

**Elastodynamics - Fundamentals of wave propagation,
scattering, imaging, inversion and computation**

by

Adrianus T. de Hoop

Delft University of Technology

Laboratory of Electromagnetic Research

Faculty of Electrical Engineering, Mathematics and Computer Science

Mekelweg 4 • 2628 CD Delft • the Netherlands

T: +31 15 2785203 / +31 15 2788613

E: a.t.dehoop@tudelft.nl **W:** www.atdehoop.com

Presentation given at:

Karlsruhe Institute of Technology, Germany, 2010 November 23

Synopsis:

- Preliminaries (tensors, subscript notation, summation convention)
- Elastodynamic wave field and source quantities
- Elastodynamic wave field equations
- Elastodynamic constitutive relations
- Elastic waves radiated by sources, elastodynamic Green's tensors
- Elastodynamic scattering theory (contrast-source formulation)
- The elastodynamic inverse source (imaging) problem
- The elastodynamic inverse scatterer constitution problem
- Elastodynamic wave field computation (space-time integrated field equations)

- **Observational spacetime:** $\mathbb{R}^3 \times \mathbb{R}$
- **Position:** $\mathbf{x} \in \mathbb{R}^3$
- **(Elapsed) time:** $t \in \mathbb{R}$

- **Tensor subscript notation x :** x_m ($m = 1, 2, 3$)
- **Spatial differentiation $\partial/\partial x_m$:** ∂_m ($m = 1, 2, 3$)
- **Temporal differentiation $\partial/\partial t$:** ∂_t (reserved symbol)
- **Kronecker unit tensor $\delta_{m,n}$:** $\delta_{m,n} = 1$ for $m = n$, $\delta_{m,n} = 0$ for $m \neq n$
- **Summation convention (repeated subscripts):** $\delta_{p,n} \delta_{n,q} = \sum_{n=1}^3 \delta_{p,n} \delta_{n,q} = \delta_{p,q}$
- **Examples:**
 - $\partial_m x_n = \delta_{m,n}$
 - $\delta_{n,n} = 3$

- **Elastodynamic unit tensors of rank four:**

- $\Delta_{i,j,p,q}^+$

- $\Delta_{i,j,p,q}^\delta$

- $\Delta_{i,j,p,q}^\Delta$

- **Symmetrical unit tensor:** $\Delta_{i,j,p,q}^+ = (\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p})/2$

- **Diagonal unit tensor:** $\Delta_{i,j,p,q}^\delta = \delta_{i,j}\delta_{p,q}/3$

- **Deviatoric unit tensor:** $\Delta_{i,j,p,q}^\Delta = \Delta_{i,j,p,q}^+ - \Delta_{i,j,p,q}^\delta$

- **Properties:**

- $\Delta_{i,j,m,n}^\delta \Delta_{m,n,p,q}^\delta = \Delta_{i,j,p,q}^\delta$

- $\Delta_{i,j,m,n}^\Delta \Delta_{m,n,p,q}^\Delta = \Delta_{i,j,p,q}^\Delta$

- $\Delta_{i,j,m,n}^\delta \Delta_{m,n,p,q}^\Delta = 0$

- $\Delta_{i,j,m,n}^\Delta \Delta_{m,n,p,q}^\delta = 0$

- **\implies MUTUALLY ORTHOGONAL UNIT TENSORS!**

[DeHoop, A. T., Abubakar, A., Habashy, T. M., JASA, 126 (2009) 1095–1100]

Elastodynamic field and source quantities

<ul style="list-style-type: none"> • $-\tau_{p,q}$ ($= -\tau_{q,p}$) : dynamic stress • v_r : particle velocity 	intensive^[1] field quantities
<ul style="list-style-type: none"> • $-e_{i,j}$ ($= -e_{j,i}$) : dynamic strain • Φ_k : mass flow density 	extensive^[2] field quantities
<ul style="list-style-type: none"> • f_k : volume density of force • $h_{i,j}$ ($= h_{j,i}$) : volume density of deformation rate 	source quantities

- ^[1] **Area density of power flow:**

$$\bullet S_m^{\text{elast}} = -\Delta_{m,p,q,r}^+ \tau_{p,q} v_r = -\tau_{m,q} v_q$$

- ^[2] **Volume density of wave momentum:**

$$\bullet G_n^{\text{elast}} = -\Delta_{n,i,j,k}^+ e_{i,j} \Phi_k = -e_{n,j} \Phi_j$$

- **WAVEFIELDS:**
 - Changes in space ($\partial_m = \partial/\partial x_m$) **counterbalance**
 - Changes in time ($\partial_t = \partial/\partial t$)

Elastodynamic wave equations

<ul style="list-style-type: none"> • $-\Delta_{k,m,p,q}^+ \partial_m \tau_{p,q} + \partial_t \Phi_k = f_k$ 	equation of motion
<ul style="list-style-type: none"> • $\Delta_{i,j,n,r}^+ \partial_n v_r - \partial_t e_{i,j} = h_{i,j}$ 	deformation rate equation

