COMPUTATIONAL MODELING OF ELECTROMAGNETIC WAVEFIELDS IN STRONGLY HETEROGENEOUS MEDIA - THE DOMAIN INTEGRATION APPROACH

Adrianus T. de Hoop

Delft University of Technology
Faculty of Information Technology & Systems
Laboratory of Electromagnetic Research
Mekelweg 4
2628 CD DELFT, the Netherlands

Synopsis:

- Introduction
- Strongly heterogeneous media
- Domain integrated field equations
- Domain integrated compatibility relations
- Interface boundary conditions
- Constitutive relations
- Local field representations
- The discretized field problem

 ©1999

EM field equations (Maxwell):

$$\nabla\times\boldsymbol{H}-\partial_t\boldsymbol{D} = \boldsymbol{J}$$
$$\nabla\times\boldsymbol{E}+\partial_t\boldsymbol{B} = -\boldsymbol{K}$$

Compatibility relations ($\nabla\cdot(\nabla\times\ ) = 0$):

$$\nabla\cdot\partial_t\boldsymbol{D} = -\nabla\cdot\boldsymbol{J}$$
$$\nabla\cdot\partial_t\boldsymbol{B} = -\nabla\cdot\boldsymbol{K}$$

Constitutive relations (linear, time-invariant, passive, locally reacting medium):

$$\boldsymbol{D}(\boldsymbol{x},t) = \boldsymbol{\epsilon}(\boldsymbol{x},t) \overset{(t)}{*} \boldsymbol{E}(\boldsymbol{x},t)$$
$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{\mu}(\boldsymbol{x},t) \overset{(t)}{*} \boldsymbol{H}(\boldsymbol{x},t)$$
$$\overset{(t)}{*} = \text{time convolution}$$

Causality conditions

EM field problem (differentiable fields & sources)

EM field quantities:

$$\boldsymbol{E} = \text{electric field strength}$$

$$\boldsymbol{H} = \text{magnetic field strength}$$

$$\boldsymbol{D} = \text{electric flux density}$$

$$\boldsymbol{B} = \text{magnetic flux density}$$

EM source quantities:

$$\boldsymbol{J} = \text{volume source density of electric current}$$

$$\boldsymbol{K} = \text{volume source density of magnetic current}$$

EM constitutive parameters:

$$\boldsymbol{\epsilon} = \text{permittivity relaxation function}$$

$$\boldsymbol{\mu} = \text{permeability relaxation function}$$

EM field quantities

 ©1999

Strongly heterogeneous medium:

- a **strongly heterogeneous** medium is **inhomogeneous up to the scale of the computationally discretized geometry**. Its constitutive properties may change abruptly from any cell of the spatial discretization to any of its neighbors.

Consequence:

- All **field quantities jump** across the boundaries of any cell and **none of them** is **differentiable** with respect to the spatial coordinates.

Field boundedness & continuity properties:

- Even in strongly heterogeneous media the **field** and **source quantities** remain **bounded**
- Although all field quantities jump across an interface, some field **components** remain **continuous**

Computational philosophy:

- Formulate the field problem in terms of **integral relations** (to avoid differentiations)
- The **field components to be computed** are the ones that are **continuous across an interface**

Computational methodology:

- Construct field & compatibility (integral) relations from which the continuity requirements across interfaces are manifest
- Construct local field expansions in terms of their continuous components
- Substitute the local expansions in the field & compatibility relations applied to each cell of the discretized geometry
- Enforce interface continuity requirements
- Invoke constitutive properties of each cell

Domain integrated field equations:

$$\int_{\partial\mathcal{D}} \boldsymbol{\nu}\times\boldsymbol{H}\,\mathrm{d}A - \int_{\mathcal{D}} \partial_t\boldsymbol{D}\,\mathrm{d}V = \int_{\mathcal{D}} \boldsymbol{J}\,\mathrm{d}V$$

$$\int_{\partial\mathcal{D}} \boldsymbol{\nu}\times\boldsymbol{E}\,\mathrm{d}A + \int_{\mathcal{D}} \partial_t\boldsymbol{B}\,\mathrm{d}V = -\int_{\mathcal{D}} \boldsymbol{K}\,\mathrm{d}V$$

for any bounded domain $\mathcal{D}$ with closed boundary surface $\partial\mathcal{D}$ ($\boldsymbol{\nu}$ = unit vector along the outward normal to $\partial\mathcal{D}$)

