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Fields and waves excited by impulsive point sources in motion—The general 3D time-domain Doppler effect

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Abstract

The pulse shapes of fields and waves excited by impulsive point sources in motion are analyzed through the application of a general Green's function procedure. Time-domain solutions to the scalar wave equation, the scalar dissipative wave equation (amongst which the relativistic heat conduction equation) and the scalar diffusion equation are discussed. The results express the general 3D time-domain Doppler effect on the field transfer from the generating source to some point elsewhere in the medium.

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1. Introduction

In several technical applications there is a need for an investigation into the detailed pulse shapes of fields and waves that are excited by impulsive sources in motion. Examples of such applications are: marine geophysical prospecting for oil and gas reservoirs with pulsed, towed, transmitters and receivers resting on the sea bottom [10], pulsed radiowave telemetry between space probes as used in the NASA-ESA Cassini–Huygens mission to the Saturn moon Titan [9] and relativistic heat conduction in the pulsed-laser cutting and welding manufacturing processes [1]. Due to the motion, the pulse shapes of the excited fields and waves differ from the ones that the sources are impressing. In the frequency-domain plane-wave theory this effect is known as the *Doppler effect*, while the motional change in the frequency of operation is known as the *Doppler shift*. In fact, the theory presented

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in this paper is a general 3D time-domain version of the Doppler effect pertaining to point-source excitation. A time-domain Green's function representation is used to express the generated field and wave values in space–time in terms of the signatures of the exciting point sources that are understood to move along a specified trajectory. As to the governing scalar space–time partial differential equations, three cases are discussed: the scalar (lossless) wave equation (Section 4), the scalar dissipative wave equation (that includes the relativistic heat conduction equation) (Section 5) and the scalar diffusion equation (Section 6).

The Green's function method has, be it in a somewhat different setting, been applied to the electromagnetic radiation by moving sources in [6], where in particular the relation with the theory of bicharacteristics in the theory of partial differential equations is emphasized. In [4], the determination of the electromagnetic fields generated by moving dipoles and multipoles was considered by an approach that runs via the 4D formulation of relativistic electrodynamics.

In electromagnetic theory, the case of point sources in *uniform rectilinear motion* in free space is most easily handled through the use of the Lorentz transformation in Einstein's special theory of relativity [11, Chapter 3; 8]. Our restriction to point sources circumvents the difficulty about the applicability of the relativistic Lorentz contraction of the possible spatial support of an extended source for non-uniform motions and for non-electromagnetic waves and fields. In relativistic heat conduction theory, the case of point sources in uniform rectilinear motion can adequately be handled by using the Galilei transformation of Galilean relativity [1]. For non-uniform motions, the Green's function procedure seems to be the only alternative.

2. Description of the configuration

The field and waves are present in an unbounded, homogeneous, isotropic medium with constant, positive constitutive coefficients. The meanings of these coefficients as they apply to the different branches of physics, will be given at a later stage. Position in the configuration is specified by the coordinates $\{x, y, z\}$ with respect to an orthogonal, Cartesian reference frame with the origin $\mathcal{O}$ and the three mutually perpendicular base vectors $\{\boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_z\}$, each of which is of unit length. In the indicated order, the base vectors form a right-handed system. Whenever appropriate, the position is also specified by the position vector $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z$. The time coordinate is t. Partial differentiation is denoted by ∂, supplied with the relevant subscript. To focus on the essentials of the constituents contributing to the changes in pulse shape, the fields and waves under consideration are characterized by the scalar wave function $u(\boldsymbol{r}, t) = u(x, y, z, t)$. Additional vector and tensor operations are needed in the cases of acoustic, electromagnetic and elastodynamic fields and waves [3]. The trajectory of any of the point sources follows from the specification of the vectorial position $\boldsymbol{r}_0(t)$ in the course of time. The relevant velocity of the source is $\boldsymbol{v}_0(t) = \partial_t\boldsymbol{r}_0(t)$. The pertaining source signature is denoted by $Q_0(t)$. Then, the associated volume source density is

$$q(\boldsymbol{r}, t) = Q_0(t)\delta[\boldsymbol{r} - \boldsymbol{r}_0(t)] \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{1}$$

where $\delta[\boldsymbol{r} - \boldsymbol{r}_0(t)]$ is the 3D Dirac distribution operative at $\boldsymbol{r} = \boldsymbol{r}_0(t)$.

