm{i}\alpha$ with

$$\mathrm{i}\alpha = p. \tag{3.16}$$

Through an application of Cauchy's theorem and Jordan's lemma, the integration with respect to p along the imaginary p-axis is replaced with one along the modified Cagniard path:

$$px + q(c_{\mathrm{S}}^{-2} - p^2)^{1/2} = \tau \tag{3.17}$$

with τ real and positive. This path is the hyperbolic arc $\{p = p_\parallel\} \cup \{p = p_\parallel^*\}$, where

$$p_\parallel = \frac{x}{x^2+q^2}\tau + \mathrm{i}\frac{q}{x^2+q^2}\left(\tau^2 - \frac{x^2+q^2}{c_{\mathrm{S}}^2}\right)^{1/2} \quad \text{for } c_{\mathrm{S}}^{-1}(x^2+q^2)^{1/2} < \tau < \infty. \tag{3.18}$$

Taking together the contributions from $p = p_\parallel$ and $p = p_\parallel^*$, applying Schwarz's reflection principle of complex function theory and employing the relation

$$\mathrm{Im}\left[\frac{\partial p_\parallel}{\partial \tau}\right] = \frac{q}{x^2+q^2}\frac{\tau}{[\tau^2 - c_{\mathrm{S}}^{-2}(x^2+q^2)]^{1/2}}, \tag{3.19}$$

it follows that

$$\frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\exp[-\mathrm{i}s\alpha x - sq(c_S^{-2}+\alpha^2)^{1/2}]\,\mathrm{d}\alpha$$
$$= \frac{1}{\pi}\int_{\tau=c_S^{-1}(x^2+q^2)^{1/2}}^{\infty}\exp(-s\tau)\frac{q}{x^2+q^2}\frac{\tau}{[\tau^2-c_S^{-2}(x^2+q^2)]^{1/2}}\,\mathrm{d}\tau. \tag{3.20}$$

In the resulting expression for $\hat{u}_z^{\downarrow}$ the integrations with respect to q and τ are interchanged. This leads to

$$\int_{q=r_\perp}^{\infty} dq \int_{\tau=c_S^{-1}(x^2+q^2)^{1/2}}^{\infty}\cdots\mathrm{d}\tau = \int_{\tau=R/c_S}^{\infty} d\tau \int_{q=r_\perp}^{(c_S^2\tau^2-x^2)^{1/2}}\cdots\mathrm{d}q \tag{3.21}$$

in which

$$R = \left[x^2+y^2+\left(\frac{c_S}{c_P}\right)^2 z^2\right]^{1/2} > 0. \tag{3.22}$$

In the inner integration with respect to q the substitution

$$q^2 = r_\perp^2\cos^2(\psi) + (c_S^2\tau^2 - x^2)\sin^2(\psi) \quad \text{for } 0 \le \psi \le \tfrac{1}{2}\pi \tag{3.23}$$

and use of the relations

$$(q^2-r_\perp^2)^{1/2} = (c_S^2\tau^2-R^2)^{1/2}\sin(\psi), \tag{3.24}$$

$$(c_S^2\tau^2-x^2-q^2)^{1/2} = (c_S^2\tau^2-R^2)^{1/2}\cos(\psi), \tag{3.25}$$

$$q\,\mathrm{d}q = (c_S^2\tau^2-R^2)\cos(\psi)\sin(\psi)\,\mathrm{d}(\psi), \tag{3.26}$$

$$x^2+q^2 = R^2\cos^2(\psi) + c_S^2\tau^2\sin^2(\psi) \tag{3.27}$$

that follow from it, carry the integral over into

$$\int_{q=r_\perp}^{(c_S^2\tau^2-x^2)^{1/2}}\frac{1}{(q^2-r_\perp^2)^{1/2}}\frac{q}{x^2+q^2}\frac{1}{[\tau^2-c_S^{-2}(x^2+q^2)]^{1/2}}\,\mathrm{d}q$$
$$= c_S\int_{\psi=0}^{\pi/2}\frac{1}{R^2\cos^2(\psi)+c_S^2\tau^2\sin^2(\psi)}\,\mathrm{d}\psi = \frac{1}{\tau R}\frac{\pi}{2}, \tag{3.28}$$

where for the evaluation of the integration with respect to ψ we have used the results of Appendix A. Combining all results, it is found that

$$\hat{u}_z^{\downarrow} = \frac{Mg}{\lambda+2\mu}\frac{c_P}{c_S}\frac{1}{2\pi R}\int_{\tau=R/c_S}^{\infty}\exp(-s\tau)\,\mathrm{d}\tau = \frac{Mg}{\lambda+2\mu}\frac{c_P}{c_S}\frac{1}{2\pi R}\frac{\exp(-sR/c_S)}{s}. \tag{3.29}$$