Constitutive relations (linear, time-invariant, causal, locally reacting):

- $\Phi_k(\mathbf{x}, t) = \rho_{k,r}(\mathbf{x}, t) \overset{(t)}{*} v_r(\mathbf{x}, t)$
 - $\rho_{k,r}(\mathbf{x}, t) = 0$ for $t < 0$ (causality)
- $\rho_{k,r}(\mathbf{x}, t) =$ **inertia relaxation function**
 - $\overset{(t)}{*} =$ **time convolution**
- $e_{i,j}(\mathbf{x}, t) = S_{i,j,p,q}(\mathbf{x}, t) \overset{(t)}{*} \tau_{p,q}(\mathbf{x}, t)$
 - $S_{i,j,p,q}(\mathbf{x}, t) = 0$ for $t < 0$ (causality)
- $S_{i,j,p,q}(\mathbf{x}, t) =$ **compliance relaxation function**

[De Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, 1995, 2008]

06

Elastodynamic wave equations, constitutive relations

MATERIAL PROPERTIES

- **Lossless** (instantaneously reacting)

$$\rho_{k,r}(\mathbf{x}, t) = \rho_{k,r}(\mathbf{x}) \delta(t)$$

$$S_{i,j,p,q}(\mathbf{x}, t) = S_{i,j,p,q}(\mathbf{x}) \delta(t)$$

- **Homogeneous** (spatially shift invariant)

$$\rho_{k,r}(\mathbf{x}, t) = \rho_{k,r}(t)$$

$$S_{i,j,p,q}(\mathbf{x}, t) = S_{i,j,p,q}(t)$$

- **Isotropic** (orientation invariant)

$$\rho_{k,r}(\mathbf{x}, t) = \rho(\mathbf{x}, t) \delta_{k,r}$$

$$S_{i,j,p,q}(\mathbf{x}, t) = S^\delta(\mathbf{x}, t) \Delta_{i,j,p,q}^\delta + S^\Delta(\mathbf{x}, t) \Delta_{i,j,p,q}^\Delta$$

ELASTIC TENSOR PARTITIONING

- **Dynamic stress** (dynamic strain: $\tau_{p,q} \implies e_{i,j}$)

- $\tau_{p,q} = [\tau_{p,q}]^+ = [\tau_{p,q}]^\delta + [\tau_{p,q}]^\Delta$

- $[\tau_{p,q}]^+ = \Delta_{p,q,i,j}^+ \tau_{i,j} = (\tau_{p,q} + \tau_{q,p})/2$

- $[\tau_{p,q}]^\delta = \Delta_{p,q,i,j}^\delta \tau_{i,j} = (\tau_{i,i}/3)\delta_{p,q}$

- $[\tau_{p,q}]^\Delta = \Delta_{p,q,i,j}^\Delta \tau_{i,j} = \Delta_{p,q,i,j}^+ \tau_{i,j} - \Delta_{p,q,i,j}^\delta \tau_{i,j} = (\tau_{p,q} + \tau_{q,p})/2 - (\tau_{i,i}/3)\delta_{p,q}$

- **Compliance**

- $$\begin{bmatrix} [e_{i,j}]^\delta \\ [e_{i,j}]^\Delta \end{bmatrix} (\mathbf{x}, t) = \begin{bmatrix} [S_{i,j,p,q}]^{\delta,\delta} & [S_{i,j,p,q}]^{\delta,\Delta} \\ [S_{i,j,p,q}]^{\Delta,\delta} & [S_{i,j,p,q}]^{\Delta,\Delta} \end{bmatrix} (\mathbf{x}, t) \stackrel{(t)}{*} \begin{bmatrix} [\tau_{p,q}]^\delta \\ [\tau_{p,q}]^\Delta \end{bmatrix} (\mathbf{x}, t)$$

[DeHoop, A. T., Abubakar, A., Habashy, T. M., JASA, 126 (2009) 1095–1100]

ELASTIC WAVE PROPAGATION

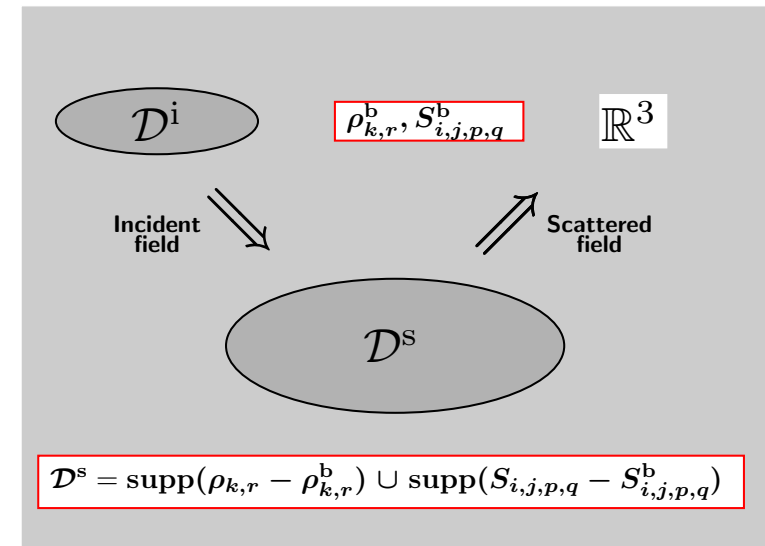
- General structure: source $\xrightarrow{\text{propagator}}$ receiver (intensive field)