3. The general source-type field or wave representation

Let the space–time partial differential equation to be satisfied by $u(\boldsymbol{r}, t)$ in general be given by

$$\mathsf{D}u = -q \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{2}$$

where $\mathsf{D} = \mathsf{D}(\boldsymbol{r}, t)$ is the pertaining space–time differential operator. Then, the corresponding Green's function $G(\boldsymbol{r}, \boldsymbol{r}', t)$ is the causal solution to the differential equation

$$\mathsf{D}G = -\delta(\boldsymbol{r} - \boldsymbol{r}', t) \quad \text{for } \boldsymbol{r}' \in \mathbb{R}^3,\ \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{3}$$

where $\delta(\boldsymbol{r}-\boldsymbol{r}', t)$ is the 4D Dirac distribution operative at $\{\boldsymbol{r}=\boldsymbol{r}', t=0\}$. Assuming the operator $\mathsf{D}(\boldsymbol{r}, t)$ to be shift-invariant in t (time-invariant background configuration), the pertaining source-type representation of $u(\boldsymbol{r}, t)$ is given by the time convolution

$$u(\boldsymbol{r}, t) = \int_{t'=0}^{\infty} \mathrm{d}t' \int_{\boldsymbol{x}' \in \mathrm{supp}(q)} G(\boldsymbol{r}, \boldsymbol{r}', t') q(\boldsymbol{r}', t-t') \, \mathrm{d}V(\boldsymbol{r}') \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{4}$$

where $\mathrm{supp}(q)$ is the spatial support of $q(\boldsymbol{r}, t)$. Substitution of Eq. (1) in Eq. (4) leads to

$$u(\boldsymbol{r}, t) = \int_{t'=0}^{\infty} G[\boldsymbol{r}, \boldsymbol{r}_0(t-t'), t'] Q_0(t-t') \, \mathrm{d}t' \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}. \tag{5}$$

Note that the right-hand side of this expression is no longer a time convolution. For the special case of the Dirac pulse source signature operative at $t = t_0$ and of magnitude Q_0, we have

$$Q_0(t) = Q_0 \delta(t - t_0), \tag{6}$$

in which case Eq. (5) reduces to

$$u(\boldsymbol{r}, t) = Q_0 G[\boldsymbol{r}, \boldsymbol{r}_0(t_0), t - t_0] \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}. \tag{7}$$

Several cases of D are discussed in subsequent sections.

4. The (lossless) scalar wave equation

For the lossless scalar wave equation, D is of shape

$$\mathsf{D} = \partial_x^2 + \partial_y^2 + \partial_z^2 - c^{-2}\partial_t^2, \tag{8}$$

and the corresponding differential equation is

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u - c^{-2}\partial_t^2 u = -q \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{9}$$

in which $c > 0$ is the wave speed (Fig. 1).

The corresponding Green's function $G_c(\boldsymbol{r}, \boldsymbol{r}', t)$ is the causal solution to

$$\partial_x^2 G_c + \partial_y^2 G_c + \partial_z^2 G_c - c^{-2}\partial_t^2 G_c = -\delta(\boldsymbol{r}-\boldsymbol{r}', t) \quad \text{for } \boldsymbol{r}' \in \mathbb{R}^3,\ \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}. \tag{10}$$

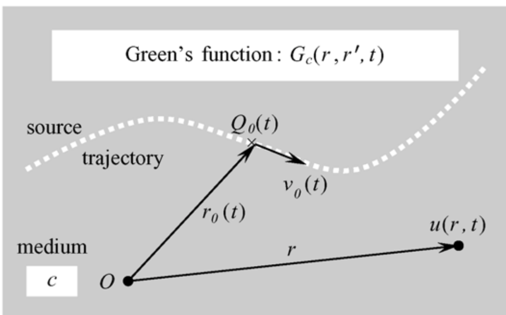

Fig. 1. Moving point source in lossless medium with wave speed c.