The corresponding time-domain expression for the particle displacement finally follows as

$$u_z^{\downarrow} = \frac{Mg}{\lambda+2\mu}\frac{c_P}{c_S}\frac{1}{2\pi R}H(t-R/c_S). \tag{3.30}$$

It is easily verified that Eq. (3.30) satisfies the partial differential equation (2.2) for $z > 0$ as well as the surface excitation condition (3.1).

4. The static point load problem

The static point load particle displacement follows from the dynamical non-moving point load result of Eq. (3.26) by taking the limit $t \to \infty$. Denoting this particle displacement by $u_z^{\downarrow\infty}$ and observing that $\lim_{t\to\infty} H(t) = 1$, the relevant expression is obtained as

$$u_z^{\downarrow\infty} = \frac{Mg}{\lambda + 2\mu} \frac{c_P}{c_S} \frac{1}{2\pi R}. \tag{4.1}$$

5. The moving transient point load problem

For the moving transient point load problem, the surface excitation is

$$\lim_{z\downarrow 0} \tau_{z,z}^{\rightarrow} = -Mg\delta(x - vt, y)H(t) \tag{5.1}$$

with $v > 0$. To determine the corresponding particle displacement $u_z^{\rightarrow}$ in space–time, the same method as in Section 3 is applied. With the aid of

$$\delta(x - vt) = v^{-1}\delta(t - x/v) \tag{5.2}$$

and

$$\int_{t=0}^{\infty} \exp(-st)\delta(t - x/v)\mathrm{d}t = \exp\left(-\frac{sx}{v}\right) H(x), \tag{5.3}$$

the surface excitation condition in the time Laplace transform domain follows from Eq. (5.1) as

$$\lim_{z\downarrow 0} \hat{\tau}_{z,z}^{\rightarrow} = -Mgv^{-1}\exp\left(-\frac{sx}{v}\right) \delta(y)H(x). \tag{5.4}$$

The use of

$$\hat{u}_z^{\rightarrow}(x, y, z, s) = \int_{t=0}^{\infty} \exp(-st)u_z^{\rightarrow}(x, y, z, t)\,\mathrm{d}t \quad \text{for } s \in \mathcal{R}, s > 0 \tag{5.5}$$

and

$$\hat{u}_z^{\rightarrow}(x, y, z, s) = \left(\frac{1}{2\pi}\right)^2 \int_{\alpha=-\infty}^{\infty} \exp(-\mathrm{i}s\alpha x)\,\mathrm{d}\alpha \int_{\beta=-\infty}^{\infty} \exp(-\mathrm{i}s\beta y)\tilde{u}_z^{\rightarrow}(\alpha, \beta, z, s)\,\mathrm{d}\beta \tag{5.6}$$

in Eq. (2.2) yields

$$c_P^2\partial_z^2\tilde{u}_z^{\rightarrow} - c_S^2 s^2(\alpha^2 + \beta^2)\tilde{u}_z^{\rightarrow} - s^2\tilde{u}_z^{\rightarrow} = 0. \tag{5.7}$$

The solution of Eq. (5.7) that is bounded as $z \to \infty$ is

$$\tilde{u}_z^{\rightarrow} = \tilde{A}^{\rightarrow} \exp(-s\gamma z) \tag{5.8}$$

in which γ is given by Eq. (3.6). Substitution of this expression in the boundary condition

$$\tilde{\tau}_{z,z}^{\rightarrow} = Mgv^{-1} \frac{s}{\mathrm{i}\alpha - v^{-1}} \tag{5.9}$$

that follows from Eq. (5.4) and the horizontal slowness representation $\tilde{\delta}(\beta, s) = s$ of $\delta(y)$, together with

$$\exp\left(-\frac{sx}{v}\right) H(x) = -\frac{1}{2\pi} \int_{\alpha=-\infty}^{\infty} \exp(-\mathrm{i}s\alpha x)\frac{1}{\mathrm{i}\alpha - v^{-1}}\,\mathrm{d}\alpha, \tag{5.10}$$

which relation can be derived by applying Jordan's lemma and the residue theorem to the integral on the right-hand side, yields

$$\tilde{A} = -\frac{Mg}{\lambda + 2\mu}\frac{1}{\gamma}\frac{v^{-1}}{\mathrm{i}\alpha - v^{-1}}. \tag{5.11}$$