- $[F](\mathbf{x}, t) = \int_{\mathbf{x}' \in \text{supp}(Q)} [G](\rho, S, \mathbf{x}, \mathbf{x}', t) \overset{(t)}{*} [Q](\mathbf{x}', t) dV(\mathbf{x}')$

Field	Green's tensor	Sources
<ul style="list-style-type: none"> • $[F] = \begin{bmatrix} -[\tau_{p,q}]^\delta \\ -[\tau_{p,q}]^\Delta \\ v_r \end{bmatrix}$ 	<ul style="list-style-type: none"> • $[G] = \begin{bmatrix} [G_{p,q,k}]^\delta & [G_{p,q,i,j}]^{\delta,\delta} & [G_{p,q,i,j}]^{\delta,\Delta} \\ [G_{p,q,k}]^\Delta & [G_{p,q,i,j}]^{\Delta,\delta} & [G_{p,q,i,j}]^{\Delta,\Delta} \\ [G_{r,k}] & [G_{r,i,j}]^\delta & [G_{r,i,j}]^\Delta \end{bmatrix}$ 	<ul style="list-style-type: none"> • $[Q] = \begin{bmatrix} f_k \\ [h_{i,j}]^\delta \\ [h_{i,j}]^\Delta \end{bmatrix}$

[DeHoop, A. T., Abubakar, A., Habashy, T. M., JASA, 126 (2009) 1095–1100]

• Total field		support
Wave field	$-\tau_{p,q}, v_r$	\mathbb{R}^3
Actual medium	$\rho_{k,r}, S_{i,j,p,q}$	\mathbb{R}^3
Sources	$f_k^i, h_{i,j}^i$	$\mathcal{D}^i \subset \mathbb{R}^3$
• Incident field		support
Wave field	$-\tau_{p,q}^i, v_r^i$	\mathbb{R}^3
Background medium	$\rho_{k,r}^b, S_{i,j,p,q}^b$	\mathbb{R}^3
Sources	$f_k^i, h_{i,j}^i$	$\mathcal{D}^i \subset \mathbb{R}^3$
• Scattered field		support
Wave field	$-\tau_{p,q}^s, v_r^s$	\mathbb{R}^3
Background medium	$\rho_{k,r}^b, S_{i,j,p,q}^b$	\mathbb{R}^3
Sources	$f_k^s, h_{i,j}^s$	$\mathcal{D}^s \subset \mathbb{R}^3$



**SCATTERED FIELD =
TOTAL FIELD –
INCIDENT FIELD**

Scattering configuration

$$\text{SCATTERED FIELD} = \text{TOTAL FIELD} - \text{INCIDENT FIELD}$$

- Total field**

$$\begin{aligned} -\Delta_{k,m,p,q}^+ \partial_m \tau_{p,q} + \rho_{k,r}^{(t)} * \partial_t v_r &= f_k^i \\ \Delta_{i,j,n,r}^+ \partial_n v_r - S_{i,j,p,q}^{(t)} * \partial_t \tau_{p,q} &= h_{i,j}^i \end{aligned}$$

- Incident field**

$$\begin{aligned} -\Delta_{k,m,p,q}^+ \partial_m \tau_{p,q}^i + \rho_{k,r}^b * \partial_t v_r^i &= f_k^i \\ \Delta_{i,j,n,r}^+ \partial_n v_r^i - S_{i,j,p,q}^b * \partial_t \tau_{p,q}^i &= h_{i,j}^i \end{aligned}$$

- Scattered field**

$$\begin{aligned} -\Delta_{k,m,p,q}^+ \partial_m \tau_{p,q}^s + \rho_{k,r}^b * \partial_t v_r^s &= f_k^s \\ \Delta_{i,j,n,r}^+ \partial_n v_r^s - S_{i,j,p,q}^b * \partial_t \tau_{p,q}^s &= h_{i,j}^s \end{aligned}$$

- Contrast sources**

$$\begin{aligned} \bullet f_k^s &= -(\rho_{k,r} - \rho_{k,r}^b) * \partial_t v_r \\ \bullet h_{i,j}^s &= (S_{i,j,p,q} - S_{i,j,p,q}^b) * \partial_t \tau_{p,q} \end{aligned}$$

Scattering problem (contrast source formulation)

FORWARD/DIRECT SCATTERING PROBLEM

• **Input:** medium: $\{\rho_{k,r}, S_{i,j,p,q}\}$, background: $\{\rho_{k,r}^b, S_{i,j,p,q}^b\}$,
 irradiating sources: $\{f_k^i, h_{i,j}^i\}$