It is known to be [3, p. 94]

$$G_c(\boldsymbol{r}, \boldsymbol{r}', t) = \frac{\delta(t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \quad \text{for } \boldsymbol{r} \neq \boldsymbol{r}'. \tag{11}$$

Substitution of Eq. (11) in Eq. (5) leads to

$$u(\boldsymbol{r}, t) = \int_{t'=0}^{\infty} \frac{\delta[t' - |\boldsymbol{r} - \boldsymbol{r}_0(t - t')|/c]}{4\pi|\boldsymbol{r} - \boldsymbol{r}_0(t - t')|} Q_0(t - t')\,\mathrm{d}t'. \tag{12}$$

To exploit the properties of the Dirac delta distribution, we introduce in Eq. (12) as the variable of integration

$$\tau = t' - \frac{|\boldsymbol{r} - \boldsymbol{r}_0(t - t')|}{c}. \tag{13}$$

The corresponding Jacobian follows from

$$\frac{\partial \tau}{\partial t'} = 1 - \frac{\boldsymbol{v}_0(t - t') \cdot [\boldsymbol{r} - \boldsymbol{r}_0(t - t')]}{c|\boldsymbol{r} - \boldsymbol{r}_0(t - t')|}. \tag{14}$$

In view of the Cauchy–Schwarz inequality

$$|\boldsymbol{v}_0(t - t') \cdot [\boldsymbol{r} - \boldsymbol{r}_0(t - t')]| \leq |\boldsymbol{v}_0(t - t')||\boldsymbol{r} - \boldsymbol{r}_0(t - t')|, \tag{15}$$

it follows that

$$\frac{\partial \tau}{\partial t'} > 0 \quad \text{for } |\boldsymbol{v}_0| < c, \tag{16}$$

which condition we assume to hold in our further considerations. Let, under this condition, the (unique) solution to Eq. (13) be denoted as

$$t' = T(\boldsymbol{r}, t, \tau), \tag{17}$$

then Eq. (12) changes into

$$u(\boldsymbol{r}, t) = \int_{\tau=\tau_{\min}}^{\infty} \frac{\delta(\tau)}{4\pi(|\boldsymbol{r} - \boldsymbol{r}_0[t - T(\boldsymbol{r}, t, \tau)]| - \boldsymbol{v}_0[t - T(\boldsymbol{r}, t, \tau)] \cdot \{\boldsymbol{r} - \boldsymbol{r}_0[t - T(\boldsymbol{r}, t, \tau)]\}/c)} \times Q_0\,[t - T(\boldsymbol{r}, t, \tau)]\,\mathrm{d}\tau, \tag{18}$$

in which (cf. Eq. (13))

$$\tau_{\min} = -\frac{|\boldsymbol{r} - \boldsymbol{r}_0(t)|}{c} \leq 0. \tag{19}$$

Employing the sifting property of the Dirac delta distribution, the final expression is obtained as

$$u(\boldsymbol{r}, t) = \frac{Q_0[t - T(\boldsymbol{r}, t, 0)]}{4\pi(|\boldsymbol{r} - \boldsymbol{r}_0[t - T(\boldsymbol{r}, t, 0)]| - \boldsymbol{v}_0[t - T(\boldsymbol{r}, t, 0)] \cdot \{\boldsymbol{r} - \boldsymbol{r}_0[t - T(\boldsymbol{r}, t, 0)]\}/c)}, \tag{20}$$

in which $T(\boldsymbol{r}, t, 0)$ is the unique solution of

$$T(\boldsymbol{r}, t, 0) - \frac{|\boldsymbol{r} - \boldsymbol{r}_0[t - T(\boldsymbol{r}, t, 0)]|}{c} = 0. \tag{21}$$

For a time-independent source signature, this expression reduces to the well-known Liénard (1898) and Wiechert (1900) potentials of the electromagnetic field emitted by a moving electric point charge [11, p. 88; 5, pp. 181–187].

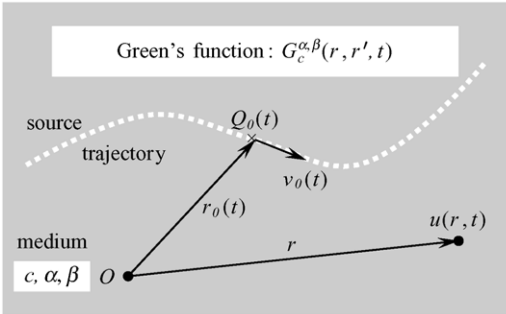

Fig. 2. Moving point source in dissipative medium with wave speed c and dissipation coefficients α and β.