As in Section 3, the first step towards arriving at the corresponding time-domain result consists of rewriting the integration with respect to β in Eq. (5.6) as an integration with respect to Ω via the change of variables expressed by Eq. (3.9) and carrying out the integration along the modified Cagniard path (3.12). Using Eq. (3.15) in Eq. (5.6), replacing, again as in Section 3, in the integration with respect to α the variable iα with p according to Eq. (3.16) and carrying out the integration with respect to p along the modified Cagniard path expressed by Eq. (3.18), the procedure now leads to (cf. Eq. (3.20))

$$\begin{aligned}&\frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\exp[-\mathrm{i}s\alpha x - sq(c_S^{-2}+\alpha^2)^{1/2}]\frac{v^{-1}}{\mathrm{i}\alpha - v^{-1}}\,\mathrm{d}\alpha\\ &\quad = \frac{1}{\pi}\int_{\tau=c_S^{-1}(x^2+q^2)^{1/2}}^{\infty}\exp(-s\tau)\mathrm{Im}\left[\frac{v^{-1}}{p_\parallel - v^{-1}}\frac{\partial p_\parallel}{\partial\tau}\right]\mathrm{d}\tau \quad \text{for } v < c_S.\end{aligned} \tag{5.12}$$

As indicated, this result unconditionally holds for the case $v < c_S$ where the simple pole $p = v^{-1}$ lies to the right of the branch point $p = c_S^{-1}$, since then the part of the complex p-plane in between the original path of integration (the imaginary p-axis) and the modified path the integrand is free from singularities and satisfies the conditions for the application of Jordan's lemma.

For the case $v > c_S$, the process of path deformation leads to a contribution from the pole $p = v^{-1}$ in those regions of space and for those values of q where the point of intersection of the modified Cagniard path (3.18) with the real p-axis lies to the right of $p = v^{-1}$, i.e. for $x(x^2+q^2)^{-1/2}c_S^{-1} > v^{-1}$. For this case, the procedure leads to

$$\begin{aligned}&\frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty}\exp[-\mathrm{i}s\alpha x - sq(c_S^{-2}+\alpha^2)^{1/2}]\frac{v^{-1}}{\mathrm{i}\alpha - v^{-1}}\,\mathrm{d}\alpha\\ &\quad = -\left\{0,\ \ \frac{1}{2},\ \ 1\right\}v^{-1}\exp\left[-\frac{sx}{v} - sq(c_S^{-2}-v^{-2})^{1/2}\right]\\ &\qquad + \frac{1}{\pi}\int_{\tau=c_S^{-1}(x^2+q^2)^{1/2}}^{\infty}\exp(-s\tau)\mathrm{Im}\left[\frac{v^{-1}}{p_\parallel - v^{-1}}\frac{\partial p_\parallel}{\partial\tau}\right]\mathrm{d}\tau \quad \text{for } v > c_S \text{ and } x(x^2+q^2)^{-1/2}c_S^{-1}\{\langle,=,\rangle\}v^{-1}.\end{aligned} \tag{5.13}$$