• **Output:** incident field: $\{-\tau_{p,q}^i, v_r^i\}$, scattered field: $\{-\tau_{p,q}^s, v_r^s\}$

$$\bullet [F^{i,s}] (\mathbf{x}, t) = \int_{\mathbf{x}' \in \text{supp}(Q^{i,s})} [G] (\rho^b, S^b, \mathbf{x}, \mathbf{x}', t) \stackrel{(t)}{*} [Q^{i,s}] (\mathbf{x}', t) dV(\mathbf{x}')$$

$$\begin{bmatrix} f_k^s \\ h_{i,j}^s \end{bmatrix} = \begin{bmatrix} -(\rho_{k,r} - \rho_{k,r}^b) \stackrel{(t)}{*} \partial_t (v_r^i + v_r^s) \\ (S_{i,j,p,q} - S_{i,j,p,q}^b) \stackrel{(t)}{*} \partial_t (\tau_{p,q}^i + \tau_{p,q}^s) \end{bmatrix}$$

• $\{\mathbf{x}, t\} \in \{\text{supp}([Q^s]), \mathbb{R}\}$ \implies integral equation of the 2nd kind

in $[F](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\text{supp}([Q^s]), \mathbb{R}\}}$ or in $[Q^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\text{supp}([Q^s]), \mathbb{R}\}}$

INVERSE SOURCE (IMAGING) PROBLEM

- **Input:** background medium: $\{\rho_{k,r}^b, S_{i,j,p,q}^b\}$; irradiating sources: $\{f_k^i, h_{i,j}^i\}$;
 support(scatterer): \mathcal{D}^s ; observational domain: $\mathcal{D}^\Omega \not\subset \mathcal{D}^s$;
 incident wave field: $[F]^i(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathbb{R}^3, \mathbb{R}\}}$;
 scattered wave field: $[F]^s(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^\Omega, \mathbb{R}\}}$ (data)

- **Desired output:** scattering sources: $[Q^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}}$ (model)

MODEL/DATA EQUATION:

$$\bullet [F^s](\mathbf{x}, t) = \int_{\mathbf{x}' \in \text{supp}(Q^s)} [G](\rho^b, S^b, \mathbf{x}, \mathbf{x}', t) \overset{(t)}{*} [Q^s](\mathbf{x}', t) dV(\mathbf{x}')$$

NOTE: $\bullet [F^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^\Omega, \mathbb{R}\}} \xrightarrow{\text{1-to-1}} [Q^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}}$

$$[Q^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}} \simeq [Q^s]^{[\infty]}(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}} \text{ (minimum norm solution)}$$

INVERSE SCATTERER CONSTITUTION PROBLEM

- **Input:** background medium: $\{\rho_{k,r}^b, S_{i,j,p,q}^b\}$; irradiating sources: $\{f_k^i, h_{i,j}^i\}$;
 support(scatterer): \mathcal{D}^s ; observational domain: $\mathcal{D}^\Omega \not\subset \mathcal{D}^s$;
 incident wave field: $[F]^i(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathbb{R}^3, \mathbb{R}\}}$;
 scattered wave field: $[F]^s(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^\Omega, \mathbb{R}\}}$ (data)

- **Desired output:** scattering sources: $[Q^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}}$ (model) \implies

$$[X](\mathbf{x}, t) = \{\rho_{k,r} - \rho_{k,r}^b, S_{i,j,p,q} - S_{i,j,p,q}^b\}(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}} \text{ (object contrast)}$$

$$\bullet [F^s](\mathbf{x}, t) = \int_{\mathbf{x}' \in \text{supp}(Q^s)} [G](\rho^b, S^b, \mathbf{x}, \mathbf{x}', t) \overset{(t)}{*} [Q^s](\mathbf{x}', t) dV(\mathbf{x}') \xrightarrow{\mathbf{x} \in \mathcal{D}^\Omega}$$

$$\bullet \text{MODEL/DATA EQUATION: } \implies [Q^s]^{[\infty]}(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}}$$

$$(\text{= minimum norm solution}) \implies [F^s](\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathbb{R}^3, \mathbb{R}\}}$$

OBJECT EQUATION

• Object equation:

- $[Q^s]^{[\infty]}(\mathbf{x}, t) = [X](\mathbf{x}, t) \stackrel{(t)}{*} [[F^i](\mathbf{x}, t) + [F^s](\mathbf{x}, t)]$ for $\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}$
- $[X](\mathbf{x}, t) = \{\rho_{k,r} - \rho_{k,r}^b, S_{i,j,p,q} - S_{i,j,p,q}^b\}(\mathbf{x}, t)|_{\{\mathbf{x}, t\} \in \{\mathcal{D}^s, \mathbb{R}\}}$ (object contrast)

• Object equation (properties):

- pointwise equation highly unstable
- reconstructed contrast parameters may be outside physical range

• 'Remedies' (?):