5. The dissipative scalar wave equation and the relativistic heat conduction equation

For the dissipative scalar wave equation and the relativistic heat conduction equation, D is of shape

$$\mathsf{D} = \partial_x^2 + \partial_y^2 + \partial_z^2 - c^{-2}(\alpha + \partial_t)(\beta + \partial_t), \tag{22}$$

and the corresponding differential equation is

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u - c^{-2}(\alpha + \partial_t)(\beta + \partial_t)u = -q \quad \text{for } \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{23}$$

in which $c > 0$ is the wave speed and $\alpha \geq 0$ and $\beta \geq 0$ are dissipation coefficients (Fig. 2). (For $\alpha > 0$ and $\beta > 0$, α^{-1} and β^{-1} are the corresponding relaxation times [7, p. 104]).

The corresponding Green's function $G_c^{\alpha,\beta}(\boldsymbol{r}, \boldsymbol{r}', t)$ is the causal solution to

$$\partial_x^2 G_c^{\alpha,\beta} + \partial_y^2 G_c^{\alpha,\beta} + \partial_z^2 G_c^{\alpha,\beta} - c^{-2}(\alpha + \partial_t)(\beta + \partial_t)G_c^{\alpha,\beta} = -\delta(\boldsymbol{r} - \boldsymbol{r}', t) \quad \text{for } \boldsymbol{r}' \in \mathbb{R}^3,\ \boldsymbol{r} \in \mathbb{R}^3,\ t \in \mathbb{R}. \tag{24}$$

It is known to be [3, p. 101]

$$G_c^{\alpha,\beta}(\boldsymbol{r}, \boldsymbol{r}', t) = G^{\delta}(\boldsymbol{r}, \boldsymbol{r}', t) + G^{H}(\boldsymbol{r}, \boldsymbol{r}', t), \tag{25}$$

where

$$G^{\delta}(\boldsymbol{r}, \boldsymbol{r}', t) = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \exp\left[-\frac{(\alpha + \beta)t}{2}\right] \delta\left(t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) \quad \text{for } \boldsymbol{r} \neq \boldsymbol{r}', \tag{26}$$

and

$$G^{H}(\boldsymbol{r}, \boldsymbol{r}', t) = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \exp\left[-\frac{(\alpha + \beta)t}{2}\right] \frac{|\beta - \alpha||\boldsymbol{r} - \boldsymbol{r}'|/2c}{(t^2 - |\boldsymbol{r} - \boldsymbol{r}'|^2/c^2)^{1/2}} I_1\left[\frac{|\beta - \alpha|(t^2 - |\boldsymbol{r} - \boldsymbol{r}'|^2/c^2)^{1/2}}{2}\right] \times H\left(t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) \quad \text{for } \boldsymbol{r} \neq \boldsymbol{r}', \tag{27}$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind and order one and $H(\cdot)$ is the Heaviside unit step function. Upon substitution of Eqs. (25)–(27) in Eq. (5), introducing the variable of integration τ defined by Eq. (13) and using the sifting property of the Dirac delta distribution and the switching property of the Heaviside unit step function, the result can be written as

$$u(\boldsymbol{r}, t) = u^{\delta}(\boldsymbol{r}, t) + u^{H}(\boldsymbol{r}, t), \tag{28}$$

with

$$u^{\delta}(\boldsymbol{r},t)=\frac{\exp[-(\alpha+\beta)T(\boldsymbol{r},t,0)/2]Q_0[t-T(\boldsymbol{r},t,0)]}{4\pi(|\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,0)]|-\boldsymbol{v}_0[t-T(\boldsymbol{r},t,0)]\cdot\{\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,0)]\}/c)},\tag{29}$$

and

$$\begin{aligned}u^{H}(\boldsymbol{r},t)=&\int_{\tau=0}^{\infty}\frac{\exp[-(\alpha+\beta)T(\boldsymbol{r},t,\tau)/2]Q_0[t-T(\boldsymbol{r},t,\tau)]}{4\pi(|\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,\tau)]|-\boldsymbol{v}_0[t-T(\boldsymbol{r},t,\tau)]\cdot\{\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,\tau)]\}/c)}\\&\times\frac{|\beta-\alpha||\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,\tau)]|^2/2c}{(t^2-|\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,\tau)]|/c^2)^{1/2}}I_1[|\beta-\alpha|(t^2-|\boldsymbol{r}-\boldsymbol{r}_0[t-T(\boldsymbol{r},t,\tau)]|^2/c^2)^{1/2}/2]\,\mathrm{d}\tau.\end{aligned}\tag{30}$$