To construct the time-domain equivalent of $u_z^{\rightarrow}$ corresponding to the first term on the right-hand side of Eq. (5.13), the relevant expression for $\hat{u}_z^{\rightarrow}$ is, through the transformation

$$\frac{x}{v} + q(c_S^{-2} - v^{-2})^{1/2} = \tau \tag{5.14}$$

rewritten as

$$\begin{aligned}&-\frac{1}{\pi v}\int_{q=r_\perp}^{x(v^2/c_S^2-1)^{1/2}}\exp\left[-\frac{sx}{v} - sq(c_S^{-2}-v^{-2})^{1/2}\right]\frac{1}{(c_S/c_P)(q^2-r_\perp^2)^{1/2}}\,\mathrm{d}q\\ &\quad = -\frac{1}{\pi v}\int_{\tau=T_{CW}}^{T_{CW}^{+}}\exp(-s\tau)\frac{1}{(c_S/c_P)[(\tau - x/v)^2 - r_\perp^2(c_S^{-2}-v^{-2})]^{1/2}}\,\mathrm{d}\tau\end{aligned} \tag{5.15}$$

in which

$$T_{CW} = \frac{x}{v} + r_\perp(c_S^{-2} - v^{-2})^{1/2} \tag{5.16}$$

is the value of τ that corresponds to $q = r_\perp$ in Eq. (5.14) and

$$T^+_{\mathrm{CW}} = \frac{xv}{c_S^2} \tag{5.17}$$

is the value that corresponds to $q = x(v^2/c_S^2 - 1)^{1/2}$ in Eq. (5.14), which value is a consequence of the condition stated in Eq. (5.13). In the terms in the expression for $\hat{u}_z^{\rightarrow}$ resulting from the second term on the right-hand side of Eq. (5.13) and the one resulting from Eq. (5.12), the factor in the integrand multiplying $\exp(-s\tau)$ is evaluated as follows:

$$\mathrm{Im}\left[\frac{v^{-1}}{p_\parallel - v^{-1}}\frac{\partial p_\parallel}{\partial \tau}\right] = v^{-1}\mathrm{Im}\{\partial_\tau[\log(p_\parallel - v^{-1})]\} = v^{-1}\partial_\tau\{\mathrm{Im}[\log(p_\parallel - v^{-1})]\}$$
$$= v^{-1}\partial_\tau\left\{\arctan\left[\frac{\mathrm{Im}(p_\parallel - v^{-1})}{\mathrm{Re}(p_\parallel - v^{-1})}\right]\right\} = v^{-1}c_S\frac{(-\tau/v + x/c_S^2)}{P + q^2 Q}\frac{q}{[c_S^2\tau^2 - (x^2+q^2)]^{1/2}} \tag{5.18}$$

with

$$P = (\tau - x/v)^2, \tag{5.19}$$

$$Q = v^{-2} - c_S^{-2}. \tag{5.20}$$

For later reference, it is observed that, from the way Eq. (5.18) is constructed, it is clear that the denominator $P + q^2 Q$ in the right-hand side is positive for all q in the interval of interest $r_\perp < q < (c_S^2\tau^2 - x^2)^{-1/2}$. Next, the integrations with respect to q and τ are interchanged. This leads to

$$\int_{q=r_\perp}^{\infty} \mathrm{d}q \int_{\tau=c_S^{-1}(x^2+q^2)^{1/2}}^{\infty} \ldots \mathrm{d}\tau = \int_{\tau=R/c_S}^{\infty} \mathrm{d}\tau \int_{q=r_\perp}^{(c_S^2\tau^2-x^2)^{1/2}} \ldots \mathrm{d}q \tag{5.21}$$

in which R is given by Eq. (3.22). In the inner integration with respect to q the substitution (3.23) is carried out and the results (3.24)–(3.26) are used. This leads to

$$\int_{q=r_\perp}^{(c_S^2\tau^2-x^2)^{1/2}} \frac{1}{(q^2 - r_\perp^2)^{1/2}}\frac{q}{P + q^2 Q}\frac{1}{[c_S^2\tau^2 - (x^2+q^2)]^{1/2}}\,\mathrm{d}q$$
$$= \int_{\psi=0}^{\pi/2} \frac{1}{A\cos^2(\psi) + B\sin^2(\psi)}\,\mathrm{d}\psi = (AB)^{-1/2}\frac{\pi}{2} \tag{5.22}$$

with

$$A = P + r_\perp^2 Q, \tag{5.23}$$

$$B = P + (c_S^2\tau^2 - x^2)Q = c_S^2\left(\frac{\tau}{v} - \frac{x}{c_S^2}\right)^2, \tag{5.24}$$

where for the evaluation of the integration with respect to ψ we have used the results of Appendix A. Combining all results, it is found that