- replace object equation with (some?) minimum norm solution
- put bounds on parameter values (e.g., via

$$\chi \longrightarrow \bar{\chi} = B \log \left[\frac{\chi_{\max} - \chi}{\chi - \chi_{\min}} \right] \text{ for } \chi_{\min} < \chi < \chi_{\max}$$

and use $\bar{\chi}^{[\infty]}$ to evaluate $\chi^{[\infty]}$)

WAVEFIELD MODELING (ANALYTICAL TOOLS)

Wavefield modeling (analytical tools)

16

TIME LAPLACE TRANSFORMATION (CAUSAL FUNCTIONS)

• Causality condition:

- $\text{supp}[u(\mathbf{x}, t)] = \mathcal{T} = \{t \in \mathbb{R}, t_0 < t < \infty\}$

• Time Laplace transformation:

- $\hat{u}(\mathbf{x}, s) = \int_{t \in \mathcal{T}} \exp(-st) u(\mathbf{x}, t) dt$ for $s \in \mathcal{L}$

- $\mathcal{L} = \{s \in \mathbb{R}, s = s_0 + nh, s_0 > 0, h > 0, n = 0, 1, 2, \dots\}$ (Lerch sequence)

• Properties:

- $\hat{u}(\mathbf{x}, s)|_{s \in \mathcal{L}} \stackrel{1-t_0-1}{\iff} u(\mathbf{x}, t)|_{t \in \mathcal{T}}$

- $s \hat{u}(\mathbf{x}, s) \iff \partial_t u(\mathbf{x}, t)$

- $\hat{u}(\mathbf{x}, s) \hat{v}(\mathbf{x}, s) \iff u(\mathbf{x}, t) \stackrel{(t)}{*} v(\mathbf{x}, t)$ [$\stackrel{(t)}{*}$ = time convolution]

• Application: analysis and synthesis of linear, time-invariant, causal systems

THE POWER EXPONENTIAL PULSE

• Pulse shape:

- $f(t) = A \left(\frac{t}{t_r}\right)^\nu \exp\left[-\nu \left(\frac{t}{t_r} - 1\right)\right] H(t)$

• Parameters:

- pulse amplitude $A = f(t_r)$

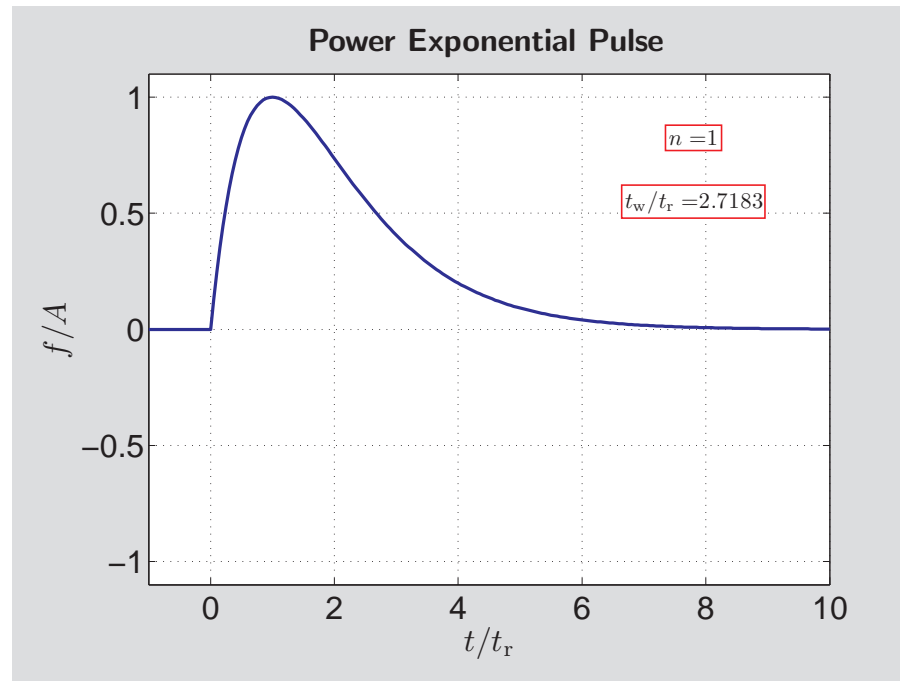
- pulse rise time t_r $[\partial_t f(t_r) = 0]$

- pulse time width $t_w = t_r \frac{\Gamma(\nu + 1)}{\nu^{\nu+1}} \exp(\nu)$ $\left[A t_w = \int_{t=0}^{\infty} f(t) dt \right]$

• Time Laplace transform:

- $\hat{f}(s) = A \frac{\Gamma(\nu + 1)}{(s + \nu/t_r)^{\nu+1}} \frac{\exp(\nu)}{t_r^\nu}$ $[\Gamma(\cdot) = \text{Euler gamma function}]$

POWER EXPONENTIAL PULSES:



The power exponential pulse

SPATIAL WAVENUMBER REPRESENTATION

- Spatial wavenumber representation:**

- $$\hat{u}(\mathbf{x}, s) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbf{k} \in \mathbb{R}^3} \exp(-ik_m x_m) \tilde{u}(\mathbf{k}, s) d\mathbf{k} \text{ for } \mathbf{x} \in \mathbb{R}^3$$

- Properties:**

- $$\tilde{u}(\mathbf{k}, s) |_{\mathbf{k} \in \mathbb{R}^3} \xleftrightarrow{1\text{-to-}1} \hat{u}(\mathbf{x}, s) |_{\mathbf{x} \in \mathbb{R}^3}$$

- $$-ik_m \tilde{u}(\mathbf{k}, s) \iff \partial_m \hat{u}(\mathbf{x}, s)$$

- $$\tilde{u}(\mathbf{k}, s) \tilde{v}(\mathbf{k}, s) \iff \hat{u}(\mathbf{x}, s) \overset{(\mathbf{x})}{*} \hat{v}(\mathbf{x}, s) \quad [* = \text{spatial convolution}]$$

- Application:**

analysis of wave motion in linear, time-invariant, shift-invariant configurations

- Dirac delta distribution:**

$$\delta(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbf{k} \in \mathbb{R}^3} \exp(-ik_m x_m) d\mathbf{k}$$

SPATIAL WAVE SLOWNESS REPRESENTATION

• Spatial wave slowness representation:

- $\hat{u}(\mathbf{x}, s) = \left(\frac{s}{2\pi}\right)^3 \int_{\alpha \in \mathbb{R}^3} \exp(-is\alpha_m x_m) \tilde{u}(\boldsymbol{\alpha}, s) d\boldsymbol{\alpha}$ for $\boldsymbol{\alpha} \in \mathbb{R}^3$, $s \in \mathbb{R}$, $s > 0$

• Properties:

- $\tilde{u}(\boldsymbol{\alpha}, s) |_{\boldsymbol{\alpha} \in \mathbb{R}^3} \stackrel{1\text{-to-1}}{\iff} \hat{u}(\mathbf{x}, s) |_{\mathbf{x} \in \mathbb{R}^3}$

- $-is\alpha_m \tilde{u}(\boldsymbol{\alpha}, s) \iff \partial_m \hat{u}(\mathbf{x}, s)$

- $\tilde{u}(\boldsymbol{\alpha}, s) \tilde{v}(\boldsymbol{\alpha}, s) \iff \hat{u}(\mathbf{x}, s) \overset{(\mathbf{x})}{*} \hat{v}(\mathbf{x}, s)$ [$\overset{(\mathbf{x})}{*}$ = spatial convolution]

• Application:

Cagniard-like methods (e.g., waves in layered media)

- **Dirac delta distribution:** • $\delta(\mathbf{x}) = \left(\frac{s}{2\pi}\right)^3 \int_{\alpha \in \mathbb{R}^3} \exp(-is\alpha_m x_m) d\boldsymbol{\alpha}$

GREEN'S FUNCTION (DISSIPATIVE SCALAR WAVE EQUATION)

- Dissipative scalar wave equation:**

- $$\partial_m \partial_m G - \frac{1}{c^2} (\alpha + \partial_t) (\beta + \partial_t) G = -\delta(\mathbf{x}, t)$$

- Spatial wavenumber representation:**

- $$\hat{G}(\mathbf{x}, s) = \left(\frac{1}{2\pi} \right)^3 \int_{\mathbf{k} \in \mathbb{R}^3} \exp(-i k_m x_m) \tilde{G}(\mathbf{k}, s) d\mathbf{k} \text{ with}$$

- $$\tilde{G}(\mathbf{k}, s) = \frac{1}{k_m k_m + (s + \alpha)(s + \beta)/c^2}$$

- Space time Green's function:**

- $$G(\mathbf{x}, s) = \frac{U_1(\alpha, \beta, |\mathbf{x}|/c, t)}{4\pi|\mathbf{x}|} \quad \bullet \quad U_1(\alpha, \beta, T, t) = \exp[-(\alpha + \beta)t/2] \times$$

$$\left\{ \delta(t - T) + \frac{(|\beta - \alpha|/2)}{(t^2 - T^2)^{1/2}} I_1 \left[(|\beta - \alpha|/2)(t^2 - T^2)^{1/2} \right] \right\} H(t - T)$$

[De Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, 1995, 2008]

PARTICLE VELOCITY WAVE EQUATION

- **For homogeneous, isotropic, perfectly elastic solid:**

- $-\partial_m([\tau_{m,k}]^\delta + [\tau_{m,k}]^\Delta) + \rho \partial_t v_k = f_k$
- $[\partial_i v_j]^\delta - S^\delta \partial_t [\tau_{i,j}]^\delta = [h_{i,j}]^\delta$
- $[\partial_i v_j]^\Delta - S^\Delta \partial_t [\tau_{i,j}]^\Delta = [h_{i,j}]^\Delta$