Domain integrated compatibility relations:

$$\int_{\mathcal{S}} \boldsymbol{\nu}\cdot\partial_t\boldsymbol{D}\,\mathrm{d}A = -\int_{\mathcal{S}} \boldsymbol{\nu}\cdot\boldsymbol{J}\,\mathrm{d}A$$

$$\int_{\mathcal{S}} \boldsymbol{\nu}\cdot\partial_t\boldsymbol{B}\,\mathrm{d}A = -\int_{\mathcal{S}} \boldsymbol{\nu}\cdot\boldsymbol{K}\,\mathrm{d}A$$

for any closed surface $\mathcal{S}$ ($\boldsymbol{\nu}$ = unit vector along the outward normal to $\mathcal{S}$)

Domain integrated field & compatibility equations

 ©1999

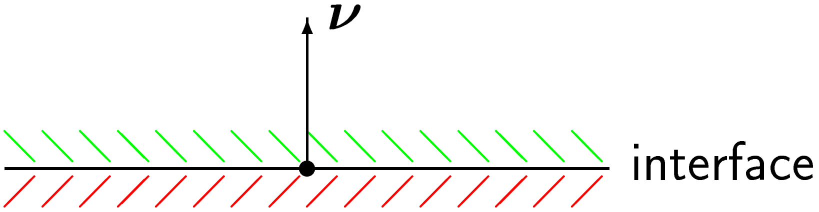
Interface boundary conditions:

- **Domain integrated field equations** $\Longrightarrow$

$$\boldsymbol{\nu} \times \boldsymbol{H} = \text{continuous}$$
$$\boldsymbol{\nu} \times \boldsymbol{E} = \text{continuous}$$

- **Domain integrated compatibility relations** $\Longrightarrow$

$$\boldsymbol{\nu} \cdot \boldsymbol{D} = \text{continuous}$$
$$\boldsymbol{\nu} \cdot \boldsymbol{B} = \text{continuous}$$



Interface boundary conditions

On a **simplicial mesh** (tetrahedra (3D)) **consistently linear expansions** exist which admit **at any vertex** the **field representation**:

$$\begin{aligned} \boldsymbol{F} &= \sum_{i=1}^{3} \alpha_i^F \boldsymbol{b}_i^F \\ \alpha_i^F &= \text{local expansion coefficient of } \boldsymbol{F} \\ \boldsymbol{b}_i^F &= \text{local base for } \boldsymbol{F} \end{aligned}$$

in which

$$\begin{aligned} \alpha_i^F = {} & \text{projection of } \boldsymbol{F} \text{ on the base vector} \\ & \boldsymbol{a}_i^F \textbf{ reciprocal} \text{ to } \boldsymbol{b}_i^F \end{aligned}$$

[**Reciprocal bases** $\Longrightarrow \boldsymbol{a_i} \cdot \boldsymbol{b}_j = \delta_{i,j}$]

[**Consistently linear** $\Longrightarrow$ linear along edges, across faces & throughout interior]

Local field expansions on simplicial mesh

For any **vertex** of a **simplex** (tetrahedron $\mathcal{T}$ in $\mathcal{R}^3$):

- {vectorial edges leaving a vertex}

  $\Longleftarrow$ reciprocal to $\Longrightarrow$

  {vectorial faces meeting at that vertex}

- **Edge expansions:**

  α_i^F = projection of $\boldsymbol{F}$ on **edges**

  $\Longrightarrow$ expansion base $\boldsymbol{b}_i^F$ = {faces}

- **Face expansions:**

  α_i^F = projection of $\boldsymbol{F}$ on **faces**

  $\Longrightarrow$ expansion base $\boldsymbol{b}_i^F$ = {edges}

Edge & face expansions

In view of the **continuity requirements** we take:

- **edge expansions** for $\boldsymbol{E}$ and $\boldsymbol{H}$
- **face expansions** for $\boldsymbol{D}$ and $\boldsymbol{B}$

and - **to model computationally the interface boundary conditions** -

- **impose equality** of the **expansion coefficients** pertaining to any **simplices** having **edges** or **faces in common**