$$\hat{u}_z^{\rightarrow} = \frac{Mg}{\lambda + 2\mu}\frac{1}{\pi v}\frac{c_P}{c_S}\left\{0,\quad \frac{1}{2},\quad 1\right\}\int_{\tau=T_{\mathrm{CW}}}^{T^+_{\mathrm{CW}}} \exp(-s\tau)\frac{1}{[P + r_\perp^2 Q]^{1/2}}\,\mathrm{d}\tau$$
$$+\frac{Mg}{\lambda + 2\mu}\frac{c_P}{v}\frac{1}{2\pi}\int_{\tau=R/c_S}^{\infty}\exp(-s\tau)\frac{\tau/v - x/c_S^2}{\{(P + r_\perp^2 Q)[P + (c_S^2\tau^2 - x^2)Q]\}^{1/2}}\,\mathrm{d}\tau, \tag{5.25}$$

where the first term on the right-hand side is only present if $v > c_S$, while it has the spatial support $x > r_\perp(v^2/c_S^2 - 1)^{-1/2}$.

The case $v < c_{\mathrm{S}}$.

For the case $v < c_{\mathrm{S}}$, only the second term on the right-hand side of Eq. (5.25) is present and in it the relation $\tau/v - x/c_{\mathrm{S}}^2 > 0$ holds for all $\tau > R/c_{\mathrm{S}}$. Using Eq. (5.24), we end up with

$$\hat{u}_z^{\rightarrow} = \frac{Mg}{\lambda + 2\mu} \frac{c_{\mathrm{P}}}{c_{\mathrm{S}} v} \frac{1}{2\pi} \int_{\tau=R/c_{\mathrm{S}}}^{\infty} \exp(-s\tau) \frac{1}{(P + r_{\perp}^2 Q)^{1/2}} \, \mathrm{d}\tau. \tag{5.26}$$

In view of Lerch's uniqueness theorem of the one-sided Laplace transformation [21], the time-domain expression for $u_z^{\rightarrow}$ in this case follows as

$$u_z^{\rightarrow}(x, y, z, t) = \frac{Mg}{\lambda + 2\mu} \frac{c_{\mathrm{P}}}{c_{\mathrm{S}} v} \frac{1}{2\pi} \frac{1}{(P + r_{\perp}^2 Q)^{1/2}} H(t - R/c_{\mathrm{S}}). \tag{5.27}$$

The case $v > c_{\mathrm{S}}$.

For the case $v > c_{\mathrm{S}}$, both the first and the second term on the right-hand side of Eq. (5.25) are present, be it that the first term is restricted to its spatial support $x > r_{\perp}(v^2/c_{\mathrm{S}}^2 - 1)^{-1/2}$. In this region, we have $T_{\mathrm{CW}} < R/c_{\mathrm{S}} < T_{\mathrm{CW}}^{+}$ and the relations $\tau/v - x/c_{\mathrm{S}}^2 < 0$ for $\tau < T_{\mathrm{CW}}^{+}$ and $\tau/v - x/c_{\mathrm{S}}^2 > 0$ for $\tau > T_{\mathrm{CW}}^{+}$ hold, while outside the region we have $\tau/v - x/c_{\mathrm{S}}^2 > 0$ for $\tau > R/c_{\mathrm{S}}$. Subdividing the intervals of the τ integration, using Eq. (5.24) and combining the results, we arrive at

$$\begin{aligned} \hat{u}_z^{\rightarrow} = {} & \frac{Mg}{\lambda + 2\mu} \frac{1}{\pi v} \frac{c_{\mathrm{P}}}{c_{\mathrm{S}}} \left\{ 0, \quad \frac{1}{2}, \quad 1 \right\} \int_{\tau=T_{\mathrm{CW}}}^{R/c_{\mathrm{S}}} \exp(-s\tau) \frac{1}{[P + r_{\perp}^2 Q]^{1/2}} \, \mathrm{d}\tau \\ & + \frac{Mg}{\lambda + 2\mu} \frac{c_{\mathrm{P}}}{c_{\mathrm{S}}} \frac{1}{2\pi v} \int_{\tau=R/c_{\mathrm{S}}}^{\infty} \exp(-s\tau) \frac{1}{(P + r_{\perp}^2 Q)^{1/2}} \, \mathrm{d}\tau. \end{aligned} \tag{5.28}$$