- **Particle velocity wave equation:**

- $(c_P^2 - c_S^2) \partial_k \partial_m v_m + c_S^2 \partial_m \partial_m v_k - \partial_t^2 v_k =$
 $\rho^{-1} \partial_t f_k - (\rho S^\delta)^{-1} \partial_m [h_{m,k}]^\delta - (\rho S^\Delta)^{-1} \partial_m [h_{m,k}]^\Delta$

- ***P*-wave speed:**

$$c_P = \left(\frac{2S^\delta + S^\Delta}{3\rho S^\delta S^\Delta} \right)^{1/2}$$

- ***S*-wave speed:**

$$c_S = \left(\frac{1}{2\rho S^\Delta} \right)^{1/2}$$

For further details, see

[De Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, 1995, 2008]

WAVEFIELD MODELING (COMPUTATIONAL TOOLS)

THE SPACE-TIME INTEGRATED FIELD EQUATIONS

- **General form:**

- $$\int_{t \in \mathcal{T}} dt \int_{\mathbf{x} \in \mathcal{D}} [\mathbf{LHS}] dV(\mathbf{x}) = \int_{t \in \mathcal{T}} dt \int_{\mathbf{x} \in \mathcal{D}} [\mathbf{RHS}] dV(\mathbf{x})$$

- **Ingredients (domain of computation):**

- $$\int_{\mathbf{x} \in \mathcal{D}} \partial_m [\dots] dV(\mathbf{x}) = \int_{\mathbf{x} \in \partial \mathcal{D}} \nu_m [\dots] dV(\mathbf{x}) \quad (\text{3D Gauss})$$

- $$\int_{t \in \mathcal{T}} \partial_t [\dots] dt = [\dots] \Big|_{t \in \partial \mathcal{T}} \quad (\text{'1D Gauss'})$$

- **implementation on simplicial grid \implies**

- $$\int_{\mathbf{x} \in \mathcal{D}} [\dots] dV(\mathbf{x}) \simeq [\dots]_{\text{av}} \int_{\mathbf{x} \in \mathcal{D}} dV(\mathbf{x}) \quad (\text{Linear interpolation over } \mathcal{D})$$

- $$\int_{t \in \mathcal{T}} [\dots] dt \simeq [\dots]_{\text{av}} \int_{t \in \mathcal{T}} dt \quad (\text{Linear interpolation over } \mathcal{T})$$

- **constitutive coefficients constant over $\mathcal{D} \times \mathcal{T}$**

THE SPACE-TIME INTEGRATED FIELD EQUATIONS

• Properties:

- Discretized system of equations contains only quantities that are **continuous** across interfaces of spatial (jump) discontinuities in material
- For differentiable fields the procedure **converges to exact solution** as $\text{maxdiam}(\mathcal{D}) \downarrow 0$ and $\text{maxdiam}(\mathcal{T}) \downarrow 0$

• Perfectly matched embedding:

- $\partial_{x_n} \longmapsto \partial_{\hat{X}_n} = \frac{1}{\hat{\chi}_n(x_n, s)} \partial_{x_n} \implies$
 - $\hat{X}_n(x_n, s) = \int_{\xi_n=a_n}^{x_n} \hat{\chi}_n(\xi_n, s) d\xi_n \quad (n = 1, 2, 3)$
 - $\hat{\chi}_n(x_n, s) = \mathbf{1}$ for $n = 1, 2, 3; \mathbf{x} \in \mathcal{D}$
- Time-dependent, causal

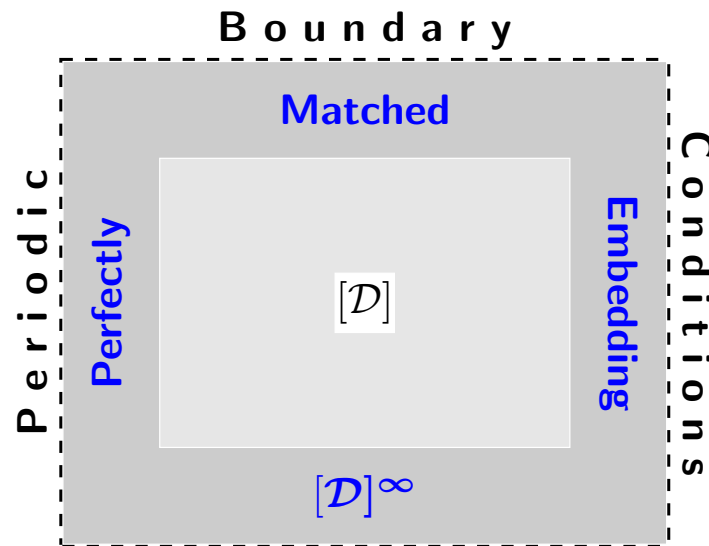
Cartesian coordinate stretching

[De Hoop, Remis, Van den Berg, Journal of Computational Physics, 221 (2007) pp. 88–105]

THE PERFECTLY MATCHED EMBEDDING

• Periodic boundary conditions:

• $\text{FIELD}(x_m + L_m) = \text{FIELD}(x_m) \quad (m = 1, 2, 3)$



[De Hoop, Remis, Van den Berg, Journal of Computational Physics, 221 (2007) pp. 88–105]

THANK YOU !