Note: For $\beta=0$, Eq. (23) reduces to the relativistic heat conduction equation [1]. The corresponding temperature distribution due to a moving, pulsating heat source follows by substituting $\beta=0$ in Eqs. (28)–(30).

6. The diffusion equation

For the diffusion equation, D is of shape

$$\mathsf{D}=\partial_x^2+\partial_y^2+\partial_z^2-D^{-1}\partial_t,\tag{31}$$

and the corresponding differential equation is

$$\partial_x^2u+\partial_y^2u+\partial_z^2u-D^{-1}\partial_tu=-q\quad\text{for }\boldsymbol{r}\in\mathbb{R}^3,\ t\in\mathbb{R},\tag{32}$$

in which $D>0$ is the diffusion coefficient (Fig. 3).

The corresponding Green's function $G_D(\boldsymbol{r},\boldsymbol{r}',t)$ is the causal solution to

$$\partial_x^2G_D+\partial_y^2G_D+\partial_z^2G_D-D^{-1}\partial_tG_D=-\delta(\boldsymbol{r}-\boldsymbol{r}',t)\quad\text{for }\boldsymbol{r}'\in\mathbb{R}^3,\ \boldsymbol{r}\in\mathbb{R}^3,\quad t\in\mathbb{R}.\tag{33}$$

It is known to be [2, pp. 90–91]

$$G_D(\boldsymbol{r},\boldsymbol{r}',t)\frac{1}{D^{1/2}(4\pi t)^{3/2}}\exp\left[-\frac{|\boldsymbol{r}-\boldsymbol{r}'|^2}{4Dt}\right]H(t).\tag{34}$$

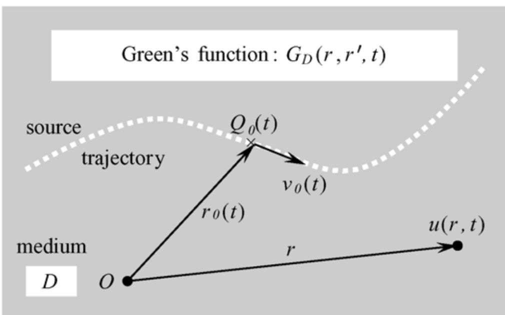

Fig. 3. Moving point source in diffusing medium with diffusion coefficient D.

Substitution of Eq. (34) in Eq. (5) leads to

$$u_D(\boldsymbol{r}, t) = \int_{t'=0}^{\infty} \frac{1}{D^{1/2}(4\pi t')^{3/2}} \exp\left[-\frac{|\boldsymbol{r} - \boldsymbol{r}_0(t - t')|^2}{4Dt'}\right] Q_0(t - t') \, \mathrm{d}t'. \tag{35}$$

7. Conclusion

A general Green's function procedure is presented for generating expressions for the fields and waves excited by impulsive, moving point sources. The results express the general 3D time-domain Doppler effect. The lossless scalar wave equation, the dissipative scalar wave equation, the relativistic heat conduction equation and the diffusion equation are considered. Various applications are indicated. For extended sources, the corresponding expressions can, within the validity of the Galilean transformation, be obtained by integrating the pertaining point-source solutions over the support of the extended source. Hereby, relativistic effects like the Lorentz contraction in the special theory of relativity [11, p. 16] are ignored. The latter can be argued to be a second-order effect in the ratio (source speed/wave speed), whereas the Doppler effect is a first-order effect in this ratio. This argument is, however, not a rigorous one since the pertaining Lorentz transformation only holds for electromagnetic fields in vacuum and sources in uniform, rectilinear motion, while for the other cases transformations of a similar nature have not been established.

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