Application of Lerch's theorem leads to the time-domain expression for $u_z^{\rightarrow}$ in this case

$$\begin{aligned} u_z^{\rightarrow}(x, y, z, t) = {} & \frac{Mg}{\lambda + 2\mu} \frac{1}{\pi v} \frac{c_{\mathrm{P}}}{c_{\mathrm{S}}} \left\{ 0, \quad \frac{1}{2}, \quad 1 \right\} \frac{1}{[P + r_{\perp}^2 Q]^{1/2}} [H(t - T_{\mathrm{CW}}) - H(t - R/c_{\mathrm{S}})] \\ & + \frac{Mg}{\lambda + 2\mu} \frac{c_{\mathrm{P}}}{c_{\mathrm{S}}} \frac{1}{2\pi v} \frac{1}{(P + r_{\perp}^2 Q)^{1/2}} H(t - R/c_{\mathrm{S}}). \end{aligned} \tag{5.29}$$

6. Discussion of the results

The expression for the total particle displacement is given by

$$u_z(x, y, x, t) = u_z^{\downarrow\infty}(x, y, z) - u_z^{\downarrow}(x, y, z, t) + u_z^{\rightarrow}(x, y, z, t). \tag{6.1}$$

In this expression, the first term $u_z^{\downarrow\infty}$ is the solution to the static surface point load problem for the equation of soil dynamics (elastodynamic wave equation in the vertical particle displacement approximation). The anisotropy in this equation manifests itself in the anisotropic distance function R as given by Eq. (3.22). This particle displacement is shown in Fig. 2.

The second term contains the dynamic response $u_z^{\downarrow}$ to the sudden application of a non-moving surface point load and, through the minus sign, represents the annihilation of the static point load displacement, propagating away from the point of application, arriving at the instant R/c_{S} and, hence propagating with the wave speed c_{S} in the horizontal direction and the wave speed c_{P} in the vertical direction. The third term $u_z^{\rightarrow}$ is generated by the application and the subsequent motion of the moving surface load. The expression for this contribution differs for the cases $v < c_{\mathrm{S}}$ and $v > c_{\mathrm{S}}$.

For a load speed $v < c_{\mathrm{S}}$, Fig. 3 shows a snapshot in space of the total particle displacement near the surface; Fig. 4 shows the total particle displacement at some point near the surface as a function of time. For this case, only

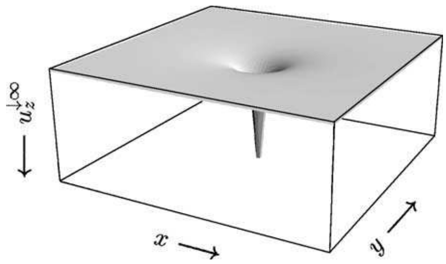

Fig. 2. Near-surface vertical particle displacement from surface point load at rest ($t < 0$).

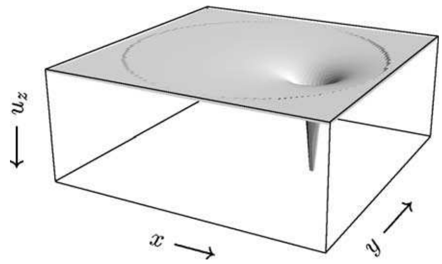

Fig. 3. Near-surface vertical particle displacement from surface point load set into motion ($v < c_S$, $t > 0$).

a finite jump at the wave front occurs that is composed of the annihilating contribution of $-u_z^{\downarrow}$ and the arriving contribution of $u_z^{\rightarrow}$. The relevant wave front arrives prior to the peak associated with the load being overhead.

For a load speed $v > c_S$, Fig. 5 shows a snapshot in space of the total particle displacement near the surface; Fig. 6 shows the total particle displacement at some point near the surface as a function of time. For this case,

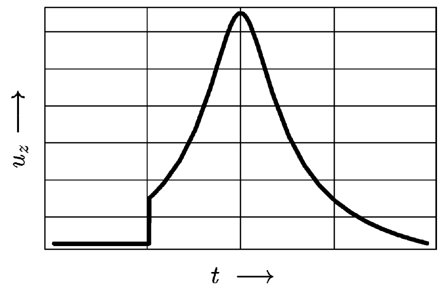

Fig. 4. Near-surface vertical particle displacement from surface point load set into motion ($v < c_S$).