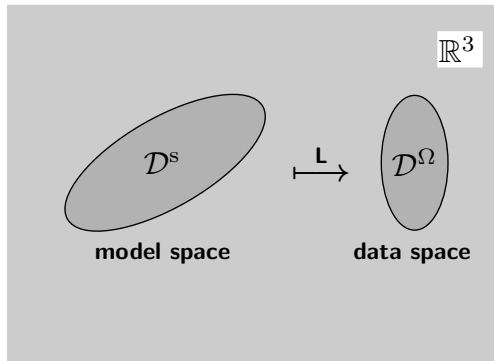
Thank you

APPENDIX

SPACE-TIME MODEL/DATA EQUATION

• Configuration:

- | | |
|--|--|
| <ul style="list-style-type: none"> • $\mathcal{D}^s = \text{supp}(\text{model}) \subset \mathbb{R}^3$ • $\mathcal{D}^\Omega = \text{supp}(\text{data}) \subset \mathbb{R}^3$ | <ul style="list-style-type: none"> • $\mathcal{D}^\Omega \cap \mathcal{D}^s = \emptyset$ ('remote sensing') |
| <ul style="list-style-type: none"> • $m(\mathbf{x}, t) \in \mathcal{D}^s \times \mathbb{R}$ (model) • $d(\mathbf{x}, t) \in \mathcal{D}^\Omega \times \mathbb{R}$ (data) | <ul style="list-style-type: none"> • $\{\mathcal{D}^s \times \mathbb{R}\} \xrightarrow{\mathbf{L}} \{\mathcal{D}^\Omega \times \mathbb{R}\}$ (linear operator) |



• Model/data equation:

$$\bullet d = \mathbf{L}m$$

$$d \xrightarrow{?} m$$

'SOLVING' MODEL/DATA EQUATION

Model/data equation: • $d = \mathbf{L}m$ for $\{x, t\} \in \{\mathcal{D}^\Omega, \mathbb{R}\} \implies d \xrightarrow{?} m$

Iterative 'solution' (computational procedure):

- $m \in \{m^{[0]}, m^{[1]}, m^{[2]}, \dots\}; m^{[0]} = \mathbf{0}$ (sequence of models)
- $r^{[i]} = d - \mathbf{L}m^{[i]}$ for $i = 0, 1, 2, \dots; r^{[0]} = d$ (r = residual)
- $\langle \cdot, \cdot \rangle$ symmetric inner product in (functional) Hilbert space

Mismatch error: • $ERR^{[i]} = \langle r^{[i]}, r^{[i]} \rangle = \|r^{[i]}\|^2$ ($\|\cdot\| = \text{norm} \geq 0$)

Improvement condition: $ERR^{[i+1]} < ERR^{[i]}$ for $i = 0, 1, 2, \dots$

Model update: • $m^{[i+1]} = m^{[i]} + \delta m^{[i]}$ for $i = 0, 1, 2, \dots$

\implies improved model

‘SOLVING’ MODEL/DATA EQUATION

- $$ERR^{[i+1]} = ERR^{[i]} - 2\langle \mathbf{L}\delta\mathbf{m}^{[i]}, \mathbf{r}^{[i]} \rangle + \|\mathbf{L}\delta\mathbf{m}^{[i]}\|^2$$

$$= ERR^{[i]} - 2\langle \delta\mathbf{m}^{[i]}, \mathbf{L}^*\mathbf{r}^{[i]} \rangle + \|\mathbf{L}\delta\mathbf{m}^{[i]}\|^2 \quad (\mathbf{L}^* = \text{adjoint to } \mathbf{L})$$

Steepest descent: • $\delta\mathbf{m}^{[i]} = \alpha^{[i]}\mathbf{L}^*\mathbf{r}^{[i]}$

- $$\alpha^{[i]} = \frac{\|\mathbf{L}^*\mathbf{r}^{[i]}\|^2}{\|\mathbf{L}\mathbf{L}^*\mathbf{r}^{[i]}\|^2} \implies \bullet \quad ERR^{[i+1]} = ERR^{[i]} - \frac{\|\mathbf{L}^*\mathbf{r}^{[i]}\|^4}{\|\mathbf{L}\mathbf{L}^*\mathbf{r}^{[i]}\|^2} < ERR^{[i]} \implies$$

- $$\{m^{[0]}, m^{[1]}, m^{[2]}, \dots\} \xrightarrow{\text{converges}} m^{[\infty]}$$

- $\mathbf{L}^*\mathbf{L} m^{[\infty]} = \mathbf{L}^*d$

BUT

- $\mathbf{L} m^{[\infty]} \stackrel{?}{=} d$

NO!

De Hoop, M. V., De Hoop, A. T., Proceedings of the Royal Society of London, Series A, 456 (2000) pp. 641–682.