On simplicial mesh $\{\cup_{m=1}^{M} \mathcal{T}_m\}$

- **substitute edge/face expansions in domain integrated field equations applied to each $\mathcal{T}_m \Longrightarrow$**
  - **equations relating $\{\alpha_i^H\}$ and $\{\alpha_i^D\}$ to $\boldsymbol{J}$**
  - **equations relating $\{\alpha_i^E\}$ and $\{\alpha_i^B\}$ to $\boldsymbol{K}$**
- **substitute face expansions in domain integrated compatibility relations applied to each $\mathcal{T}_m \Longrightarrow$**
  - **equations relating $\{\alpha_i^D\}$ to $\boldsymbol{J}$**
  - **equations relating $\{\alpha_i^B\}$ to $\boldsymbol{K}$**

On **simplicial mesh** $\{\cup_{m=1}^{M}\mathcal{T}_m\}$ and **time window** of interest $\mathcal{W}$:

- **minimize** $||\boldsymbol{D} - \boldsymbol{\Omega}_{\boldsymbol{\epsilon}}[\boldsymbol{E}]||_{\boldsymbol{D}}$, where $\boldsymbol{\Omega}_{\boldsymbol{\epsilon}} : \boldsymbol{E} \longrightarrow \boldsymbol{D}$ follows from $\boldsymbol{\Omega}_{\boldsymbol{\epsilon}}[\boldsymbol{E}] = \boldsymbol{\epsilon} \overset{(t)}{*} \boldsymbol{E}$ for $\boldsymbol{x} \in \mathcal{T}_m$ and $t \in \mathcal{W} \Longrightarrow$ **equations relating** $\{\alpha_i^E\}$ **to** $\{\alpha_i^D\}$
- **minimize** $||\boldsymbol{B} - \boldsymbol{\Omega}_{\boldsymbol{\mu}}[\boldsymbol{H}]||_{\boldsymbol{B}}$, where $\boldsymbol{\Omega}_{\boldsymbol{\mu}} : \boldsymbol{H} \longrightarrow \boldsymbol{B}$ follows from $\boldsymbol{\Omega}_{\boldsymbol{\mu}}[\boldsymbol{H}] = \boldsymbol{\mu} \overset{(t)}{*} \boldsymbol{H}$ for $\boldsymbol{x} \in \mathcal{T}_m$ and $t \in \mathcal{W} \Longrightarrow$ **equations relating** $\{\alpha_i^B\}$ **to** $\{\alpha_i^H\}$

Here, $||\cdot||_{\boldsymbol{D}}$ and $||\cdot||_{\boldsymbol{B}}$ are **any suitable norms** over the **spaces of** $\boldsymbol{D}$ and $\boldsymbol{B}$, respectively.

Note:

At several steps of the procedure, the edge/face field representations also had to be **continued into the interior of each** $\mathcal{T}_m$. The **consistently linear expansions** on a **simplicial mesh** provide these continuations. The use of these continuations implies that at the sub-mesh scale we refrain from computing the field in detail.

It is important to notice that the relevant continuations are still in terms of the **edge** and **face expansion coefficients** used to satisfy the continuity across interfaces (in machine precision).

Modeling of the radiation into the embedding:

Any computational modeling of EM wavefields in unbounded domains must include conditions that simulate the (causal) radiation into the **embedding**. For the latter, one standardly takes the one with a homogeneous, isotropic medium, for which the point-source solutions (Green's functions) are analytically known. Computationally, the field values at the boundary of the embedding are then related to the field values in the interior via the contrast sources that express the presence of the inhomogeneous domain of computation as the action of a radiator in the embedding.