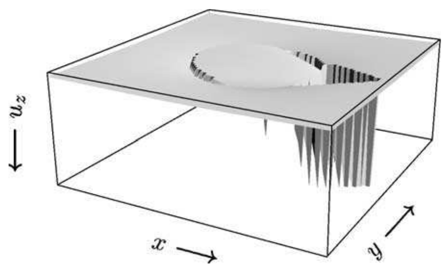

Fig. 5. Near-surface vertical particle displacement from surface point load set into motion ($v > c_S$, $t > 0$).

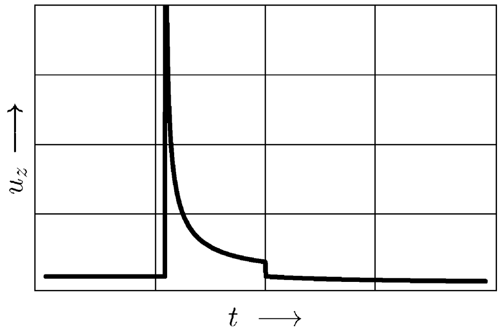

Fig. 6. Near-surface vertical particle displacement from surface point load set into motion ($v > c_S$).

outside the region of space where a conical wave propagates, only a finite jump discontinuity occurs at the wave front associated with the annihilating contribution of $-u_z^{\downarrow}$ and the arriving contribution of $u_z^{\rightarrow}$. In the region of space, however, where a conical wave propagates, this wave shows an inverse square root singularity at the arrival of its wave front. The relevant arrival time T_{CW} is given by Eq. (5.16).

In a practical situation, like the one of a high-speed train traveling on a poorly consolidated soil with low shear wave speed c_S, this singularity is a main concern of the safety in the operation along the relevant railroad track. The singularity in the particle displacement disappears as soon as losses in the elastic medium are taken into account. Some procedures for this are briefly discussed in Section 7.

7. The incorporation of loss mechanisms

The incorporation of dissipation or loss mechanisms in the model is accomplished by replacing the volume density of mass in the equation of motion by an inertia relaxation operator and the compliance in the deformation rate equation by a viscosity relaxation operator. For linear, time-invariant media the relaxation operators take the form of a Boltzmann type time convolution. The relaxation functions themselves have to satisfy certain conditions as to causality and passivity of the medium, but are otherwise arbitrary. Some parametrized form of them (for example, a rational or Padé approximation in the time differentiation operator) allows the relevant parameters to be adjusted to an experimentally observed behavior of the wave motion. Simple models in this category include the frictional-force mechanism in the inertia relaxation function and the Maxwell–Voigt viscosity mechanism in the compliance relaxation function [22].

A particular case occurs if the space–time dependence of the relaxation functions decomposes into the product of a function of the spatial coordinates and a function of time. This type of relaxation function we denote as a *global relaxation function*, since its time relaxation behavior is the same throughout the spatial configuration. For this type of relaxation, a *correspondence theorem* applies by which the quantities associated with the wave motion in the dissipative case can be constructed from their lossless counterparts through a relatively simple time operation [23,24].

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Appendix A. Evaluation of integrals (3.28) and (5.22)

In this appendix, we show how to evaluate integrals of the type

$$I(A,B)=\int_{\psi=0}^{\pi/2}\frac{1}{A\cos^2(\psi)+B\sin^2(\psi)}\,\mathrm{d}\psi \tag{A.1}$$

where $A>0$ and $B>0$. Through the substitution $\tan(\psi)=w$, the integral is carried over into

$$\int_{\psi=0}^{\pi/2}\frac{1}{A\cos^2(\psi)+B\sin^2(\psi)}\,\mathrm{d}\psi=\frac{1}{B}\int_{w=0}^{\infty}\frac{1}{A/B+w^2}\,\mathrm{d}w. \tag{A.2}$$

With

$$\int_{w=0}^{\infty}\frac{1}{A/B+w^2}\,\mathrm{d}w=\frac{(B/A)^{1/2}\pi}{2}, \tag{A.3}$$

the result

$$I(A,B)=\tfrac{1}{2}(AB)^{-1/2}\pi \tag{A.4}$$

follows.

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