The **system of equations** in $\{\alpha_i^E\}$, $\{\alpha_i^H\}$, $\{\alpha_i^D\}$, $\{\alpha_i^B\}$ is **overdetermined** of the type

$$\mathsf{L}\alpha \simeq q,$$

where L is an algebraic operator. The system is "solved" by

$$\text{minimizing } ||q - \mathsf{L}\alpha||,$$

where

$$||\cdot||$$

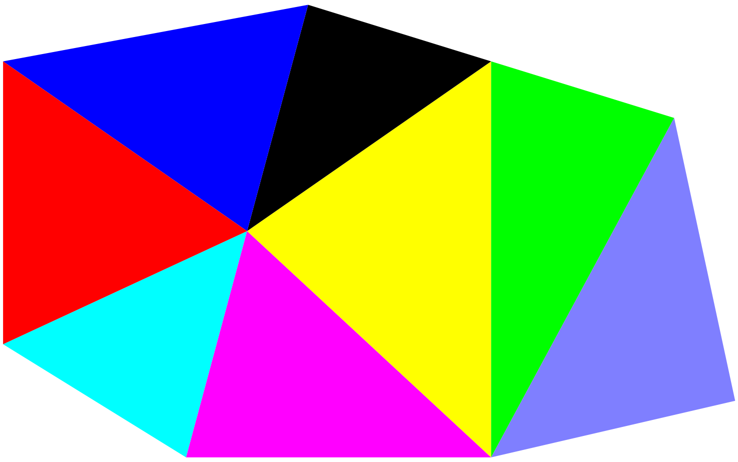
is an appropriate norm. The relevant standard procedures (of numerical linear algebra) are applicable.

The system of equations to be solved

The **domain integration method** with **edge** and **face expansions:**

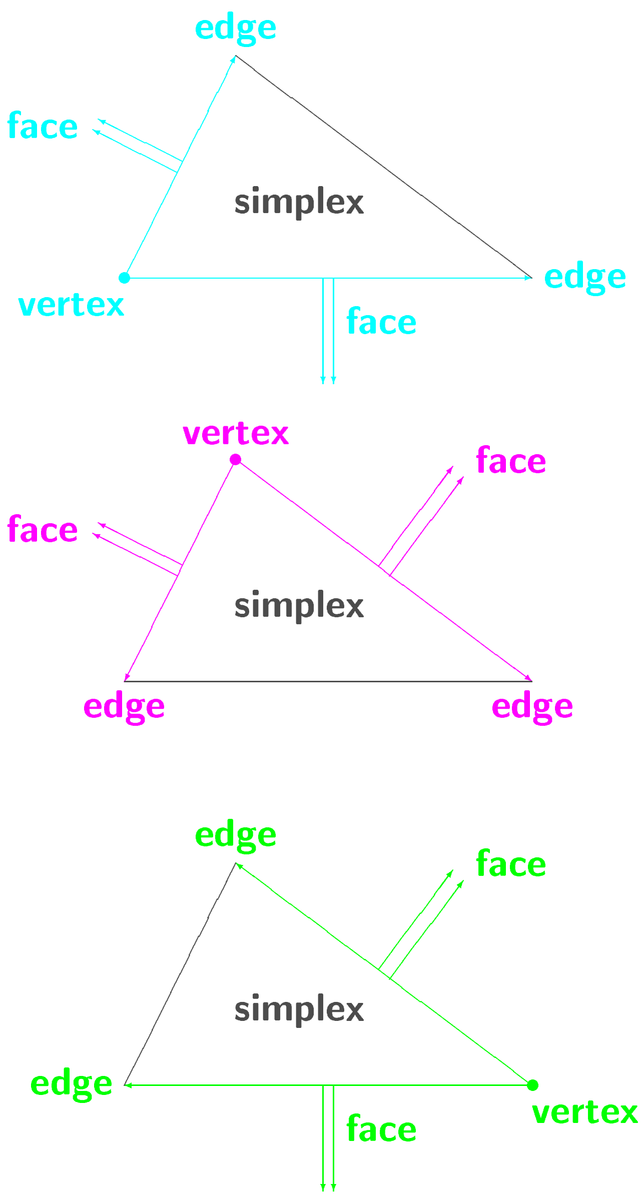
- **models the physics** of **strongly heterogeneous media**
- requires **all field quantities** to be **expanded**
- **enforces** the **continuity requirements across interfaces**
- satisfies **constitutive relations** by a **minimization procedure** (which - for a strongly heterogeneous medium - is again modeling the physics)
- can handle **large constitutive contrasts**

 ©1999



- **Strongly heterogeneous medium:** Constitutive properties jump across any interfaces between adjacent cells of the discretized geometry

Strongly heterogeneous medium



Expansion bases on simplicial mesh